

## Week 5: Group actions and the class formula

### Practice Problems

1. Show that any group with class formula  $60 = 1 + 12 + 12 + 15 + 20$  has no nontrivial normal subgroups.
2. For each positive integer  $n$ , construct an interesting action of the dihedral group of order  $2n$  on  $\mathbb{R}^2$ .
3. Construct an interesting action of the quaternion group  $Q_8$  on the set  $\{\{\pm 1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}\}$ . Construct a homomorphism  $Q_8 \rightarrow S_4$ . Is this homomorphism injective?

### Presentation Problems

1. Let  $G$  be a finite group and let  $H$  be a proper subgroup of  $G$ . Show that  $G \neq \bigcup_{g \in G} gHg^{-1}$ .
2. Let  $G$  be a finite group acting on a set  $X$ . For each  $g \in G$ , let  $X^g = \{x \in X : gx = x\}$  be the set of the elements in  $X$  fixed by  $g$ . Let  $X/G$  denote the set of orbits of  $X$  under the action of  $G$ . Show that

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

by counting the set  $\{(g, x) \in G \times X : gx = x\}$  in 2 different ways. This is known as Burnside's lemma.

3. Let  $G$  be a finite  $p$ -group and let  $H$  be a nontrivial normal subgroup of  $G$ . Show that  $H \cap Z(G) \neq 1$ . Give a counterexample if  $H$  is not a normal subgroup of  $G$ .
4. Let  $H$  and  $K$  be finite subgroups of a group  $G$ .
  - (a) Construct a transitive group action of  $H \times K$  on  $HK$ .
  - (b) Use the orbit-stabilizer theorem to show that  $|HK| = |H||K|/|H \cap K|$ .

### Bonus: Frobenius' Theorem

Let  $G$  be a finite group and let  $n$  be a divisor of  $|G|$ . Frobenius' theorem states that the number of solutions to  $g^n = 1$  is a multiple of  $n$ . A proof of this result using Burnside's lemma and some combinatorics can be found at <https://sbseminar.wordpress.com/2015/09/05/a-counting-argument-for-frobenius-theorem/>. By “ $d$ -torsion” elements, the post means elements  $g$  such that  $g^d = 1$ . Here are a couple applications of Frobenius' theorem:

1. Let  $G$  be a finite group of order  $p^k m$  where  $p$  is a prime not dividing  $m$ . Show that the number of elements of  $G$  of order a power of  $p$  is congruent to  $p^k \pmod{p^{k+1}}$ .
2. Show that if  $p$  is a prime then  $p$  divides  $(p-1)! + 1$ .

This last result is known as Wilson's theorem (2/4).

### Tricky Problems

1. Let  $G$  be a group and let  $H$  be a subgroup of  $G$  of finite index. Show that  $G \neq \bigcup_{g \in G} gHg^{-1}$ . Give a counterexample if  $H$  is not assumed to be of finite index in  $G$ .
2. Let  $G$  be a finite group of order  $p^k m$  where  $p$  is a prime possibly dividing  $m$ . Let  $X$  be the collection of subsets of  $G$  of order  $p^k$ . Let  $G$  act on  $X$  by left multiplication. For each  $0 \leq j \leq k$ , let  $n_j$  count the number of orbits of cardinality  $p^j m$ .
  - (a) Show that if  $S \in X$  then  $S$  is a (disjoint) union of right cosets of the stabilizer subgroup  $\text{Stab}_G(S)$ .

- (b) Show that every orbit of  $X$  has cardinality  $p^j m$  for some  $0 \leq j \leq k$ .
- (c) Show that  $n_0$  counts the number of subgroups of  $G$  of order  $p^k$ .
- (d) Show that

$$\binom{p^k m - 1}{p^k - 1} = \frac{1}{m} |X| = \sum_{j=0}^k n_j p^j \equiv n_0 \pmod{p}.$$

- (e) Show that the number of subgroups of  $G$  of order  $p^k$  is congruent to 1 modulo  $p$  by applying parts (c) and (d) to both  $G$  and  $\mathbb{Z}/p^k m \mathbb{Z}$ . In particular,  $G$  contains a subgroup of order  $p^k$ .

Here are a couple applications of this result:

- (f) Let  $2^p - 1$  be a Mersenne prime and let  $G$  be a finite group of order  $2^p(2^p - 1)$ . Show that  $G$  contains a normal subgroup of order  $2^p - 1$  or  $2^p$ .
- (g) Show that if  $p$  is a prime then  $p$  divides  $(p - 1)! + 1$ .

This last result is known as Wilson's theorem (3/4).