Chapter 1 (Part 2)
Predicates & Quantifiers

Outline

- Predicate Logic
  - The Language of Quantifiers
  - Logical Equivalences
  - Nested Quantifiers
  - Translation from Predicate Logic to English
  - Translation from English to Predicate Logic

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: $P(x)$, $M(x)$
  - Quantifiers: ??
- *Propositional functions* are a generalization of propositions.
  - They contain variables and a predicate, e.g., $P(x)$
  - Variables can be replaced by elements from their *domain*.

Propositional Functions

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier).
- The statement $P(x)$ is said to be the value of the propositional function P at x.
- For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

  P(-3) is false.

  P(0) is false.

  P(3) is true.
- Often the domain is denoted by U. So in this example U is the integers.

Examples: Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers.

  Find these truth values:

  R(2,-1,5)

  Solution: F

  R(3,4,7)

  Solution: T

  R(x, 3, z)

  Solution: Not a Proposition

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers.

  Find these truth values:

  Q(2,-1,3)

  Solution: T

  Q(3,4,7)

  Solution: F

  Q(x, 3, z)

  Solution: Not a Proposition

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If $P(x)$ denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ **Solution**: T

  $P(3) \wedge P(-1)$ **Solution**: F

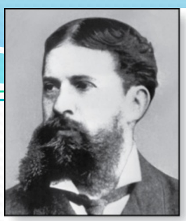
  $P(3) \rightarrow P(-1)$ **Solution**: F

  $P(3) \rightarrow P(1)$ **Solution**: T
- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$

  $P(x) \rightarrow P(y)$
- When used with quantifiers, these expressions (propositional functions) become propositions.

Quantifiers



Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier*, "For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.
- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

- $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall\, x\, P(x)$ is false.

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Thinking about Quantifiers

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step P(x) is true, then $\forall x\, P(x)$ is true.
  - If at a step P(x) is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, P(x) is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which P(x) is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
- **Examples**:
  1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

- **Solution 1**: If *U* is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.
- **Solution 2**: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 1: But if U is all people, then translate as

$\exists x\, (S(x) \wedge J(x))$

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example**: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers
- An existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
- If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \land P(2) \land P(3)$$

$$\exists x P(x) \equiv P(1) \lor P(2) \lor P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and

  the domain is students in your class.

- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

  Symbolically $\neg \forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions (*continued*)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

  Symbolically $\neg \exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important. You will use these.

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x \ (S(x) \land M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

   $$\forall x \ (S(x) \rightarrow (M(x) \lor C(x)))$$

Translating from English into Logical Expressions

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  Translate “Everything is a fleegle”

  Solution: $\forall x\, F(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"Nothing is a snurd."

Solution: $\neg \exists x\, S(x)$ What is this equivalent to?

Solution: $\forall x \neg S(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

 "All fleegles are snurds."

Solution: $\forall x\ (F(x) \rightarrow S(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"Some fleegles are thingamabobs."

Solution: $\exists x\, (F(x) \land T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "No snurd is a thingamabob."

  Solution: $\neg \exists x\ (S(x) \wedge T(x))$ What is this equivalent to?

  Solution: $\forall x\ (\neg S(x) \vee \neg T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): x is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

"If any fleegle is a snurd then it is also a thingamabob."

Solution: $\forall x\ ((F(x) \wedge S(x)) \rightarrow T(x))$

Example: System Specification

- Predicate logic is used for specifying properties that systems must satisfy.
- For example, translate into predicate logic:
  - "Every mail message larger than one megabyte will be compressed."
  - "If a user is active, at least one network link will be available."
- Decide on predicates and domains (left implicit here) for the variables:
  - Let $L(m, y)$ be "Mail message m is larger than y megabytes."
  - Let $C(m)$ denote "Mail message m will be compressed."
  - Let $A(u)$ represent "User u is active."
  - Let $S(n, x)$ represent "Network link n is state x.
- Now we have:

$$\forall m(L(m, 1) \rightarrow C(m))$$

$$\exists u\, A(u) \rightarrow \exists n\, S(n, available)$$

Charles Lutwidge Dodgson
(AKA Lewis Caroll)
(1832-1898)

Lewis Carroll Example

- The first two are called *premises* and the third is called the *conclusion*.
    1. "All lions are fierce."
    2. "Some lions do not drink coffee."
    3. "Some fierce creatures do not drink coffee."
- One way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.
    1. $\forall x\, (P(x) \rightarrow Q(x))$
    2. $\exists x\, (P(x) \land \neg R(x))$
    3. $\exists x\, (Q(x) \land \neg R(x))$
- Later we will see how to prove that the conclusion follows from the premises.

Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is

  $\forall x \; \exists y(x + y = 0)$

  where the domains of x and y are the real numbers.
- We can also think of nested propositional functions:

  $\forall x \; \exists y(x + y = 0)$ can be viewed as $\forall x \, Q(x)$ where $Q(x)$ is $\exists y \, P(x, y)$ where $P(x, y)$ *is* $(x + y = 0)$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y\, P(x,y)$ is true, loop through the values of x :
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x\ \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x\ \forall y\ P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x\ \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x\ \exists y P(x,y)$ has been shown to be false.

    $\forall x\ \exists y\ P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x\ \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: True

3. $\exists x \forall y\ P(x,y)$

   Answer: True

4. $\exists x\ \exists\ y\ P(x,y)$

   Answer: True

Questions on Order of Quantifiers

Example 2: Let U be the real numbers,

Define $P(x,y) : x / y = 1$

What is the truth value of the following:

1. $\forall x \, \forall y P(x,y)$

   Answer: False

2. $\forall x \, \exists y P(x,y)$

   Answer: True

3. $\exists x \, \forall y \, P(x,y)$

   Answer: False

4. $\exists x \;\; \exists \; y \, P(x,y)$

   Answer: True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x \ (C(x) \vee \exists y \ (C(y) \wedge F(x, y)))$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 1: Translate the statement

$$\exists x \forall y \ \forall z \ ((F(x, y) \wedge F(x,z) \ \wedge \ (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: Every student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables x and y, and specify the domain, to obtain:

   "For all positive integers x and y, $x + y$ is positive."

3. The result is:

$$\forall x \ \forall \ y \ ((x > 0) \wedge \ (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w, f)$ be "w has taken f " and $Q(f, a)$ be "f is a flight on a ."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w \, \forall a \, \exists f \, (P(w, f) \wedge Q(f, a))$$

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(\mathrm{x,y}) \rightarrow S(\mathrm{x,y}))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: "Everyone loves their ownself"

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$$\exists w \, \forall a \, \exists f \, (P(w,f) \wedge Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg \exists w \, \forall a \, \exists f \, (P(w,f) \wedge Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg \exists w \, \forall a \, \exists f \, (P(w,f) \wedge Q(f,a))$
2. $\forall w \neg \forall a \, \exists f \, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \, \exists a \neg \exists f \, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \, \exists a \, \forall f \neg (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \, \exists a \, \forall f (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"