Chapter 1 (Part 3)
Rules of Inference

Outline

- Valid Arguments
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments

Valid Arguments

- We will show how to construct valid arguments in propositional logic
- The rules of inference are the essential building block in the construction of valid arguments.
  - Propositional Logic
    - Inference Rules

Arguments in Propositional Logic

- An *argument* is a sequence of propositions.
- All but the final proposition are called *premises*. The last statement is the *conclusion*.
- The argument is valid if the premises imply the conclusion.
- If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then

  $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.
- Inference rules are all simple argument forms that will be used to construct more complex argument forms.

Rules of Inference: Modus Ponens

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$(p \wedge (p \to q)) \to q$

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$(\neg q \wedge (p \to q)) \to \neg p$

Example:

Let p be "it is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Corresponding Tautology:
$((p \to q) \land (q \to r)) \to (p \to r)$

Example:
Let p be "it snows."
Let q be "I will study discrete math."
Let r be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$p \rightarrow (p \vee q)$

Example:

Let p be "I will study discrete math."

Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Simplification

$$\frac{p \wedge q}{\therefore\ p}$$

Corresponding Tautology:

$(p \wedge q) \rightarrow p$

Example:

Let p be "I will study discrete math."

Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:

$((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:

Let p be "I will study discrete math."

Let q be "I will study English literature."

"I will study discrete math."

"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

Resolution plays an important role in AI and is used in Prolog.

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:
$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Example:
Let p be "I will study discrete math."
Let r be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will English literature."

Rules of Inference: Modus Ponens

$$
\begin{array}{l}
p \to q \\
q \\
\hline
\therefore p
\end{array}
$$

(1) If Margaret Thatcher is the president of the U.S., then she is at least 35 years old.
(2) Margaret Thatcher is at least 35 years old.
(3) Therefore, Margaret Thatcher is the president of the US.

Rule of Inference: Modus Ponens

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

(1) If the sum of the digits of 371,487 is divisible by 3, then 371,487 is divisible by 3.

(2) The sum of the digits of 371,487 is divisible by 3.

(3) 371,487 is divisible by 3.

Rules of Inference: Modus Tollens

$$
\begin{array}{l}
p \rightarrow q \\
\neg q \\
\hline
\therefore \neg p
\end{array}
$$

(1) If Zeus is human, then Zeus is mortal.
(2) Zeus is not mortal.
(3) Therefore, Zeus is not human.

Modus ponens and Modus Tollens

a) If there are more pigeons than there are pigeonholes , then at least two pigeons roost in the same hole.

There are more pigeons than there are pigeonholes.

Therefore,___________________________________

b) If 870,232 is divisible by 6, then it is divisible by 3.

870,232 is not divisible by 3.

Therefore, ___________________________________

Valid Arguments

Example 1: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that q is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \rightarrow q)$	Premise
2. p	Conjunction using (1)
3. $p \rightarrow q$	Conjunction using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments

Example 2:

- With these hypotheses:
  "It is not sunny this afternoon and it is colder than yesterday."
  "We will go swimming only if it is sunny."
  "If we do not go swimming, then we will take a canoe trip."
  "If we take a canoe trip, then we will be home by sunset."
- Using the inference rules, construct a valid argument for the conclusion:
  "We will be home by sunset."

Solution:

1. Choose propositional variables:
   p : "It is sunny this afternoon."
   r : "We will go swimming."
   t : "We will be home by sunset."
   q : "It is colder than yesterday."
   s : "We will take a canoe trip."
2. Translation into propositional logic:

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Hypotheses: $\neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$
Conclusion: t

Validity of Argument

Our first example demonstrates the validity of the argument

$$
\begin{array}{r}
p \rightarrow \mathrm{r} \\
\neg p \rightarrow q \\
q \rightarrow s \\
\therefore \neg r \rightarrow s
\end{array}
$$

$$
\begin{tikzcd}
p \rightarrow r \arrow[r] & \neg r \rightarrow \neg p \arrow[dr] & & \\
\neg p \rightarrow q \arrow[rr] & & \neg r \rightarrow q \arrow[dr] & \\
q \rightarrow s \arrow[rrr] & & & \neg r \rightarrow s
\end{tikzcd}
$$

$$
\frac{\begin{array}{c} p \rightarrow r \\ \neg p \rightarrow q \\ q \rightarrow s \end{array}}{\therefore \neg r \rightarrow s}
$$

Steps	**Reasons**
1) $p \rightarrow r$	Premise
2) $\neg r \rightarrow \neg p$	Step (1) and $p \rightarrow r \equiv \neg r \rightarrow \neg p$
3) $\neg p \rightarrow q$	Premise
4) $\neg r \rightarrow q$	Steps (2) and (3) and transitivity
5) $q \rightarrow s$	Premise
6) $\therefore \neg r \rightarrow \mathrm{s}$	Steps (4) and (5) and the transitivity

Validity of the argument

Establish the validity of the argument

$p \rightarrow q$
$q \rightarrow (r \wedge s)$
$\neg r \vee (\neg t \vee u)$
$p \wedge t$
$\therefore u$

Steps	**Reasons**
1) $p \rightarrow q$	Premise
2) $q \rightarrow (r \wedge s)$	Premise
3) $p \rightarrow (r \wedge s)$	Steps (1) and (2) and transitivity
4) $p \wedge t$	Premise
5) p	Step (4) and the specialisation
6) $r \wedge s$	Steps (5) and (3) and the modus ponens
7) r	Step (6) and the specialisation
8) $\neg r \vee (\neg t \vee u)$	Premise
9) $\neg t \vee u$	Step (7) and (8), and elimination
10) t	Step (4) and specialisation
11) $\therefore u$	Steps (9) and (10) and elimination

Rule of Inference	Tautology	Name
p $p \rightarrow q$ $\therefore q$	$(p \land (p \rightarrow q)) \rightarrow q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$(\neg q \land (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \rightarrow q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \rightarrow (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \rightarrow p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \rightarrow (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	Resolution

Class Activity 1

You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

Solution: Class Activity 1

Let RK = I was reading the newspaper in the kitchen.
GK = My glasses are on the kitchen table.
SB = I saw my glasses at breakfast.
RL = I was reading the newspaper in the living room.
GC = My glasses are on the coffee table.

1.	$RK \to GK$	by (a)
	$GK \to SB$	by (b)
	$\therefore RK \to SB$	by transitivity
2.	$RK \to SB$	by the conclusion of (1)
	$\sim SB$	by (c)
	$\therefore \sim RK$	by modus tollens
3.	$RL \vee RK$	by (d)
	$\sim RK$	by the conclusion of (2)
	$\therefore RL$	by elimination
4.	$RL \to GC$	by (e)
	RL	by the conclusion of (3)
	$\therefore GC$	by modus ponens

Thus the glasses are on the coffee table.

Class Activity 2

If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been canceled and Alicia would have been angry. If the party were canceled, then refunds would have had to be made. No refunds were made. The above statement imply the conclusion the band could play rock music.

First we convert the given argument into symbolic form by using the following statement assignments:

p: The band could play rock music.

q: The refreshments were delivered on time.

r: The New Year's party was canceled.

s: Alicia was angry.

t: Refunds had to be made.

Solution: Class Activity 2

The argument becomes:

$$(\neg p \vee \neg q) \rightarrow (r \wedge s)$$
$$r \rightarrow t$$
$$\neg t$$
$$\therefore p$$

Therefore the band could play rock music.

Class Activity 3

What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

a) Addition b) Simplification c) Modus ponens d) Modus tollens
e) Hypothetical syllogism

Class Activity 4

Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

Solution: Class Activity 4

Let w be the proposition "Randy works hard," let d be the proposition "Randy is a dull boy," and let j be the proposition "Randy will get the job." We are given premises w, $w \to d$, and $d \to \neg j$. We want to conclude $\neg j$. We set up the proof in two columns, with reasons, as in Example 6.

Step	**Reason**
1. w	Hypothesis
2. $w \to d$	Hypothesis
3. d	Modus ponens using (2) and (3)
4. $d \to \neg j$	Hypothesis
5. $\neg j$	Modus ponens using (3) and (4)

Class Activity 5

Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

Solution: Class Activity 5

Let r be the proposition "It rains," let f be the proposition "It is foggy," let s be the proposition "The sailing race will be held," let l be the proposition "The life saving demonstration will go on," and let t be the proposition "The trophy will be awarded." We are given premises $(\neg r \vee \neg f) \to (s \wedge l)$, $s \to t$, and $\neg t$. We want to conclude r. We set up the proof in two columns, with reasons, as in Example 6. Note that it is valid to replace subexpressions by other expressions logically equivalent to them.

Step	Reason
1. $\neg t$	Hypothesis
2. $s \to t$	Hypothesis
3. $\neg s$	Modus tollens using (1) and (2)
4. $(\neg r \vee \neg f) \to (s \wedge l)$	Hypothesis
5. $(\neg(s \wedge l)) \to \neg(\neg r \vee \neg f)$	Contrapositive of (4)
6. $(\neg s \vee \neg l) \to (r \wedge f)$	De Morgan's law and double negative
7. $\neg s \vee \neg l$	Addition, using (3)
8. $r \wedge f$	Modus ponens using (6) and (7)
9. r	Simplification using (8)