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NPTEL ONLINE CERTIFICATION COURSES

NPTEL

Soft Computing

Introduction to Soft Computing

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INTRODUCTION TO SOFT COMPUTING

- Concept of computation
- Hard computing
- Soft computing
- How soft computing?
- Hard computing vs. Soft computing
- Hybrid computing

CONCEPT OF COMPUTATION

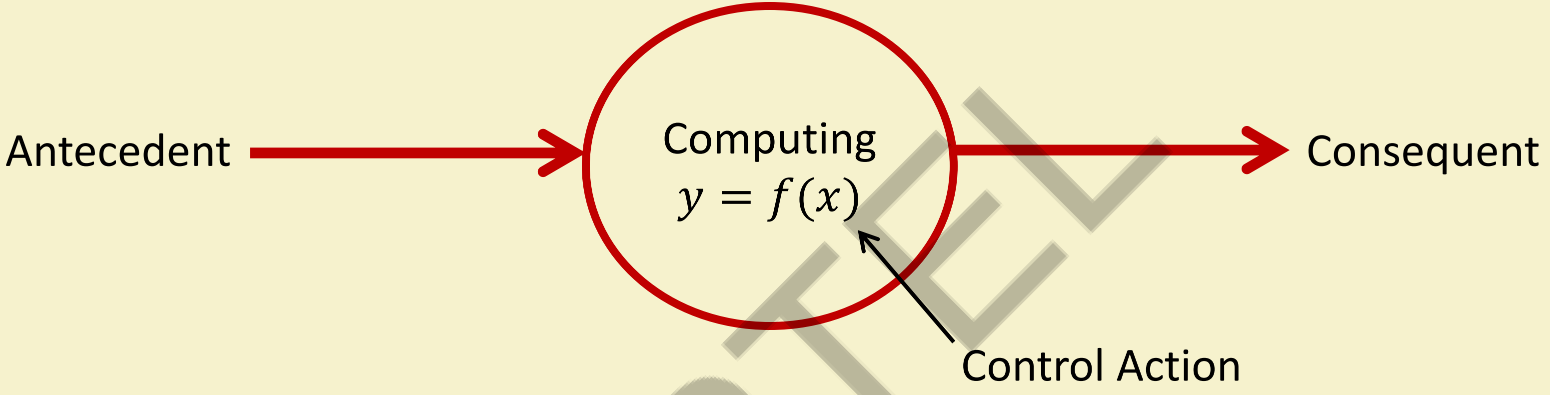


Figure: Basic of computing

$y = f(x), f$ is a mapping function.

f is also called a formal method or an algorithm to solve a problem.

Important characteristics of computing

- **Should provide precise solution.**
- **Control action should ne unambiguous and accurate.**
- **Suitable for problem, which is easy to model mathematically.**

Hard computing

- In 1996, L. A. Zade (LAZ) introduced the term hard computing.
- According to LAZ: We term a computing as Hard computing, if
  - ✓ Precise result is guaranteed.
  - ✓ Control action is unambiguous.
  - ✓ Control action is formally defined (i.e., with mathematical model or algorithm).

Examples of hard computing

- Solving numerical problems (e.g., roots of polynomials, integration, etc.).
- Searching and sorting techniques.
- Solving computational geometry problems (e.g., shortest tour in a graph, finding closet pair of points given a set of points, etc.).

- many more…

Soft computing

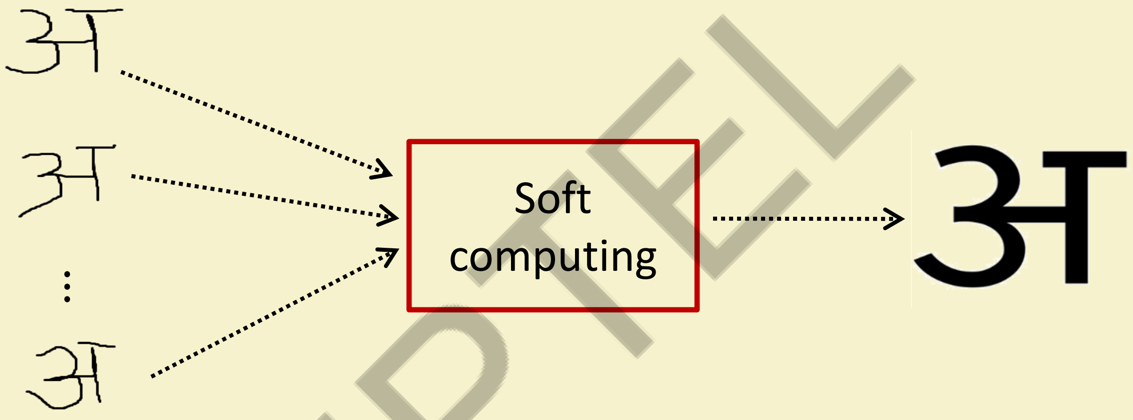
- **The term soft computing was proposed by the inventor of fuzzy logic, Lotfi A. Zadeh. He describes it as follows.**

Definition 1: Soft computing
Soft computing is a collection of methodologies that aim to exploit the tolerance for imprecision and uncertainty to achieve tractability, robustness, and low solution cost. Its principal constituents are fuzzy logic, neuro-computing, and probabilistic reasoning. The role model for soft computing is the human mind.

Characteristics of soft computing

- It does not require any mathematical modeling of problem solving.
- It may not yield the precise solution.
- Algorithms are adaptive (i.e., it can adjust to the change of dynamic environment).
- Use some biological inspired methodologies such as genetics, evolution, Ant's behaviors, particles swarming, human nervous system, etc.).

Examples of soft computing



Example: Hand written character recognition
(Artificial Neural Networks)

Examples of soft computing

सेन्ट्रल बैंक ऑफ़ इंडिया
Central Bank of India
1911 से आपके लिए "केंद्रित" "CENTRAL" TO YOU SINCE 1911

State Bank of India

⋮

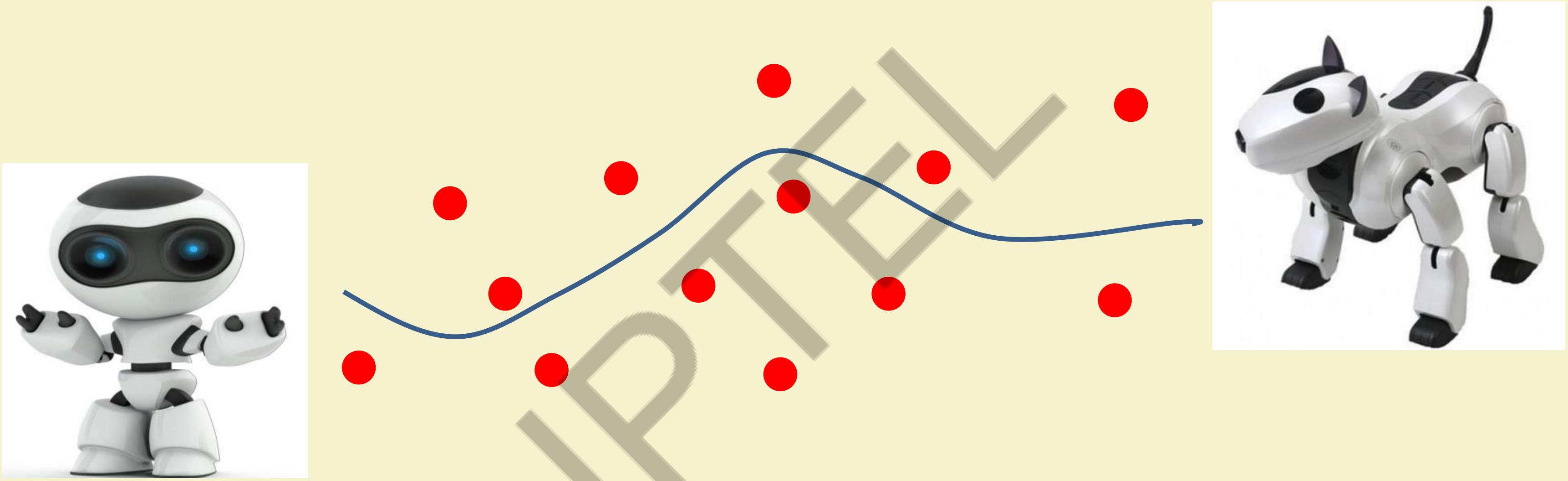
बैंक ऑफ़ इंडिया
Bank of India
Relationships beyond banking.

Soft computing

Bank with maximum return

Example: Money allocation problem
(Evolutionary Computing)

Examples of soft computing



Example: Robot movement
(Fuzzy Logic)

How soft computing?

- How a student learns from his teacher?
  - Teacher asks questions and tell the answers then.
  - Teacher puts questions and hints answers and asks whether the answers are correct or not.
  - Student thus learn a topic and store in his memory.
  - Based on the knowledge he solves new problems.
- This is the way how human brain works.
- Based on this concept Artificial Neural Network is used to solve problems.

How soft computing?

- How world selects the best?
  - It starts with a population (random).
  - Reproduces another population (next generation).
  - Rank the population and selects the superior individuals.
- Genetic algorithm is based on this natural phenomena.
  - Population is synonymous to solutions.
  - Selection of superior solution is synonymous to exploring the optimal solution.

How soft computing?

- How a doctor treats his patient?
  - Doctor asks the patient about suffering.
  - Doctor find the symptoms of diseases.
  - Doctor prescribed tests and medicines.
- This is exactly the way Fuzzy Logic works.
  - Symptoms are correlated with diseases with uncertainty .
  - Doctor prescribes tests/medicines fuzzily.

Hard computing vs. Soft computing

Hard computing	Soft computing
▪ It requires a precisely stated analytical model and often a lot of computation time.	▪ It is tolerant of imprecision, uncertainty, partial truth, and approximation.
▪ It is based on binary logic, crisp systems, numerical analysis and crisp software.	▪ It is based on fuzzy logic, neural nets and probabilistic reasoning.
▪ It has the characteristics of precision and categoricity.	▪ It has the characteristics of approximation and dispositionality.

Hard computing vs. Soft computing

Hard computing	Soft computing
▪ It is deterministic.	▪ It incorporates stochasticity.
▪ It requires exact input data.	▪ It can deal with ambiguous and noisy data.
▪ It is strictly sequential.	▪ It allows parallel computations.
▪ It produces precise answers.	▪ It can yield approximate answers

Hybrid computing

- It is a combination of the conventional hard computing and emerging soft computing.

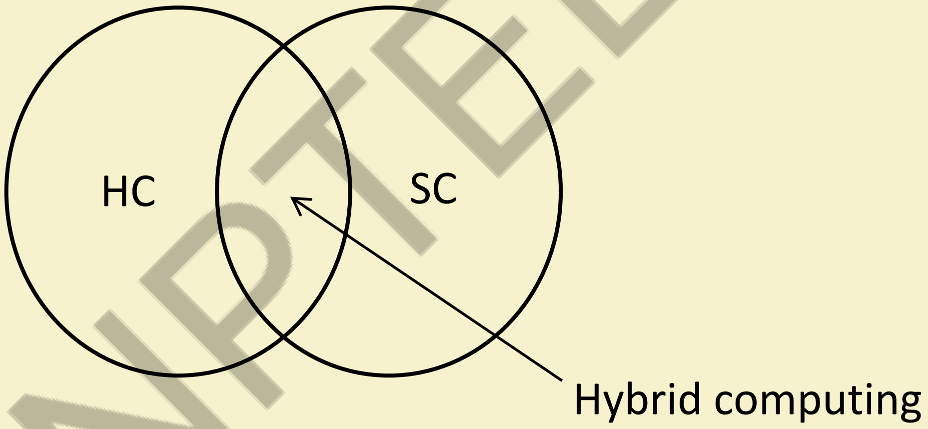


Figure: Concept of Hybrid Computing

In this course...

- You will be able to learn
  - Basic concepts of Fuzzy algebra and then how to solve problems using Fuzzy logic.
  - The framework of Genetic algorithm and solving varieties of optimization problems.
  - How to build an artificial neural network and train it with input data to solve a number of problems, which are not possible to solve with hard computing.

Thank You!!

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Soft Computing

Introduction to Fuzzy Logic

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What is Fuzzy logic?

- Fuzzy logic is a mathematical language to express something.
  - This means it has grammar, syntax, semantic like a language for communication.

- There are some other mathematical languages also known
  - **Relational algebra** (operations on sets)
  - **Boolean algebra** (operations on Boolean variables)
  - **Predicate algebra** (operations on well formed formulae (wff), also called predicate propositions)

- **Fuzzy logic deals with Fuzzy set or Fuzzy algebra.**

What is fuzzy?

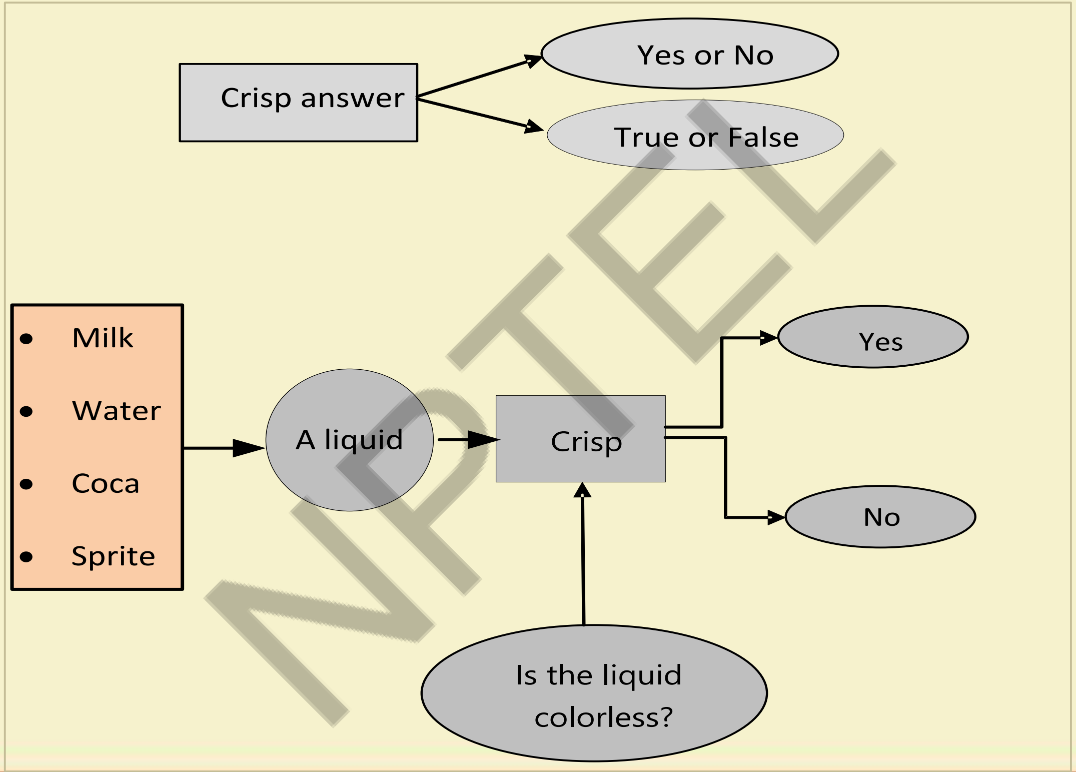
- Dictionary meaning of **fuzzy** is not clear, noisy, etc.

  Example: Is the picture on this slide is fuzzy?

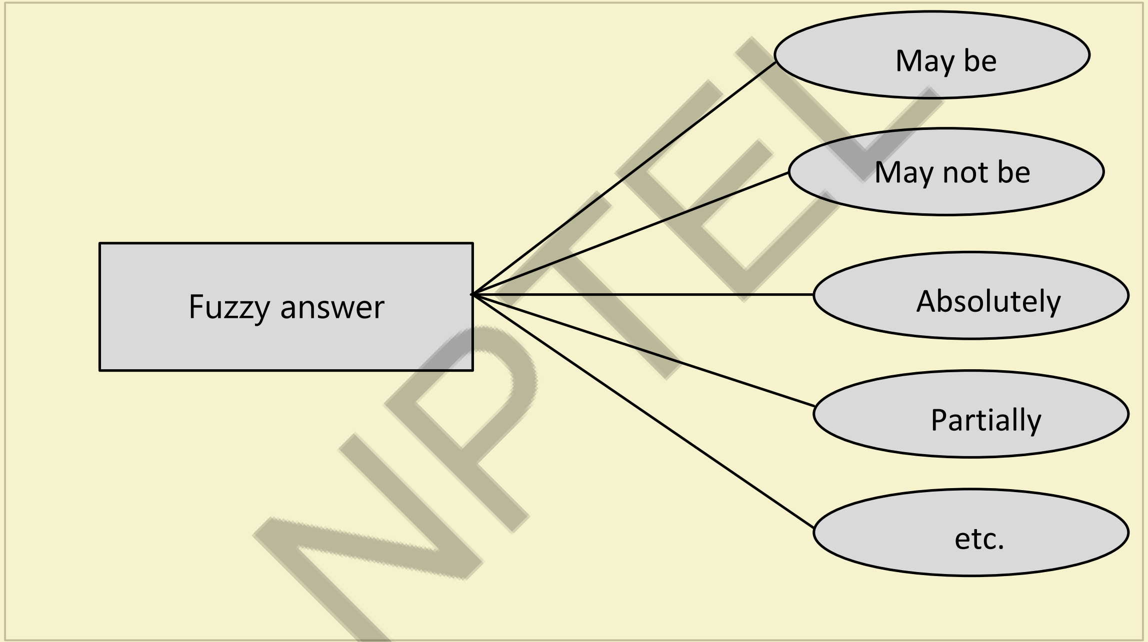
- Antonym of fuzzy is **crisp**

  Example: Are the chips crisp?

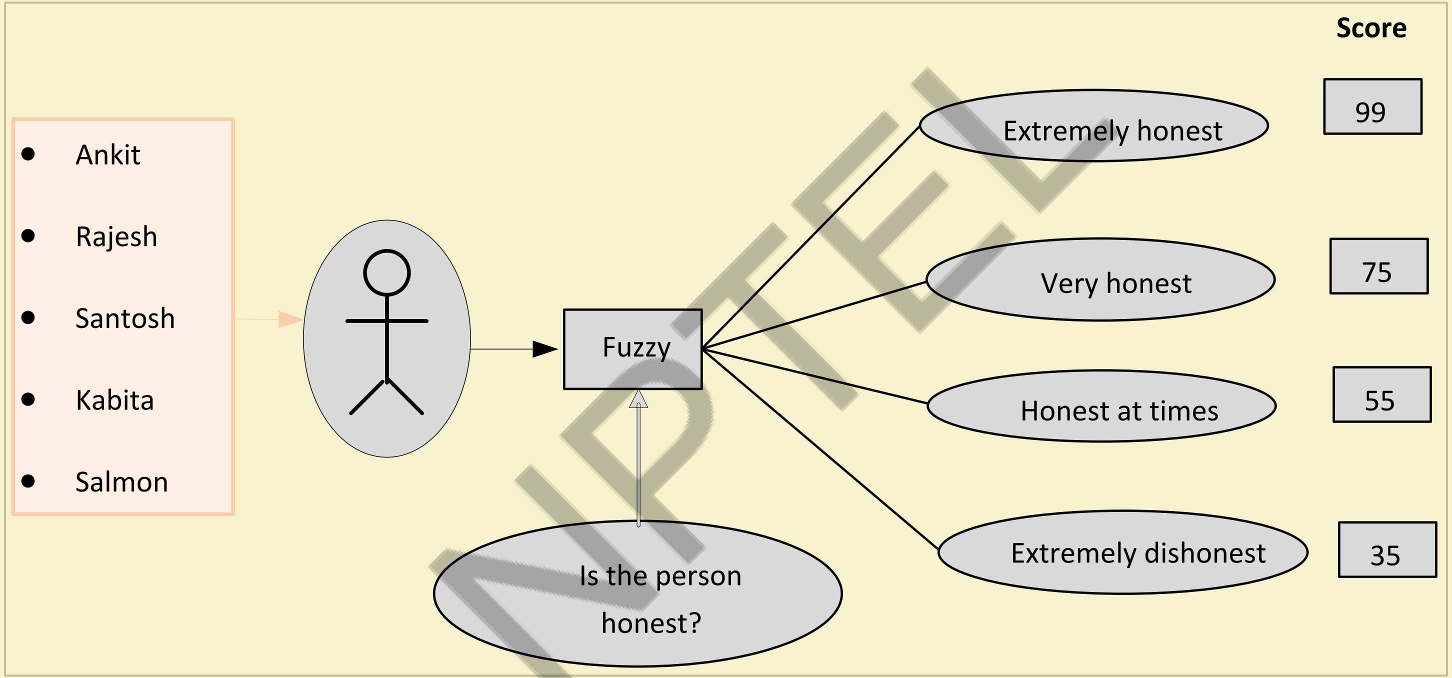
Example : Fuzzy logic vs. Crisp logic



Example : Fuzzy logic vs. Crisp logic

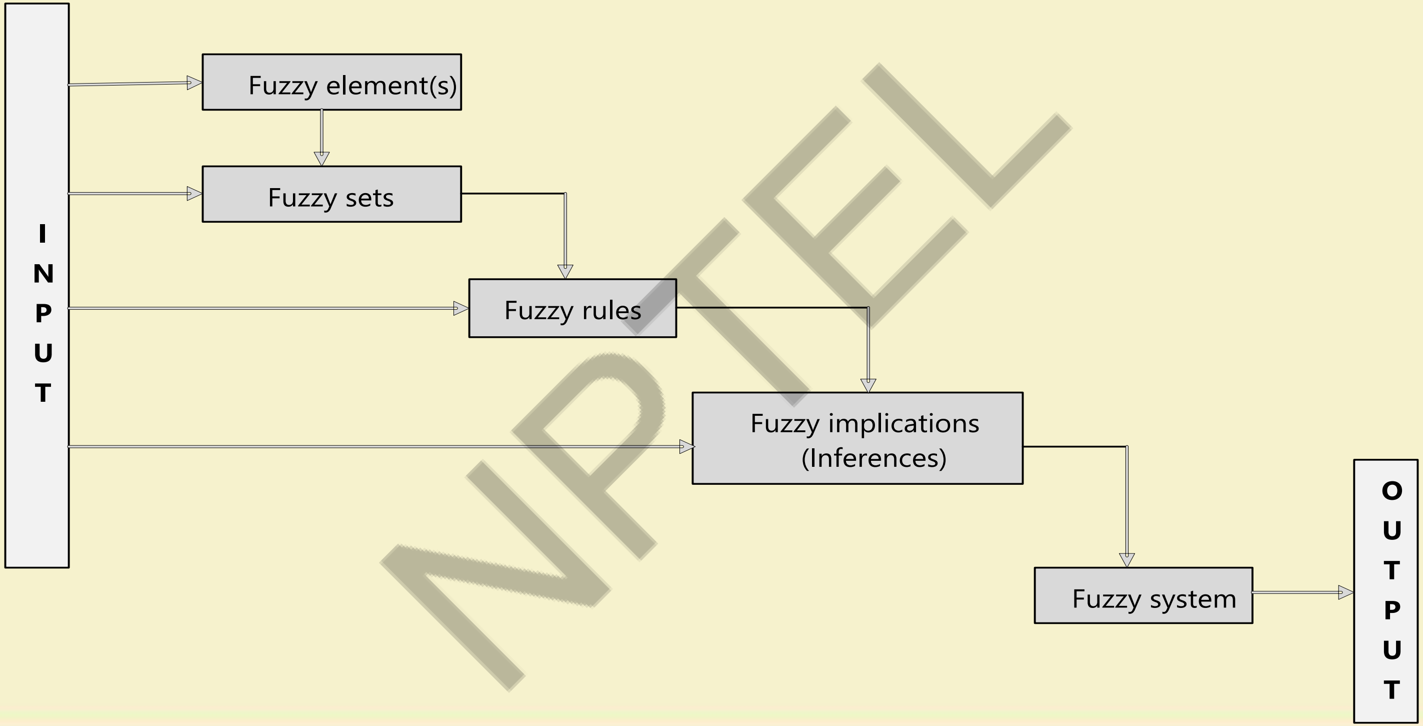


Example : Fuzzy logic vs. Crisp logic



World is fuzzy!

Concept of fuzzy system



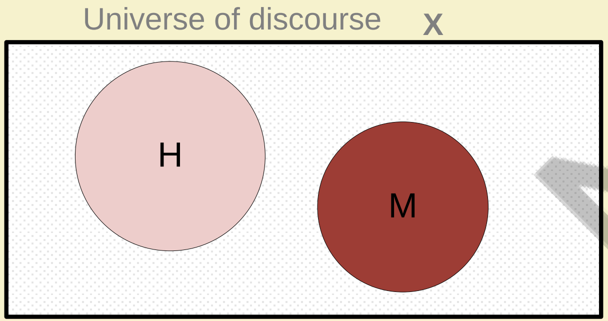
Concept of fuzzy set

To understand the concept of **fuzzy set** it is better, if we first clear our idea of **crisp set**.

X = The entire population of India.

H = All Hindu population = $\{h_1, h_2, h_3, \ldots \ldots, h_L,\}$

M = All Muslim population =$\{m_1, m_2, m_3, \ldots \ldots, m_N,\}$



Here, All are the sets of finite numbers of individuals. Such a set is called crisp set.

Example of fuzzy set

Let us discuss about fuzzy set.

X = All students in NPTEL.

S = All **Good students**.

$S = \{(s, g(s)) \mid s \in X\}$ and $g(s)$ is a measurement of goodness of the student s.

Example:

$S = \{ (Rajat, 0.8), (Kabita, 0.7), (Salman, 0.1), (Ankit, 0.9)\}$, etc.

Fuzzy set vs. Crisp set

Crisp set	Fuzzy set
▪ $S = \{s \vert s \in X\}$	▪ $F = (s, \mu(s)) \vert s \in X$ and $\mu(s)$ is the degree of s.
▪ It is a collection of elements.	▪ It is a collection of ordered pairs.
▪ Inclusion of an element $s \in X$ into S is crisp, that is, has strict boundary yes or no.	▪ Inclusion of an element $s \in X$ into F is fuzzy, that is, if present, then with a degree of membership.

Fuzzy set vs. Crisp set

Note: A crisp set is a fuzzy set, but, a fuzzy set is not necessarily a crisp set.

Example:

$$\mathrm{H} = \{(h_1, 1), (h_2, 1) \ldots \ldots \ldots, (h_L, 1)\}$$

$$\mathrm{Person} = \{(p_1, 0), (p_2, 0) \ldots \ldots \ldots, (p_N, 0)\}$$

In case of a crisp set, the elements are with extreme values of degree of membership namely either 1 or 0.

Degree of membership

How to decide the degree of memberships of elements in a fuzzy set?

City	Bangalore	Bombay	Hyderabad	Kharagpur	Madras	Delhi
μ	0.95	0.90	0.80	0.01	0.65	0.75

How the cities of comfort can be judged?

Example: Course evaluation in a crisp way

EX : Marks ≥ 90

A : $80 \leq$ Marks < 90

B : $70 \leq$ Marks < 80

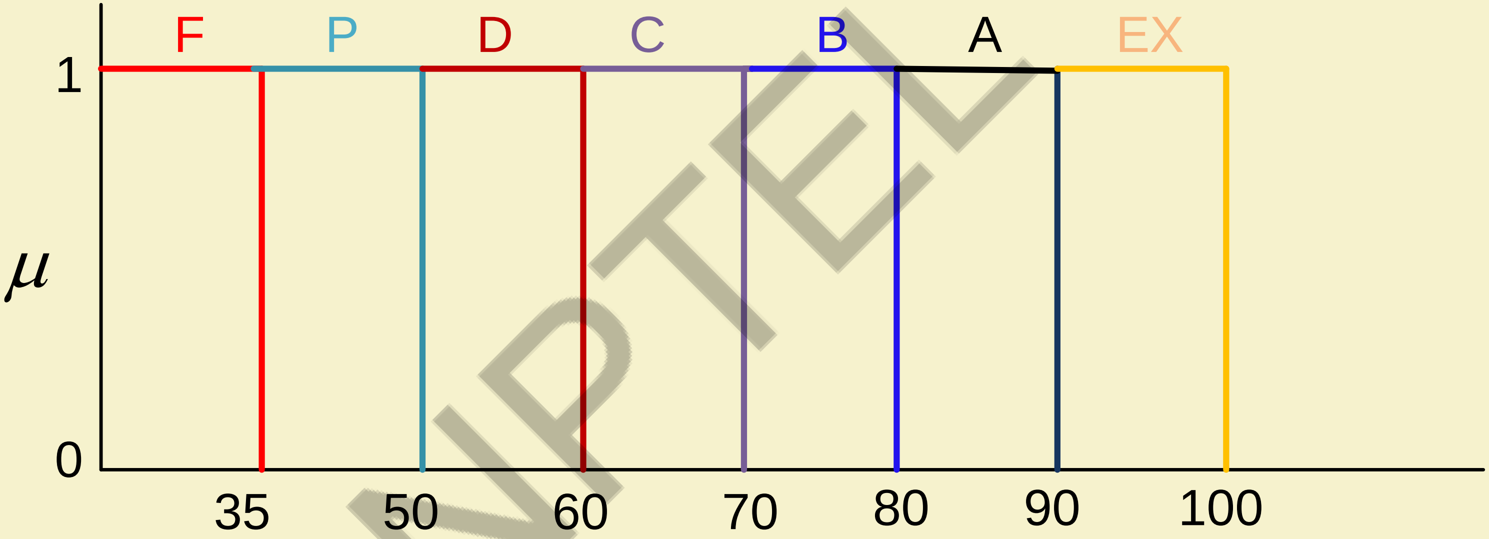
C : $60 \leq$ Marks < 70

D : $50 \leq$ Marks < 60

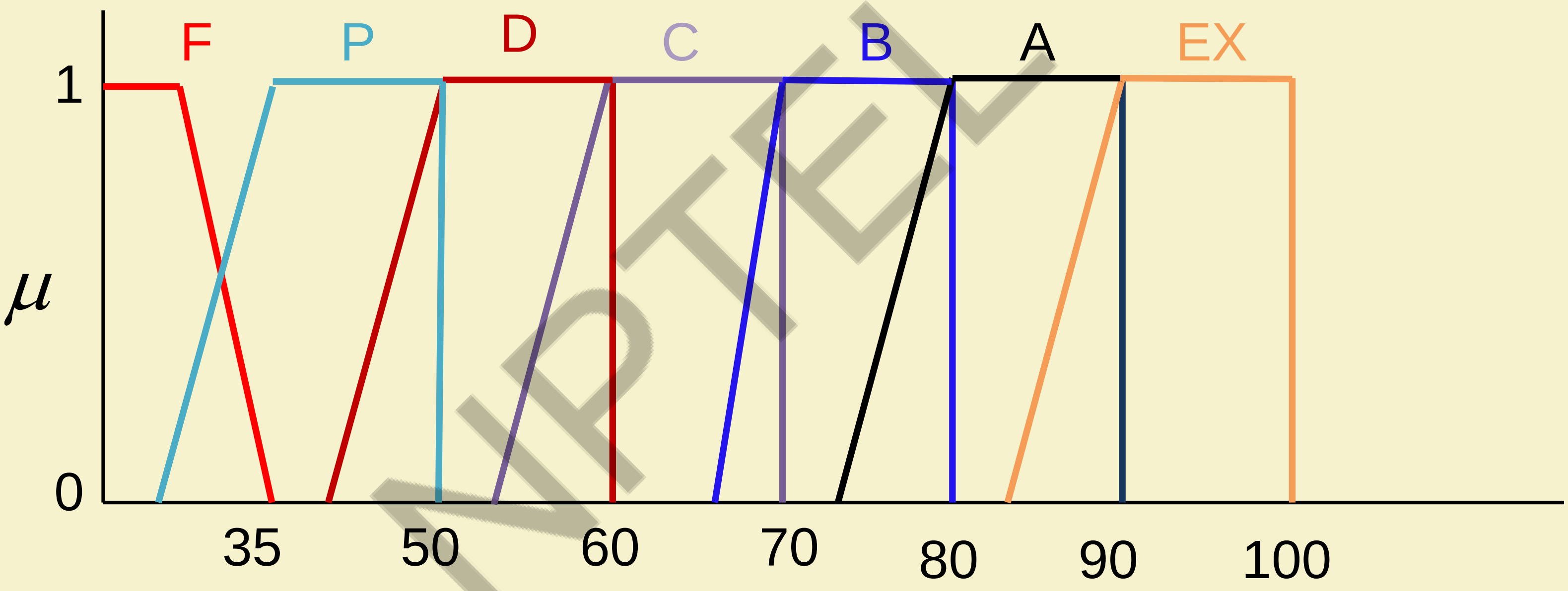
P : $35 \leq$ Marks < 50

F : Marks ≤ 35

Example: Course evaluation in a crisp way



Example: Course evaluation in a fuzzy way



Few examples of fuzzy set

- High Temperature
- Low Pressure
- Colour of Apple
- Sweetness of Orange
- Weight of Mango

Note: Degree of membership values lie in the range [0...1].

Some basic terminologies and notations

Definition 1: Membership function (and Fuzzy set)

If X is a universe of discourse and $x \in X$, then a fuzzy set A in X is defined as a set of ordered pairs, that is
$A = \{(x, \mu_A(x)) | x \in X\}$ where $\mu_A(x)$ is called the membership function for the fuzzy set A.

Note: $\mu_A(x)$ map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

Question: How (and who) decides $\mu_A(x)$ for a fuzzy set A in X?

Some basic terminologies and notations

Example:

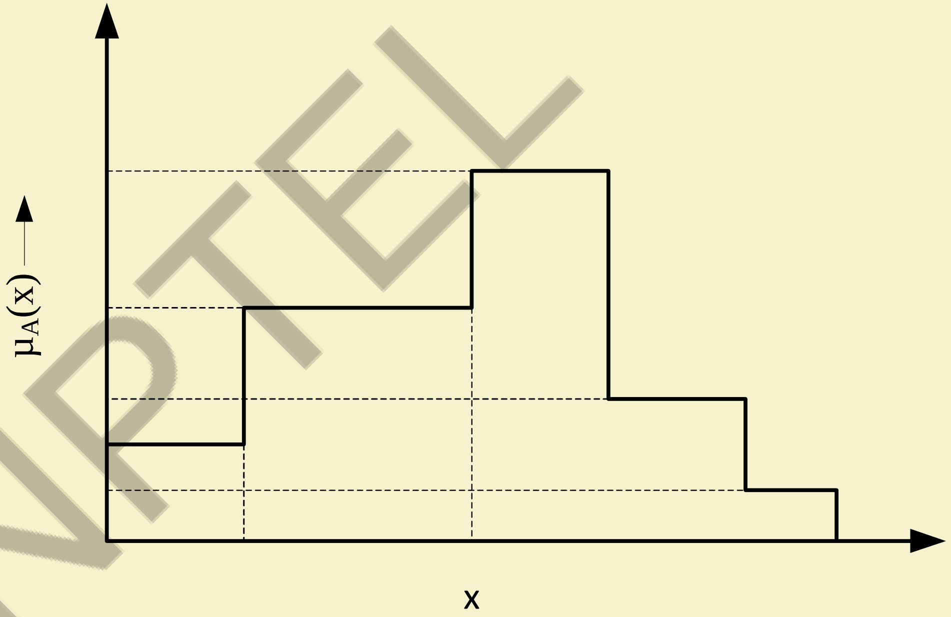
X = All cities in India

A = City of comfort

A={(New Delhi, 0.7), (Bangalore, 0.9), (Chennai, 0.8), (Hyderabad, 0.6), (Kolkata, 0.3), (Kharagpur, 0)}

Membership function with discrete membership values

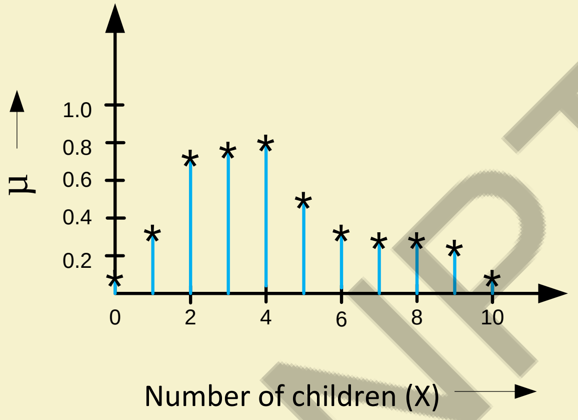
The membership values may be of discrete values.



A fuzzy set with discrete values of μ

Membership function with discrete membership values

Either elements or their membership values (or both) also may be of discrete values.



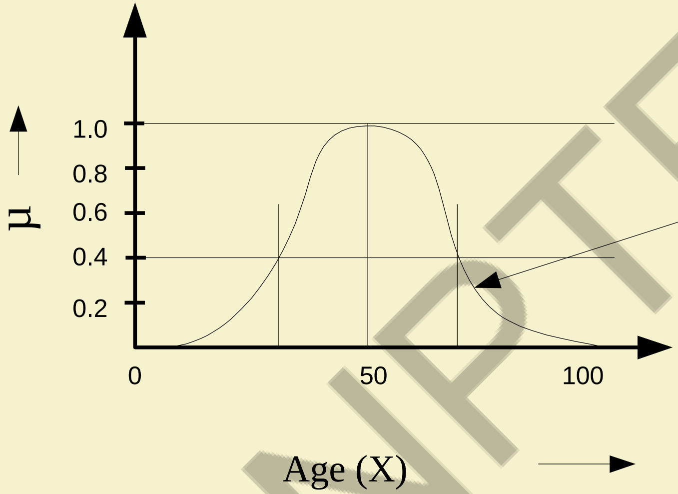
A = “Happy family”

A ={(0,0.1),(1,0.30),(2,0.78)……(10,0.1)}

Note : X = discrete value

How you measure happiness ??

Membership function with continuous membership values



$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$

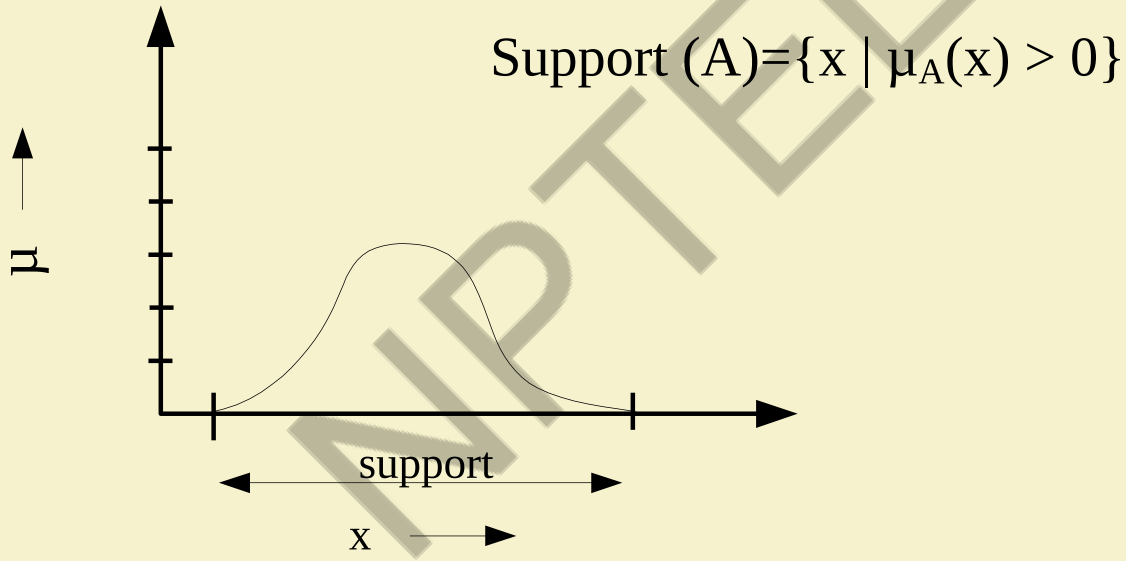
$$B=\{(x,\mu_B(x))\}$$

B = "Middle aged"

Note : x = real value = R^+

Fuzzy terminologies: Support

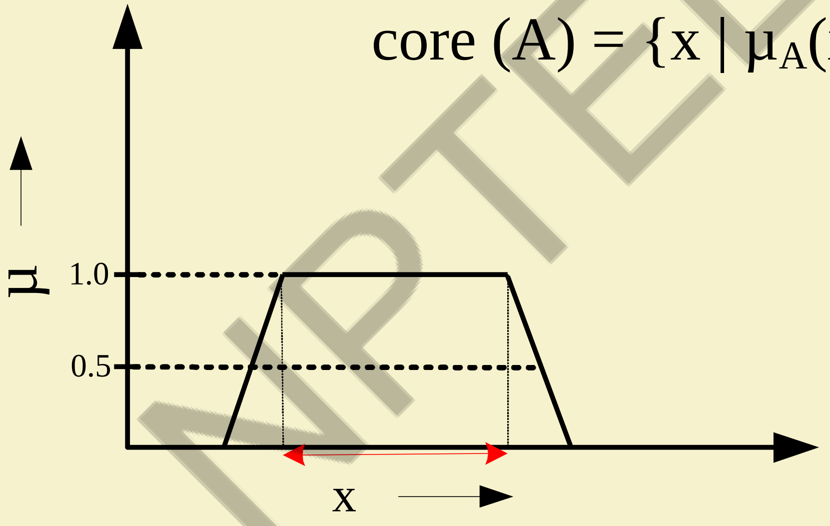
Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(X) > 0$



Fuzzy terminologies: Core

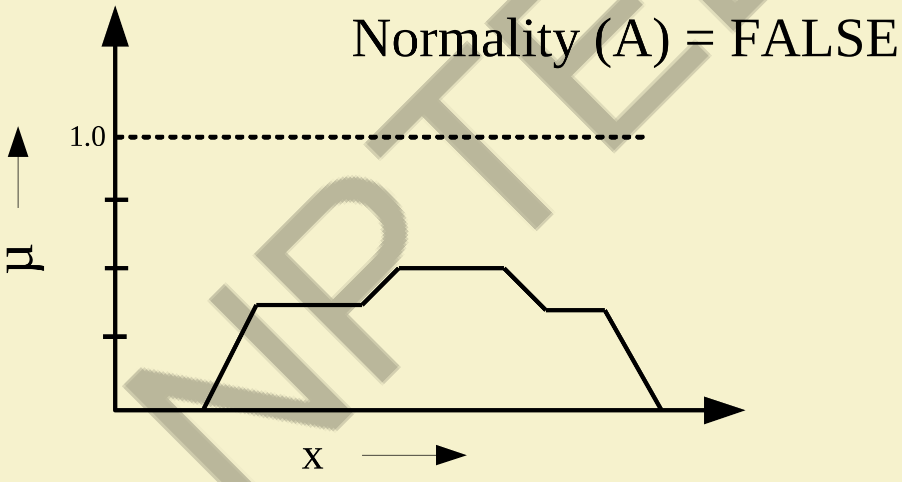
Core: The core of a fuzzy set A is the set of all points x in X such that $\mu_A(X) = 1$

$$\text{core}(A) = \{x \mid \mu_A(x) = 1\}$$



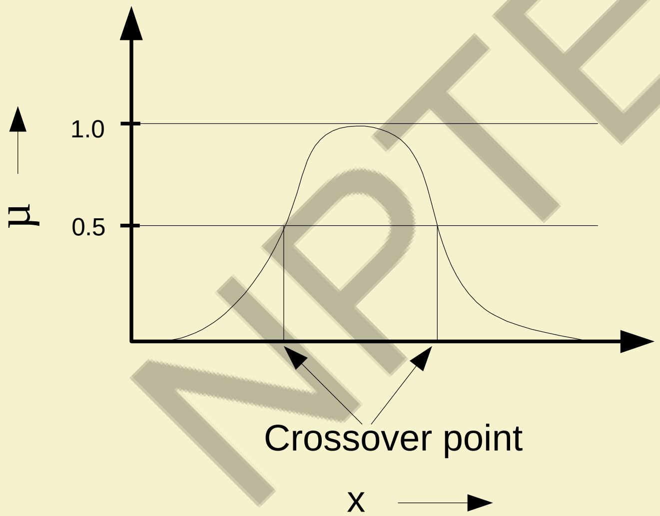
Fuzzy terminologies: Normality

Normality : A fuzzy set A is a normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(X) = 1$



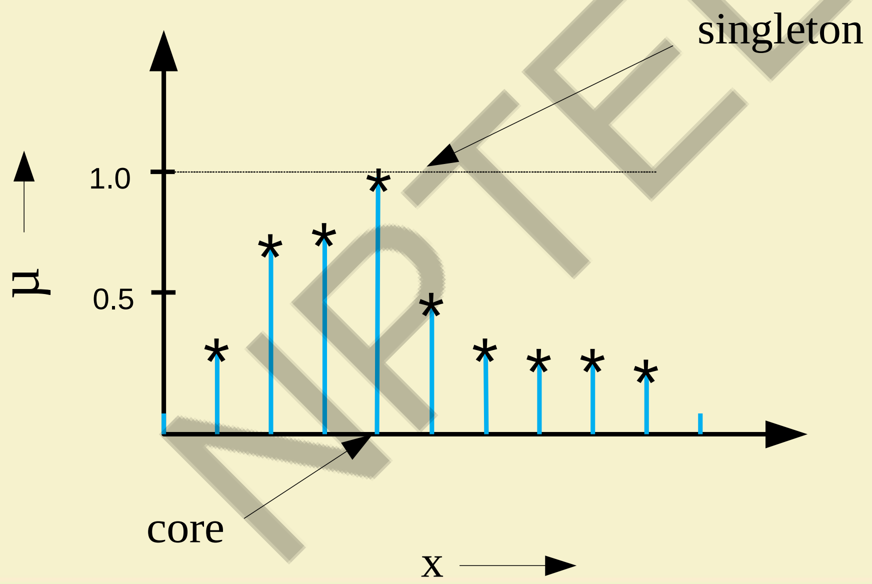
Fuzzy terminologies: Crossover points

Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(X) = 0.5$. That is Crossover (A) $=\{x|\mu_A(x) = 0.5\}$



Fuzzy terminologies: Fuzzy Singleton

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = \{x|\mu_A(x) = 1\}$



Fuzzy terminologies: α-cut and strong α-cut

α-cut and strong α-cut :

- ✓ The α-cut of a fuzzy set A is a crisp set defined by

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\}$$

- ✓ Strong α-cut is defined similarly :

$$A'_\alpha = \{x | \mu_A(x) > \alpha\}$$

Note : Support (A) = A_0' and Core (A) = A_1.

Fuzzy terminologies: Bandwidth

Bandwidth :
For a fuzzy set, the bandwidth (or width) is defined as the distance between the two unique crossover points:

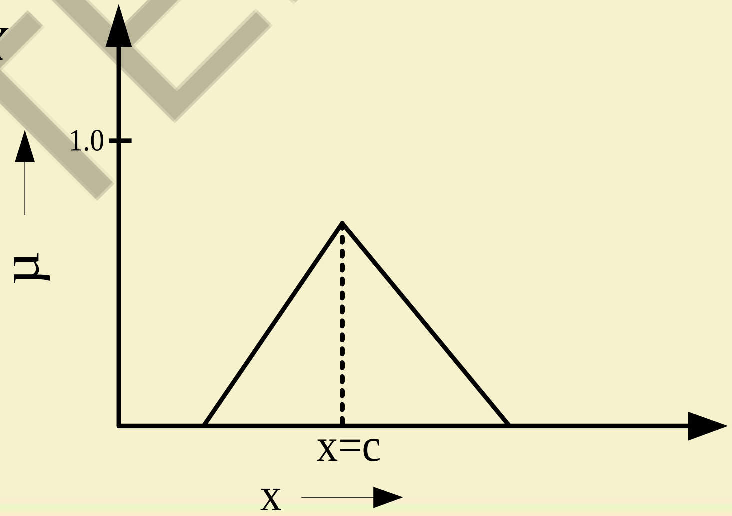
$$\text{Bandwidth}\ (A) = |x_1 - x_2|$$

where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Fuzzy terminologies: Symmetry

Symmetry :

A fuzzy set A is symmetric if its membership function around a certain point $x = c$, namely $\mu_A(x + c) = \mu_A(x - c)$ for all $x \in X$



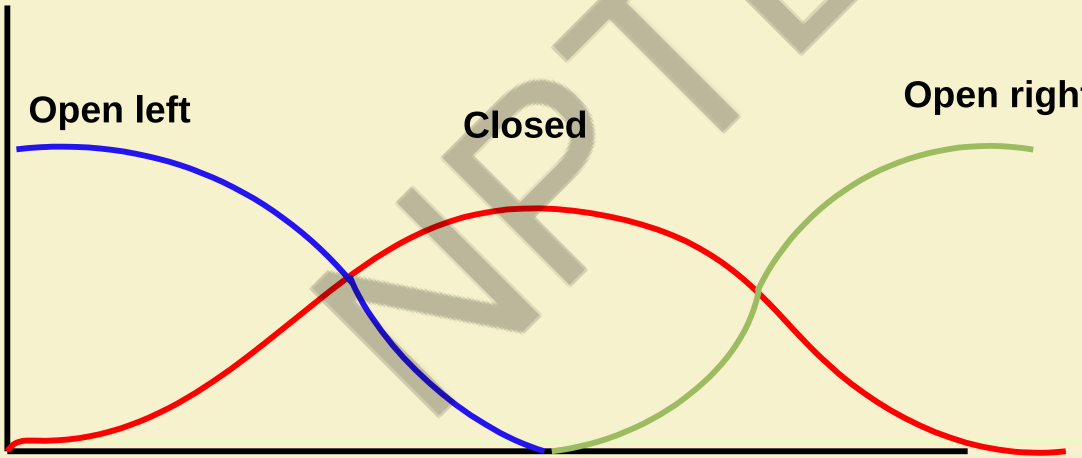
Fuzzy terminologies: Open and Closed

A fuzzy set A is

Open left : If $\lim x \to -\infty\ \mu_A(x) = 1$and $\lim x \to +\infty\ \mu_A(x) = 0$

Open right: If $\lim x \to -\infty\ \mu_A(x) = 0$and $\lim x \to +\infty\ \mu_A(x) = 1$

Closed: If $\lim x \to -\infty\ \mu_A(x) = \lim x \to +\infty\ \mu_A(x) = 0$



Fuzzy vs. Probability

Fuzzy : When we say about certainty of a thing

Example: A patient come to the doctor and he has to diagnose so that medicine can be prescribed.

Doctor prescribed a medicine with certainty 60% that the patient is suffering from flue. So, the disease will be cured with certainty of 60% and uncertainty 40%. Here, in stead of flue, other diseases with some other certainties may be.

Probability: When we say about the chance of an event to occur

Example: India will win the T20 tournament with a chance 60% means that out of 100 matches, India own 60 matches.

Prediction vs. Forecasting

The Fuzzy vs. Probability is analogical to Prediction vs. Forecasting

Prediction : When you start guessing about things.

Forecasting : When you take the information from the past job and apply it to new job.

The main difference:

Prediction is based on the best guess from experiences.

Forecasting is based on data you have actually recorded and packed from previous job.

Thank You!!

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Introduction to Soft Computing

Fuzzy membership functions

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Fuzzy membership functions

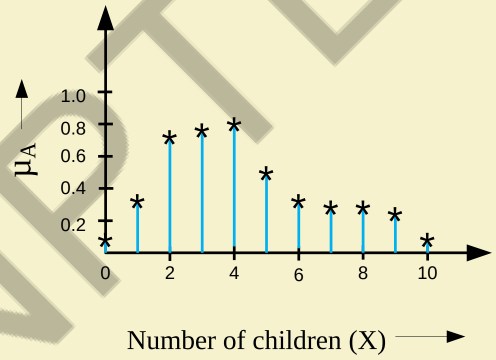
A fuzzy set is completely characterized by its membership function (sometimes abbreviated as MF and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

Example:

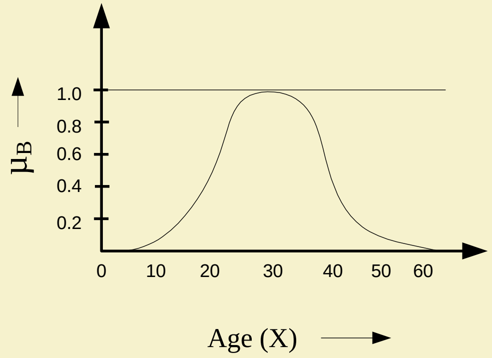
Note: A membership function can be on

a) a discrete universe of discourse and

b) a continuous universe of discourse.



A = Fuzzy set of “Happy family”

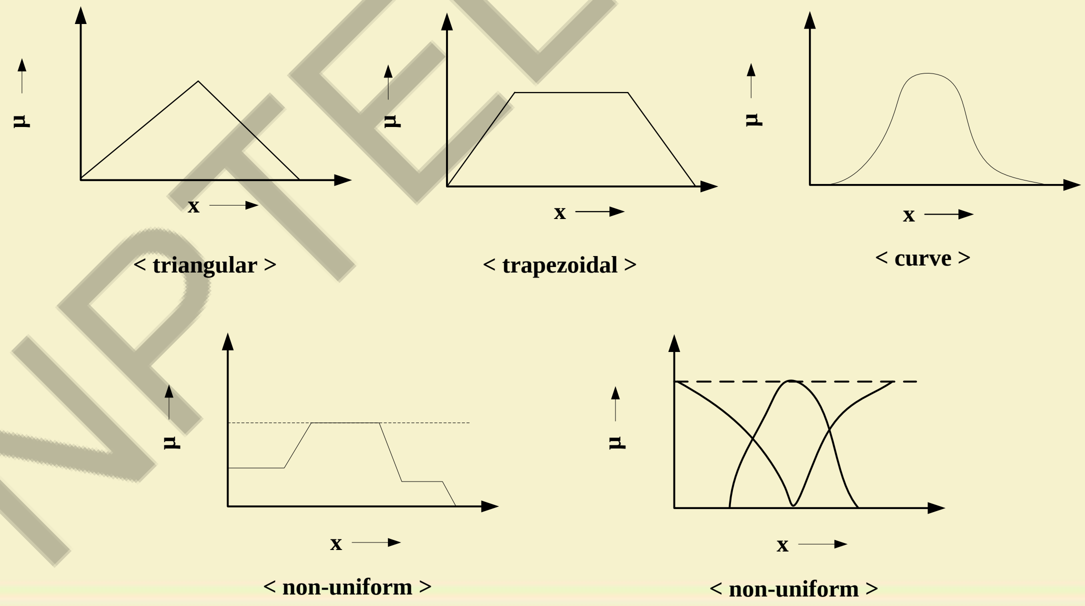


B = “Young age”

Fuzzy membership functions

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

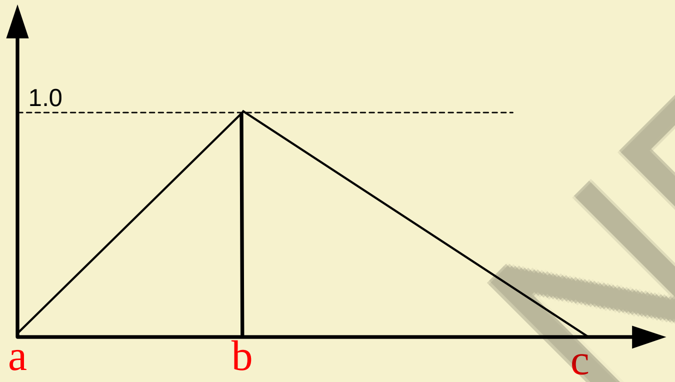
Following figures shows typical examples of membership functions.



Fuzzy MFs : Formulation and parameterization

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

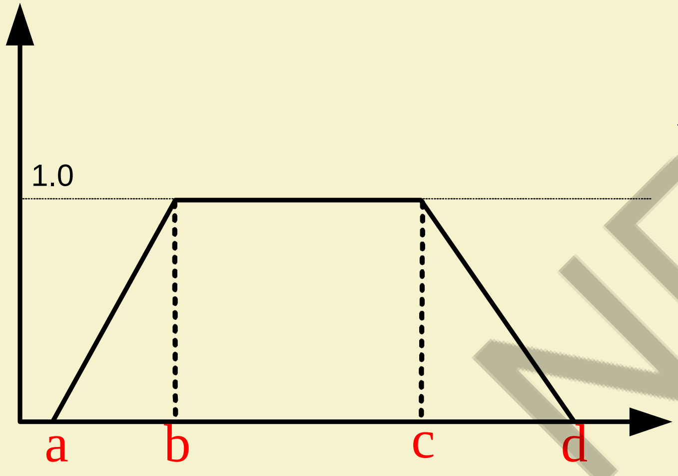
Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.



$$\text{triangle}\,(x; a, b, c) = \begin{cases} 0 & if\ x \leq a \\ \dfrac{x-a}{a-b} & if\ a \leq x \leq b \\ \dfrac{c-x}{c-b} & if\ b \leq x \leq c \\ 0 & if\ c \leq x \end{cases}$$

Fuzzy MFs: Trapezoidal

A **trapezoidal MF** is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

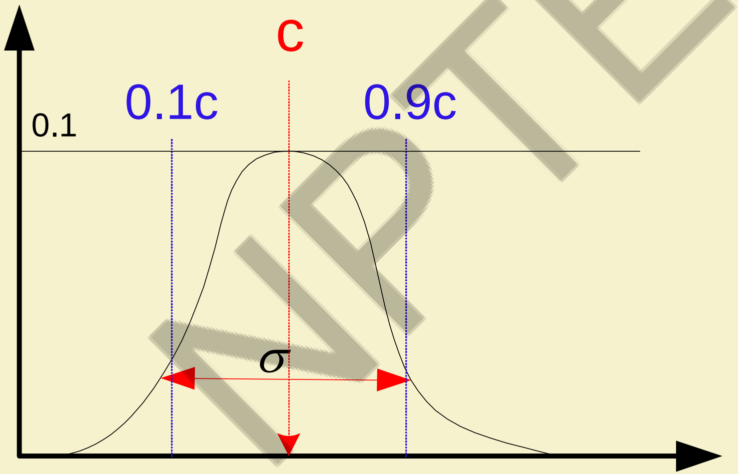


$$\text{trapozoid}\,(x; a, b, c, d) = \begin{cases} 0 & if\ x \leq a \\ \dfrac{x-a}{b-a} & if\ a \leq x \leq b \\ 1 & if\ b \leq x \leq c \\ \dfrac{d-x}{d-c} & if\ c \leq x \leq d \\ 0 & if\ d \leq x \end{cases}$$

Fuzzy MFs: Gaussian

A **Gaussian MF** is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

$$gaussian\ (x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$



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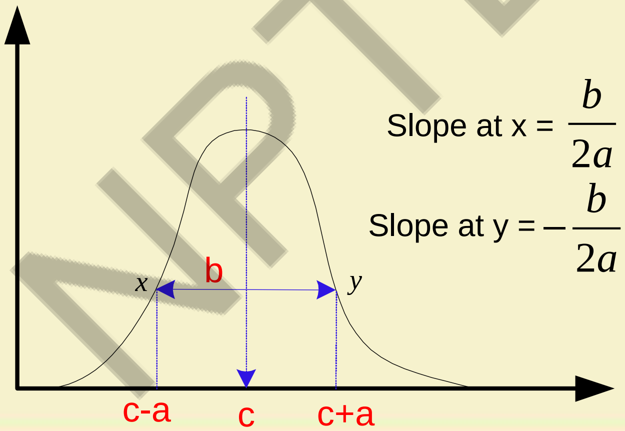
Fuzzy MFs: Generalized bell

It is also called Cauchy MF. A generalized bell MF is specified by three parameters $\{a, b, c\}$ and is defined as:

$$bell(x;\ a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$



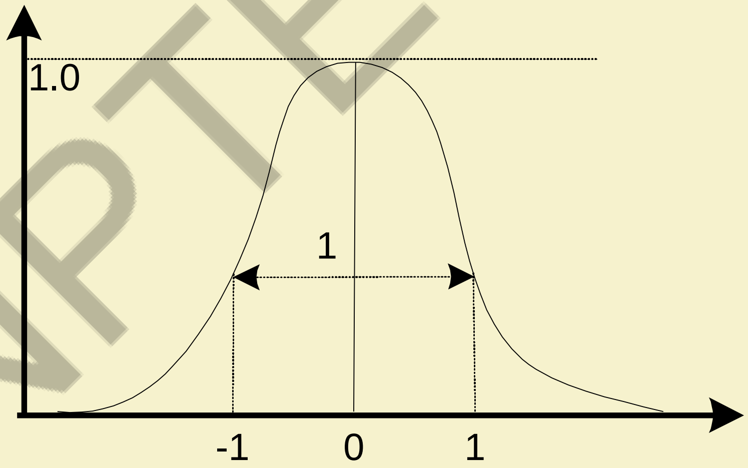
Slope at x = $\frac{b}{2a}$

Slope at y = $-\frac{b}{2a}$

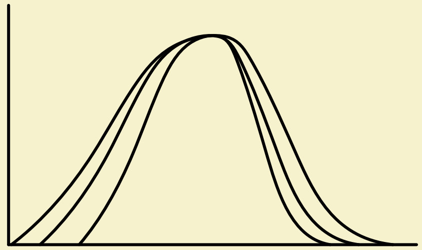
Example: Generalized bell MFs

Example: $\mu(x) = \frac{1}{1+|x|^2}$;

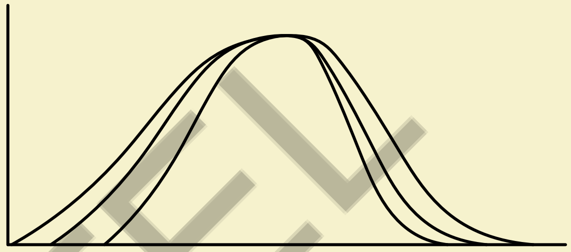
a = b = 1 and c = 0;



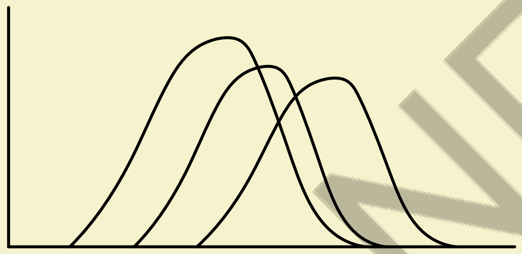
Generalized bell MFs: Different shapes



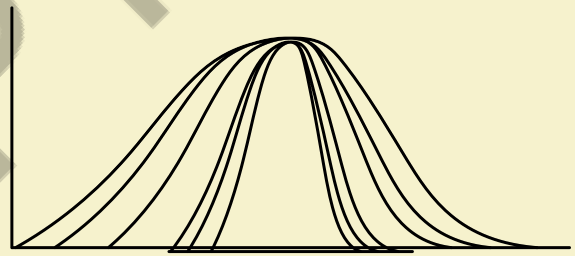
Changing a



Changing b



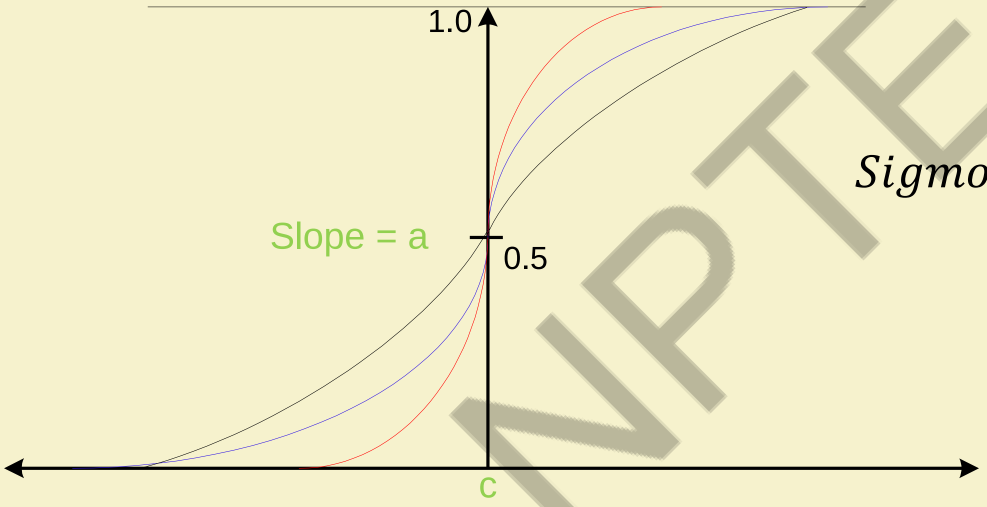
Changing a



Changing a and b

Fuzzy MFs: Sigmoidal MFs

Parameters: $\{a, c\}$; where c = crossover point and a = slope at c;



$$Sigmoid(x;\ a, c) = \frac{1}{1 + e^{-\left[\frac{x-c}{a}\right]}}$$

Fuzzy MFs : Example

Example : Consider the following grading system for a course.

Excellent = Marks $\leq$ 90

Very good = 75 $\leq$ Marks $\leq$ 90

Good = 60 $\leq$ Marks $\leq$ 75

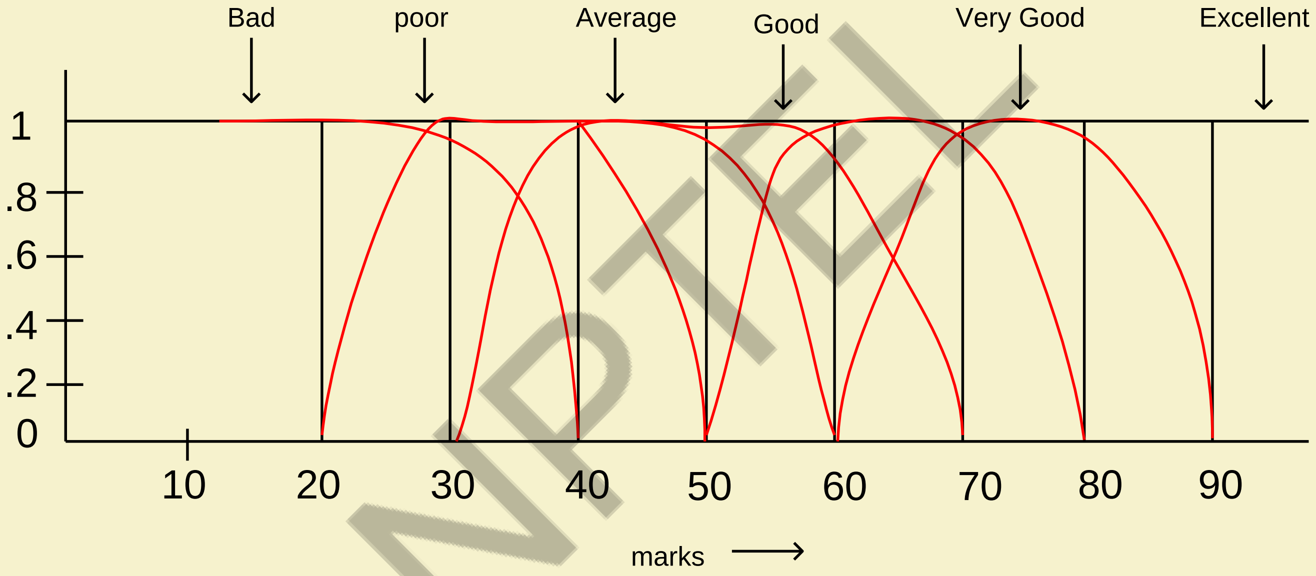
Average = 50 $\leq$ Marks $\leq$ 60

Poor = 35 $\leq$ Marks $\leq$ 50

Bad= Marks $\leq$ 35

Grading System

A fuzzy implementation will look like the following.



You can decide a standard fuzzy MF for each of the fuzzy grade.

Few More on Membership Functions

Generation of MFs

Given a membership function of a fuzzy set representing a linguistic hedge, we can derive many more MFs representing several other linguistic hedges using the concept of Concentration and Dilation.

1. **Concentration:** $A^k = [\mu_A(x)]^k; k > 1$
2. **Dilation:** $A^k = [\mu_A(x)]^k; k < 1$

Example : Age = { Young, Middle-aged, Old }

Thus, corresponding to Young, we have : Not young, Very young, Not very young and so on.

Similarly, with Old we can have : Not old, Very old, Very very old, Extremely old, etc.

Thus, $\mu_{Extremely\ old}(x) = (((\mu_{Old}(x))^2)^2)^2$ and so on

Or, $\mu_{More\ or\ less\ old}(x) = A^{0.5} = (\mu_{Old}(x))^{0.5}$

Linguistic variables and values

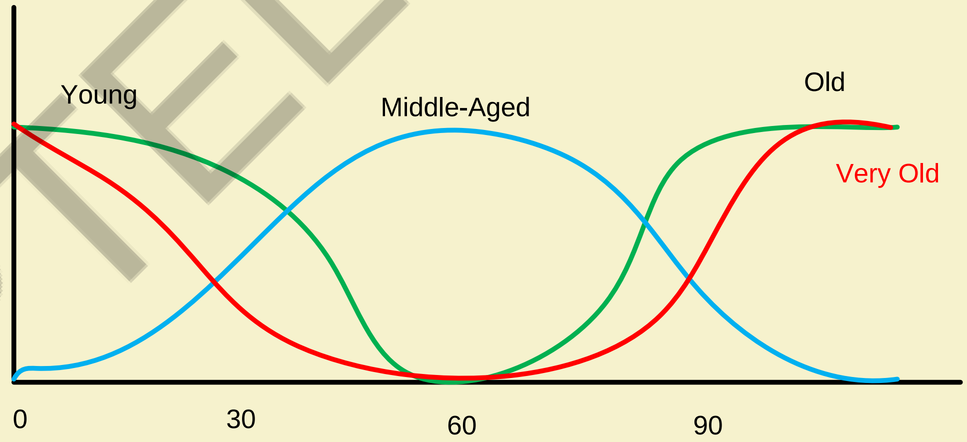
$\mu_{young}(x)$ =bell(x,20,2,0) $= \frac{1}{1+(\frac{x}{20})^4}$

$\mu_{old}(x)$ =bell(x,30,3,100) $= \frac{1}{1+(\frac{x-100}{30})^6}$

$\mu_{middle-aged}(x)$ =bell(x,30,60,50)

Not young=$\overline{\mu_{young}(x)} = 1 - \mu_{young}(x)$

Young but not too young = $\mu_{young}(x) \cap \overline{\mu_{young}(x)}$



Thank You!!

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Introduction to Soft Computing

Operations on Fuzzy sets

Prof. Debasis Samanta

Department of Computer Science and Engineering

IIT Kharagpur

Basic fuzzy set operations: Union

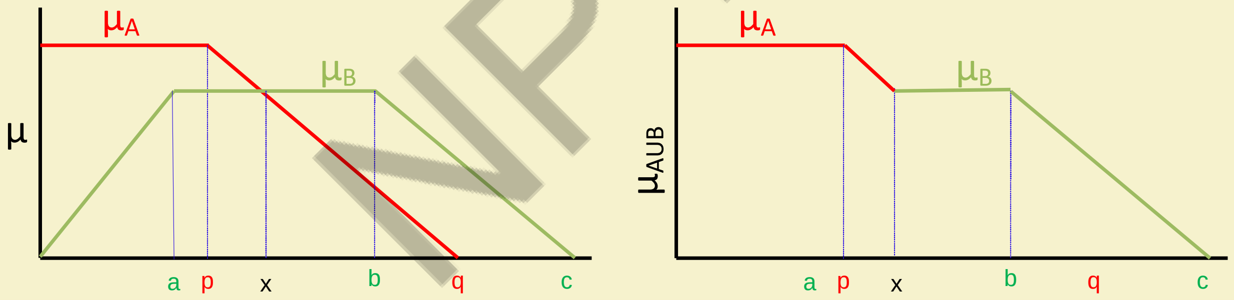
Union (A ∪ B): $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\};$

$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$



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Basic fuzzy set operations: Intersection

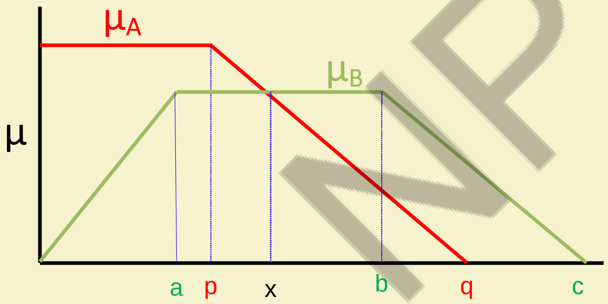
Intersection (A ∩ B): $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$

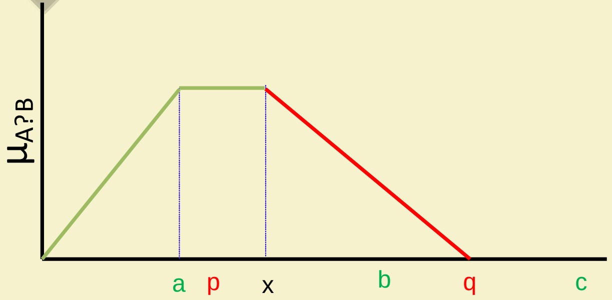
Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;

$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$





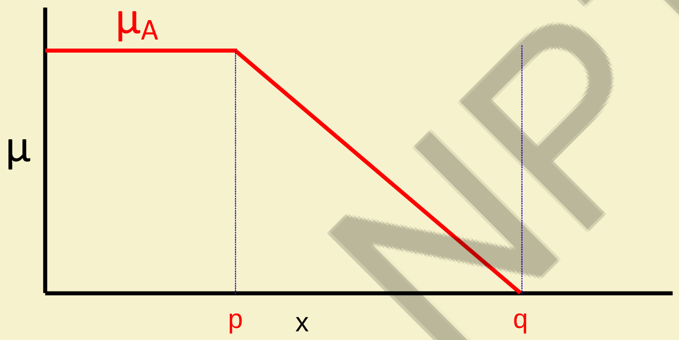
Basic fuzzy set operations: Complement

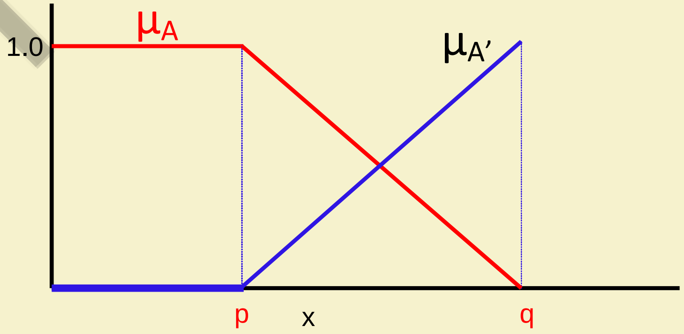
Complement (A^c): $\mu_{A^c}(x) = 1 - \mu_A(x)$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$C = A^c = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$





Basic fuzzy set operations: Products

Algebric product or Vector product ($A \cdot B$):

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Scalar product ($\alpha \times A$):

$$\mu_{\alpha A}(x) = \alpha \times \mu_A(x)$$

Basic fuzzy set operations: Sum and Difference

Sum (A + B)**:**

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)$$

Difference ($A - B = A \cap B^C$) **:**

$$\mu_{A-B}(x) = \mu_{A\cap B^C}(x)$$

Disjunctive sum:

$$A \oplus B = (A^C \cap B) \cup (A \cap B^C)$$

Bounded Sum:

$$| A(x) \oplus B(x)| = \mu_{|A(x)\oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Bounded Difference:

$$| A(x) \ominus B(x)| = \mu_{|A(x)\ominus B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Basic fuzzy set operations: Equality and Power

Equality ($A = B$):

$$\mu_A(x) = \mu_B(x)$$

Power of a fuzzy set A^α:

$$\mu_{A^\alpha}(x) = (\mu_A(x))^\alpha$$

- ✓ If $\alpha < 1$, then it is called dilation
- ✓ If $\alpha > 1$, then it is called concentration

Basic fuzzy set operations: Cartesian product

Caretsian Product ($A \times B$): $\mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

Example:

$A(x) = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$
$B(y) = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$

$$\mathrm{A} \times B = \min\big(\mu_A(x), \mu_B(y)\big) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix} \end{array}$$

Properties of fuzzy sets

Commutativity :

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Properties of fuzzy sets

Idempotence :

$$A \cup A = A$$
$$A \cap A = \emptyset\,;$$
$$A \cup \emptyset; = A$$
$$A \cap \emptyset\,; = \emptyset;$$

Transitivity :

If $A \subseteq B; B \subseteq C$ then $A \subseteq C$

Involution :

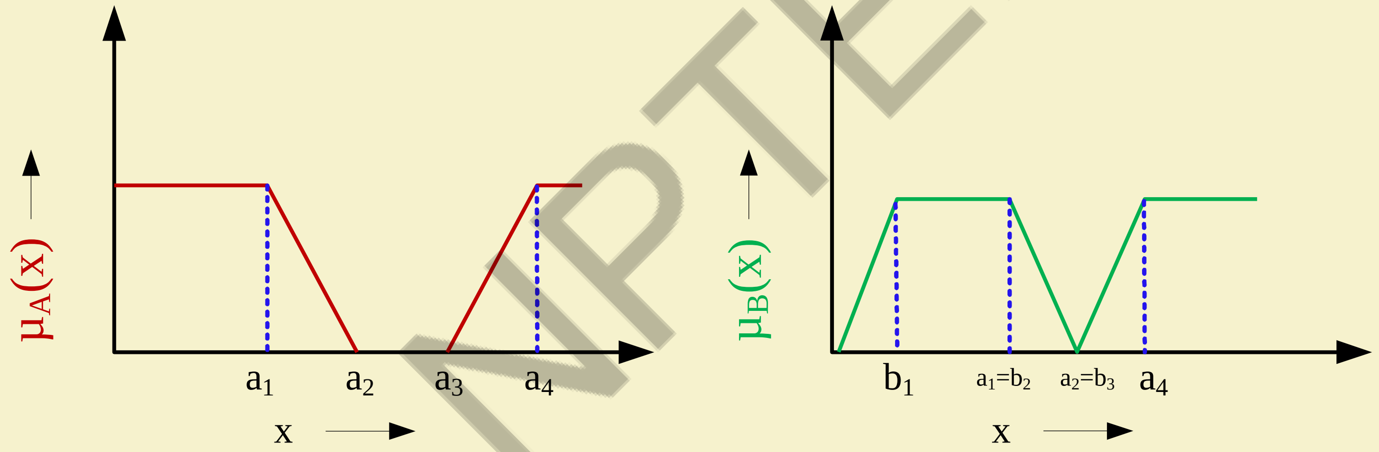
$$(A^c)^c = A$$

De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

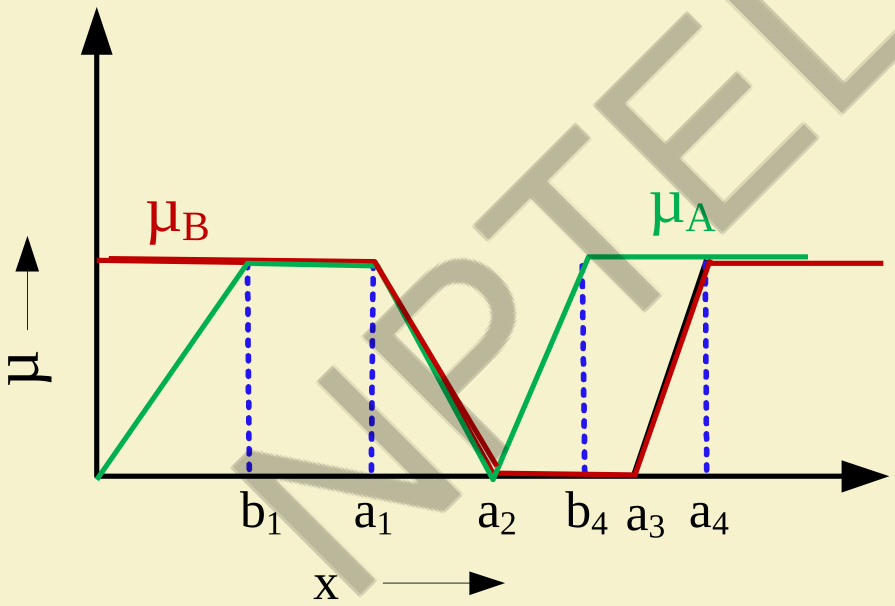
Example 1: Fuzzy Set Operations

Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Two MFs $\mu_A(x)$ and $\mu_B(x)$ are shown graphically.



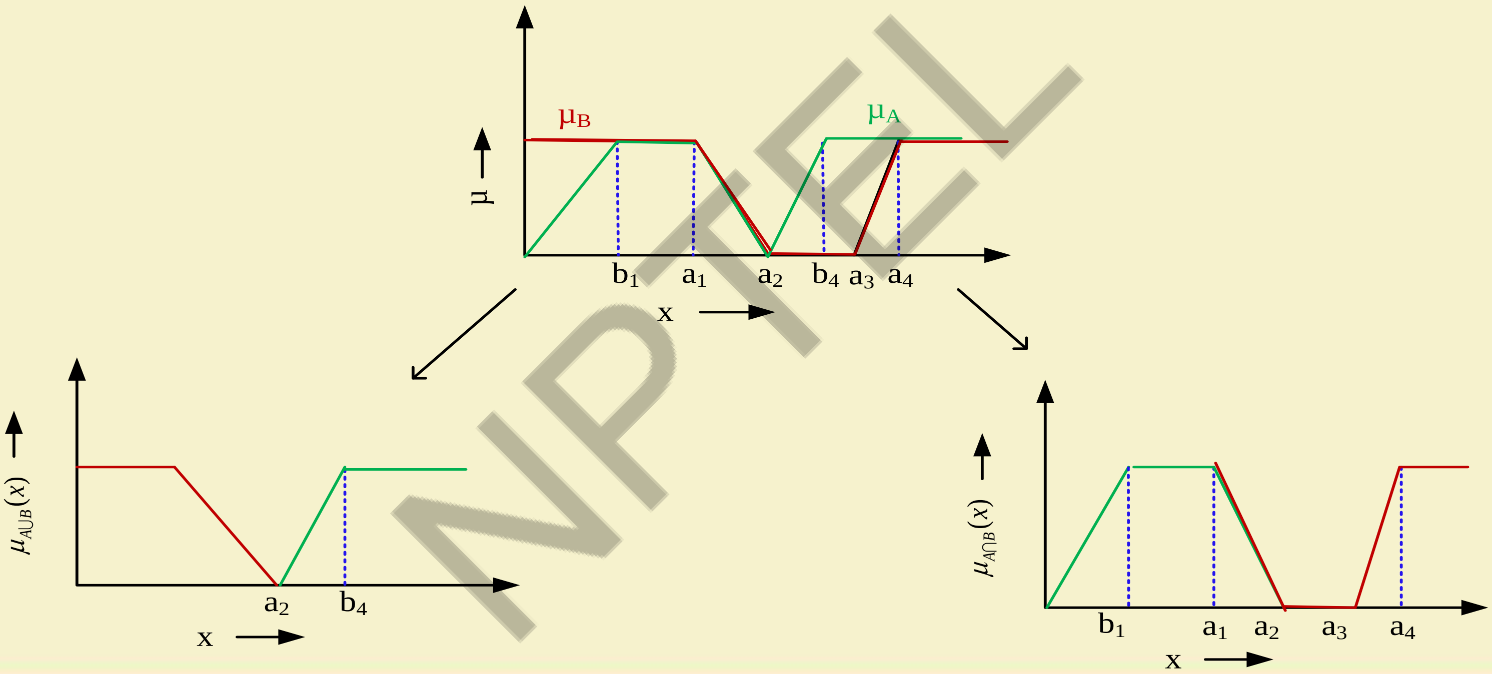
Example 1: Plotting two sets on the same graph

Let's plot the two membership functions on the same graph



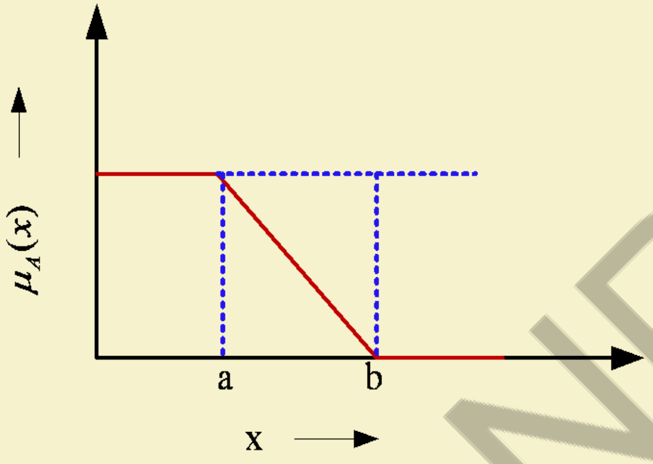
Example 1: Union and Intersection

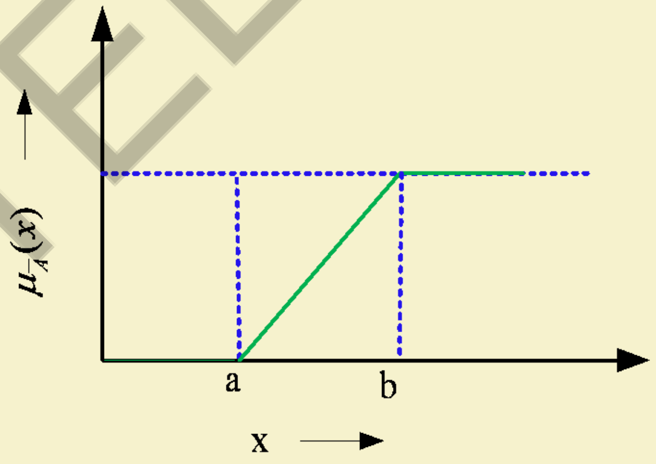
The plots of union $A \cup B$ and intersection $A \cap B$ are shown in the following.



Example 1: Complementation

The plots of union $\mu_{\bar{A}}(x)$ of the fuzzy set A is shown in the following.





Fuzzy set operations: Practice

Consider the following two fuzzy sets A and B defined over a universe of discourse [0,5] of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

Determine the membership functions of the following and draw them graphically.

I. $\bar{A}, \bar{B}$
II. $A \cup B$
III. $A \cap B$
IV. $(A \cup B)^c$

[Hint: Use De' Morgan law]

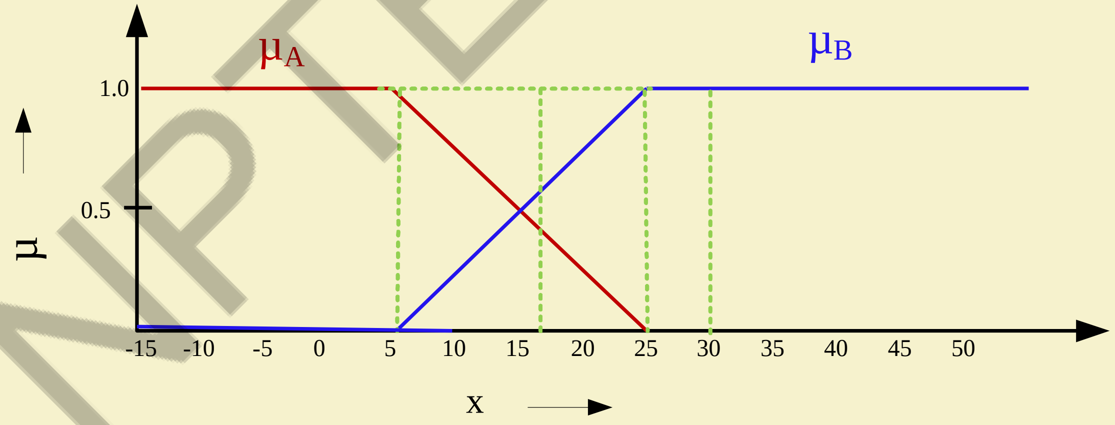
Example 2: A real-life example

Two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.

A = **Cold climate** with $\mu_A(x)$ as the MF.

B = **Hot climate** with $\mu_B(x)$ as the M.F.

Here, X being the universe of discourse representing entire range of temperatures.



Example 2: A real-life example

What are the fuzzy sets representing the following?

1. **Not cold climate**
2. **Not hot climate**
3. **Extreme climate**
4. **Pleasant climate**

Note: Note that ”Not cold climate” $\neq$ ”Hot climate” and vice-versa.

Example 2: A real-life example

Answer would be the following.

- ✓ **Not cold climate**

  $\bar{A}$ with $1 - \mu_A(x)$ as the MF.

- ✓ **Not hot climate**

  $\bar{B}$ with $1 - \mu_B(x)$ as the MF.

- ✓ **Extreme climate**

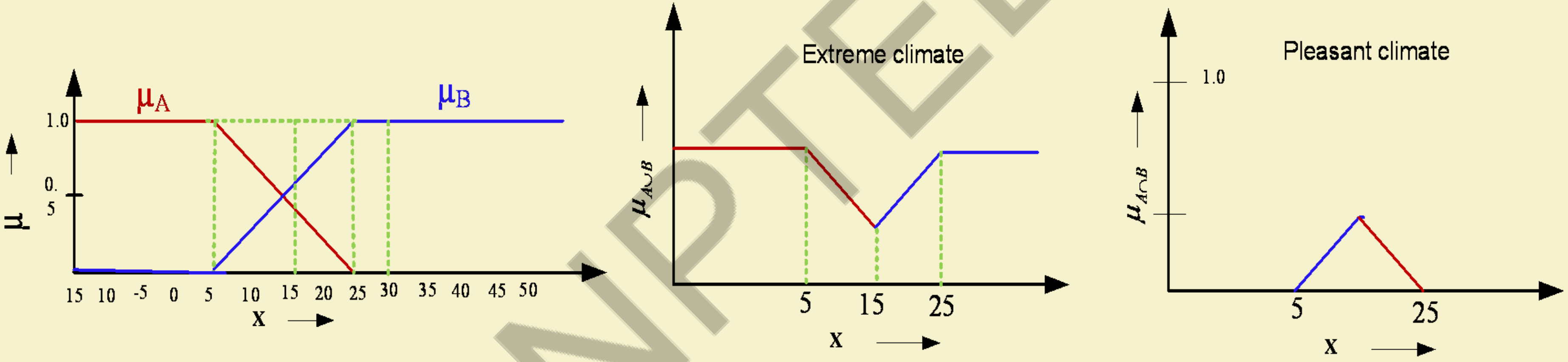
  $A \cup B$ with $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ as the MF.

- ✓ **Pleasant climate**

  $A \cap B$ with $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ as the MF.

Example 2: A real-life example

The plot of the MFs of $A \cup B$ and $A \cap B$ are shown in the following.



Thank You!!

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Soft Computing

Fuzzy Relations

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Fuzzy Relations

- Crisp relations
- Operations on crisp relations
- Examples on crisp relations
- Fuzzy relations
- Operations on fuzzy relations
- Examples on fuzzy relations

Crisp relations

- **Order pairs:**

  Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Note :

(1) $A \times B \neq B \times A$ (2) $|A \times B| = |A| \times |B|$

(3) $A \times B$ provides a mapping from $a \in A$ to $b \in B$.

A particular mapping so mentioned is called a relation.

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Crisp relations

Example:

Consider the two crisp sets A and B as given below.
$A = \{1, 2, 3, 4\}$ $B = \{3, 5, 7\}$.

Then, $A \times B = \{(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (3,3), (3,5), (3,7), (4,3), (4,5), (4,7)\}$

Let us define a relation as $R = \{(a, b) | b = a + 1, (a, b) \in A \times B\}$

Then, $R = \{(2,3), (4,5)\}$ in this case.

Crisp relations

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

- **Union:** $R(x, y) \cup S(x, y) = max(R(x, y), S(x, y));$
- **Intersection:** $R(x, y) \cap S(x, y) = min(R(x, y), S(x, y));$
- **Complement:** $\overline{R(x, y)} = 1 - R(x, y)$

Example: Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the following

- $R \cup S$
- $R \cap S$
- $\bar{R}$

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z. Then, $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x, z) | (x, y) \in R \; and \; (y, z) \in S \; and \; \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S, the max-min composition is defined as $T = R \circ S$;

$$T(x, z) = max\{min\{R(x, y), S(y, z) \; and \; \forall y \in Y\}\}$$

Composition: Composition

Example : Given $X = \{1,3,5\}$; $Y = \{1,3,5\}$; $R = \{(x,y)|y = x + 2\}$;
$S = \{(x,y)|x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have $R = \{(1,3),(3,5)\}$, $S = \{(1,3),(1,5),(3,5)\}$

$$R = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad and \quad S = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Using max-min composition

$$R \circ S = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X_1, X_2, \ldots, X_n$
- Here, n-tuples $(x_1, x_2, \ldots, x_n)$ may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Fuzzy relations

Example:

$X = \{typhoid, viral, cold\}, Y = \{running\ nose, high\ temp, shivering\}$

The fuzzy relation R is defined as

$$R = \begin{array}{c c} & \begin{array}{ccc} running\ nose & high\ temperature & shivering \end{array} \\ \begin{array}{c} typhoid \\ viral \\ cold \end{array} & \begin{bmatrix} 0.1 & 0.9 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

Fuzzy Cartesian product

Suppose

- A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$
- B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given

by $\mu_R(x,y) = \mu_{AxB}(x,y) = min\{\mu_A(x), \mu_B(y)\}$

Fuzzy Cartesian product

Example :

$A = \{(a1, 0.2), (a2, 0.7), (a3, 0.4)\}$ and $B = \{(b1, 0.5), (b2, 0.6)\}$

$$R = A \times B = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

- **Union:** $\mu_{R\cup S}(a,b) = max\{\mu_R(a,b), \mu_S(a,b)\}$
- **Intersection:** $\mu_{R\cap S}(a,b) = min\{\mu_R(a,b), \mu_S(a,b)\}$
- **Complement:** $\mu_{\bar{R}}(a,b) = 1 - \mu_R(a,b)$
- **Composition:** $T = R \circ S$

$$\mu_{R\circ S} = max_{y\in Y}\{min\big(\mu_R(x,y), \mu_S(y,z)\big)\}$$

Operations on Fuzzy relations: Example

Example : $X = (x_1, x_2, x_3),\ \ Y = (y_1, y_2),\ \ Z = (z_1, z_2, z_3),$

$$R = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array} \quad and \quad S = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

$$R \circ S = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$\mu_{R\circ S}\,(x_1, y_1) = max\{min(\mu_R(x_1, y_1), \mu_S(y_1, z_1))\,, min(\mu_R(x_1, y_2)\,, \mu_S(y_2, z_1))\}$

$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = \ 0.5\ and\ so\ on.$

Fuzzy relation : An example

Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants

$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants.

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, which is stated as

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : An example

Also, consider T be the another relation on $D \times S$, which is given by

$$S = \begin{array}{c c} & \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} & \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 1.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition**.

Hint: Find $R \circ T$, and verify that

$$R \circ S = \begin{array}{c c} & \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy relation : Another example

Let, $R = x$ is relevant to y

and $S = y$ is relevant to z

be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively, where $X = \{1,2,3\}$, $Y = \{\alpha, \beta, \gamma, \delta\}$ and $Z = \{a, b\}$. Assume that R and S can be expressed with the following relation matrices :

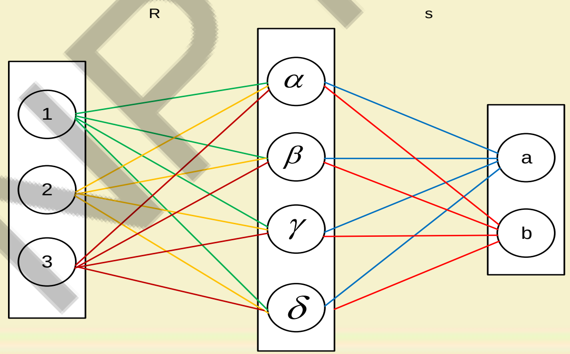
$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \end{array} \quad and \quad S = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : Another example

Now, we want to find $R \circ S$, which can be interpreted as a derived fuzzy relation x **is relevant to** z.

Suppose, we are only interested in the degree of relevance between $2 \in X$ and $a \in Z$. Then, using max-min composition,

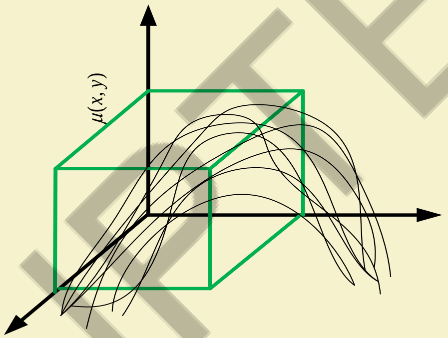
$$\mu_{R \circ S}(2, a) = \max\{(0.4 \wedge 0.9), (0.2 \wedge 0.2), (0.8 \wedge 0.5), (0.9 \wedge 0.7)\}$$
$$= \max\{0.4, 0.2, 0.5, 0.7\} = 0.7$$



2D Membership functions : Binary fuzzy relations

(Binary) fuzzy relations are fuzzy sets $A \times B$ which map each element in $A \times B$ to a membership grade between 0 and 1 (both inclusive).

Note that a membership function of a binary fuzzy relation can be depicted with a 3D plot.



Important: Binary fuzzy relations are fuzzy sets with two dimensional MFs and so on.

2D membership function : An example

Let, $X = R^{+} = y$ (the positive real line) and

$R = X \times Y =$ "y is much greater than x"

The membership function of $\mu_R(x, y)$ is defined as

$$\mu_R(x, y) = \begin{cases} \dfrac{(y - x)}{4} & if\ y > x \\ 0 & if\ y \leq x \end{cases}$$

Suppose, $X = \{3,4,5\}$ and $Y = \{3,4,5,6,7\}$, then

$$R = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0.25 & 0.5 & 0.75 & 1.0 \\ 0 & 0 & 0.25 & 0.5 & 0.75 \\ 0 & 0 & 0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

Example:

How you can derive the following?

If x is A <u>or</u> y is B then z is C;

Given that

- R1: If x is A then z is C $[R1 \in A \times C]$
- R2: If y is B then z is C $[R2 \in B \times C]$

Hint:

- ✓ You have given two relations R1 and R2.
- ✓ Then, the required can be derived using the union operation of R1 and R2

Thank You!!