Week 8: The symmetric group

Practice Problems

1. Compute the size of the conjugacy class of $(12)(3456)$ in S_6 and the size of the conjugacy class of (123) in A_4.

2. Show that A_4 has a unique Sylow 2-subgroup P. Show that P is a normal subgroup of S_4. Find a normal subgroup N of P such that N is not a normal subgroup of S_4. This gives another example that normality is not transitive.

3. Compute the center of A_n.

Presentation Problems

1. Determine the derived series of S_n.

2. Let G be a finite simple group with a proper subgroup of index n. Show that G is isomorphic to a subgroup of A_n unless G is cyclic of order 2.

3. (a) Let $n \geq 5$. Show that A_n is the only nontrivial normal subgroup of S_n.
   (b) Let $n \geq 2$. Show that A_n is the only subgroup of S_n of index 2.

4. A transitive subgroup of S_n is a subgroup that acts transitively on $\{1, \ldots, n\}$.

   (a) Let G be a transitive subgroup of S_n. Show that the order of G is divisible by n.
   (b) Let p be a prime, let G be a transitive subgroup of S_p, and suppose that G contains a transposition. Show that $G = S_p$.

   This problem will be useful when doing Galois theory. If $p(x)$ is a polynomial of degree d then we will consider a faithful action of "the Galois group" G on the d complex roots of $p(x)$. Since the action is faithful, we can view G as a subgroup of S_d. This problem tells us that if d is prime, G contains a transposition, and G acts transitively on the roots of $p(x)$, then G consists of every possible permutation of the roots of $p(x)$.

Tricky Problems

1. Let G be a finite simple group with exactly $p+1$ Sylow p-subgroups. Show that the order of G divides $\frac{1}{2}p(p-1)(p+1)$. Show that there is no simple group of order 264.

2. Let G be a finite group of even order with a cyclic Sylow 2-subgroup.

   (a) Show that the image of G under the Cayley embedding $G \to S_{|G|}$ is not contained in $A_{|G|}$.
   (b) Show that G contains a normal subgroup of index 2.
   (c) Show that if $|G| = 2^k m$ with m odd then the collection H of elements of G of odd order forms a normal subgroup of G of order m.
   (d) Suppose that $k = 1$ and suppose that G has exactly m involutions. Show that any two elements of G of odd order commute.