Week 4: Cannonical decomposition and Lagrange's theorem, presentations and free groups

Practice Problems

1. Show that S_3 admits the presentation $(a, b \mid a^2, b^2, (ab)^3)$.
2. Let G be a group, let H be a subgroup of G, and let N be a normal subgroup of G contained in H. Show that H is a normal subgroup of G if and only if H/N is a normal subgroup of G/N.
3. Let G be a group and let H be a subgroup of G of finite index n.
   (a) Construct a homomorphism $G \to S_n$ with kernel contained in H.
   (b) Deduce that if G is finite then there is an injective homomorphism $G \to S_{|G|}$.

   This is known as the Cayley embedding.

Presentation Problems

1. Let G be a group. Show that every subgroup of G of index 2 is normal.
2. Let G be a finite group and let p be the smallest prime dividing the order of G.
   (a) Show that every subgroup of G of index p is normal. *Hint*: Use the homomorphism $G \to S_{[G:H]}$.
   (b) Show that every normal subgroup of G of order p is central. *Hint*: Use the injective homomorphism $N_G(H)/C_G(H) \to \mathrm{Aut}(H)$.
   (c) Show that if $|G| = p^2$ then G is abelian.
   (d) Show that if $|G| = p^2$ then $G \cong \mathbb{Z}/p^2\mathbb{Z}$ or $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. *Hint*: If G has no element of order p^2 then think about a basis for G as a vector space over $\mathbb{Z}/p\mathbb{Z}$.
3. Let A be a set. Show that $F^{\mathrm{ab}}(A) \cong F(A)^{\mathrm{ab}}$, meaning that the free abelian group on A is isomorphic to the abelianization of the free group on A.
4. Let $(A \mid \mathscr{R})$ be a presentation for a group G. Let $(B \mid \mathscr{S})$ be a presentation for a group H. We may assume that A and B are disjoint. Show that the group

   $$G * H = (A \cup B \mid \mathscr{R} \cup \mathscr{S})$$

   satisfies the universal property for the coproduct of G in H in the category of groups.

Bonus: Adjoint Functors

Let C and D be categories. A functor $\mathscr{F} : \mathsf{C} \to \mathsf{D}$ consists of an assignment of an object $\mathscr{F}(A) \in \mathrm{Obj}(\mathsf{D})$ for every object $A \in \mathrm{Obj}(\mathsf{C})$ and of a function

$$\mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{D}}(\mathscr{F}(A), \mathscr{F}(B))$$

for every pair of objects $A, B \in \mathrm{Obj}(\mathsf{C})$. These functions are also all denoted by $\mathscr{F}$ and are required to preserve identities and compositions, meaning that $\mathscr{F}(\mathrm{id}_A) = \mathrm{id}_{\mathscr{F}(A)}$ for every object $A \in \mathrm{Obj}(\mathsf{C})$ and that $\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f)$ whenever the target of f agrees with the source of g.

Two functors $\mathscr{F} : \mathsf{C} \to \mathsf{D}$ and $\mathscr{G} : \mathsf{D} \to \mathsf{C}$ are said to be *adjoint* if there are bijections

$$\mathrm{Hom}_{\mathsf{C}}(A, \mathscr{G}(B)) \cong \mathrm{Hom}_{\mathsf{D}}(\mathscr{F}(A), B)$$

for all objects $A \in \mathrm{Obj}(\mathsf{C})$ and $B \in \mathrm{Obj}(\mathsf{D})$. These bijections are also required to be "natural" but don't worry about this condition. In this case, $\mathscr{F}$ is called the left adjoint and $\mathscr{G}$ is called a right adjoint.

1. Construct six functors

$$\begin{tikzcd} & \mathsf{Set} \arrow[dl, shift right] \arrow[dr, shift left] & \\ \mathsf{Ab} \arrow[ur, shift right] \arrow[rr, shift left] & & \mathsf{Grp} \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd}$$

such that the following three conditions are satisfied:

(a) The inner triangle commutes.

(b) The outer triangle commtues.

(c) Each edge of the triangle is a pair of adjoint functors.

Tricky Problems

1. Let G be a group.

   (a) Show that if G has a non-normal subgroup of index 3 then G has a normal subgroup of index 2.

   (b) Show that if G has a subgroup of index 4 then G has a normal subgroup of index 2 or 3.

2. Let G and H be finite groups.

   (a) Construct a surjective group homomorphism $\varphi\colon G * H \to G \times H$.

   (b) Show that $\ker \varphi$ is a free group of rank $(|G|-1)(|H|-1)$.