

Week 4: Canonical decomposition and Lagrange's theorem, presentations and free groups

Practice Problems

1. Show that S_3 admits the presentation $(a, b \mid a^2, b^2, (ab)^3)$.
2. Let G be a group, let H be a subgroup of G , and let N be a normal subgroup of G contained in H . Show that H is a normal subgroup of G if and only if H/N is a normal subgroup of G/N .
3. Let G be a group and let H be a subgroup of G of finite index n .
 - (a) Construct a homomorphism $G \rightarrow S_n$ with kernel contained in H .
 - (b) Deduce that if G is finite then there is an injective homomorphism $G \rightarrow S_{|G|}$.

This is known as the Cayley embedding.

Presentation Problems

1. Let G be a group. Show that every subgroup of G of index 2 is normal.
2. Let G be a finite group and let p be the smallest prime dividing the order of G .
 - (a) Show that every subgroup of G of index p is normal. *Hint:* Use the homomorphism $G \rightarrow S_{[G:H]}$.
 - (b) Show that every normal subgroup of G of order p is central. *Hint:* Use the injective homomorphism $N_G(H)/C_G(H) \rightarrow \text{Aut}(H)$.
 - (c) Show that if $|G| = p^2$ then G is abelian.
 - (d) Show that if $|G| = p^2$ then $G \cong \mathbb{Z}/p^2\mathbb{Z}$ or $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. *Hint:* If G has no element of order p^2 then think about a basis for G as a vector space over $\mathbb{Z}/p\mathbb{Z}$.
3. Let A be a set. Show that $F^{\text{ab}}(A) \cong F(A)^{\text{ab}}$, meaning that the free abelian group on A is isomorphic to the abelianization of the free group on A .
4. Let $(A \mid \mathcal{R})$ be a presentation for a group G . Let $(B \mid \mathcal{S})$ be a presentation for a group H . We may assume that A and B are disjoint. Show that the group

$$G * H = (A \cup B \mid \mathcal{R} \cup \mathcal{S})$$

satisfies the universal property for the coproduct of G in H in the category of groups.

Bonus: Adjoint Functors

Let \mathbf{C} and \mathbf{D} be categories. A functor $\mathcal{F}: \mathbf{C} \rightarrow \mathbf{D}$ consists of an assignment of an object $\mathcal{F}(A) \in \text{Obj}(\mathbf{D})$ for every object $A \in \text{Obj}(\mathbf{C})$ and of a function

$$\text{Hom}_{\mathbf{C}}(A, B) \rightarrow \text{Hom}_{\mathbf{D}}(\mathcal{F}(A), \mathcal{F}(B))$$

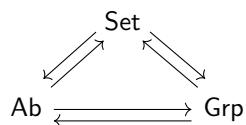
for every pair of objects $A, B \in \text{Obj}(\mathbf{C})$. These functions are also all denoted by \mathcal{F} and are required to preserve identities and compositions, meaning that $\mathcal{F}(\text{id}_A) = \text{id}_{\mathcal{F}(A)}$ for every object $A \in \text{Obj}(\mathbf{C})$ and that $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ whenever the target of f agrees with the source of g .

Two functors $\mathcal{F}: \mathbf{C} \rightarrow \mathbf{D}$ and $\mathcal{G}: \mathbf{D} \rightarrow \mathbf{C}$ are said to be *adjoint* if there are bijections

$$\text{Hom}_{\mathbf{C}}(A, \mathcal{G}(B)) \cong \text{Hom}_{\mathbf{D}}(\mathcal{F}(A), B)$$

for all objects $A \in \text{Obj}(\mathbf{C})$ and $B \in \text{Obj}(\mathbf{D})$. These bijections are also required to be “natural” but don’t worry about this condition. In this case, \mathcal{F} is called the left adjoint and \mathcal{G} is called a right adjoint.

1. Construct six functors



such that the following three conditions are satisfied:

- (a) The inner triangle commutes.
- (b) The outer triangle commutes.
- (c) Each edge of the triangle is a pair of adjoint functors.

Tricky Problems

1. Let G be a group.
 - (a) Show that if G has a non-normal subgroup of index 3 then G has a normal subgroup of index 2.
 - (b) Show that if G has a subgroup of index 4 then G has a normal subgroup of index 2 or 3.
2. Let G and H be finite groups.
 - (a) Construct a surjective group homomorphism $\varphi: G * H \rightarrow G \times H$.
 - (b) Show that $\ker \varphi$ is a free group of rank $(|G| - 1)(|H| - 1)$.