

Chapter 1 (Part I)

Fundamentals of logic

Learning Outcome

Students should be able to:

- Understand the rules of logic and know how to construct correct mathematical arguments
- Distinguish between valid and invalid mathematical argument
- Solve related problems of logic
- Discuss applications of logic to computer science

Outline

- Propositional Logic
 - The Language of Propositions
 - Applications
 - Logical Equivalences
- Predicate Logic
 - The Language of Quantifiers
 - Logical Equivalences
- Rules of Inference

Propositions

- A *proposition* is a declarative sentence that is either **True** or **False**.
- Examples of propositions:
 - a) The Moon is made of green cheese.
 - b) Trenton is the capital of New Jersey.
 - c) Toronto is the capital of Canada.
 - d) $1 + 0 = 1$
 - e) $0 + 0 = 2$
- Examples that are not propositions.
 - a) Sit down!
 - b) What time is it?
 - c) $x + 1 = 2$
 - d) $x + y = z$

Propositional Logic

- Constructing Propositions
 - Propositional Variables: p, q, r, s, \dots
 - The proposition that is always **true** is denoted by **T** and the proposition that is always **false** is denoted by **F**.
 - Compound Propositions; constructed from logical connectives and other propositions
 - Negation \neg
 - Conjunction \wedge
 - Disjunction \vee
 - Implication \rightarrow
 - Bi-conditional \leftrightarrow

Compound Propositions: Negation

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

- **Example:** If p denotes “The earth is round.”, then $\neg p$ denotes “It is not the case that the earth is round,” or more simply “The earth is not round.”

Conjunction

- The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Example:** If p denotes “I am at home.” and q denotes “It is raining.” then $p \wedge q$ denotes “I am at home and it is raining.”

Disjunction

- The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Example:** If p denotes “I am at home.” and q denotes “It is raining.” then $p \vee q$ denotes “I am at home or it is raining.”

The Connective Or in English

- In English “or” has two distinct meanings.
 - “**Inclusive Or**” - In the sentence “Students who have taken CS202 or Math120 may take this class,” we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \vee q$ to be true, **either one or both of p and q must be true**.
 - “**Exclusive Or**” - When reading the sentence “Soup or salad comes with this entrée,” we do not expect to be able to get both soup and salad. This is the meaning of **Exclusive Or (Xor)**. In $p \oplus q$, **one of p and q must be true, but not both**. The truth table for \oplus is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as “if p , then q ” and has this truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example:** If p denotes “I am at home.” and q denotes “It is raining.” then $p \rightarrow q$ denotes “If I am at home then it is raining.”
- In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication

- In $p \rightarrow q$ there does not need to be any connection between the antecedent or the consequent.
- The “meaning” of $p \rightarrow q$ depends only on the truth values of p and q .
- These implications are perfectly fine, but would not be used in ordinary English.
 - “If the moon is made of green cheese, then I have more money than Bill Gates. ”
 - “If the moon is made of green cheese then I’m on welfare.”
 - “If $1 + 1 = 3$, then your grandma wears combat boots.”

Understanding Implication (cont..)

- A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract.
 - “If I am elected, then I will lower taxes.”
 - “If you get 100% on the final, then you will get an A.”
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge.

Converse, Contrapositive, and Inverse

- From $p \rightarrow q$ we can form new conditional statements .
 - $q \rightarrow p$ is the **converse** of $p \rightarrow q$
 - $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$
 - $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$

Example: Find the converse, inverse, and contrapositive of “It raining is a sufficient condition for my not going to town.”

Solution:

converse: If I do not go to town, then it is raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

Biconditional

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as “ p if and only if q ”.
- The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes “I am at home.” and q denotes “It is raining.” then $p \leftrightarrow q$ denotes “I am at home if and only if it is raining.”

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are *equivalent* if they always have the same truth value.
- Example:** Show using a truth table that $p \rightarrow q$ is equivalent to the contrapositive $\neg q \rightarrow \neg p$

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Precedence of Logical Operators

Operator	Precedence
\neg	1
\wedge	2
\vee	3
\rightarrow	4
\leftrightarrow	5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Applications of Propositional Logic

- Translating English to Propositional Logic
- System Specifications
- Boolean Searching
- Logic Circuits

Applications of Propositional Logic

- Logic has many important applications in computer science and numerous other disciplines
- For example:
 - Logic is used in the specification of software and hardware
 - Rule of logic can be used to design computer circuits, to construct computer programs, to verify the correctness of programs, and to build expert systems
 - Logic can be used to analyze and solve many familiar puzzles

Translating English Sentences

- English and other human language is often ambiguous
- Steps to convert an English sentence into a propositional logic
 - Identify atomic propositions and represent using propositional variables.
 - Determine appropriate logical connectives
- “If I go to Harry’s or to the country, I will not go shopping.”
 - p : I go to Harry’s
 - q : I go to the country.
 - r : I will go shopping.

If p or q then not r .

$$(p \vee q) \rightarrow \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

Solution: Let a , c , and f represent respectively “You can access the internet from campus,” “You are a computer science major,” and “You are a freshman.”

$$a \rightarrow (c \vee \neg f)$$

System Specifications

- Translating sentences from natural language into logical expressions is an essential part of specifying both hardware and software systems
- System specification should be consistent, that is they should not contain conflicting requirements that could be used to derive a contradiction.
- When specifications are not consistent, there would be no way to develop a system that satisfies all specifications

Example: Express in propositional logic:

“The automated reply cannot be sent when the file system is full”

Solution: One possible solution: Let p denote “The automated reply can be sent” and q denote “The file system is full.”

$$q \rightarrow \neg p$$

Consistent System Specifications

Definition: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.

Exercise: Are these specifications consistent?

- “The diagnostic message is stored in the buffer or it is retransmitted.”
- “The diagnostic message is not stored in the buffer.”
- “If the diagnostic message is stored in the buffer, then it is retransmitted.”

Solution: Let

p : “The diagnostic message is not stored in the buffer.”

q : “The diagnostic message is retransmitted”

The specification can be written as: $p \vee q, p \rightarrow q, \neg p$.

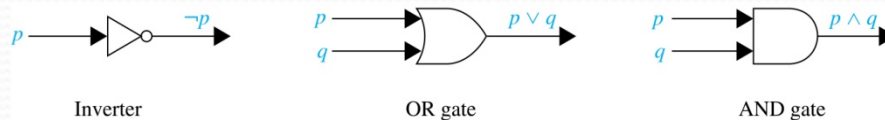
When p is false and q is true all three statements are true. So the specification is consistent.

- What if “The diagnostic message is not retransmitted is added.”

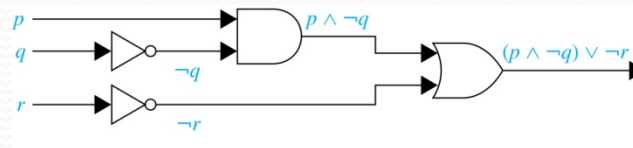
Solution: Now we are adding $\neg q$ and there is no satisfying assignment. So the specification is not consistent.

Logic Circuits

- Electronic circuits; each input/output signal can be viewed as a 0 or 1.
 - 0 represents **False**
 - 1 represents **True**
- Complicated circuits are constructed from three basic circuits called gates.



- The inverter (**NOT gate**) takes an input bit and produces the negation of that bit.
 - The **OR gate** takes two input bits and produces the value equivalent to the disjunction of the two bits.
 - The **AND gate** takes two input bits and produces the value equivalent to the conjunction of the two bits.
- More complicated digital circuits can be constructed by combining these basic circuits to produce the desired output given the input signals by building a circuit for each piece of the output expression and then combining them. For example:



Logic Puzzles



Raymond
Smullyan
(Born 1919)

- An island has two kinds of inhabitants, *knight*s, who always tell the truth, and *knave*s, who always lie.
- You go to the island and meet A and B.
 - A says “B is a knight.”
 - B says “The two of us are of opposite types.”

Example: What are the types of A and B?

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively. So, then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.

- If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
- If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

Propositional Equivalences

Tautologies, Contradictions, and Contingencies

- A **tautology** is a proposition which is always true.
 - Example: $p \vee \neg p$
- A **contradiction** is a proposition which is always false.
 - Example: $p \wedge \neg p$
- A **contingency** is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table show $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$



Augustus De Morgan

1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

- Identity Laws: $p \wedge T \equiv p, \quad p \vee F \equiv p$
- Domination Laws: $p \vee T \equiv T, \quad p \wedge F \equiv F$
- Idempotent laws: $p \vee p \equiv p, \quad p \wedge p \equiv p$
- Double Negation Law: $\neg(\neg p) \equiv p$
- Negation Laws: $p \vee \neg p \equiv T, \quad p \wedge \neg p \equiv F$

Key Logical Equivalences (*cont*)

- Commutative Laws: $p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$
- Associative Laws:
 $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
 $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive Laws:
 $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$
 $(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$
- Absorption Laws:
 $p \vee (p \wedge q) \equiv p$
 $p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

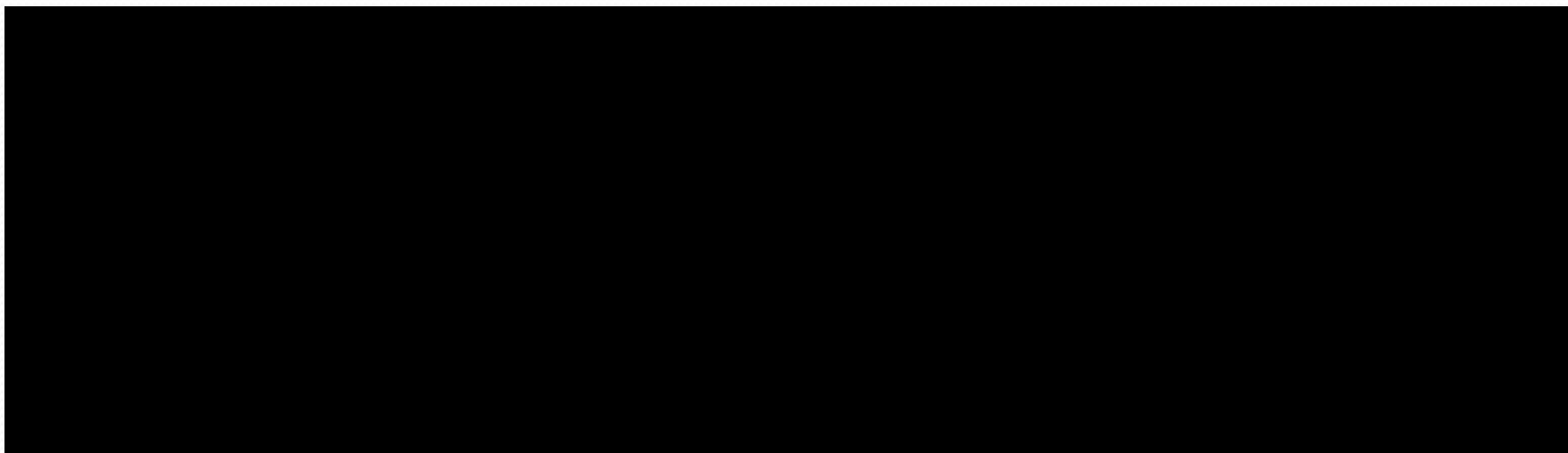
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$
is logically equivalent to $\neg p \wedge \neg q$

Solution:



Tautology

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$
is a tautology.

Solution:

$$\begin{aligned}(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\ &\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and commutative laws} \\ &\equiv T \vee T && \text{laws for disjunction} \\ &\equiv T && \text{by truth tables} \\ &&& \text{by the domination law}\end{aligned}$$

Truth Table

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Example

s : Phyllis goes out for a walk.

t : The moon is out.

u : It is snowing.

$(t \wedge \neg u) \rightarrow s$: If the moon is out and it is not snowing, then
Phyllis goes out for a walk.

If it is snowing and the moon is not out, then Phyllis
will not go out for a walk. $(u \wedge \neg t) \rightarrow \neg s$

Example: Logical Equivalence

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

$s_1 \Leftrightarrow s_2$

logically equivalent

Example

Negate and simplify the compound statement

$$(p \vee q) \rightarrow r$$

Solution:

$$\begin{aligned}\neg[(p \vee q) \rightarrow r] &\Leftrightarrow \neg[\neg(p \vee q) \vee r] \Leftrightarrow \\ \neg[(\neg p \wedge \neg q) \vee r] &\Leftrightarrow \neg(\neg p \wedge \neg q) \wedge \neg r \Leftrightarrow \\ (p \vee q) \wedge \neg r\end{aligned}$$

Simplification Compound Statement

$$(p \vee q) \wedge \neg(\neg p \wedge q)$$

Demorgan's Law

$$\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$$

Law of Double Negation

$$\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$$

Distributive Law

$$\Leftrightarrow p \vee (q \wedge \neg q)$$

Inverse Law and
Identity Law

$$\Leftrightarrow p \vee F_0 \Leftrightarrow p$$