Week 1: Ring homomorphisms, quotient rings, and ideals

Let R be a commutative ring with identity.

Practice Problems

1. Let R be the collection of continuous functions on $[0,1]$. Show that R is a commutative ring with identity. Let $x \in [0,1]$ and let $I = \{f \in R : f(x) = 0\}$. Show that I is a prime ideal of R.

2. Let I and J be ideals of R.

   (a) Show that $I + J$ and $I \cap J$ and IJ are ideals of R.

   (b) Show that $IJ \subseteq I \cap J \subseteq I + J$.

   (c) Give an example where $IJ \neq I \cap J$.

3. Let $R = \mathbb{C}[x]$ and let I be the collection of polynomials in R with no constant or linear term. Show that I is an ideal of R. Show that R is an integral domain but R/I is not an integral domain.

Presentation Problems

1. (a) Let I, J, and K be ideals of R. Show that if $I \subseteq J \cup K$ then $I \subseteq J$ or $I \subseteq K$.

   (b) Let P be a prime ideal of R and I, J be ideals of R. Show that if $IJ \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

2. (a) Show that if I is an ideal of R then $\sqrt{I} = \{x \in R : x^m \in I \text{ for some } m \geq 1\}$ is an ideal of R.

   (b) The *nilradical* of R is defined by $\mathfrak{N}(R) = \sqrt{0}$. A ring S satisfying $\mathfrak{N}(S) = 0$ is called *reduced*. Show that $R/\mathfrak{N}(R)$ is reduced.

   We now show that $\mathfrak{N}(R)$ is the intersection of all prime ideals of R.

   (c) Show that $\mathfrak{N}(R)$ is contained in the intersection of the prime ideals of R.

   (d) Let $x \in R \setminus \mathfrak{N}(R)$ and let $S = \{x^n : n \geq 1\}$. Apply Zorn's Lemma to the collection of ideals of R disjoint from S to obtain a prime ideal P of R with $x \notin P$.

   (e) Deduce that $\mathfrak{N}(R)$ is the intersection of the prime ideals of R.

   (f) Let I be an ideal of R. Show that $\sqrt{I}$ is the intersection of the prime ideals of R containing I.

3. Let $J, I_1, \ldots, I_n$ be ideals of R. Suppose that I_k is a prime ideal of R for all $k \geq 3$. The prime avoidence lemma states that if $J \subseteq I_1 \cup \cdots \cup I_n$ then $J \subseteq I_k$ for some k.

   (a) Show that the prime avoidence lemma holds when $n = 2$.

   Now suppose that $n \geq 3$ and inductively assume that the prime avoidence lemma holds for all smaller values of n. Suppose for contradiction that $J \subseteq I_1 \cup \cdots \cup I_n$ but $J \not\subseteq I_k$ for all k.

   (b) Use the inductive hypothesis to find $x_k \in J \setminus (I_1 \cup \cdots \cup I_{k-1} \cup I_{k+1} \cup \cdots \cup I_n)$.

   (c) Show that $x_k \in I_k$.

   (d) Use the primality of I_n to show that $x_1 \cdots x_{n-1} + x_n \in J \setminus (I_1 \cup \cdots \cup I_n)$. Derive a contradiction.

   This proves the prime avoidence lemma.

4. Let R be the ring of continuous functions on $[0,1]$. For each $x \in [0,1]$, let $I_x = \{f \in R : f(x) = 0\}$.

   (a) Prove that each I_x is a maximal ideal of R.

   (b) Prove that every maximal ideal of R is of the form I_x for some $x \in I$.

Module Theory Problem

1. A sequence of R-modules is a diagram of R-modules of the form

$$\cdots \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} M_{-1} \xrightarrow{f_{-1}} \cdots.$$

   If only finitely many M_n are nonzero, we only include one of the zero terms on each side. For example, if $M_n = 0$ for $n > 2$ and $n < -2$, then we would draw the above as

$$0 \longrightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} M_{-1} \xrightarrow{f_{-1}} M_{-2} \longrightarrow 0.$$

   A sequence of R-modules is called a chain complex if $f_{n-1} \circ f_n = 0$ for all n, i.e. $\operatorname{im} f_n \subseteq \ker f_{n-1}$. A sequence if called exact if $\operatorname{im} f_n = \ker f_{n-1}$.

   Let $R = \mathbb{R}$, and let $C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ denote the vector space (R-module) of infinitely differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^k$. Come up with an exact sequence of R-modules

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(\mathbb{R}^3, \mathbb{R}) \longrightarrow C^\infty(\mathbb{R}^3, \mathbb{R}^3) \longrightarrow C^\infty(\mathbb{R}^3, \mathbb{R}^3) \longrightarrow C^\infty(\mathbb{R}^3, \mathbb{R}) \longrightarrow 0.$$

   Hint: Where have you heard the term "exact" before?

2. A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

   What property will the map f have? What property will the map g have? Show that A is isomorphic to a submodule of B and that C is isomorphic to a quotient of B.

3. We say a short exact sequence (as above) left-splits if there is a map $r\colon B \to A$ such that $r \circ f = \mathrm{id}_A$. We say it right-splits if there is a map $s\colon C \to B$ such that $g \circ s = \mathrm{id}_C$. Prove the following are equivalent:

   (a) The short exact sequence left-splits.

   (b) The short exact sequence right-splits.

   (c) There is an isomorphism $\varphi : B \to A \oplus C$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "id"] & B \arrow[r, "g"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\iota"] & A \oplus C \arrow[r, "\pi"] & C \arrow[r] & 0 \end{tikzcd}$$

   where ι and π are the inclusion and projection maps.

   This is known as the splitting lemma.

4. Let k be a field and let V and W be finite dimensional vector spaces over k. Let $L\colon V \to W$ be a linear transformation. Prove that $\dim V = \operatorname{rk} L + \operatorname{null} L$.

   This is known as the rank-nullity theorem.

Tricky Problems

1. Ideals I and J of R are called coprime if $I + J = R$.

   (a) Show that if I and J are coprime, then $I \cap J = IJ$.

   (b) Show that if I and J are coprime, then $R/(I \cap J) \cong R/I \times R/J$.

   (c) Show that if $I_1, \ldots, I_n$ are pairwise coprime then $R/(I_1 \cap \ldots \cap I_k) \cong R/I_1 \times \ldots \times R/I_k$.

   This is known as the Chinese remainder theorem (2/2).

   Now let n be a positive integer and let $n = p_1^{a_1} \ldots p_k^{a_k}$ be the prime factorization of n.

   (d) Show that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}$.

   (e) Show that $(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/p_1^{a_1}\mathbb{Z})^\times \times \ldots \times (\mathbb{Z}/p_k^{a_k}\mathbb{Z})^\times$.

2. Let $\operatorname{Spec} R$ denote the set of prime ideals of R. Given an ideal I of R, let $V(I) = \{P \in \operatorname{Spec} R : I \subseteq P\}$. Prove that the following are equivalent.

   (a) There are ideals I and J of R such that $V(I)$ and $V(J)$ are disjoint and $V(I) \cup V(J) = R$.

   (b) There exist nonzero idempotents $e_1, e_2 \in R$ such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.

   (c) R is isomorphic to a direct product $R_1 \times R_2$ of two nonzero rings (both commutative with identity).

   If you know about topological spaces, show that we can put a topology on $\operatorname{Spec} R$ by defining closed sets to be sets of the form $V(I)$. What does condition (a) say topologically?