Week 9: Semidirect products and the extension problem

Practice Problems

1. Let N and H be groups. Prove that $(N \rtimes H)/N \cong H$.
2. Show that the dihedral group of order $2n$ is isomorphic to the semidirect product $C_n \rtimes_\varphi C_2$ for some homomorphism $\varphi\colon C_2 \to \mathrm{Aut}(C_n)$. What is the map $\varphi\colon C_2 \to \mathrm{Aut}(C_n)$?

   Disclaimer: For some values of n, there might be multiple choices for φ. However, there is one natural choice of φ that works for all values of n.
3. Classify groups of order 6 up to isomorphism.

Presentation Problems

1. Show that the quaternion group Q_8 is not a nontrivial semidirect product.
2. Let $G = C_2 \times C_2$ and let $\varphi\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ be the identity map. Show that $G \rtimes_\varphi \mathrm{Aut}(G) \cong S_4$. In general, the semidirect product $G \rtimes_\varphi \mathrm{Aut}(G)$ is called the holomorph of G.
3. Classify groups of order 28 up to isomorphism.
4. Let $G = N \rtimes P$ where P is a p-group and where N has order indivisible by p. Show that

   $$N = \{g \in G : |g| \text{ is indivisible by } p\}.$$

   Deduce that N is a characteristic subgroup of G.

Tricky Problems

1. Classify groups of order 315 up to isomorphism.
2. Let $n = p_1^{a_1} \dots p_k^{a_k}$ be the prime factorization of a positive integer n.

   (a) Show that every group of order n is cyclic if and only if $a_i = 1$ and $p_i \nmid p_j - 1$.

   (b) Show that every group of order n is abelian if and only if $1 \le a_i \le 2$ and $p_i \nmid p_j^{a_j} - 1$.

   (c) Show that every group of order n is nilpotent if and only if $p_i \nmid p_j^b - 1$ for all $1 \le b \le a_j$.

   (d) Show that every group of order n is solvable if and only if n is not divisible by the order of a nonabelian finite simple group.

   Hint: Apply part (g) of tricky problem 2 from Week 7 to a minimal counterexample to part (c). Then use part (c) for parts (a) and (b).