Data 605 - HW3

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Problem Set 1

1) Rank of given matrix A. For a square matrix the rank will be equal to the number of rows so long as the Upper Triangular (I'll call U) form of the matrix has a non-zero determinant. If $\det(U) = 0$, then we know the rank is less than the dimension and will need to repeat this process for N-1 dimension submatrix until a non-zero determinant is found. For some cases, visual inspection might be better but those are rare cases and do not scale well.

I'm going to bring in a function from last weeks HW to calculate the Upper Triangular matrix.

```
matrix.factorize<- function(input_mat){
  mat_L<-diag(dim(input_mat)[1])
  row_idx<-1
  for(j in 1:(dim(input_mat)[2]-1)){
    for(i in 1:(dim(input_mat)[1]-row_idx)){
      mat_L[i+row_idx,j]<-(input_mat[i+row_idx,j] /
                             input_mat[j,j])
      input_mat[i+row_idx,]<-((-1*mat_L[i+row_idx,j]) *
                                input_mat[row_idx,]) +
                                input_mat[i+row_idx,]
      }
  row_idx<-row_idx+1
    }

  return(list(input_mat, mat_L))
}
```

```
A<- matrix(c(1,2,3,4,-1,0,1,3,0,1,-2,1,5,4,-2,-3), byrow=T, nrow=4)
soln<-matrix.factorize(A)
det(soln[[1]])
```

```
## [1] -9
```

Thus the rank of matrix A is 4, since it has a non=zero determinant. I'm going to try another example where I know the answer, just to be sure this works. A 4x4 checkerboard matrix of 1s and -1s should have a rank of 1.

```
C<-matrix(c(1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1), byrow = T, nrow = 4)
soln<- matrix.factorize(C)
det(soln[[1]])
```

```
## [1] 0
```

This is the expected answer for this matrix. Let's take one step further and find the rank of square matrix. I'll return a vector of all the determinants as well for redundacy and transparency.

```
find.matrix.rank<- function(mat){
  mat_dim<-dim(mat)[1]-1
  rank<-dim(mat)[1]
  mat_det_vec<-rep(NA,mat_dim)
  for (i in 1:mat_dim){
    mat_det_vec[i]<-det(matrix.factorize(mat)[[1]])
    if (mat_det_vec[i]!=0){
      return(list(rank, mat_det_vec))
  } else {
    rank<-rank-1
    mat<-mat[1:dim(mat)[1]-1, 1:dim(mat)[2]-1]
  }
  }
  return(list(rank, mat_det_vec))
}
find.matrix.rank(C)
```

```
## [[1]]
## [1] 1
##
## [[2]]
## [1] 0 0 0
```

As expected. Let's re-try problem 1.1 with the same matrix A.

```
find.matrix.rank(A)
```

```
## [[1]]
## [1] 4
##
## [[2]]
## [1] -9 NA NA
```

Excellent.

2) The minimum rank for any non-zero matrix is rank =1. The maximum is no greater than the lesser of the dimensions. In this case, n.

3) Rank of a given matrix can be found with out handy function from 1.1

```
B<-matrix(c(1,2,1,3,6,3,2,4,2), byrow=T, nrow=3)
find.matrix.rank(B)
```

```
## [[1]]
## [1] 1
##
## [[2]]
## [1] 0 0
```

This makes sense as this represents three parallel lines, which is a dependent system.

Problem 2 - Eigenvectors and values

We are to calculate the eigenvector and values for a 3x3 matrix in Upper Triangular form, which is helpful. I'll do most of this work by-hand but with some machine computation. First set up the $det(A-\lambda I)$ equation.

$$det(\begin{bmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{bmatrix} - \begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}) = 0$$

Which expands to the following:

$$(1-\lambda)det(\begin{bmatrix}4-\lambda & 5\\ 0 & 6-\lambda\end{bmatrix}) - (2)det(\begin{bmatrix}0 & 5\\ 0 & 6-\lambda\end{bmatrix}) + (3)det(\begin{bmatrix}0 & 4-\lambda\\ 0 & 0\end{bmatrix}) = 0$$

Thankfully, the last two terms of this expansion reduce to zero. Thus we are left with the following:

$$(1-\lambda)(24-10\lambda+\lambda^2) = 0$$

This expands to the following,

$$24 - 34\lambda + 11\lambda^2 - \lambda^3 = 0$$

Which can be factored via the fundamental theorem of algebra (luckily $\lambda = 1$ is a root!). We see that this reduces to,

$-\lambda^2 + 10\lambda - 24 = 0$. Which factors our last two remaining eignevalues and our set of eigenvalues are as follows:

$\lambda_1 = 1, \lambda_2 = 4, and \lambda_3 = 6.$

If we return the first eigenvalue in the original matrix A we apply $A - \lambda * I$ as the following,

$$\varepsilon(1) = \begin{bmatrix}0 & 2 & 3\\ 0 & 3 & 5\\ 0 & 0 & 5\end{bmatrix} \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = 0$$

The second eigenvalue can be used to obtain the eigenvector in the same way subtract $\lambda = 4$ from the diagonal of the original matrix A and the same is true for the the last eigenvalue.

$$\varepsilon(4) = \begin{bmatrix}-3 & 2 & 3\\ 0 & 0 & 5\\ 0 & 0 & 2\end{bmatrix} \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = 0$$

$$\varepsilon(6) = \begin{bmatrix}-5 & 2 & 3\\ 0 & -2 & 5\\ 0 & 0 & 0\end{bmatrix} \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = 0$$

Solving these by hand and scaling them to unit vectors using the square root of the sum of the squares we get that;

$$\varepsilon(1) = \begin{bmatrix}0\\ .55\\ 83\end{bmatrix}, \varepsilon(6) = \begin{bmatrix}.51\\ .30\\ .79\end{bmatrix}, \text{ and } \varepsilon(4) = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}.$$

This is the span of vectors for the Eigenspace that are characteristic to martix A.