Week 2: Group homomorphisms and the category of groups

Practice Problems

1. (a) Determine which of the groups $(\mathbb{R}, +), (\mathbb{R}^{>0}, \times), (\mathbb{R} \setminus \{0\}, \times)$ are isomorphic.
   (b) Determine which of the groups $(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{Q}^{>0}, \times), (\mathbb{Q} \setminus \{0\}, \times)$ are isomorphic.
   (c) Are any of the groups in part (a) isomorphic to any of the groups in part (b)?
2. Show that $\mathrm{Aut}(C_2 \times C_2) \cong S_3$. Show that $\mathrm{Aut}(C_2 \times C_2) \ncong \mathrm{Aut}(C_2) \times \mathrm{Aut}(C_2)$.
3. Let G be a group. Show that the function $g \mapsto (h \mapsto ghg^{-1})$ is a homomorphism $G \to \mathrm{Aut}(G)$.

   More precisely, for each $g \in G$, define the function $\theta_g \colon G \to G$ by $\theta_g(h) = ghg^{-1}$. Show that for each $g \in G$, θ_g is an automorphism of G. Define the function $\varphi \colon G \to \mathrm{Aut}(G)$ by $\varphi(g) = \theta_g$. Show that φ is a homomorphism.

   This is known as the conjugation action. In the case where A is an invertible matrix, θ_A is the change of basis by A.

Presentation Problems

1. Let G be a group. Show that the following are equivalent:
   - G is abelian.
   - The function $g \mapsto g^2$ is a homomorphism from $G \to G$.
   - The function $g \mapsto g^{-1}$ is an automorphism of G.
   - The function $g \mapsto (h \mapsto ghg^{-1})$ is the trivial homomorphism from $G \to \mathrm{Aut}(G)$.
2. Show that $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$.
3. Let G and H be finite groups of coprime order.
   (a) Let φ be an automorphism of $G \times H$. Let $\widetilde{G} = G \times \{e_H\}$ be the copy of G in $G \times H$. Let $\widetilde{H} = \{e_G\} \times H$ be the copy of H in $G \times H$. Show that φ takes $\widetilde{G}$ to $\widetilde{G}$ and that φ takes $\widetilde{H}$ to $\widetilde{H}$.
   (b) Show that $\mathrm{Aut}(G \times H) \cong \mathrm{Aut}(G) \times \mathrm{Aut}(H)$.

   Compare this to practice problem 2.
4. An automorphism φ of a group G is said to be *fixed-point-free* if $\varphi(g) \neq g$ for all $g \in G \setminus \{1\}$. Find a group G of order 12 and a fixed-point-free automorphism φ of G of order 6 (as an element of $\mathrm{Aut}(G)$).

Bonus: Chinese Remainder Theorem

Let n be a positive integer and let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n.

1. Construct an injective homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{a_k}$.
2. Show that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}$.
3. Show that $(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/p_1^{a_1}\mathbb{Z})^\times \times (\mathbb{Z}/p_2^{a_2}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_k^{a_k}\mathbb{Z})^\times$.

This is known as the Chinese remainder theorem (1/2).

Tricky Problems

1. Let φ be a fixed-point-free automorphism of a finite group G.

   (a) Show that every element of G is of the form $g^{-1}\varphi(g)$ for some $g \in G$ and that every element of G is of the form $\varphi(g)g^{-1}$ for some $g \in G$.

   (b) Suppose that $|\varphi| = k$ and let $g \in G$. Show that

   $$g\varphi(g)\dots\varphi^{k-1}(g) = \varphi^{k-1}(g)\dots\varphi(g)g = 1.$$

   (c) Show that if $|\varphi| = 2$ then G is abelian and $\varphi(g) = g^{-1}$ for all $g \in G$.

2. Let G be a group. For each integer k, G is called k-*abelian* if $g^k h^k = (gh)^k$ for all $g, h \in G$.

   (a) Show that if G is abelian then G is k-abelian for all integers k.

   (b) Show that G is k-abelian if and only if G is $(1-k)$-abelian.

   (c) Show that if G is k-abelian and $(k+1)$-abelian and $(k+2)$-abelian, then G is abelian.

   (d) Suppose that $\gcd(k-1, |G|) = \gcd(k, |G|) = 1$. Show that if G is k-abelian then G is abelian.