

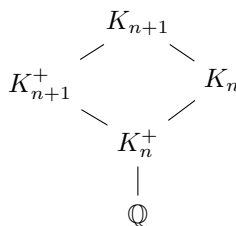
## Week 8: Field extensions and geometry (14.4, 14.5)

### Practice Problems

1. Find a Galois extension  $K/\mathbb{Q}$  with  $\text{Gal}(K/\mathbb{Q}) \cong C_3$ .
2. Without appealing to the fundamental theorem of algebra, show that every polynomial in  $\mathbb{R}[x]$  of odd degree has a root in  $\mathbb{R}$ . Deduce that there are no nontrivial odd-degree extensions of  $\mathbb{R}$ .
3. Without appealing to the fundamental theorem of algebra, show that every polynomial in  $\mathbb{C}[x]$  of degree 2 has a root in  $\mathbb{C}$ . Deduce that there are no quadratic extensions of  $\mathbb{C}$ .

### Presentation Problems

1. Let  $p$  be an odd prime and let  $\zeta_p = e^{2\pi i/p}$ . Show that there exists a unique quadratic extension  $K/\mathbb{Q}$  with  $K \subseteq \mathbb{Q}(\zeta_p)$ . What is this quadratic extension?
2. Show that  $\sqrt[3]{2}$  is not contained in any cyclotomic field over  $\mathbb{Q}$ .
3. For each integer  $n \geq 1$ , set  $K_n = \mathbb{Q}(\zeta_{2^{n+2}})$  and  $K_n^+ = \mathbb{Q}(\alpha_n)$  where  $\alpha_n = \zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1}$ .
  - (a) Show that  $K_n^+ = K_n \cap K_{n+1}^+$  and determine the degrees of the extensions in the diagram



- (b) Determine the minimal polynomials for  $\zeta_{2^{n+2}}$  and  $\alpha_{n+1}$  over  $K_n^+$ , with coefficients in terms of  $\alpha_n$ .
  - (c) Inductively give an explicit formula for  $\alpha_n$  using nested square roots.
4. For each  $n \geq 1$ , determine  $\text{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n)$  and  $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n)$ . Your answer will depend on the prime factorization of  $n$ .

### Tricky Problems

1.
  - (a) For each  $n \geq 1$ , compute the lattice of subfields for the extension  $\mathbb{Q}(\zeta_{2^{n+2}})/\mathbb{Q}$ .
  - (b) Compute  $\text{Gal}(K/\mathbb{Q})$  for each intermediate field  $K$  of the extension  $\mathbb{Q}(\zeta_{2^{n+2}})/\mathbb{Q}$ .
2. Let  $f(x) \in \mathbb{C}[x]$  and let  $g(x) = f(x)\bar{f}(x)$  where  $\bar{f}(x)$  is given by taking the complex conjugate of the coefficients of  $f$ .
  - (a) Show that  $g(x) \in \mathbb{R}[x]$ .

Let  $K$  be the splitting field of  $g(x)$  over  $\mathbb{R}$ .

- (b) Show that  $K(i)$  is a Galois extension of  $\mathbb{R}$ .

Let  $G = \text{Gal}(K(i)/\mathbb{R})$  and let  $P$  be a Sylow 2-subgroup of  $G$ .

- (c) Show that the fixed field of  $P$  is an extension of  $\mathbb{R}$  of odd degree. Deduce that  $G$  is a 2-group.
- (d) Show that if  $\text{Gal}(K(i)/\mathbb{C}) \neq 1$  then there exists a quadratic extension of  $\mathbb{C}$ . Deduce that  $K(i) = \mathbb{C}$ .

- (e) Show that if  $g(x)$  is nonconstant then  $g(x)$  has a root in  $\mathbb{C}$ .
- (f) Show that if  $f(x)$  is nonconstant then  $f(x)$  has a root in  $\mathbb{C}$ .

This is known as the fundamental theorem of algebra.