Week 0: Sets, Functions, and Quotients

Locate and understand an explanation of Russell's Paradox and Cantor's Diagonalization argument.

Practice Problems

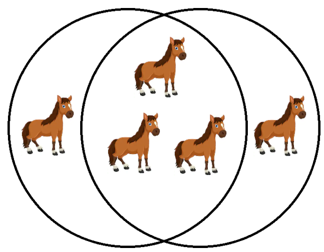
1. Identify the error in each of the following proofs and give a counterexample to its statement.

   Claim. *Let X be a set and let R be a symmetric and transitive relation on X. Then R is reflexive.*

   Proof. For any $x, y \in X$, xRy implies yRx by symmetry, and then xRx follows from transitivity. Thus, xRx for any $x \in X$ which shows that R is reflexive. □

   Claim. *For any nonempty finite set S of horses, all of the horses in S have the same color.*

   Proof. We will prove the claim by induction on $|S|$. The base case of $|S| = 1$ is trivially true, since any single horse has the same color as itself. Now suppose that the result holds when $|S|$ is one less. Then in the picture below, all of the horses in the left circle have the same color, and all of the horses in the right circle have the same color.

   By transitivity of "same color", all horses in S have the same color. □

2. (a) Define the relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if $x - y$ is an integer. What are the equivalence classes of $\sim$?

   (b) Let $f\colon \mathbb{R} \to \mathbb{R}^2$ be the function $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Describe all sets and functions in the canonical decomposition of f, and interpret the set $\mathbb{R}/{\sim_f}$ geometrically (where $x \sim_f y$ if and only if $f(x) = f(y)$).

3. Suppose that $\sim$ is an equivalence relation on a set X, and let $\pi\colon X \to X/{\sim}$ be the natural projection. When is there a function $f\colon X/{\sim} \to X$ such that $f \circ \pi = id_X$?

Presentation Problems

1. Let X be a set.

   (a) Given an equivalence relation $\sim$ on X, construct a partition of X.

   (b) Given a partition P of X, construct an equivalence relation on X.

   (c) Parts (a) and (b) still aren't enough to tell us that there is a one-to-one correspondence between partitions of X and equivalence relations on X. What more needs to be checked?

   (d) Conclude that there is a bijection between the sets $\{R \subseteq X \times X : R \text{ is an equivalence relation on } X\}$ and $\{P \subseteq 2^X : P \text{ is a partition of } X\}$ (the set 2^X is the *powerset* of X, the set of subsets of X).

2. Suppose that $f : X \to Y$ is a surjection. Prove that there is a function $g : Y \to X$ such that $f \circ g = id_Y$.

   Note: If you know what the axiom of choice is, you may assume it. In fact, you should prove that this is equivalent to that axiom. If you don't, don't worry about it.

3. Let A and B be finite sets. What is the cardinality of A^B? You must prove your answer correct.

4. (a) Let X be a set. Show that for any element $x_0 \in X$ and any function $h\colon X \times \mathbb{N} \to X$, there exists a unique function $f\colon \mathbb{N} \to X$ such that $f(0) = x_0$ and $f(n+1) = h(f(n), n)$ for all $n \in \mathbb{N}$.

   Recall that a "function" from X to Y is a subset Γ of $X \times Y$ such that for every element $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in \Gamma$.

   (b) Show that there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ that satisfies $f(0) = 0$ and $f(n+1) = (n+1)f(n)$.

   (c) Let X be a set and let $p\colon X \to X$ be a function. Show that there is a sequence of functions $p^n\colon X \to X$ for $n \in \mathbb{N}$, such that $p^0(x) = x$ and $p^{n+1}(x) = p(p^n(x))$ for each $n \in \mathbb{N}$.

   This is known as the principle of recursion.

Tricky Problems

1. Let X and Y be arbitrary sets and let $f\colon X \to Y$ and $g\colon Y \to X$ be injective functions.

   (a) Show that the function $h\colon X \to Y$ given by

$$h(x) = \begin{cases} (f \circ g)^n(a) & \text{if } x = g((f \circ g)^n(a)) \text{ for some } n \in \mathbb{N} \text{ and } a \in Y \setminus \operatorname{im} f \\ f(x) & \text{otherwise} \end{cases}$$

   is well defined.

   Note: The term "well defined" is context dependent. In this case, the worry is that for a fixed x, there might be multiple possible choices for n and a, potentially resulting in multiple possible values for $h(x)$.

   (b) Show that h is bijective.

   This is known as the Cantor-Schröder-Bernstein Theorem.

2. Let X be a countable set and let S be a subset of 2^X (the power set of X) such that for any two distinct elements $A, B \in S$, the intersection $A \cap B$ is finite. Is S necessarily countable? Give a proof or a counterexample.