Week 5: Group actions and the class formula

Practice Problems

1. Show that any group with class formula $60 = 1+12+12+15+20$ has no nontrivial normal subgroups.
2. For each positive integer n, construct an interesting action of the dihedral group of order $2n$ on $\mathbb{R}^2$.
3. Construct an interesting action of $\mathrm{Aut}(Q_8)$ on the set $\{\{\pm i\}, \{\pm j\}, \{\pm k\}\}$. Construct a homomorphism $\mathrm{Aut}(Q_8) \to S_3$. Is this homomorphism injective?

Presentation Problems

1. Let G be a finite group and let H be a proper subgroup of G. Show that $G \neq \bigcup_{g\in G} gHg^{-1}$.
2. Let G be a finite group acting on a set X.
   (a) For each $g \in G$, let $X^g = \{x \in X : gx = x\}$ be the set of the elements in X fixed by g. Let X/G denote the set of orbits of X under the action of g. Show that
   $$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$
   by counting the set $\{(g, x) \in G \times X : gx = x\}$ in two different ways.
   This is known as Burnside's lemma.
   (b) Suppose that G acts transitively on X and that $|X| \geq 2$. Show that there is some $g \in G$ with no fixed points (i.e. $gx \neq x$ for all $x \in X$).
3. Let G be a finite p-group and let H be a nontrivial normal subgroup of G. Show that $H \cap Z(G) \neq 1$. Give a counterexample if H is not a normal subgroup of G.
4. Let H and K be finite subgroups of a group G.
   (a) Construct a transitive group action of $H \times K$ on HK.
   (b) Use the orbit-stabilizer theorem to show that $|HK| = |H||K|/|H \cap K|$.

Bonus: Frobenius' Theorem

Let G be a finite group and let n be a divisor of $|G|$. Frobenius' theorem states that the number of solutions to $g^n = 1$ is a multiple of n. A proof of this result using Burnside's lemma and some combinatorics can be found at `https://sbseminar.wordpress.com/2015/09/05/a-counting-argument-for-frobenius-theorem/`. By "d-torsion" elements, the post means elements g such that $g^d = 1$. Here are a couple applications of Frobenius' theorem:

1. Let G be a finite group of order $p^k m$ where p is a prime not dividing m. Show that the number of elements of G of order a power of p is congruent to $p^k \pmod{p^{k+1}}$.
2. Show that if p is a prime then p divides $(p-1)! + 1$.

This last result is known as Wilson's theorem (2/4).

Tricky Problems

1. Let G be a group and let H be a subgroup of G of finite index. Show that $G \neq \bigcup_{g \in G} gHg^{-1}$. Give a counterexample if H is not assumed to be of finite index in G.

2. Let G be a finite group of order $p^k m$ where p is a prime possibly dividing m. Let X be the collection of subsets of G of order p^k. Let G act on X by left multiplication. For each $0 \leq j \leq k$, let n_j count the number of orbits of cardinality $p^j m$.

   (a) Show that if $S \in X$ then S is a (disjoint) union of right cosets of the stabilizer subgroup $\mathrm{Stab}_G(S)$.

   (b) Show that every orbit of X has cardinality $p^j m$ for some $0 \leq j \leq k$.

   (c) Show that n_0 counts the number of subgroups of G of order p^k.

   (d) Show that
   $$\binom{p^k m - 1}{p^k - 1} = \frac{1}{m}|X| = \sum_{j=0}^{k} n_j p^j \equiv n_0 \pmod{p}.$$

   (e) Show that the number of subgroups of G of order p^k is congruent to 1 modulo p by applying parts (c) and (d) to both G and $\mathbb{Z}/p^k m\mathbb{Z}$. In particular, G contains a subgroup of order p^k.

   Here are a couple applications of this result:

   (f) Let $2^p - 1$ be a Mersenne prime and let G be a finite group of order $2^p(2^p - 1)$. Show that G contains a normal subgroup of order $2^p - 1$ or 2^p.

   (g) Show that if p is a prime then p divides $(p-1)! + 1$.

   This last result is known as Wilson's theorem (3/4).