



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

# Soft Computing

## Fuzzy Propositions

Prof. Debasis Samanta

Department of Computer Science & Engineering  
IIT Kharagpur

# Fuzzy Propositions

- Two-valued logic vs. Multi-valued logic
- Examples of Fuzzy proposition
- Fuzzy proposition vs. Crisp proposition
- Canonical representation of Fuzzy proposition
- Graphical interpretation of Fuzzy proposition

# Two-valued logic vs. Multi-valued logic

- The basic assumption upon which crisp logic is based - that every proposition is either **TRUE** or **FALSE**.
- The classical two-valued logic can be extended to multi-valued logic.
- As an example, three valued logic to denote true(1), false(0) and indeterminacy ( 1/2 ).

# Two-valued logic vs. Multi-valued logic

Different operations with three-valued logic can be extended as shown in the truth table:

a	b	0	0	0	0	1	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	
0	1	0	1	1	1	1	0	
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	
$\frac{1}{2}$	1							
$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	
1	0	0	1	0	0	0	0	
1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$		
1	1	1	1	0	1	1		

Fuzzy connectives used in the above table are:

- AND ( $\wedge$ )
- OR ( $\vee$ )
- NOT ( $\neg$ )
- IMPLICATION ( $\Rightarrow$ ) and
- EQUAL (=)

# Three-valued logic

Fuzzy connectives defined for such a three-valued logic better can be stated as follows:

Symbol	Connective	Usage	Definition
	NOT		
	OR		
	AND		
	IMPLICATION		
	EQUALITY		

# Fuzzy proposition: Example 1

P: Ram is honest

- |              |                             |
|--------------|-----------------------------|
| $T(P) = 0.0$ | : Absolutely false          |
| $T(P) = 0.2$ | : Partially false           |
| $T(P) = 0.4$ | : May be false or not false |
| $T(P) = 0.6$ | : May be true or not true   |
| $T(P) = 0.8$ | : Partially true            |
| $T(P) = 1.0$ | : Absolutely true.          |

# Fuzzy proposition: Example 2

P : Mary is efficient ;  $T(P) = 0.8$

Q : Ram is efficient ;  $T(Q) = 0.6$

- **Mary is not efficient.**

$$T(\neg P) = 1 - T(P) = 0.2$$

- **Mary is efficient and so is Ram.**

$$T(P \wedge Q) = \min\{T(P), T(Q)\} = 0.6$$



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Fuzzy proposition: Example 2

P : Mary is efficient ;  $T(P) = 0.8$

Q : Ram is efficient ;  $T(Q) = 0.6$

- Either Mary or Ram is efficient

$$T(P \vee Q) = \max\{T(P), T(Q)\} = 0.8$$

- If Mary is efficient then so is Ram

$$T(P \Rightarrow Q) = \max\{1 - T(P), T(Q)\} = 0.6$$



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Fuzzy proposition vs. Crisp proposition

- The fundamental difference between crisp (classical) proposition and fuzzy propositions is in the range of their truth values.
- While each classical proposition is required to be either true or false, the truth or falsity of fuzzy proposition is a matter of degree.
- The degree of truth of each fuzzy proposition is expressed by a value in the interval  $[0,1]$  both inclusive.

# Canonical representation of Fuzzy proposition

- Suppose,  $X$  is a universe of discourse of five persons. Intelligent of  $x \in X$  is a fuzzy set as defined below.

Intelligent:  $\{(x_1, 0.3), (x_2, 0.4), (x_3, 0.1), (x_4, 0.6), (x_5, 0.9)\}$

- We define a fuzzy proposition as follows:

$P$  :  $x$  is Intelligent

- The canonical form of fuzzy proposition of this type,  $P$  is expressed by the sentence  $P$  :  $v$  is  $F$ .

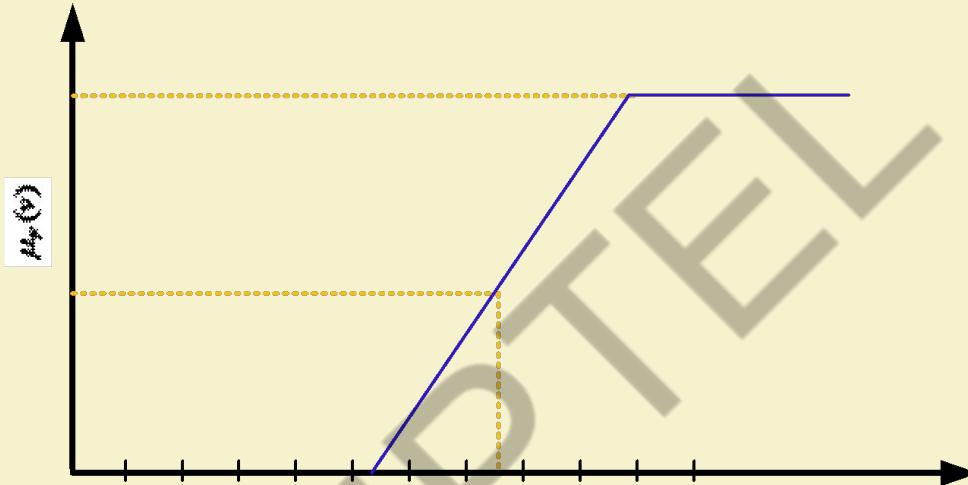
# Canonical representation of Fuzzy proposition

- Predicate in terms of fuzzy set.

$P : v \text{ is } F$  ; where  $v$  is an element that takes values  $v$  from some universal set  $V$  and  $F$  is a fuzzy set on  $V$  that represents a fuzzy predicate.

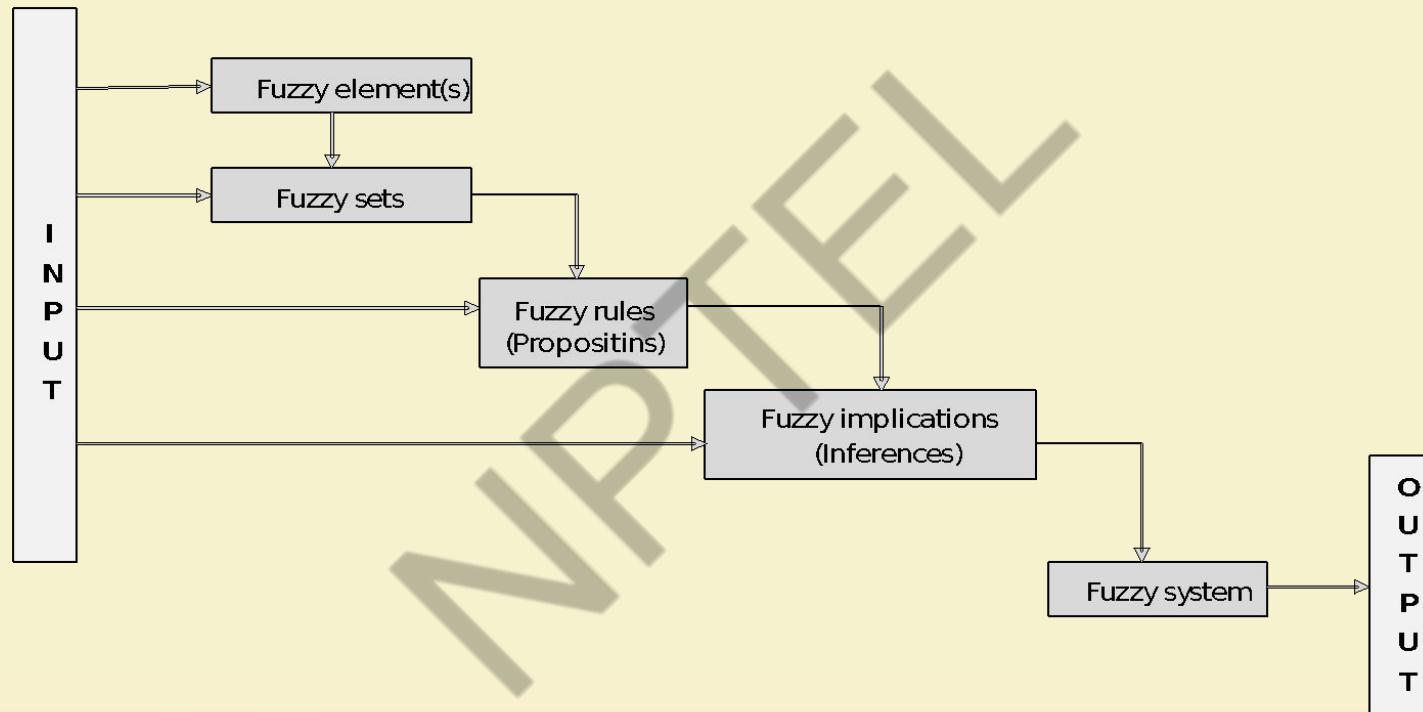
- In other words, given, a particular element  $v$ , this element belongs to  $F$  with membership grade  $\mu_F(v)$ .

# Graphical interpretation of fuzzy proposition



- ✓ For a given value  $v$  of variable  $V$  in proposition  $P$ ,  $T(P)$  denotes the degree of truth of proposition  $P$ .

# Fuzzy system



# Thank You!!

NPTEL



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

# Soft Computing

## Fuzzy Implication

Prof. Debasis Samanta

Department of Computer Science & Engineering  
IIT Kharagpur

# Fuzzy implications

- Fuzzy rule
- Examples of fuzzy implications
- Interpretation of fuzzy rules
- Product operators
- Zadeh's Max-Min rule and some examples

# Fuzzy rule

- A fuzzy implication (also known as fuzzy **If-then rule**, fuzzy rule, or fuzzy conditional statement) assumes the form :

**If  $x$  is  $A$  then  $y$  is  $B$**

where,  $A$  and  $B$  are two linguistic variables defined by fuzzy sets  $A$  and  $B$  on the universe of discourses  $X$  and  $Y$ , respectively.

- Often,  $x$  is  $A$  is called the **antecedent** or premise, while  $y$  is  $B$  is called the **consequence** or conclusion.

# Fuzzy implication : Example 1

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High
- The fuzzy implication is denoted as  $R : A \rightarrow B$
- In essence, it represents a binary fuzzy relation  $R$  on the (Cartesian) product of  $A \times B$

# Fuzzy implication : Example 2

- Suppose, P and T are two universes of discourses representing pressure and temperature, respectively as follows.

$$P = \{1,2,3,4\} \text{ and } T = \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$$

- Let the linguistic variable **High temperature** and **Low pressure** are given as

$$T_{HIGH} = \{(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)\}$$

$$P_{LOW} = \{(1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)\}$$

# Fuzzy implication : Example 2

Then the fuzzy implication If temperature is High then pressure is Low can be defined as

$$R : T_{HIGH} \rightarrow P_{LOW}$$

where,  $R =$

	1	2	3	4
20	0.2	0.2	0.2	0.2
25	0.4	0.4	0.4	0.4
30	0.6	0.6	0.6	0.4
35	0.6	0.6	0.6	0.4
40	0.7	0.7	0.6	0.4
45	0.8	0.8	0.6	0.4
50	0.8	0.8	0.6	0.4

**Note :** If temperature is 40 then what about low pressure?

# Interpretation of fuzzy rules

In general, there are two ways to compute the fuzzy rule  $A \rightarrow B$  as

- A coupled with B
- A entails B

# Interpretation as A coupled with B

$R : A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x) * \mu_B(y) |(x, y)$ ; where  $*$  is called a T-norm operator.

The most frequently used T-norm operators are:

- **Minimum** :  $T_{min}(a, b) = min(a, b) = a \wedge b$
- **Algebraic product** :  $T_{ap}(a, b) = ab$
- **Bounded product** :  $T_{bp}(a, b) = 0 \vee (a + b - 1)$
- **Drastic product** :  $T_{dp} = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{if } a, b < 1 \end{cases}$

Here,  $a = \mu_A(x)$  and  $b = \mu_B(y)$ .  $T$  is called the function of T-norm operator.

# Interpretation as A coupled with B

Based on the T-norm operator as defined, we can automatically define the fuzzy rule  $R : A \rightarrow B$  as a fuzzy set with two-dimensional MF:

$\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b)$  with  $a = \mu_A(x)$ ,  $b = \mu_B(y)$  and  $f$  is the fuzzy implication function.

# Interpretation as A coupled with B

In the following, few implications of  $R : A \rightarrow B$

**Min operator:**

$$R_m = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) |(x, y) \quad \text{or} \quad f_{min}(a, b) = a \wedge b$$

[Mamdani rule]

**Algebraic product operator**

$$R_{ap} = A \times B = \int_{X \times Y} \mu_A(x) \cdot \mu_B(y) |(x, y) \quad \text{or} \quad f_{ap}(a, b) = ab$$

[Larsen rule]

# Product Operators

## Bounded product operator

$$\begin{aligned} R_{bp} &= A \times B = \int_{X \times Y} \mu_A(x) \odot \mu_B(y) |(x, y) \\ &= \int_{X \times Y} 0 \vee (\mu_A(x) + \mu_B(y) - 1) |(x, y) \text{ or } f_{bp}(a, b) = 0 \vee (a + b - 1) \end{aligned}$$

## Drastic product operator

$$R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \widehat{\bullet} \mu_B(y) |(x, y) \text{ or } f_{dp}(a, b) = \begin{cases} a \text{ if } b = 1 \\ b \text{ if } a = 1 \\ 0 \text{ if otherwise} \end{cases}$$

# Interpretation of A entails B

There are three main ways to interpret such implication:

**Material implication :**

$$R : A \rightarrow B = \bar{A} \cup B$$

**Propositional calculus :**

$$R : A \rightarrow B = \bar{A} \cup (A \cup B)$$

**Extended propositional calculus :**

$$R : A \rightarrow B = (\bar{A} \cap \bar{B}) \cup B$$

# Interpretation of A entails B

With the above mentioned implications, there are a number of fuzzy implication functions that are popularly followed in fuzzy rule-based system.

## Zadeh's arithmetic rule :

$$R_{za} = \bar{A} \cup B = \int_{X \times Y} 1 \wedge (1 - \mu_A(x) + \mu_B(y)) |(x, y) \text{ or } f_{za}(a, b) = 1 \wedge (1 - a + b)$$

## Zadeh's max-min rule :

$$R_{mm} = \bar{A} \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y)) |(x, y) \text{ or } f_{mm}(a, b) = (1 - a) \vee (a \wedge b)$$

# Interpretation of A entails B

Boolean fuzzy rule:

$$R_{bf} = \bar{A} \cup B = \int_{X \times Y} (1 - \mu_A(x)) \vee \mu_B(x) |(x, y) \text{ or}$$
$$f_{bf}(a, b) = (1 - a) \vee b$$

Goguen's fuzzy rule:

$$R_{gf} = \int_{X \times Y} \mu_A(x) * \mu_B(y) |(x, y) \text{ where } a * b = \begin{cases} 1 & \text{if } a \leq b \\ \frac{b}{a} & \text{if } a > b \end{cases}$$

## Example 3: Zadeh's Max-Min rule

If  $x$  is  $A$  then  $y$  is  $B$  with the implication of Zadeh's max-min rule can be written equivalently as :

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y)$$

Here,  $Y$  is the universe of discourse with membership values for all  $y \in Y$  is 1, that is,  $\mu_Y(y) = 1 \forall y \in Y$ :

Suppose  $X = \{a, b, c, d\}$  and  $Y = \{1, 2, 3, 4\}$  and

$$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\},$$

$$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$$
 are two fuzzy sets.

We are to determine  $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

## Example 3: Zadeh's Max-Min rule

The computation of  $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$  is as follows

$$A \times B =$$

$$\begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ a \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \end{array} \right] \\ b \left[ \begin{array}{cccc} 0.2 & 0.8 & 0.8 & 0 \end{array} \right] \\ c \left[ \begin{array}{cccc} 0.2 & 0.6 & 0.6 & 0 \end{array} \right] \\ d \left[ \begin{array}{cccc} 0.2 & 1.0 & 0.8 & 0 \end{array} \right] \end{array} \text{ and}$$

$$\bar{A} \times Y =$$

$$\begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ a \left[ \begin{array}{cccc} 1 & 1 & 1 & 1 \end{array} \right] \\ b \left[ \begin{array}{cccc} 0.2 & 0.2 & 0.2 & 0.2 \end{array} \right] \\ c \left[ \begin{array}{cccc} 0.4 & 0.4 & 0.4 & 0.4 \end{array} \right] \\ d \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

## Example 3: Zadeh's Max-Min rule

Therefore,  $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

	1	2	3	4
a	1	1	1	1
b	0.2	0.8	0.8	0.2
c	0.4	0.6	0.6	0.4
d	0.2	1.0	0.8	0

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) =$$



IIT KHARAGPUR



NPTEL  
ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

## Example 4:

IF x is A THEN y is B ELSE y is C. The relation R is equivalent to

$$R = (A \times B) \cup (\bar{A} \times C)$$

The membership function of R is given by

$$\mu_R(x, y) = \max[\min\{\mu_A(x), \mu_B(y)\}, \min\{\mu_{\bar{A}}(x), \mu_C(y)\}]$$

## Example 4:

$$X = \{a, b, c, d\}$$

$$Y = \{1, 2, 3, 4\}$$

$$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$$

$$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$$

$$C = \{(1, 0.0), (2, 0.4), (3, 1.0), (4, 0.8)\}$$

Determine the implication relation :

**If x is A then y is B else y is C**



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

## Example 4:

Here,  $A \times B =$

$$\begin{array}{c} & 1 & 2 & 3 & 4 \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \left[ \begin{matrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{matrix} \right] \end{array}$$

and  $\bar{A} \times C =$

$$\begin{array}{c} & 1 & 2 & 3 & 4 \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \left[ \begin{matrix} 0 & 0.4 & 1.0 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{matrix} \right] \end{array}$$



IIT KHARAGPUR



NPTEL  
ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

## Example 4:

Here,  $A \times B =$

$$\begin{array}{c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \left[ \begin{matrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{matrix} \right] \end{array}$$

and  $\bar{A} \times C =$

$$\begin{array}{c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \left[ \begin{matrix} 0 & 0.4 & 1.0 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{matrix} \right] \end{array}$$



IIT KHARAGPUR



NPTEL  
NPTEL  
ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

## Example 4:

$$R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{matrix}$$



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Thank You!

NPTEL



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

# Soft Computing

## Fuzzy Inferences

Debasis Samanta

Department of Computer Science and Engineering  
IIT KHARAGPUR

# Fuzzy inferences

Let's start with propositional logic. We know the following in propositional logic.

**1. Modus Ponens** .  $P, P \Rightarrow Q,$

$\Leftrightarrow Q$

**2. Modus Tollens** .  $P \Rightarrow Q, \neg Q,$

$\Leftrightarrow, \neg Q$

**3. Chain rule** .  $P \Rightarrow Q, Q \Rightarrow R,$

$\Leftrightarrow, P \Rightarrow R$



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# An example from propositional logic

Given.

- 1)  $C \vee D$
- 2)  $\sim H \Rightarrow (A \wedge \sim B)$
- 3)  $C \vee D \Rightarrow \sim H$
- 4)  $(A \wedge \sim B) \Rightarrow (R \vee S)$

From the above can we infer  $R \vee S$ ?

Similar concept is also followed in fuzzy logic to infer a fuzzy rule from a set of given fuzzy rules (also called fuzzy rule base).

# Inferring procedures in Fuzzy logic

Two important inferring procedures are used in fuzzy systems.

- **Generalized Modus Ponens (GMP)**

If  $x$  is  $A$  Then  $y$  is  $B$

$x$  is  $A'$

---

$y$  is  $B'$

- **Generalized Modus Tollens (GMT)**

If  $x$  is  $A$  Then  $y$  is  $B$

$y$  is  $B'$

---

$x$  is  $A'$



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Fuzzy inferring procedures

- Here,  $A, B, A'$  and  $B'$  are fuzzy sets.
- To compute the membership function  $A'$  and  $B'$  the max-min composition of fuzzy sets  $B'$  and  $A'$ , respectively with  $R(x, y)$  (which is the known implication relation) is used.
- Thus,

$$B' = A' \circ R(x, y)$$

$$A' = B' \circ R(x, y)$$

$$\mu_{B'}(y) = \max[\min(\mu_{A'}(x), \mu_R(x, y))]$$

$$\mu_{A'}(x) = \max[\min(\mu_{B'}(y), \mu_R(x, y))]$$



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Generalized Modus Ponens : Example

$P : \text{If } x \text{ is } A \text{ then } y \text{ is } B$

Let us consider two sets of variables x and y be

$$X = \{x_1, x_2, x_3\} \text{ and } Y = \{y_1, y_2\}$$

Also, let us consider the following.

$$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$$

$$B = \{(y_1, 1), (y_2, 0.4)\}$$

# Example: Generalized Modus Ponens

## Generalized Modus Ponens (GMP)

If  $x$  is  $A$  Then  $y$  is  $B$

$x$  is  $A'$

---

$y$  is  $B'$

# Generalized Modus Ponens

$P : \text{If } x \text{ is } A \text{ then } y \text{ is } B$

Suppose, given a fact expressed by the proposition  $x$  is  $A'$ ,

where  $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

We are to derive a conclusion in the form  $y$  is  $B'$

Here, we should use generalized modus ponens (GMP).

# Example. Generalized Modus Ponens

If  $x$  is  $A$  Then  $y$  is  $B$

$x$  is  $A'$

---

$y$  is  $B'$

We are to find  $B' = A' \circ R(x, y)$ , where  $R(x, y) = \max\{A \times B, \bar{A} \times Y\}$

$$A \times B = \begin{matrix} & y_1 & y_2 \\ x_1 & [0.5 & 0.4] \\ x_2 & [1 & 0.4] \\ x_3 & [0.6 & 0.4] \end{matrix}$$

and

$$\bar{A} \times Y = \begin{matrix} & y_1 & y_2 \\ x_1 & [0.5 & 0.5] \\ x_2 & [0 & 0] \\ x_3 & [0.4 & 0.4] \end{matrix}$$

Note. For  $A \times B$ ,  $\mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

# Example. Generalized Modus Ponens

$$R(x, y) = (A \times B) \cup (\bar{A} \times Y) = \begin{matrix} & y_1 & y_2 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{matrix}$$

Now  $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

$$\text{Therefore } B' = A' \circ R(x, y) = [0.6 \quad 0.9 \quad 0.7] \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = [0.9 \quad 0.5]$$

Thus we derive that  $y$  is  $B'$  where  $B' = \{(y_1, 0.9), (y_2, 0.5)\}$

# Example. Generalized Modus Tollens

**Generalized Modus Tollens (GMT)**

If  $x$  is  $A$  Then  $y$  is  $B$

$y$  is  $B'$

---

$x$  is  $A'$

NPTEL



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Example. Generalized Modus Tollens

- Let the universe of discourses be  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_2\}$ , respectively.
- Assume that a proposition **If  $x$  is  $A$  Then  $y$  is  $B$**  given where  $A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$  And  $B = \{(y_1, 1), (y_2, 0.4)\}$
- Assume now that a fact expressed by a proposition  **$y$  is  $B'$**  is given where  $B' = \{(y_1, 0.9), (y_2, 0.7)\}$
- From the above, we are to conclude that  **$x$  is  $A'$** . That is, we are to determine  $A'$



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Example. Generalized Modus Tollens

1. We first calculate  $R(x, y) = (A \times B) \cup (\bar{A} \times Y)$

$$R(x, y) = \begin{matrix} & y_1 & y_2 \\ x_1 & [0.5 & 0.5] \\ x_2 & [1 & 0.4] \\ x_3 & [0.6 & 0.4] \end{matrix}$$

2. Next, we calculate  $A' = B' \circ R(x, y)$

$$A' = [0.9 \quad 0.7] \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = [0.5 \quad 0.9 \quad 0.6]$$

3. Hence, we calculate that  $x$  is  $A'$  where

$$A' = [(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)]$$

# Practical example

Apply the fuzzy GMP rule to deduce **Rotation is quite slow**

Given that .

1. If temperature is High then rotation is Slow.
2. temperature is Very High

Let,

$X = \{30, 40, 50, 60, 70, 80, 90, 100\}$  be the set of temperatures.

$Y = \{10, 20, 30, 40, 50, 60\}$  be the set of rotations per minute.



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Practice

The fuzzy set High( $H$ ), Very High ( $VH$ ), Slow( $S$ ) and Quite Slow ( $QS$ ) are given below.

$$H = \{(70, 1), (80, 1), (90, 0.3)\}$$

$$VH = \{(80, 0.6), (90, 0.9), (100, 1)\}$$

$$S = \{(30, 0.8), (40, 1.0), (50, 0.6)\}$$

$$QS = \{(10, 1), (20, 0.8), (30, 0.5)\}$$

1. If temperature is High then rotation is Slow.

$$R = (H \times S) \cup (\bar{H} \times Y)$$

2. temperature is Very High

Thus, to deduce “rotation is Quite Slow”, we make use the composition rule

$$QS = VH \circ R(x, y)$$



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Thank You!

NPTEL



IIT KHARAGPUR



NPTEL  
ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

# Soft Computing

## Defuzzyfication Techniques-I

Debasis Samanta

Department of Computer Science and Engineering  
IIT KHARAGPUR

# What is defuzzification?

- Defuzzification means the fuzzy to crisp conversion.

## Example 1.

Suppose,  $T_{HIGH}$  denotes a fuzzy set representing **temperature is High**.

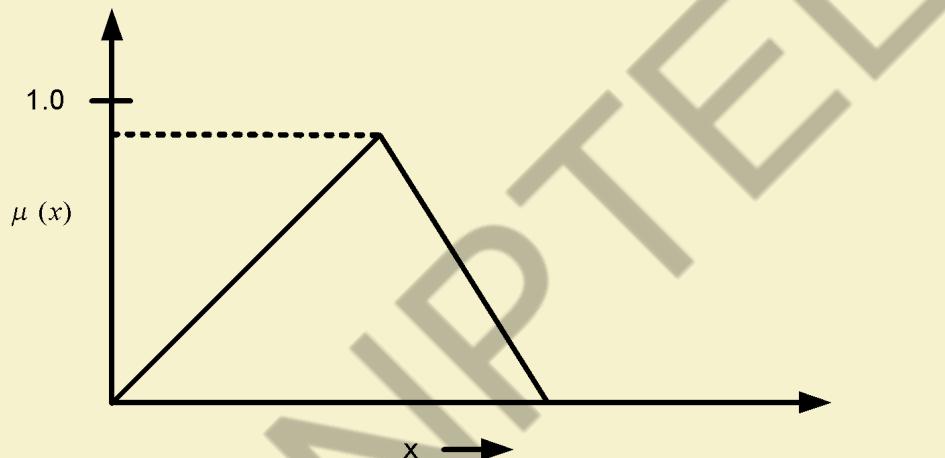
$T_{HIGH}$  is given as follows.

$$T_{HIGH} = \{(15,0.1), (20, 0.4), (25, 0.45), (30, 0.55), (35, 0.65), (40, 0.7), (45, 0.85), (50, 0.9)\}$$

- What is the crisp value that implies the high temperature?

## Example 2. Fuzzy to crisp

As an another example, let us consider a fuzzy set whose membership function is shown in the following figure.



What is the crisp value of the fuzzy set in this case?

# Example 3. Fuzzy to crisp

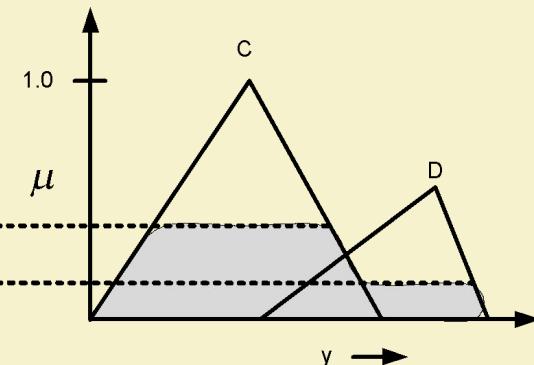
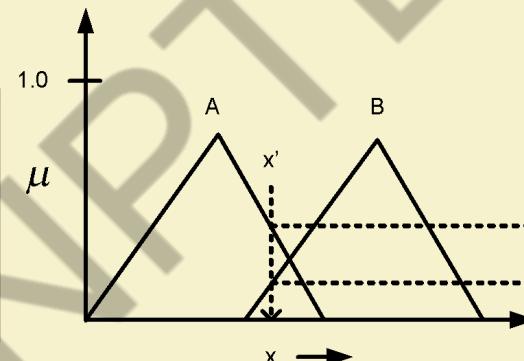
Now, consider the following two rules in the fuzzy rule base.

R1. If  $x$  is A then  $y$  is C

R2. If  $x$  is B then  $y$  is D

A pictorial representation of the above rule base is shown in the following figures.

What is the crisp value that can be inferred from the above rules given an input say  $x'$ ?



# Why defuzzification?

The fuzzy results generated can not be used in an application, where decision has to be taken only on crisp values.

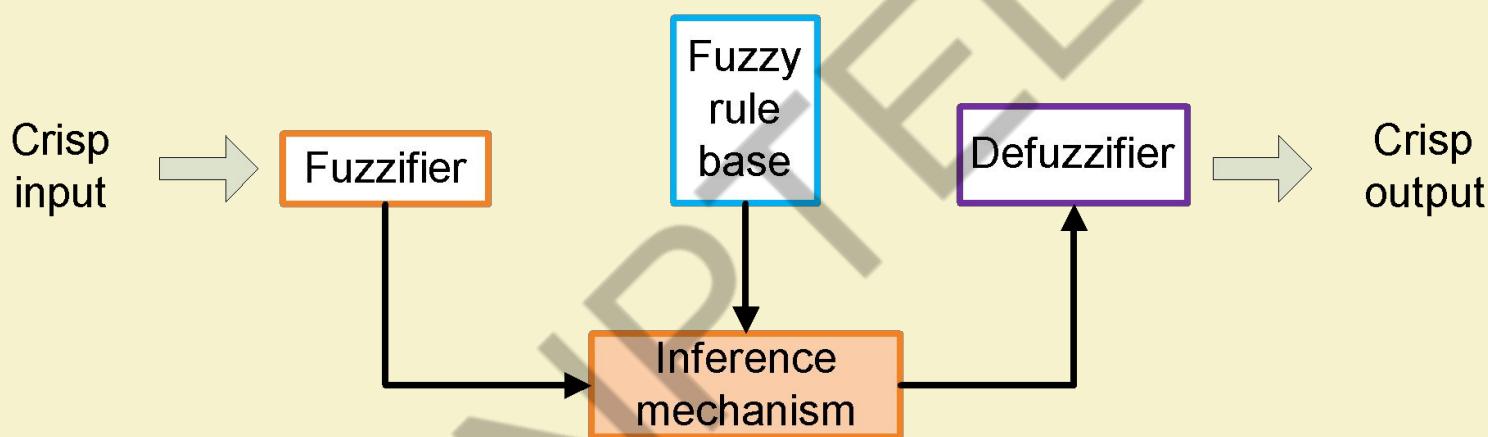
**Example.**

*If temperature is  $T_{HIGH}$  Then rotation is  $R_{FAST}$ .*

Here, may be input  $T_{HIGH}$  is **fuzzy**, but action **rotation** should be based on the crisp value of  $R_{FAST}$  .

# Generic structure of a Fuzzy system

Following figure shows a general framework of a fuzzy system.



# Defuzzification Techniques



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Defuzzification methods

A number of defuzzification methods are known. Such as

- 1) Lambda-cut method
- 2) Weighted average method
- 3) Maxima methods
- 4) Centroid methods

# Lambda-cut method



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Lambda-cut method

Lambda-cut method is applicable to derive crisp value of a fuzzy set or relation.

- Lambda-cut method for fuzzy relation

The same has been applied to Fuzzy set

- Lambda-cut method for fuzzy set

In many literature, Lambda-cut method is also alternatively termed as **Alpha-cut method**.

# Lamda-cut method for fuzzy set

- 1) In this method a fuzzy set A is transformed into a crisp set  $A_\lambda$  for a given value of  $\lambda(0 \leq \lambda \leq 1)$
- 2) In other-words,  $A_\lambda = \{x | \mu_A(x) \geq \lambda\}$
- 3) That is, the value of Lambda-cut set  $A_\lambda$  is  $x$ , when the membership value corresponding to  $x$  is greater than or equal to the specified  $\lambda$ .
- 4) This Lambda-cut set  $A_\lambda$  is also called alpha-cut set.

# Lambda-cut for a fuzzy set : Example

$$A_1 = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.2), (x_4, 0.3)\}$$

$$\lambda = 0.6$$

$$A_{0.6} = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 0)\} = \{x_1\}$$

$$A_2 = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.8), (x_4, 0.7)\}$$

$$\lambda = 0.2$$

$$A_{0.2} = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 1)\} = \{x_2, x_3, x_4\}$$

# Lambda-cut sets : Example

Two fuzzy sets P and Q are defined on  $x$  as follows.

P	0.1	0.2	0.7	0.5	0.4
Q	0.9	0.6	0.3	0.2	0.8

Find the following :

- a)  $P_{0.2}, Q_{0.3}$
- b)  $(P \cup Q)_{0.6}$
- c)  $(P \cup \bar{P})_{0.8}$
- d)  $(P \cap Q)_{0.4}$

# Lambda-cut for a fuzzy relation

The Lambda-cut method for a fuzzy set can also be extended to fuzzy relation also.

**Example:** For a fuzzy relation R

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}$$

We are to find  $\lambda$ -cut relations for the following values of

$$\lambda = 0, 0.2, 0.9, 0.5$$

$$R_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and}$$

$$R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } R_{0.5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

# Some properties of $\lambda$ -cut sets

If A and B are two fuzzy sets, defined with the same universe of discourse, then

- 1)  $(A \cup B)_\lambda = A_\lambda \cup B_\lambda$
- 2)  $(A \cap B)_\lambda = A_\lambda \cap B_\lambda$
- 3)  $\overline{(A)}_\lambda \neq \overline{A}_\lambda$  except for value of  $\lambda = 0.5$
- 4) For any  $\lambda \leq \alpha$ , where  $\alpha$  varies between 0 and 1, it is true that  $A_\alpha \subseteq A_\lambda$ ,  
where the value of  $A_0$  is the universe of discourse.

# Some properties of $\lambda$ -cut relations

If R and S are two fuzzy relations, defined with the same fuzzy sets over the same universe of discourses, then

$$1) \quad (R \cup S)_\lambda = R_\lambda \cup S_\lambda$$

$$2) \quad (R \cap S)_\lambda = R_\lambda \cap S_\lambda$$

$$3) \quad \overline{(R)}_\lambda \neq \overline{R}_\lambda$$

$$4) \quad \text{For } \lambda \leq \alpha, \text{ where } \alpha \text{ between 0 and 1, then } R_\alpha \subseteq R_\lambda$$



IIT KHARAGPUR



NPTEL  
ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Summary: Lambda-cut methods

Lambda-cut method converts a fuzzy set (or a fuzzy relation) into a crisp set (or relation).

# Output of a Fuzzy System



IIT KHARAGPUR



NPTEL  
ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Output of a fuzzy System

The output of a fuzzy system can be a single fuzzy set or union of two or more fuzzy sets.

To understand the second concept, let us consider a fuzzy system with n-rules.

$R_1: \text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1$

$R_2: \text{If } x \text{ is } A_2 \text{ then } y \text{ is } B_2$

.....

.....

$R_n: \text{If } x \text{ is } A_n \text{ then } y \text{ is } B_n$

In this case, the output  $y$  for a given input  $x = x_1$  is possibly

$$B = B_1 \cup B_2 \cup \dots \dots \dots B_n$$



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# Output fuzzy set : Illustration

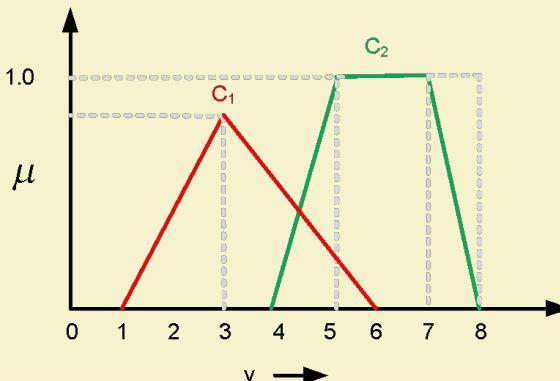
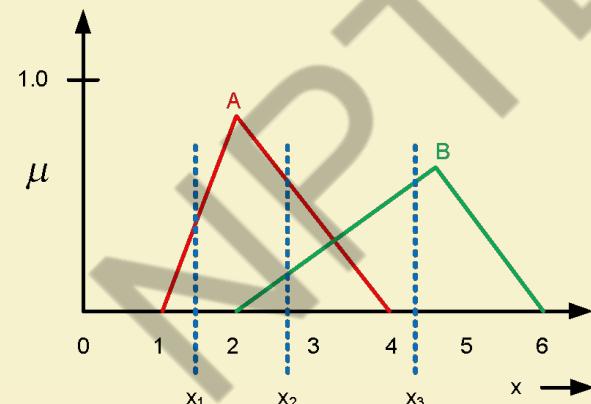
Suppose, two rules  $R_1$  and  $R_2$  are given as follows:

$R_1: \text{if } x \text{ is } A_1 \text{ then } y \text{ is } C_1$

$R_2: \text{if } x \text{ is } A_2 \text{ then } y \text{ is } C_2$

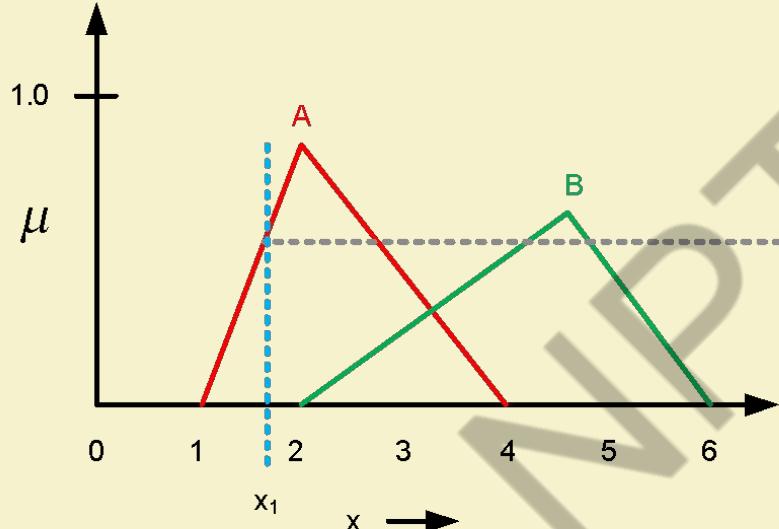
Here, the output fuzzy set  $C = C_1 \cup C_2$

For instance,  
let us consider  
the following:

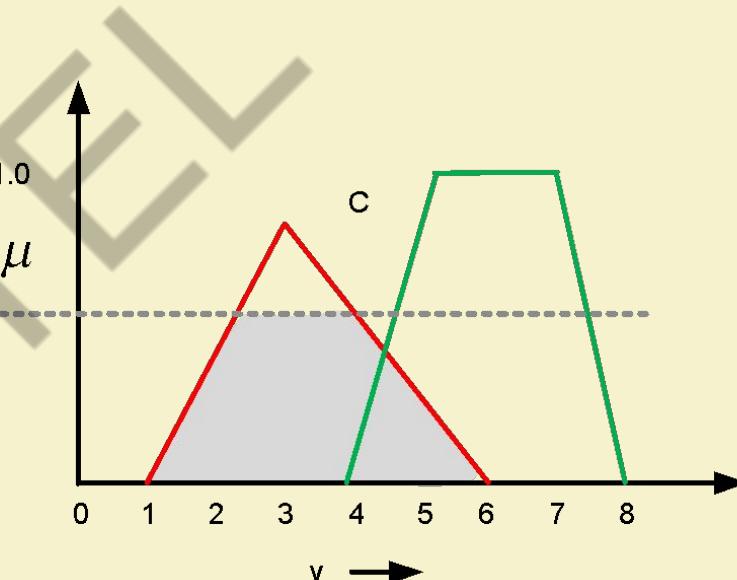


# Output fuzzy set : Illustration

The fuzzy output for  $x = x_1$  is shown below.

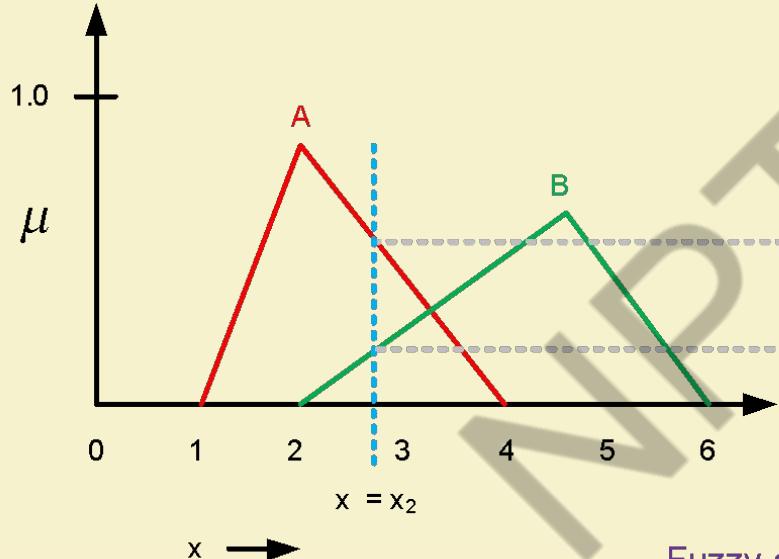


Fuzzy output for  $x = x_1$

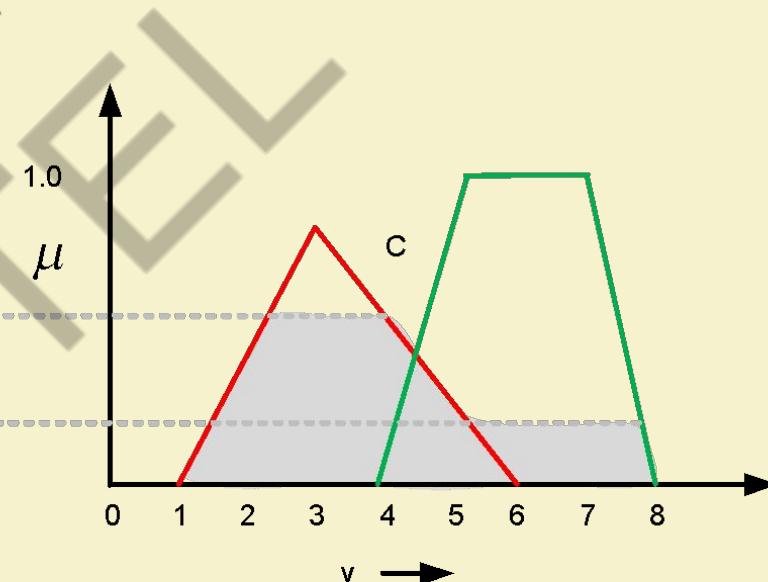


# Output fuzzy set : Illustration

The fuzzy output for  $x = x_2$  is shown below.

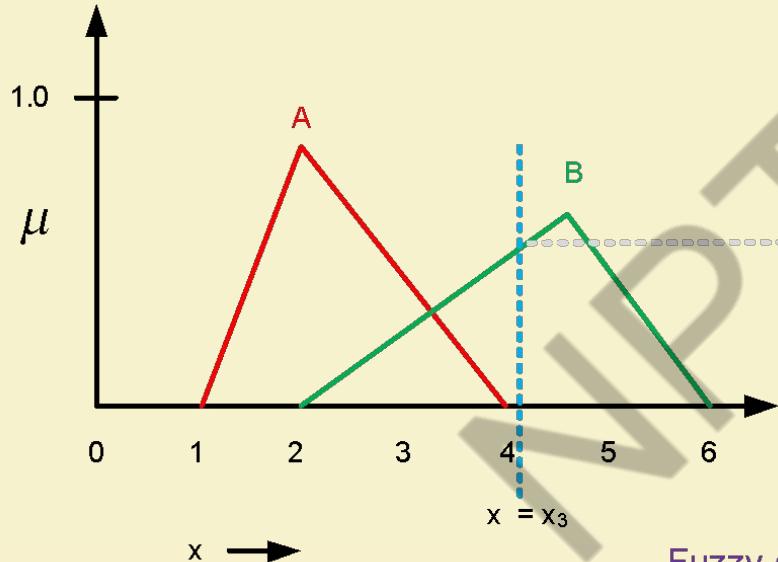


Fuzzy output for  $x = x_2$

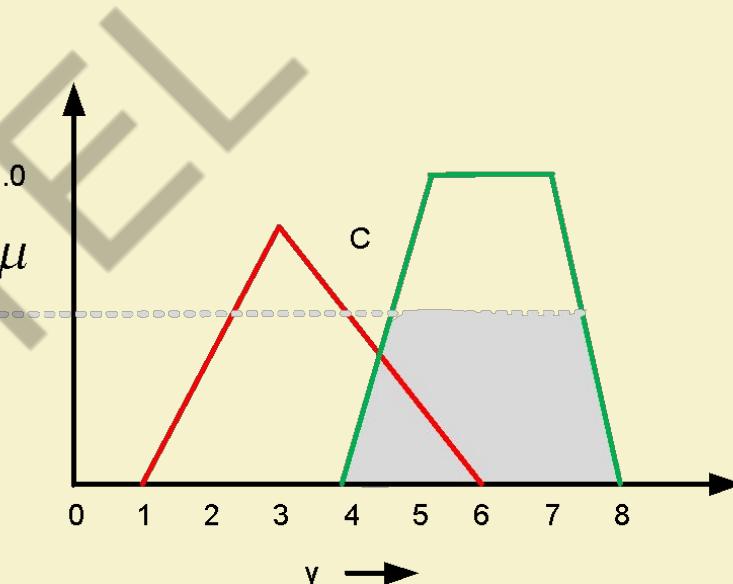


# Output fuzzy set : Illustration

The fuzzy output for  $x = x_3$  is shown below.



Fuzzy output for  $x = x_3$



# Thank You!!

NPTEL



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES

# Soft Computing

## Defuzzification Techniques-II

Debasis Samanta

Department of Computer Science and Engineering  
IIT KHARAGPUR

# Defuzzification Methods

Following defuzzification methods are known to calculate crisp output in the situations as discussed in the last lecture.

## 1. Maxima Methods

- a) Height method
- b) First of maxima (FoM)
- c) Last of maxima (LoM)
- d) Mean of maxima(MoM)

## 2. Centroid methods

- a) Centre of gravity method (CoG)
- b) Centre of sum method (CoS)
- c) Centre of area method (CoA)

## 3. Weighted average method

# Maxima methods

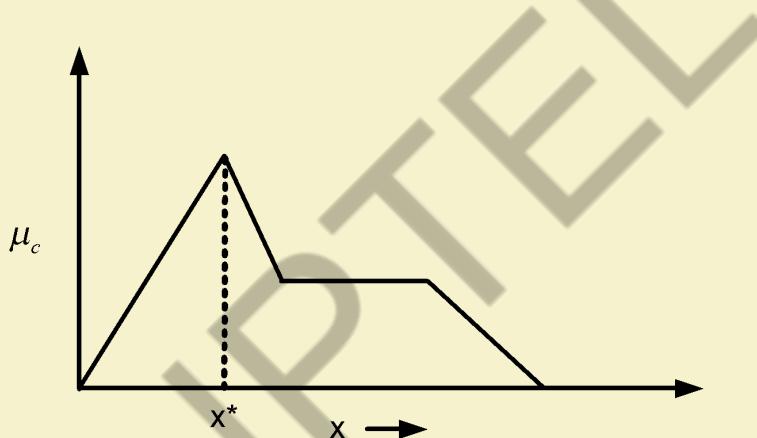
## 1. Maxima Methods

- a) Height method
- b) First of maxima (FoM)
- c) Last of maxima (LoM)
- d) Mean of maxima(MoM)

# Maxima method : Height method

This method is based on [Max-membership principle](#), and defined as follows.

$$\mu_c(x^*) \geq \mu_c(x) \text{ for all } x \in X$$

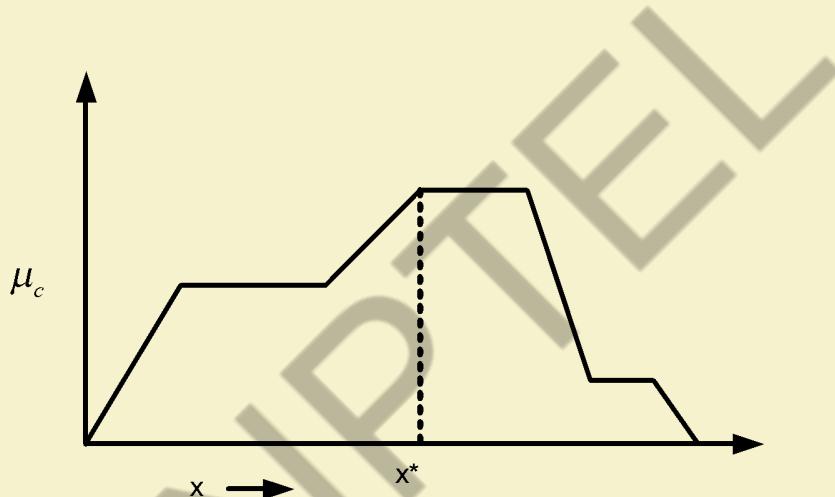


**Note:**

1. Here,  $x^*$  is the height of the output fuzzy set C.
2. [This method is applicable when height is unique.](#)

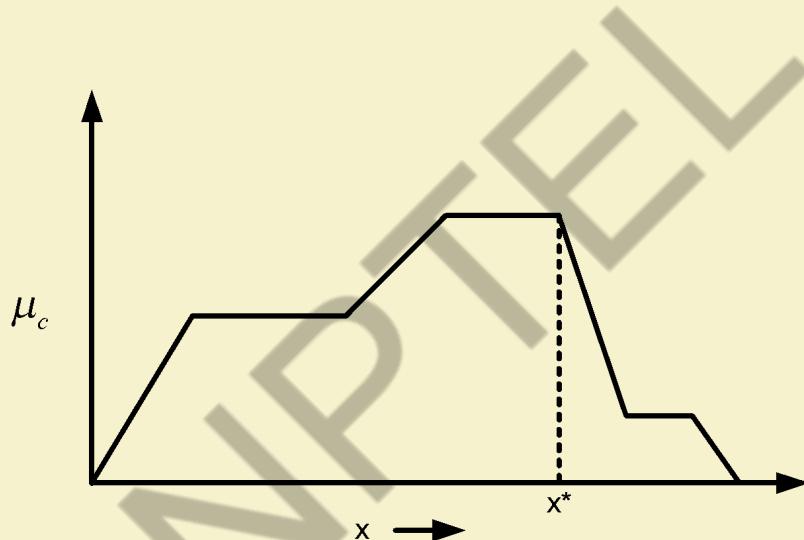
# Maxima method : FoM

FoM: First of Maxima :  $x^* = \min(x | C(x) = \max_w C\{w\})$



# Maxima method : LoM

LoM: Last of Maxima :  $x^* = \max(x | C(x) = \max_w C\{w\})$



# Maxima method : MoM

$$x^* = \frac{\sum_{x_i \in M} (x_i)}{|M|}$$

where,  $M = \{x_i | \mu(x_i) = h(C)\}$  where  $h(C)$  is the height of the fuzzy set  $C$

# MoM : Example 1

Suppose, a fuzzy set **Young** is defined as follows:

$$Young = \{(15,0.5), (20,0.8), (25,0.8), (30,0.5), (35,0.3)\}$$

Then the crisp value of **Young** using MoM method is

$$x^* = \frac{20 + 25}{2} = 22.5$$

Thus, a person of 22.5 years old is treated as young!



IIT KHARAGPUR

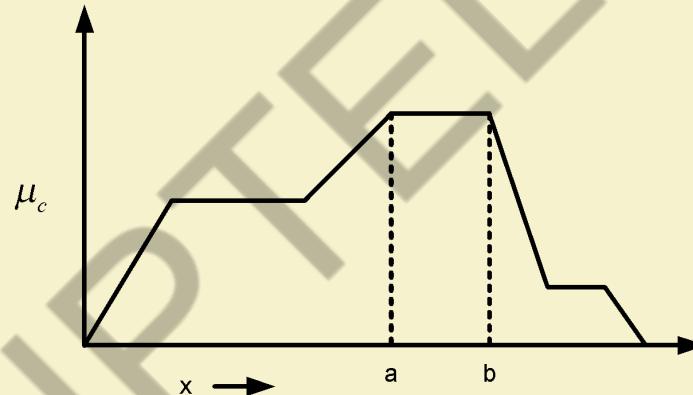


NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# MoM : Example 2

What is the crisp value of the fuzzy set using MoM in the following case?



$$x^* = \frac{a + b}{2}$$

**Note:**

- Thus, MoM is also synonymous to **middle of maxima**.
- MoM is also a general method of **Height**.

# Centroid methods

## 2. Centroid methods

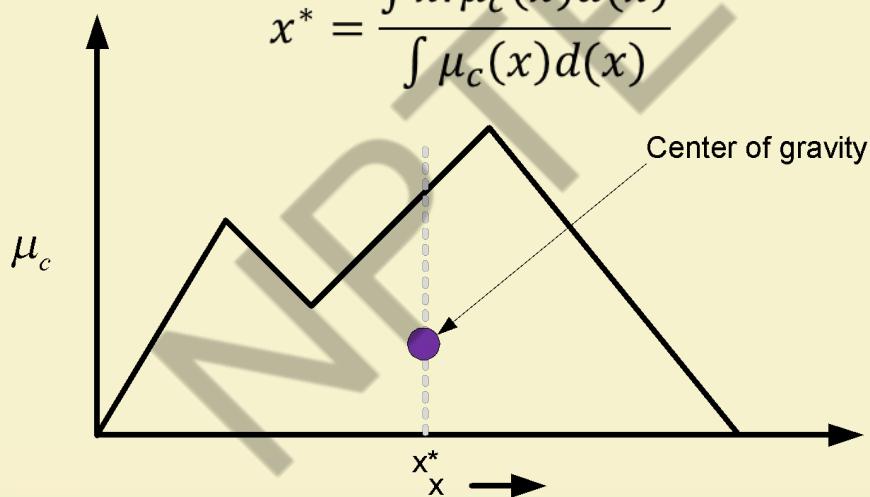
- a) Centre of gravity method (CoG)
- b) Centre of sum method (CoS)
- c) Centre of area method (CoA)

# Centroid method : CoG

- 1) The basic principle in CoG method is to find the point  $x$  where a vertical line would slice the aggregate into two equal masses.
- 2) Mathematically, the CoG can be expressed as follows :

$$x^* = \frac{\int x \cdot \mu_c(x) d(x)}{\int \mu_c(x) d(x)}$$

- 3) Graphically,



# Centroid method : CoG

## Note:

1

- 1)  $x^*$  is the x-coordinate of centre of gravity.
- 2)  $\int \mu_c(x)d(x)$  denotes the area of the region bounded by the curve  $\mu_c$
- 3) If  $\mu_c$  is defined with a discrete membership function, then CoG can be stated as :

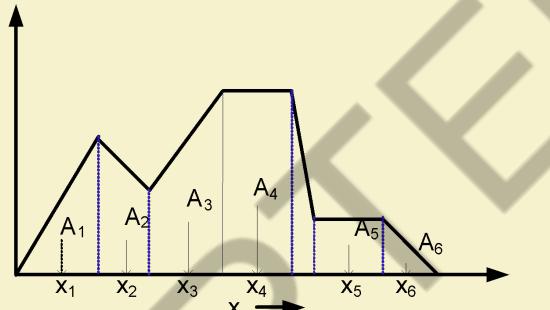
$$x^* = \frac{\sum x_i \cdot \mu_c(x_i)}{\sum \mu_c(x_i)} \quad \text{for } i = 1 \text{ to } n$$

4. Here,  $x_i$  is a sample element and  $n$  represents the number of samples in fuzzy set C.

# CoG : A geometrical method of calculation

## Steps:

- 1) Divide the entire region into a number of small regular regions (e.g. triangles, trapezoid, etc.)

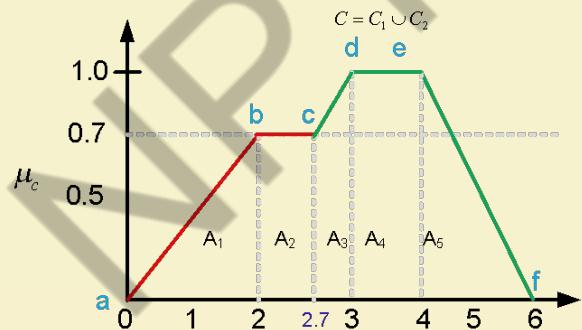
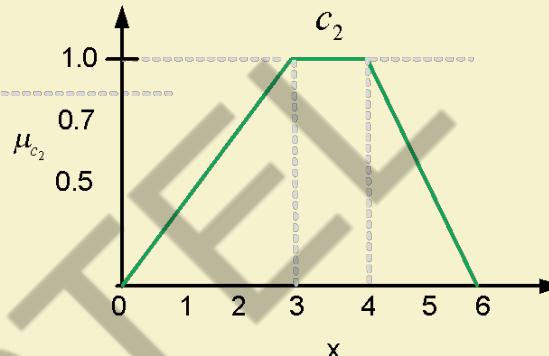
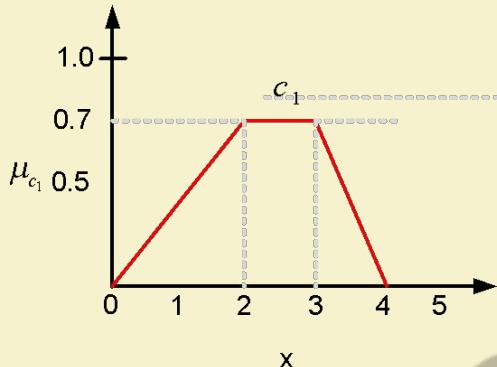


- 2) Let  $A_i$  and  $x_i$  denotes the area and c. g. of the  $i^{th}$  portion.
- 3) Then  $x^*$  according to CoG is

$$x^* = \frac{\sum_{i=1}^n x_i \cdot (A_i)}{\sum_{i=1}^n A_i}$$

where  $n$  is the number of smaller geometrical components.

# CoG: An example of integral method of calculation



# CoG: An example of integral method of calculation

$$\mu_c(x) = \begin{cases} 0.35x & 0 \leq x < 2 \\ 0.7 & 2 \leq x < 2.7 \\ x - 2 & 2.7 \leq x < 3 \\ 1 & 3 \leq x < 4 \\ (-.5x + 3) & 4 \leq x \leq 6 \end{cases}$$

For  $A_1$ :  $y - 0 = \frac{0.7}{2}(x - 0)$ , or  $y = 0.35x$

For  $A_2$ :  $y = 0.7$

For  $A_3$ :  $y - 0 = \frac{1-0}{3-2}(x - 2)$ , or  $y = x - 2$

For  $A_4$ :  $y = 1$

For  $A_5$ :  $y - 1 = \frac{0-1}{6-4}(x - 4)$ , or  $y = -0.5x + 3$



IIT KHARAGPUR



NPTEL  
NPTEL ONLINE  
CERTIFICATION COURSES

Debasis Samanta  
CSE  
IIT Kharagpur

# CoG: An example of integral method of calculation

$$\text{Thus, } x^* = \frac{\int x \cdot \mu_c(x) d(x)}{\int \mu_c(x) d(x)} = \frac{N}{D}$$

$$N = \int_0^2 0.35x^2 dx + \int_2^{2.7} 0.7x^2 dx + \int_{2.7}^3 (x^2 - 2x) dx + \int_3^4 x dx + \int_4^6 (-0.5x^2 + 3x) dx \\ = 10.98$$

$$D = \int_0^2 0.35x dx + \int_2^{2.7} 0.7x dx + \int_{2.7}^3 (x - 2) dx + \int_3^4 dx + \int_4^6 (-0.5x + 3) dx \\ = 3.445$$

$$\text{Thus, } x^* = \frac{10.98}{3.445} = 3.187$$

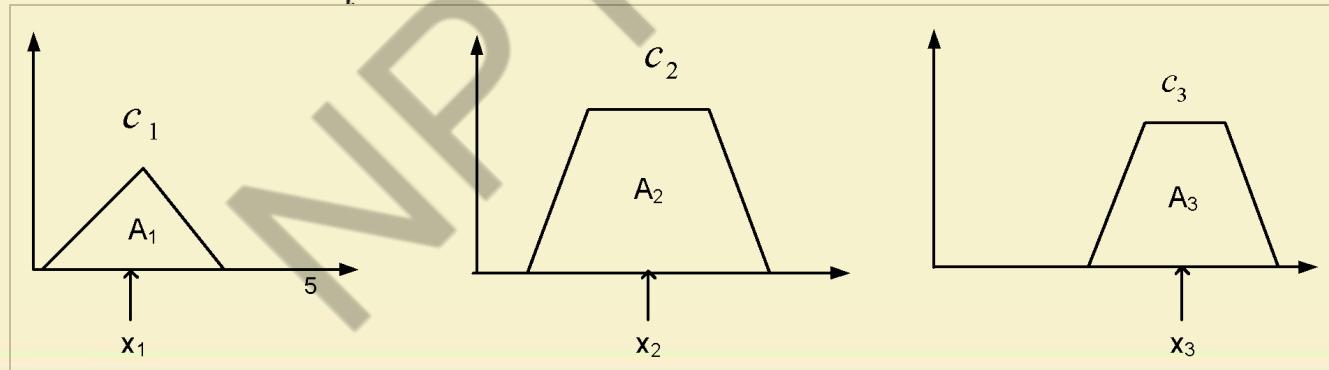
# Centroid method : CoS

If the output fuzzy set  $C = C_1 \cup C_2 \cup \dots \cup C_n$ , then the crisp value according to CoS is defined as

$$x^* = \frac{\sum_{i=1}^n x_i A_{C_i}}{\sum_{i=1}^n A_{C_i}}$$

Here,  $A_{C_i}$  denotes the area of the region bounded by the fuzzy set  $c_i$  and  $x_i$  is the geometric centre of the area  $A_{C_i}$ .

Graphically,



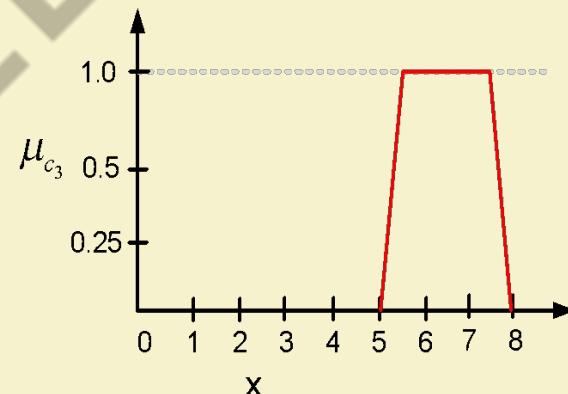
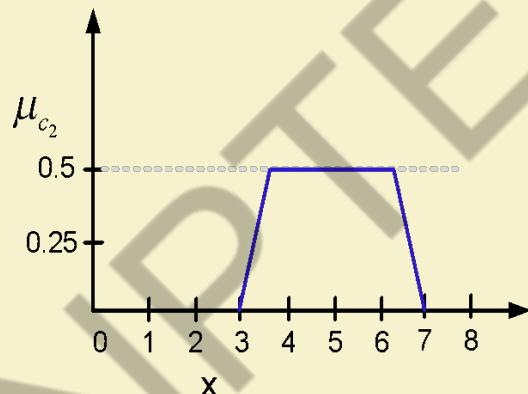
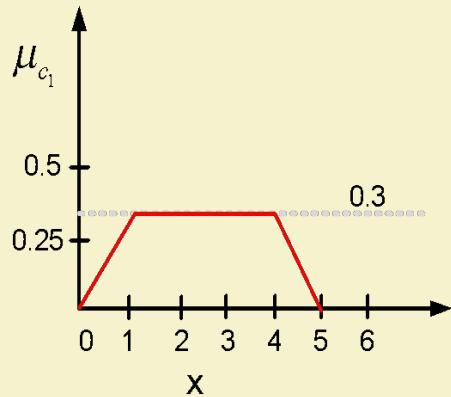
# Centroid method : CoS

## Note:

- In CoG method, the overlapping area is counted once, whereas, in CoS , the overlapping is counted twice or so.
- In CoS, we use the **centre of area** and hence, its name instead of **centre of gravity** as in CoG.

# CoS: Example

Consider the three output fuzzy sets as shown in the following plots:



# CoS: Example

In this case, we have

$$A_{c_1} = \frac{1}{2} \times 0.3 \times (3 + 5), x_1 = 2.5$$

$$A_{c_2} = \frac{1}{2} \times 0.5 \times (4 + 2), x_2 = 5$$

$$A_{c_3} = \frac{1}{2} \times 1.0 \times (3 + 1), x_3 = 6.5$$

$$\text{Thus, } x^* = \frac{\frac{1}{2} \times 0.3 \times (3+5) \times 2.5 + \frac{1}{2} \times 0.5 \times (4+2) \times 5 + \frac{1}{2} \times 1.0 \times (3+1) \times 6.5}{\frac{1}{2} \times 0.3 \times (3+5) + \frac{1}{2} \times 0.5 \times (4+2) + \frac{1}{2} \times 1.0 \times (3+1)}$$

**Note:**

The crisp value of  $C = C_1 \cup C_2 \cup C_3$  using CoG method can be found to be calculated as  $x^* = 4.9$

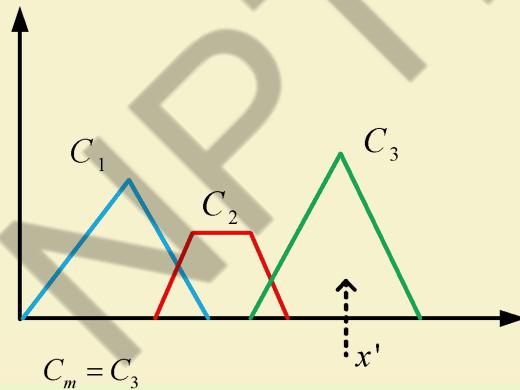
# Centroid method: Centre of largest area

If the fuzzy set has two sub regions, then the **centre of gravity of the sub region with the largest area** can be used to calculate the defuzzified value.

**Mathematically,**  $x^* = \frac{\int \mu_{C_m}(x).x'd(x)}{\int \mu_{C_m}(x) d(x)}$ ;

Here,  $C_m$  is the region with largest area,  $x'$  is the centre of gravity of  $C_m$ .

**Graphically,**



# Weighted average methods

1. Maxima Methods
  - a) Height method
  - b) First of maxima (FoM)
  - c) Last of maxima (LoM)
  - d) Mean of maxima(MoM)
2. Centroid methods
  - a) Centre of gravity method (CoG)
  - b) Centre of sum method (CoS)
  - c) Centre of area method (CoA)
3. Weighted average method

# Weighted average method

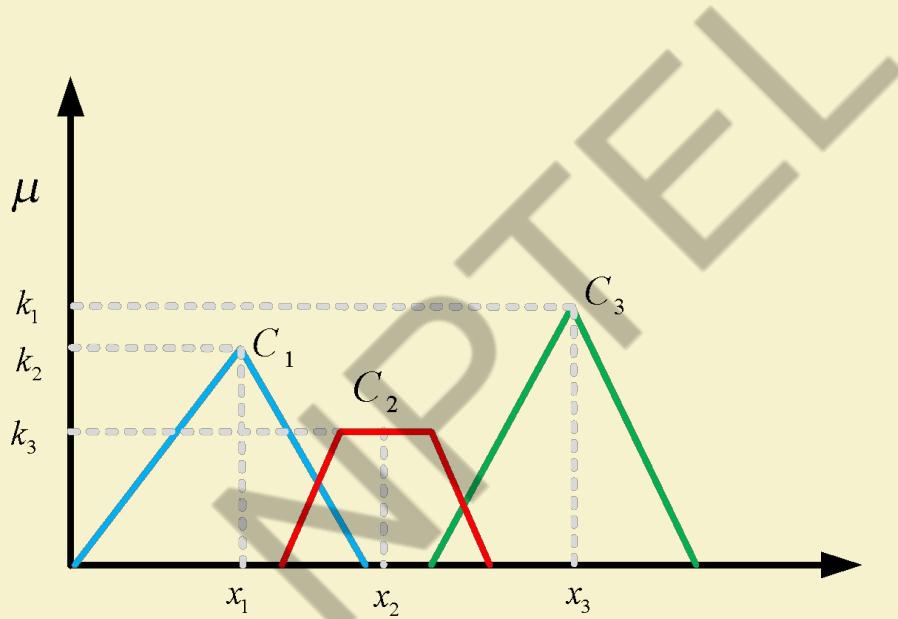
- 1) This method is also alternatively called “Sugeno defuzzification” method.
- 2) The method can be used only **for symmetrical output membership functions**.
- 3) The crisp value according to this method is

$$x^* = \frac{\sum_{i=1}^n \mu_{c_i}(x_i) \cdot x_i}{\sum_{i=1}^n \mu_{c_i}(x_i)}$$

where,  $C_1, C_2, \dots, C_n$  are the output fuzzy sets and  $(x_i)$  is the value where middle of the fuzzy set  $C_i$  is observed.

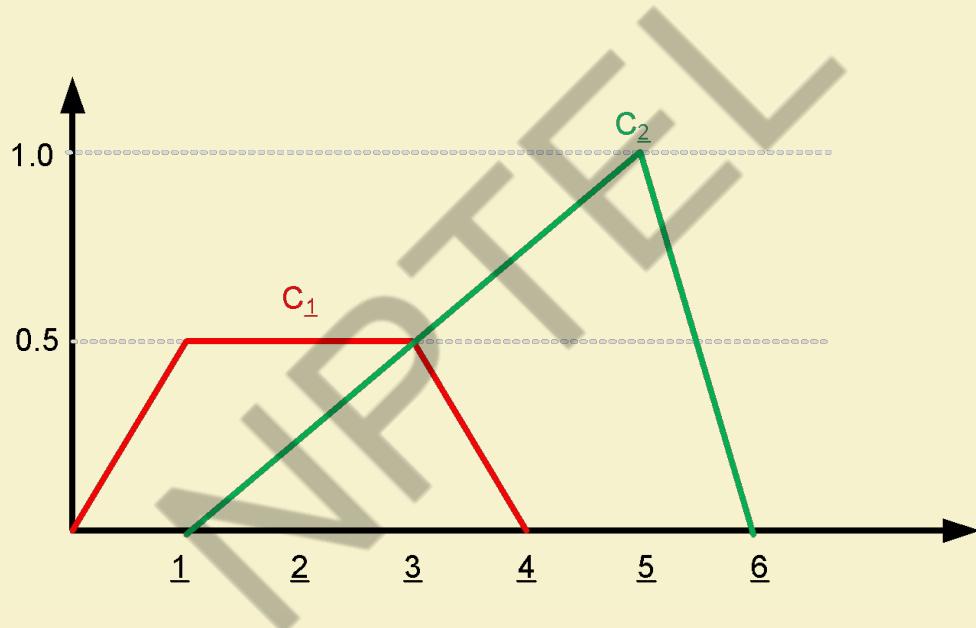
# Weighted average method

Graphically ,



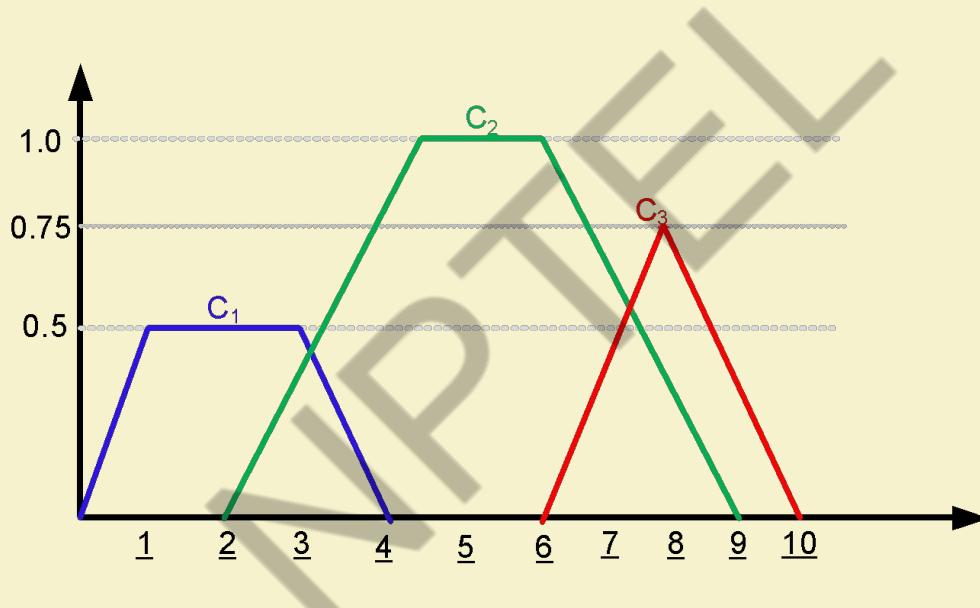
# Exercise 1

Find the crisp value of the following using all defuzzified methods.



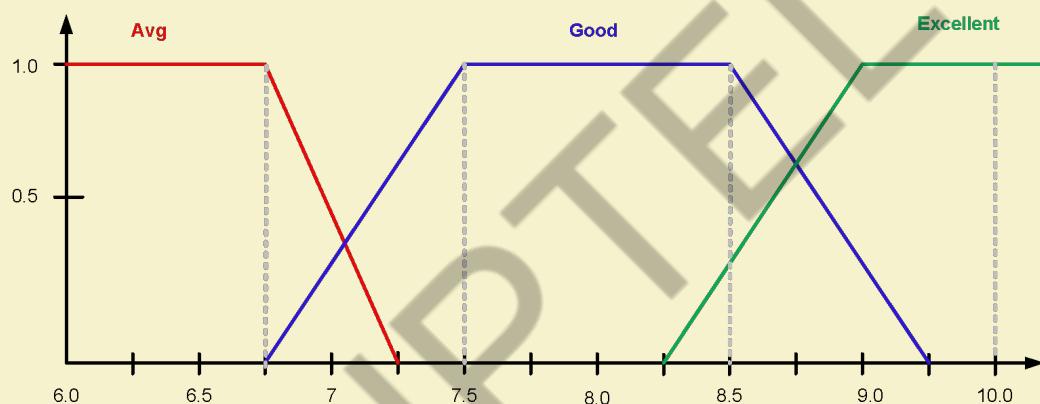
## Exercise 2

Find the crisp value of the following using all defuzzified methods.



# Exercise 3

- The membership function defining a student as *Average*, *Good*, and *Excellent* denoted by respective membership functions are as shown below.

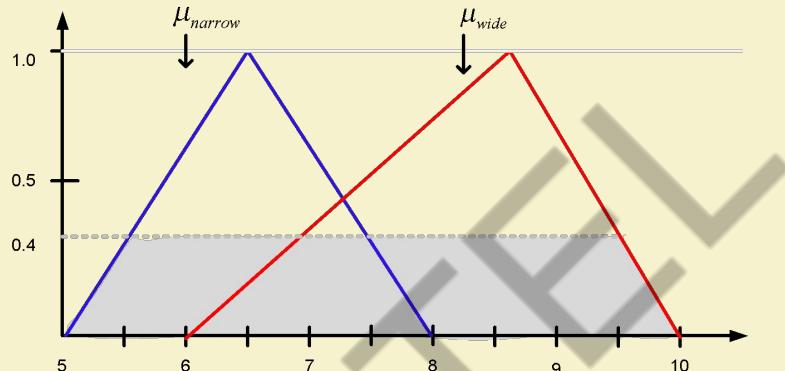


- Find the crisp value of “**Good Student**”

**Hint:**

Use CoG method to the portion *Good* to calculate it.

# Exercise 4



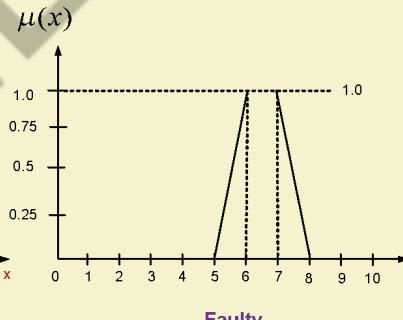
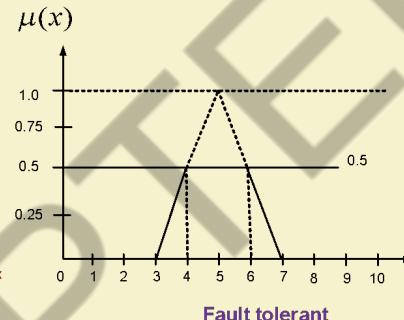
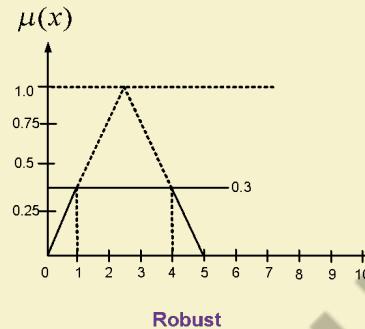
- The width of a road as narrow and wide is defined by two fuzzy sets, whose membership functions are plotted as shown above.
- If a road with its degree of membership value is 0.4 then what will be its width (in crisp) measure.

**Hint:**

Use CoG method for the shaded region

# Exercise 5

- The faulty measure of a circuit is defined fuzzily by three fuzzy sets namely *Robust(R)*, *Fault tolerant (FT)* and *Faulty(F)*, defined by three membership functions with number of faults occur as universe of discourses and is shown below.



- Reliability is measured as  $R^* = F \cup FT \cup R$  With a certain observation in testing  $(x, 0.3) \in R$ ,  $(x, 0.5) \in FT$ ,  $(x, 0.8) \in F$ .
- Calculate the reliability measure in crisp value.
- Calculate with 1) CoS 2) CoG .

# Thank You!!

NPTEL



IIT KHARAGPUR



NPTEL ONLINE  
CERTIFICATION COURSES