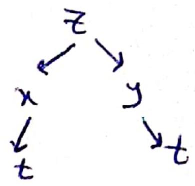
قاعدهٔ زنجیره‌ای مشتق :

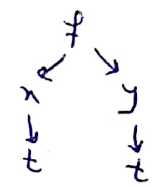
فرض کنید $z=f(x,y)$ تابعی دو متغیره برحسب x و y باشد. به طوری که $x=g(t)$ و $y=h(t)$.

در این صورت

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$



مثال : فرض کنید $f(x,y)=e^{x^2y}$ و $x=\sin^2 t$ و $y=e^{5t}$. مطلوبست $\frac{df}{dt}$.



$$\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$=\left(2xy\,e^{x^2y}\right)\left(2\sin t\cos t\right)+\left(x^2e^{x^2y}\right)\left(5e^{5t}\right)$$

$$\overset{x=\sin^2 t}{\underset{y=e^{5t}}{=}}\ \text{— — —}$$

مثال : فرض کنید شعاع (r) یک استوانه با آهنگ ($\frac{dr}{dt}$) $10\,\frac{cm}{s}$ (زمان) افزایش و ارتفاع (h) آن با آهنگ ($\frac{dh}{dt}$) $5\,\frac{cm}{s}$ کاهش می‌یابد. در لحظه‌ای که شعاع برابر 100cm و ارتفاع برابر 200cm است، حجم (V) استوانه با چه آهنگی ($\frac{dV}{dt}$) تغییر می‌کند؟

$V=\pi r^2 h$

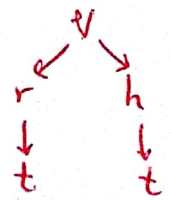
r و h تابعی t برحسب زمان

$$\frac{dV}{dt}=\frac{\partial V}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$=(2\pi rh)\left(\frac{dr}{dt}\right)+(\pi r^2)\left(\frac{dh}{dt}\right)$$

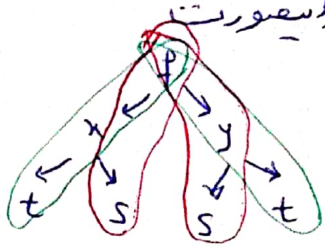
$$=\left(2\pi(100)(200)\right)(10)+\left(\pi(100)^2\right)(-5)$$

$$=\ \text{— — —}$$



تذکر: فرض کنید f تابعی مشتق‌پذیر برحسب x و y و $x=g(t,s)$ ، $y=h(t,s)$.

در این صورت



$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

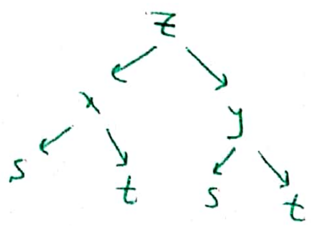
مثال: اگر $z=y^2\sin 3x$ و $x=s^2e^t$ ، $y=te^t\ln s$. مطلوبست $\frac{\partial z}{\partial s}$ ، $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\,\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\,\frac{\partial y}{\partial s}$$

$$=(3y^2\cos 3x)(2se^t)+(2y\sin 3x)\left(\frac{te^t}{s}\right)\ \underset{y=te^t\ln s}{\overset{x=s^2t}{=}}\ \cdots$$

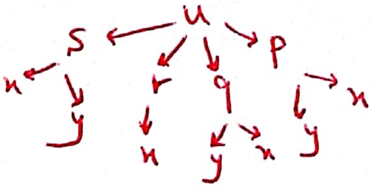
$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$=(3y^2\cos 3x)(s^2e^t)+(2y\sin 3x)(e^t\ln s+te^t\ln s)=\cdots$$



مثال: اگر $u=p^2q+rsp-q^2$ ، $p=x^2y$ ، $q=y-x$ ، $r=x^2$ و $s=2x-y^2$ ،

مطلوبست $\frac{\partial u}{\partial x}$ و $\frac{\partial u}{\partial y}$.



$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x}+\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$=(2pq+rs)(2xy)+(p^2-2q)(-1)+(sp)(2x)+(rp)(2)=\text{———}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y}+\frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y}+\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

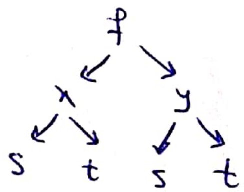
$$=(2pq+rs)(x^2)+(p^2-2q)(1)+\underbrace{(sp)(0)}_{0}+(rp)(-2y)=\text{———}$$

مثال: فرض کنید f تابعی دو متغیره و مشتق پذیر باشد و $g(t,s)=f(\underbrace{s^2-t^2}_{x},\underbrace{t^2-s^2}_{y})$.

نشان دهید $s\frac{\partial g}{\partial t}+t\frac{\partial g}{\partial s}=0$.

$x=s^2-t^2$

$y=t^2-s^2$ قرار دهید



$$\frac{\partial g}{\partial t}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$=\frac{\partial f}{\partial x}(-2t)+\frac{\partial f}{\partial y}(2t)$$

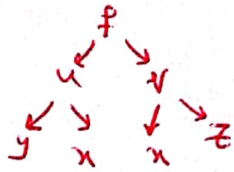
$$\frac{\partial g}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$=\frac{\partial f}{\partial x}(2s)+\frac{\partial f}{\partial y}(-2s)$$

$$s\frac{\partial g}{\partial t}+t\frac{\partial g}{\partial s}=-2ts\frac{\partial f}{\partial x}+2ts\frac{\partial f}{\partial y}+2ts\frac{\partial f}{\partial x}-2ts\frac{\partial f}{\partial y}=0$$

مثال: اگر $w=x^2 f(\underbrace{\frac{y}{x}}_{u},\underbrace{\frac{z}{x}}_{v})$. حاصل عبارت زیر را برحسب w بدست آورید.

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}$$



$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$=\frac{\partial f}{\partial u}\left(\frac{-y}{x^2}\right)+\frac{\partial f}{\partial v}\left(\frac{-z}{x^2}\right)$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}$$
$$=\frac{\partial f}{\partial u}\cdot\left(\frac{1}{x}\right)$$

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}=\frac{\partial f}{\partial v}\left(\frac{1}{x}\right)$$

$$\frac{\partial w}{\partial x}=2x\,f\left(\frac{y}{x},\frac{z}{x}\right)+x^2\frac{\partial f}{\partial x}=2x\,f\left(\frac{y}{x},\frac{z}{x}\right)+\frac{\partial f}{\partial u}(-y)+\frac{\partial f}{\partial v}(-z)$$

$$\frac{\partial w}{\partial y}=x^2\frac{\partial f}{\partial y}=x\frac{\partial f}{\partial u}$$

$$\frac{\partial w}{\partial z}=x^2\frac{\partial f}{\partial z}=x\frac{\partial f}{\partial v}$$

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=\underbrace{2x^2 f\left(\frac{y}{x},\frac{z}{x}\right)}_{w}-\cancel{xy\frac{\partial f}{\partial u}}\cancel{-zx\frac{\partial f}{\partial v}}+\cancel{xy\frac{\partial f}{\partial u}}+\cancel{xz\frac{\partial w}{\partial v}}$$
$$=2w$$

مثال: فرض کنید $f(x,y)$ تابعی دو متغیره باشد، $x=2rs$ ، $y=r^2+s^2$. مطلوبست $\dfrac{\partial^2 f}{\partial r^2}$.

$$\frac{\partial^2 f}{\partial r^2}=\frac{\partial\left(\frac{\partial f}{\partial r}\right)}{\partial r}$$

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial f}{\partial x}(2s)+\frac{\partial f}{\partial y}(2r)$$

$$\frac{\partial^2 f}{\partial r^2}=\frac{\partial\left(\frac{\partial f}{\partial x}(2s)+\frac{\partial f}{\partial y}(2r)\right)}{\partial r}=\frac{\partial\left(\frac{\partial f}{\partial x}(2s)\right)}{\partial r}+\frac{\partial\left(\frac{\partial f}{\partial y}(2r)\right)}{\partial r}$$

$$=\frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial r}(2s)+\cancel{\overset{0}{\frac{\partial(2s)}{\partial r}}\,\frac{\partial f}{\partial x}}+\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial r}(2r)+\overset{2}{\frac{\partial(2r)}{\partial r}}\cdot\frac{\partial f}{\partial y}$$

$$=\left(\frac{\partial^2 f}{\partial x^2}(2s)+\frac{\partial^2 f}{\partial x\partial y}(2r)\right)(2s)+\left(\frac{\partial^2 f}{\partial x\partial y}(2s)+\frac{\partial^2 f}{\partial y^2}(2r)\right)(2r)$$

$$+2\frac{\partial f}{\partial y}$$

$$\frac{\partial\left(\boxed{\frac{\partial f}{\partial x}}\right)}{\partial r}=\frac{\partial\left(\boxed{\frac{\partial f}{\partial x}}\right)}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial\left(\boxed{\frac{\partial f}{\partial x}}\right)}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial^2 f}{\partial x^2}(2s)+\frac{\partial^2 f}{\partial x\partial y}(2r)$$

$$\frac{\partial\left(\boxed{\frac{\partial f}{\partial y}}\right)}{\partial r}=\frac{\partial\left(\boxed{\frac{\partial f}{\partial y}}\right)}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial\left(\boxed{\frac{\partial f}{\partial y}}\right)}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial^2 f}{\partial x\partial y}(2s)+\frac{\partial^2 f}{\partial y^2}(2r)$$

تمرین: تابع $f(x,y)=\frac{(y-1)^2}{x}$ را در نظر بگیرید.

الف. معادلهٔ صفحهٔ مماس بر رویهٔ $z=f(x,y)$ در نقطهٔ $(x_0,y_0,f(x_0,y_0))$ را بنویسید.

ب. اگر این صفحه از نقطهٔ $(c,0,0)$ بگذرد، عدد c را بدست آورید.

تمرین: فرض کنید $z=f(e^s\cos t, e^s\sin t)$. حاصل $\left(\frac{\partial z}{\partial s}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2$ را بنویسید.

تمرین: اگر $z=f(r\cos\theta, r\sin\theta)$، مطلوبست $\frac{\partial z}{\partial r}$ و $\frac{\partial z}{\partial \theta}$.

تمرین: فرض کنید n عددی طبیعی و f تابعی پیوسته باشد و $f(tx,ty)=t^n f(x,y)$.

نشان دهید $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$

(راهنمایی: با استفاده از قاعدهٔ زنجیره‌ای از $f(tx,ty)$ نسبت به t مشتق بگیرید.)