Week 1: The definition of a group and examples

Practice Problems

1. In the group of invertible 2×2 real matrices, consider the elements

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

   Verify that $|g| = 4$, $|h| = 3$, and $|gh| = \infty$.

2. For a finite group G with identity element e, the *multiplication table* of G is the table

$\bullet$	e	$\cdots$	g	$\cdots$	h	$\cdots$
e	e	$\cdots$	g	$\cdots$	h	$\cdots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$	
g	g	$\cdots$	g^2	$\cdots$	$g \bullet h$	$\cdots$
$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	
h	h	$\cdots$	$h \bullet g$	$\cdots$	h^2	$\cdots$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\ddots$

   Of course, the multiplication table of G depends on the ordering of the nonidentity elements of G.

   (a) Show that every row and every column of the multiplication table of G contains all elements of the group exactly once (like a Sudoku).

   (b) Show that G is abelian if and only if the multiplication table of G is symmetric about the diagonal.

3. (a) Prove that for $n \in \{1, 2, 3\}$, there is only one multiplication table for a group with n elements.

   (b) Prove that there are two possible multiplication tables for a group with 4 elements.

   (c) Show that every group of order at most 4 is abelian.

   The noncyclic group of order 4 is called the *Klein four-group*.

Presentation Problems

An *involution* is an element of order 2.

1. Let G be a group such that $g^2 = 1$ for all $g \in G$ (or equivalently, that every nonidentity element of G is an involution). Show that G is abelian.

2. Let G be a finite abelian group.

   (a) Show that the product $\prod_{g \in G} g$ is well-defined.

   (b) Show that if G contains exactly one involution f then $\prod_{g \in G} g = f$.

   (c) Apply the previous part to a well-chosen group to show that p divides $(p-1)! + 1$.

   This last result is known as Wilson's theorem (1/4).

3. Let G be a finite group and let m be the number of involutions in G. Show that $|G| - m$ is odd. Deduce that if $|G|$ is even then G contains at least one involution.

4. Let G be a group and let $g, h \in G$. Show that

$$|gh| = |hg| = |g^{-1}h^{-1}| = |h^{-1}g^{-1}|.$$

Tricky Problems

1. Let G be a finite group, let $g \in G$, and let k be an integer with $\gcd(k, |G|) = 1$.

   (a) By partitioning G into subsets of cardinality $|g|$, show that $|g|$ divides $|G|$.

   (b) Show that $g^{|G|} = 1$.

   (c) Show that $ak = b|G| + 1$ for some integers a and b.

   (d) Show that $g^{ak} = g$.

   (e) Show that the kth power map $x \mapsto x^k$ is injective.

   (f) Deduce that the kth power map $x \mapsto x^k$ is surjective.

2. Let G be a finite group. Suppose that $gh \neq hg$ for all distinct involutions $g, h \in G$. Show that $|gh|$ is odd for all involutions $g, h \in G$.