# Week 10: The Infamous 1004913 Problem

For a finite group $G$ and a prime $p$ dividing $|G|$, let $n_p(G)$ denote the number of Sylow $p$-subgroups of $G$.

1. Let $p$ be a prime and let $h \in S_n$ have order $p$. Suppose that $g \in S_n$ satisfies $ghg^{-1} = h^k$ for some integer $k$. Show that if $gh \neq hg$ then $g$ fixes at most one point in each cycle of $h$.

2. Let $G$ be a finite simple group of order $3^3 \cdot 7 \cdot 13 \cdot 409$. Show that

$$\begin{aligned} n_7(G) &= 3^2 \cdot 13 \cdot 409, \\ n_{13}(G) &= 3^2 \cdot 7 \cdot 409, \\ n_{409}(G) &= 3^2 \cdot 7 \cdot 13. \end{aligned}$$

3. Let $P$ and $Q$ be Sylow 3-subgroups of $G$ with $|P \cap Q|$ maximal. Suppose for contradiction that $P \cap Q \neq 1$. Let $H = N_G(P \cap Q)$.
   (a) Show that $P \cap H$ and $Q \cap H$ are distinct Sylow 3-subgroups of $H$.
   (b) Show that $9 \mid |H|$ and that $|H| \mid 3^3 \cdot 7 \cdot 13$.
   (c) Show that if $7 \mid |H|$ then $n_7(H) = 351$.
   (d) Show that if $13 \mid |H|$ then $n_{13}(H) = 27$.
   (e) Show that $H$ has too many elements.

   Thus, $P \cap Q = 1$ so any two distinct Sylow 3-subgroups of $G$ intersect trivially.

4. Let $P$ be a Sylow 3-subgroup of $G$. By letting $P$ act by conjugation on the Sylow 3-subgroups of $G$, show that $n_3(G) \equiv 1 \pmod{27}$. Deduce that

$$n_3(G) = 7 \cdot 409.$$

5. Show that every nonidenty element of $G$ has prime order.

6. Compute the class equation for $G$.

7. Let $G$ act by conjugation on the Sylow 409-subgroups of $G$. Use Burnside's lemma to show that each element of $G$ of order 3 fixes exactly 9 Sylow 409-subgroups of $G$.

8. Use the first problem to obtain a contradiction.