Week 7: Composition series and solvability

Practice Problems

1. Find composition series for the dihedral groups of orders 8 and 10. Are there any other composition series for these groups? What are the composition factors of these groups?
2. Find the derived series of the quaternion group.
3. Suppose that G is a finite group with $G = G'$. Show that the order of G is divisible by the order of some nonabelian finite simple group.

Presentation Problems

1. Let G be a group.
   (a) Let N and K be normal subgroups of G with $N \cap K = \{1\}$. Prove that N and K commute, meaning that $nk = kn$ for all $n \in N$ and $k \in K$. Show that $NK \cong N \times K$.
   (b) Suppose that G is finite and let $N_1, N_2, \ldots, N_m$ be normal subgroups of G such that $|G| = \prod_j |N_j|$ and such that $N_i \cap N_j = \{1\}$ for all $i \neq j$. Show that $G \cong N_1 \times N_2 \times \cdots \times N_m$.
2. Show that the following are equivalent:
   (a) Every finite group of odd order is solvable.
   (b) Every nonabelian finite simple group has even order.
   (c) There is no nonabelian finite simple group of odd order with all proper subgroups solvable.
3. Let G be a finite solvable group and let N be a minimal normal subgroup of G.
   (a) Suppose that H is a characteristic subgroup of N, meaning that $\varphi(H) = H$ for all $\varphi \in \mathrm{Aut}(N)$. Show that $H = 1$ or $H = N$.
   (b) Show that $[N, N]$ is a characteristic subgroup of N. Deduce that N is abelian.
   (c) Show that if p is a prime dividing the order of N then the kernel and image of the pth power map are characteristic subgroups of N. Deduce that $g^p = 1$ for all $g \in N$.
   (d) Show that N is elementary abelian, meaning that $N \cong C_p \times \ldots \times C_p$ for some prime p.
4. If h and k are elements of G then we define the commutator $[h, k] = hkh^{-1}k^{-1}$. If H and K are subgroups of G then we define the commutator subgroup $[H, K]$ as the subgroup of G generated by elements of the form $[h, k]$ for $h \in H$ and $k \in K$.
   (a) Show that $[H, K] = [K, H]$.
   (b) Show that $[G, H] = 1$ if and only if H is a central subgroup of G.

   For elements x, y, z of G we define $[x, y, z] = [[x, y], z]$ and $x^y = yxy^{-1}$. For subgroups X, Y, Z of G we define $[X, Y, Z] = [[X, Y], Z]$. Let H be a subgroup of G.

   (c) Show that $[z^{-1}, x, y]^z [y^{-1}, z, x]^y [x^{-1}, y, z]^x = 1$.
   (d) Show that if $[X, Y, Z] = 1$ and $[Y, Z, X] = 1$ then $[Z, X, Y] = 1$.
   (e) Show that if $[G, G] = G$ and if $[G, H, G] = 1$ then H is a central subgroup of G.
   (f) Show that if $[G, G] = G$ then $Z(G/Z(G)) = 1$.

   This shows that if the derived series of G stabilizes at the first step then the "upper central series" of G stabilizes by the second step. The upper central series of G is related to G being nilpotent (see tricky problem 1). The "binary icosahedral group" is an example of a finite group G with $[G, G] = G$ and $Z(G) \neq 1$.

Tricky Problems

1. Let G be a finite group.

   (a) Show that the following conditions on G are equivalent:

   - H is a proper subgroup of $N_G(H)$ for every proper subgroup H of G.
   - Every maximal subgroup of G is a normal subgroup of G.
   - Every Sylow subgroup of G is a normal subgroup of G.
   - G is isomorphic to a direct product of p-groups.
   - G has a normal subgroup of order d for every divisor d of $|G|$.

   A finite group satisfying these conditions is called *nilpotent*.

   (b) Show that G is nilpotent if and only if $G/Z(G)$ is nilpotent.

   (c) Show that if $\mathrm{Aut}(G)$ is nilpotent then G is nilpotent.

   (d) Show that the intersection of all maximal subgroups of G is nilpotent.
   This subgroup is called the Frattini subgroup of G, or $\Phi(G)$.

   (e) Show that if G has a unique maximal subgroup then G is nilpotent.

   (f) Show that if G has a unique maximal subgroup then G is cyclic.

2. Let π be a collection of primes. A Hall π-subgroup is a subgroup whose order is a product of primes in π and whose index is not divisible by any prime in π. Let G be a minimal counterexample to the statement "Every finite solvable group has a Hall π-subgroup."

   (a) Let N be a minimal normal subgroup of G. Show that N is a p-group for some $p \notin \pi$.

   (b) Show that $|G| = p^k m$ where $|N| = p^k$ and where m is a product of primes in π (so that any Hall π-subgroup of G would have order m).

   (c) Let M/N be a minimal normal subgroup of G/N. Show that M/N is a q-group for some $q \in \pi$.

   (d) Let Q be a Sylow q-subgroup of M. Show that Q is not a normal subgroup of G.

   (e) Show that $N_G(Q)$ contains a Hall π-subgroup of G.

   This proves Hall's theorem that finite solvable groups have Hall π-subgroups for all collections of primes π. Here is an application of this result:

   (f) Let G be a finite group such that every proper subgroup of G is nilpotent. By considering a maximal intersection of maximal subgroups of G, show that G is solvable.

   (g) Use Hall's theorem to show that if $|G|$ is divisible by three distinct primes then G is nilpotent.