Algebra Problem Sets

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1 Week 1

1. (Exercise 2.1.2 in Aluffi) If $g^2 = 1$ for all $g \in G$ then G is abelian. Show that $g^2 = 1$ for all $g \in (\mathbb{Z}/24\mathbb{Z})^\times$.
2. (Exercise 2.1.8 in Aluffi) If G is a finite abelian group with exactly one element $f \in G$ of order 2 then $\prod_{g \in G} g = f$. Deduce that if p is a prime then p divides $(p-1)! + 1$.
3. (Exercise 2.1.9 in Aluffi) If G is a finite group with exactly m elements of order 2 then $|G| - m$ is odd. Deduce that if $|G|$ is even then G contains at least one element of order 2.
4. (Exercise 2.1.11 in Aluffi) For all $g, h \in G$, $|gh| = |hg|$.
5. (Exercise 2.1.14 in Aluffi) If g and h commute and if $\gcd(|g|, |h|) = 1$ then $|gh| = |g||h|$.
6. If G is a finite group and if $g \in G$ then $|g|$ divides $|G|$. Deduce that $g^{|G|} = 1$ for all $g \in G$.
7. If G is a finite group and if $g, h \in G$ satisfy $g^k = h^k$ where $\gcd(k, |G|) = 1$ then $g = h$.
8. If n is a prime power then how many elements of S_n have order n? What if $n = 6$?
9. For which n does $(\mathbb{Z}/n\mathbb{Z})^\times$ have order 4? Which of these groups are isomorphic?