# Week 3: Polynomial rings I

Let R be a commutative ring with identity. Let K be a field.

## **Practice Problems**

- 1. Factor the polynomial  $x^4 + 1$  in the rings  $\mathbb{C}[x]$ ,  $\mathbb{R}[x]$ ,  $\mathbb{Q}[x]$ .
- 2. Construct a surjective ring homomorphism  $K[x,y] \to K$  with kernel (x,y). Construct a surjective ring homomorphism  $K[x,y] \to K[y]$  with kernel (x). Deduce that (x,y) is maximal and that (x) is prime.
- 3. Show that  $\mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/(x^2+5)$ .

#### **Presentation Problems**

- 1. Show that K[x] contains infinitely many primes. *Hint*: Look at Euclid's proof that there are infinitely many primes in  $\mathbb{Z}$ .
- 2. Let  $I = (xy, (x y)z) \subseteq K[x, y, z]$ . Show that  $\sqrt{I} = (xy, xz, yz)$ .
- 3. (a) Show that  $K[x,y]/(y^2-x) \cong K[y]$ .
  - (b) Show that  $K[x, y]/(y^2 x) \ncong K[x, y]/(y^2 x^2)$ .
- 4. (a) Construct an injective ring homomorphism  $K[x,y]/(xy) \to K[x] \times K[y]$ .
  - (b) Show that  $K[x,y]/(xy) \ncong K[x] \times K[y]$ .

# Module Theory Problem

1. (a) Show that if  $f: M \to N$  is an R-module homomorphism then the sequence of R-modules

$$0 \longrightarrow \ker f \longrightarrow M \stackrel{f}{\longrightarrow} N \longrightarrow \operatorname{coker} f \longrightarrow 0$$

is exact, where coker  $f = N/\operatorname{im} f$ .

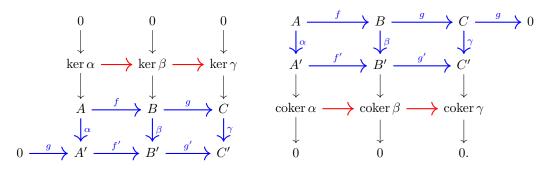
(b) Show that the blue commutative diagram of R-modules induces the red R-module homomorphisms

$$0 \longrightarrow \ker \alpha \longrightarrow A \xrightarrow{\alpha} A' \longrightarrow \operatorname{coker} \alpha \longrightarrow 0$$

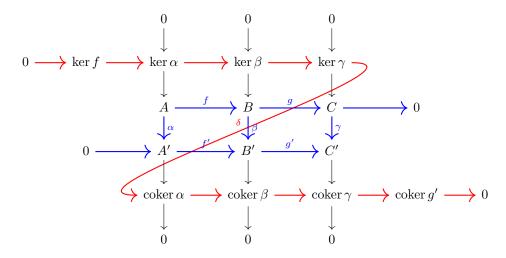
$$\downarrow \qquad \qquad \downarrow f \qquad \qquad \downarrow f' \qquad \qquad \downarrow$$

$$0 \longrightarrow \ker \beta \longrightarrow B \xrightarrow{\beta} B' \longrightarrow \operatorname{coker} \beta \longrightarrow 0.$$

(c) Show that the blue commutative diagrams of R-modules with exact rows induce the red exact sequences of R-modules



(d) Show that the blue commutative diagram of R-modules with exact rows induces the red exact sequence of R-modules



This is known as the snake lemma.

### Tricky Problems

- 1. Suppose that R is an integral domain. Let a and b be positive integers with gcd(a,b) = 1. Consider the ring homomorphism  $\varphi \colon R[x,y] \to R[t]$  defined by  $\varphi(x) = t^b$  and  $\varphi(y) = t^a$ .
  - (a) Show that  $(x^a y^b) \subseteq \ker \varphi$ .
  - (b) Let  $f(x,y) \in \ker \varphi$ . Show that we can write f(x,y) = g(x,y) + h(x,y) with  $g(x,y) \in (x^a y^b)$  and  $\deg_y h(x,y) \le b-1$ .
  - (c) Show that  $h(x, y) \in \ker \varphi$ .
  - (d) Show that the exponents of  $\varphi(x^iy^j)$  are distinct for  $0 \le j \le b-1$  and deduce that h(x,y)=0.
  - (e) Show that  $\ker \varphi = (x^a y^b)$ .
  - (f) Show that  $(x^a y^b)$  is a prime ideal of R[x, y].
- 2. (a) Show that if R is a Noetherian ring then every quotient of R is Noetherian.
  - (b) Show that if R and S are Noetherian rings then  $R \times S$  is a Noetherian ring.
  - (c) Show that R is Noetherian if and only if every ideal of R is finitely generated.

The remaining parts of this problem will show that if R is Noetherian then so is R[x].

- (d) Let I be an ideal of R[x]. Let I' denote the set of leading coefficients of polynomials in I. Prove that I' is an ideal of R, and deduce that  $I' = (a_1, a_2, \ldots, a_n)$  for some  $a_1, a_2, \ldots, a_n \in R$ . By definition of I', for each  $a_j$  there is some  $f_j \in I$  whose leading coefficient is  $a_j$ .
- (e) Let d be a positive integer. Let  $I_d$  be the set of leading coefficients of polynomials of degree d in I, as well as 0. Prove that that  $I_d$  is an ideal. Deduce that  $I_d = (a_{d,1}, a_{d,2}, \ldots, a_{d,n_d})$  for each d, and let  $f_{d,j} \in I$  be a polynomial of degree d with leading coefficient  $a_{d,j}$ .
- (f) Let  $N = \max_{1 \le i \le n} \deg f_i$ . Prove that

$$I = (f_1, f_2, \dots, f_n) + \sum_{d=1}^{N-1} (f_{d,1}, f_{d,2}, \dots, f_{d,n}).$$

Hint: If they're not equal, then there is some  $f \in I$  not in the ideal in the right of minimal degree.

(g) Deduce that if R is a Noetherian ring then R[x] is Noetherian.

This is known as Hilbert's Basis Theorem.