# Week 8: Field extensions and geometry (14.4, 14.5)

## Practice Problems

1. Find a Galois extension $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong C_3$.
2. Without appealing to the fundamental theorem of algebra, show that every polynomial in $\mathbb{R}[x]$ of odd degree has a root in $\mathbb{R}$. Deduce that there are no nontrivial odd-degree extensions of $\mathbb{R}$.
3. Without appealing to the fundamental theorem of algebra, show that every polynomial in $\mathbb{C}[x]$ of degree 2 has a root in $\mathbb{C}$. Deduce that there are no quadratic extensions of $\mathbb{C}$.

## Presentation Problems

1. Let $p$ be an odd prime and let $\zeta_p = e^{2\pi i/p}$. Show that there exists a unique quadratic extension $K/\mathbb{Q}$ with $K \subseteq \mathbb{Q}(\zeta_p)$. What is this quadratic extension?
2. Show that $\sqrt[3]{2}$ is not contained in any cyclotomic field over $\mathbb{Q}$.
3. For each integer $n \geq 1$, set $K_n = \mathbb{Q}(\zeta_{2^{n+2}})$ and $K_n^+ = \mathbb{Q}(\alpha_n)$ where $\alpha_n = \zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1}$.

   (a) Show that $K_n^+ = K_n \cap K_{n+1}^+$ and determine the degrees of the extensions in the diagram

$$\begin{tikzcd}
& K_{n+1} \arrow[dl, dash] \arrow[dr, dash] & \\
K_{n+1}^+ \arrow[dr, dash] & & K_n \arrow[dl, dash] \\
& K_n^+ \arrow[d, dash] & \\
& \mathbb{Q} &
\end{tikzcd}$$

   (b) Determine the minimal polynomials for $\zeta_{2^{n+2}}$ and $\alpha_{n+1}$ over $K_n^+$, with coefficients in terms of $\alpha_n$.

   (c) Inductively give an explict formula for $\alpha_n$ using nested square roots.
4. For each $n \geq 1$, determine $\mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n)$ and $\mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n)$. Your answer will depend on the prime factorization of $n$.

## Tricky Problems

1. (a) For each $n \geq 1$, compute the lattice of subfields for the extension $\mathbb{Q}(\zeta_{2^{n+2}})/\mathbb{Q}$.

   (b) Compute $\mathrm{Gal}(K/\mathbb{Q})$ for each intermediate field $K$ of the extension $\mathbb{Q}(\zeta_{2^{n+2}})/\mathbb{Q}$.
2. Let $f(x) \in \mathbb{C}[x]$ and let $g(x) = f(x)\overline{f}(x)$ where $\overline{f}(x)$ is given by taking the complex conjugate of the coefficients of $f$.

   (a) Show that $g(x) \in \mathbb{R}[x]$.

   Let $K$ be the splitting field of $g(x)$ over $\mathbb{R}$.

   (b) Show that $K(i)$ is a Galois extension of $\mathbb{R}$.

   Let $G = \mathrm{Gal}(K(i)/\mathbb{R})$ and let $P$ be a Sylow 2-subgroup of $G$.

   (c) Show that the fixed field of $P$ is an extension of $\mathbb{R}$ of odd degree. Deduce that $G$ is a 2-group.

   (d) Show that if $\mathrm{Gal}(K(i)/\mathbb{C}) \neq 1$ then there exists a quadratic extension of $\mathbb{C}$. Deduce that $K(i) = \mathbb{C}$.

(e) Show that if $g(x)$ is nonconstant then $g(x)$ has a root in $\mathbb{C}$.

(f) Show that if $f(x)$ is nonconstant then $f(x)$ has a root in $\mathbb{C}$.

This is known as the fundamental theorem of algebra.