NUMERICAL METHODS (EM 215)

Solutions to system of linear equations

Assignment 1(SLE)

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E/18/397

1) a) Gaussian elimination with pivoting

$$\underbrace{\begin{pmatrix} 1 & 2 & -12 & 8 \\ 3 & 4 & 7 & -2 \\ -3 & 7 & 9 & 5 \\ 6 & -12 & -8 & 3 \end{pmatrix}}_{\underline{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 27 \\ 4 \\ 11 \\ 49 \end{pmatrix}}_{\underline{b}}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -12 & 8 & 27 \\ 3 & 4 & 7 & -2 & 4 \\ -3 & 7 & 9 & 5 & 11 \\ 6 & -12 & -8 & 3 & 49 \end{array}\right) \text{ - Augmented matrix}$$

$$r_2 \leftarrow r_2 - 3r_1$$
$$r_3 \leftarrow r_3 + 3r_1$$
$$r_4 \leftarrow r_4 - 6r_1$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -12 & 8 & 27 \\ 0 & -2 & 43 & -26 & -77 \\ 0 & 13 & -27 & 29 & 92 \\ 0 & -24 & 64 & -45 & -113 \end{array}\right)$$

$$r_3 \leftarrow r_3 + \frac{13}{2} r_2$$
$$r_4 \leftarrow r_4 - 12 r_2$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -12 & 8 & 27 \\ 0 & -2 & 43 & -26 & -77 \\ 0 & 0 & 252.5 & -140 & -408.5 \\ 0 & 0 & -452 & 267 & 811 \end{array}\right)$$

$$r_4 \leftarrow r_4 + \frac{452}{252.5} r_3$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -12 & 8 & 27 \\ 0 & -2 & 43 & -26 & -77 \\ 0 & 0 & 252.5 & -140 & -408.5 \\ 0 & 0 & 0 & 1655/101 & 40271/505 \end{array}\right)$$

By backward substitution,

$$\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 33957/8275 \\ -12704/8275 \\ 8941/8275 \\ 40271/8275 \end{pmatrix} = \begin{pmatrix} 4.1034 \\ -1.5351 \\ 1.0805 \\ 4.8666 \end{pmatrix}$$

b) LU decomposition with pivoting.

$$A = \begin{pmatrix} 1 & 2 & -12 & 8 \\ 3 & 4 & 7 & -2 \\ -3 & 7 & 9 & 5 \\ 6 & -12 & -8 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -12 & 8 \\ 3 & 4 & 7 & -2 \\ -3 & 7 & 9 & 5 \\ 6 & -12 & -8 & 3 \end{pmatrix} \Bigg| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Big\downarrow \; r_1 \leftrightarrow r_4$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -12 & -8 & 3 \\ 3 & 4 & 7 & -2 \\ -3 & 7 & 9 & 5 \\ 1 & 2 & -12 & 8 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Big\downarrow \; \begin{array}{l} r_2 \leftarrow r_2 - \frac{1}{2} r_1 \\ r_3 \leftarrow r_3 - \left(-\frac{1}{2} r_1\right) \\ r_4 \leftarrow r_4 - \frac{1}{6} r_1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ 1/6 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -12 & -8 & 3 \\ 0 & 10 & 11 & -3.5 \\ 0 & 1 & 5 & 6.5 \\ 0 & 4 & -32/3 & 7.5 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Big\downarrow \; \begin{array}{l} r_3 \leftarrow r_3 - \frac{1}{10} r_2 \\ r_4 \leftarrow r_4 - \frac{2}{5} r_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -1/2 & 1/10 & 1 & 0 \\ 1/6 & 2/5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -12 & -8 & 3 \\ 0 & 10 & 11 & -3.5 \\ 0 & 0 & 3.9 & 6.85 \\ 0 & 0 & \frac{-226}{15} & 8.9 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Big\downarrow \; r_3 \leftrightarrow r_4$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/6 & 2/5 & 1 & 0 \\ -1/2 & 1/10 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -12 & -8 & 3 \\ 0 & 10 & 11 & -3.5 \\ 0 & 0 & \frac{-226}{15} & 8.9 \\ 0 & 0 & 3.9 & 6.85 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Big\downarrow \; r_4 \leftarrow r_4 - \left(\frac{-3.9 \times 15}{226}\right) r_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/6 & 2/5 & 1 & 0 \\ -1/2 & 1/10 & \frac{-117}{452} & 1 \end{pmatrix} \begin{pmatrix} 6 & -12 & -8 & 3 \\ 0 & 10 & 11 & -3.5 \\ 0 & 0 & \frac{-226}{15} & 8.9 \\ 0 & 0 & 0 & \frac{8275}{904} \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\underline{L}$ $\qquad\qquad\qquad$ $\underline{U}$ $\qquad\qquad\qquad$ $\underline{P}$

$$\underline{Pb} = \begin{pmatrix} 49 \\ 4 \\ 27 \\ 11 \end{pmatrix}$$

$$\underline{PA} = \underline{LU}$$
$$\underline{Ax} = \underline{b}$$
$$\underline{PAx} = \underline{Pb}$$
$$\underline{LUx} = \underline{Pb}$$
$$\underline{LY} = \underline{Pb}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/6 & 2/5 & 1 & 0 \\ -1/2 & 1/10 & \frac{-117}{452} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 49 \\ 4 \\ 27 \\ 11 \end{pmatrix}$$

using forward substitution

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 49 \\ -20.5 \\ 811/30 \\ 40271/904 \end{pmatrix}$$

$\underline{D}x$ $\underline{U}\,\underline{X} = \underline{Y}$

$$\begin{pmatrix} 6 & -12 & -8 & 3 \\ 0 & 10 & 11 & -3.5 \\ 0 & 0 & -226/15 & 8.9 \\ 0 & 0 & 0 & 8275/904 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 49 \\ -20.5 \\ 811/30 \\ 40271/904 \end{pmatrix}$$

using backward substitution,

$$\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 33957/8275 \\ -12704/8275 \\ 8941/8275 \\ 40271/8275 \end{pmatrix} = \begin{pmatrix} 4.1036 \\ -1.5352 \\ 1.0805 \\ 4.8666 \end{pmatrix}$$

2) a) Jacobi method

The given system of linear eq^{ns}

$$5x_1 + 10x_2 + 3x_3 + x_4 = 6.7$$
$$6x_1 + 7x_2 + 20x_3 - x_4 = 5.8$$
$$12x_1 + 2x_2 + 3x_3 - 30x_4 = 4.3$$
$$15x_1 - x_2 + x_3 + x_4 = 2.1$$

This is not diagonally dominant. Therefore rearranget syste of linear eq^{ns}

$$15x_1 - x_2 + x_3 + x_4 = 2.1$$
$$5x_1 + 10x_2 + 3x_3 + x_4 = 6.7$$
$$6x_1 + 7x_2 + 20x_3 - x_4 = 5.8$$
$$12x_1 + 2x_2 + 3x_3 - 30x_4 = 4.3$$

Python code: -

jaccobi.py > ...

```python
import math as m

# Defining equations to be solved
# in diagonally dominant form
f1 = lambda x1 ,x2, x3, x4    : (2.1+x2-x3-x4)/15
f2 = lambda x1 ,x2, x3, x4    : (6.7-5*x1-3*x3-x4)/10
f3 = lambda x1 ,x2, x3, x4    : (5.8-6*x1-7*x2+x4)/20
f4 = lambda x1 ,x2, x3, x4    : (4.3-12*x1-2*x2-3*x3)/(-30)

# Initial setup
x1_0 = 0
x2_0 = 0
x3_0 = 0
x4_0 = 0
count = 1

# Reading tolerable error
tolerance = float(input('Enter tolerable error: '))

# Implementation of Jacobi Iteration
print('\nCount\tx1\t\tx2\t\tx3\t\tx4\n')

condition = True

while condition:
    x1new = f1(x1_0,x2_0,x3_0, x4_0)
    x2new = f2(x1_0,x2_0,x3_0, x4_0)
    x3new = f3(x1_0,x2_0,x3_0, x4_0)
    x4new = f4(x1_0,x2_0,x3_0, x4_0)

    print('%d\t%0.6f\t%0.6f\t%0.6f\t%0.6f\n' %(count, x1new,x2new, x3new, x4new))

    # norm of solutions
    e1 = m.pow((x1new-x1_0),2)
    e2 = m.pow((x2new-x2_0),2)
    e3 = m.pow((x3new-x3_0),2)
    e4 = m.pow((x4new-x4_0),2)

    count += 1

    x1_0 = x1new
    x2_0 = x2new
    x3_0 = x3new
    x4_0 = x4new

    calculated_tolerance = m.sqrt(e1+e2+e3+e4)
    condition = calculated_tolerance > tolerance

print('\nSolution: x1=%0.3f, x2=%0.3f, x3 = %0.3f and x4 =%0.3f\n'% (x1new,x2new,x3new, x4new))
```

Result of the python code: -

```
Count   x1          x2          x3          x4
1       0.140000    0.670000    0.290000    -0.143333
2       0.174889    0.527333    0.006333    -0.013667
3       0.175644    0.582022    0.052283    -0.037589
4       0.177822    0.570252    0.031719    -0.029046
5       0.177839    0.574478    0.035613    -0.031016
6       0.177992    0.573498    0.034030    -0.030338
7       0.177987    0.573829    0.034361    -0.030500
8       0.177998    0.573748    0.034239    -0.030447
9       0.177997    0.573774    0.034266    -0.030460
10      0.177998    0.573768    0.034257    -0.030456
11      0.177998    0.573770    0.034259    -0.030457

Solution: x1=0.178, x2=0.574, x3 = 0.034 and x4 =-0.030
PS D:\my files\Semester 4\EM314 Numerical Methods\Part 2- Solutions to Systems of Linear Equations>
```

Values converges to,

x1 = 0.177998

x2 = 0.573770

x3 = 0.034259

x4 = −0.030457

b) Gauss-Seidel method

Python code: -

gauss.py > ...

```python
import math as m

# Defining equations to be solved
# in diagonally dominant form
f1 = lambda x1 ,x2, x3, x4    : (2.1+x2-x3-x4)/15
f2 = lambda x1 ,x2, x3, x4    : (6.7-5*x1-3*x3-x4)/10
f3 = lambda x1 ,x2, x3, x4    : (5.8-6*x1-7*x2+x4)/20
f4 = lambda x1 ,x2, x3, x4    : (4.3-12*x1-2*x2-3*x3)/(-30)

# Initial setup
x1_0 = 0
x2_0 = 0
x3_0 = 0
x4_0 = 0
count = 1

# Reading tolerable error
tolerance = float(input('Enter tolerable error: '))

# Implementation of Gauss Seidel Iteration
print('\nCount\tx1\t\tx2\t\tx3\t\tx4\n')

condition = True

while condition:
    x1new = f1(x1_0,x2_0,x3_0, x4_0)
    x2new = f2(x1new,x2_0,x3_0, x4_0)
    x3new = f3(x1new,x2new,x3_0, x4_0)
    x4new = f4(x1new,x2new,x3new, x4_0)

    print('%d\t%0.6f\t%0.6f\t%0.6f\t%0.6f\n' %(count, x1new,x2new, x3new, x4new))

    # norm of solutions
    e1 = m.pow((x1new-x1_0),2)
    e2 = m.pow((x2new-x2_0),2)
    e3 = m.pow((x3new-x3_0),2)
    e4 = m.pow((x4new-x4_0),2)

    count += 1

    x1_0 = x1new
    x2_0 = x2new
    x3_0 = x3new
    x4_0 = x4new

    calculated_tolerance = m.sqrt(e1+e2+e3+e4)
    condition = calculated_tolerance > tolerance

print('\nSolution: x1=%0.3f, x2=%0.3f, x3 = %0.3f and x4 =%0.3f\n'% (x1new,x2new,x3new, x4new))
```

Result of the python code: -

```
PS D:\my files\Semester 4\EM314 Numerical Methods\Part 2- Solutions to Systems of Linear Equations> & C:/Use
rs/ASUS/AppData/Local/Programs/Python/Python39/python.exe "d:/my files/Semester 4/EM314 Numerical Methods/Pa
rt 2- Solutions to Systems of Linear Equations/gauss.py"
Enter tolerable error: 0.00001

Count   x1              x2              x3              x4

1       0.140000        0.600000        0.038000        -0.043533

2       0.180369        0.572769        0.033244        -0.029677

3       0.177947        0.574021        0.034225        -0.030464

4       0.178017        0.573770        0.034252        -0.030450

5       0.177998        0.573770        0.034258        -0.030457

6       0.177998        0.573769        0.034259        -0.030457


Solution: x1=0.178, x2=0.574, x3 = 0.034 and x4 =-0.030
```

Values converges to,

x1 = 0.177998

x2 = 0.573769

x3 = 0.034259

x4 = −0.030457