Week 4: Polynomial rings II

Let R be a commutative ring with identity. Let K be a field.

Practice Problems

1. Show that $f(x) = x^3 + x + t$ is irreducible in $\mathbb{C}(t)[x]$. *Hint*: Gauss' Lemma.
2. Find all irreducible polynomials in $\mathbb{Z}/2\mathbb{Z}[x]$ of degree ≤ 4.
3. What are the maps from $\mathbb{Q}[t]/(t^3 - 2)$ into $\mathbb{R}$? What about into $\mathbb{C}$?

Presentation Problems

1. Define $\varphi\colon K[x, y, z] \to K[t]$ by $\varphi(x) = t$, $\varphi(y) = t^2$, and $\varphi(z) = t^3$. Find polynomials $f_1, \ldots, f_n$ in $K[x, y, z]$ such that $\ker\varphi = (f_1, \ldots, f_n)$. The curve given parametrically by $p(t) = (t, t^2, t^3)$ is called the "twisted cubic curve".
2. Let p be an odd prime of $\mathbb{Z}$. Prove $x^n - p$ is irreducible over $\mathbb{Q}(i)$ for all n.
3. (a) Show that the polynomial $(x - 1)\cdots(x - n) - 1$ is irreducible in $\mathbb{Q}[x]$ for all $n \geq 1$.
   (b) Show that the polynomial $(x - 1)\cdots(x - n) + 1$ is irreducible in $\mathbb{Q}[x]$ for all $n \geq 1$ except $n = 4$.
4. Prove that $x^a + y^b + c^z$ irreducible in $\mathbb{C}[x, y, z]$. *Hint*: Use Eisenstein's Criterion.

Module Theory Problem

Given a chain complex $\mathcal{C}$ of R-modules,

$$\cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \longrightarrow \cdots$$

we define the R-modules $Z_n(\mathcal{C}) = \ker d_n$ (the n-cycles) and $B_n(\mathcal{C}) = \operatorname{im} d_{n+1}$ (the n-boundaries). Note that $Z_n(\mathcal{C})$ and $B_n(\mathcal{C})$ are both submodules of C_n. Furthermore, since $\mathcal{C}$ is a chain complex, $B_n(\mathcal{C})$ is a submodule of $Z_n(\mathcal{C})$. Then we can form the quotient $H_n(\mathcal{C}) = Z_n(\mathcal{C})/B_n(\mathcal{C})$, which is called the nth homology group. Despite the name, $H_n(\mathcal{C})$ is actually an R-module, rather than just an abelian group. Note that $\mathcal{C}$ is exact if and only if every $B_n(\mathcal{C}) = Z_n(\mathcal{C})$ if and only if every $H_n(\mathcal{C}) = 0$. Thus, there is a sense in which the homology of a chain complex measures its failure to be exact. Problem 1 below gives an example of this.

1. Given open subsets U, V of $\mathbb{R}^n, \mathbb{R}^m$, let $C^\infty(U, V)$ denote the $\mathbb{R}$-vector space of infinitely differentiable functions from U to V. For $U \subseteq \mathbb{R}^2$, define the linear operator $D : C^\infty(U, \mathbb{R}^2) \to C^\infty(U, \mathbb{R})$ by $D(F) = \partial_y F_1 - \partial_x F_2$, where $F(x, y) = (F_1(x, y), F_2(x, y))$. Verify that

   $$0 \longrightarrow \mathbb{R} \xrightarrow{\text{const}} C^\infty(U, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^2) \xrightarrow{D} C^\infty(U, \mathbb{R}) \longrightarrow 0$$

   is a chain complex of $\mathbb{R}$-modules, where $\text{const}(c)$ is the constant function with value c. Determine the third homology group $(\ker D/\operatorname{im}\text{grad})$ of this complex when $U = \mathbb{R}^2 \setminus \{0\}$.

   Hint: Problem 5.8.4 in *Advanced Calculus*, by Folland.

2. A homomorphism of chain complexes $\alpha\colon \mathcal{A} \to \mathcal{B}$ is a commutative diagram of R-modules

   $$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & A_2 & \xrightarrow{d_2} & A_1 & \xrightarrow{d_1} & A_0 & \xrightarrow{d_0} & A_{-1} & \xrightarrow{d_{-1}} & A_{-2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_{-1}} & & \downarrow{\scriptstyle \alpha_{-2}} & & \\ \cdots & \longrightarrow & B_2 & \xrightarrow{d_2} & B_1 & \xrightarrow{d_1} & B_0 & \xrightarrow{d_0} & B_{-1} & \xrightarrow{d_{-1}} & B_{-2} & \longrightarrow & \cdots \end{array}$$

   We call α a "chain map".

(a) Show that a chain map $\alpha : \mathcal{A} \to \mathcal{B}$ induces R-module homomorphisms $(\alpha_*)_n : H_n(\mathcal{A}) \to H_n(\mathcal{B})$ for each n.

A short exact sequence of chain complexes $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ is a commutative diagram of R-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_2 \arrow[r, "d_2"] \arrow[d, "\alpha_2"] & A_1 \arrow[r, "d_1"] \arrow[d, "\alpha_1"] & A_0 \arrow[r, "d_0"] \arrow[d, "\alpha_0"] & A_{-1} \arrow[r, "d_{-1}"] \arrow[d, "\alpha_{-1}"] & A_{-2} \arrow[r] \arrow[d, "\alpha_{-2}"] & \cdots \\
\cdots \arrow[r] & B_2 \arrow[r, "e_2"] \arrow[d, "\beta_2"] & B_1 \arrow[r, "e_1"] \arrow[d, "\beta_1"] & B_0 \arrow[r, "e_0"] \arrow[d, "\beta_0"] & B_{-1} \arrow[r, "e_{-1}"] \arrow[d, "\beta_{-1}"] & B_{-2} \arrow[r] \arrow[d, "\beta_{-2}"] & \cdots \\
\cdots \arrow[r] & C_2 \arrow[r, "f_2"] \arrow[d] & C_1 \arrow[r, "f_1"] \arrow[d] & C_0 \arrow[r, "f_0"] \arrow[d] & C_{-1} \arrow[r, "f_{-1}"] \arrow[d] & C_{-2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where each column is a short exact sequence of R-modules.

Let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of chain complexes.

(b) For each n, apply part (c) of the snake lemma problem to the commutative diagram of R-modules

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "\alpha_n"] \arrow[d, "d_n"] & B_n \arrow[r, "\beta_n"] \arrow[d, "e_n"] & C_n \arrow[r] \arrow[d, "f_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha_{n-1}"] & B_{n-1} \arrow[r, "\beta_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

To get exact sequences of R-modules

$$0 \longrightarrow Z_n(\mathcal{A}) \longrightarrow Z_n(\mathcal{B}) \longrightarrow Z_n(\mathcal{C})$$

and

$$A_{n-1}/B_{n-1}(\mathcal{A}) \longrightarrow B_{n-1}/B_{n-1}(\mathcal{B}) \longrightarrow C_{n-1}/B_{n-1}(\mathcal{C}) \longrightarrow 0$$

(c) For each n, apply the snake lemma to the commutative diagram of R-modules

$$\begin{tikzcd}
 & A_n/B_n(\mathcal{A}) \arrow[r] \arrow[d, "d_n"] & B_n/B_n(\mathcal{B}) \arrow[r] \arrow[d, "e_n"] & C_n/B_n(\mathcal{C}) \arrow[r] \arrow[d, "f_n"] & 0 \\
0 \arrow[r] & Z_{n-1}(\mathcal{A}) \arrow[r, "\alpha_{n-1}"] & Z_{n-1}(\mathcal{B}) \arrow[r, "\beta_{n-1}"] & Z_{n-1}(\mathcal{C}) &
\end{tikzcd}$$

to obtain the exact sequence of R-modules

$$H_n(\mathcal{A}) \xrightarrow{\alpha_n^*} H_n(\mathcal{B}) \xrightarrow{\beta_n} H_n(\mathcal{C}) \xrightarrow{\delta} H_{n-1}(\mathcal{A}) \xrightarrow{\alpha_{n-1}^*} H_{n-1}(\mathcal{B}) \xrightarrow{\beta_{n-1}^*} H_{n-1}(\mathcal{C}) \ .$$

(d) Combine these exact sequences to obtain a long exact sequence of R-modules

$$\cdots \longrightarrow H_n(\mathcal{A}) \longrightarrow H_n(\mathcal{B}) \longrightarrow H_n(\mathcal{C}) \longrightarrow H_{n-1}(\mathcal{A}) \longrightarrow H_{n-1}(\mathcal{B}) \longrightarrow H_{n-1}(\mathcal{C}) \longrightarrow \cdots$$

Tricky Problems

1. Let I be an ideal of R. Call I Noetherian if all ascending chains of ideals of R contained in I stabilize. Note that R is a Noetherian ring if and only if R is Noetherian as an ideal of itself.

   (a) Let $I \subseteq J$ be ideals, and suppose I and J/I are Noetherian (as ideals of R and R/I). Prove that J is Noetherian

   For the remainder of this problem, suppose R is an Artinian ring with exactly one maximal ideal $\mathfrak{m}$, which satisfies $\mathfrak{m}^k = 0$ for some k. For ease of notation, let $\mathfrak{m}^0 = R$.

   (b) For each j with $0 \leq j < k$, give $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ an $R/\mathfrak{m}$-vector space structure. Show that there is an inclusion preserving bijection between subspaces of $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ and ideals I of R such that $\mathfrak{m}^{j+1} \leq I \leq \mathfrak{m}^j$.

   (c) Use the assumption that R is Artinian to show that $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is finite dimensional. Deduce that $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is Noetherian as an ideal of the quotient ring $R/\mathfrak{m}^{j+1}$.

   (d) Inductively apply parts (a) and (c) to the chain

   $$0 = \mathfrak{m}^k \leq \mathfrak{m}^{k-1} \leq \ldots \leq \mathfrak{m}^2 \leq \mathfrak{m} \leq R$$

   to show that R is Noetherian.

   (e) Combining this with Tricky Problem 2 of Week 2, and Tricky Problem 2(b) of Week 3, prove that every (commutative) Artinian ring is Noetherian.

2. Let $f(x, y)$ be an irreducible degree two polynomial in $\mathbb{C}[x, y]$. Prove that $\mathbb{C}[x, y]/(f(x, y))$ is isomorphic to $\mathbb{C}[x, y]/(y - x^2)$ or to $\mathbb{C}[x, y]/(xy - 1)$, and that these two rings are not isomorphic. For bonus points, prove this with $\mathbb{C}$ replaced by any field satisfying the fundamental theorem of algebra.