Week 9: Galois groups of polynomials (14.6)

Practice Problems

1. Compute the Galois group of $x^3 - 2x + 4$ over $\mathbb{Q}$.
2. Compute the Galois group of $x^3 - x + 1$ over $\mathbb{Q}$.
3. Compute the Galois group of $x^3 + x^2 - 2x - 1$ over $\mathbb{Q}$.

Presentation Problems

1. Let p be a prime and let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree p with exactly $p-2$ real roots. Show that the Galois group of $f(x)$ is isomorphic to S_p.
2. Compute the Galois group of $x^5 + 4x^2 - 5x - 3$ over $\mathbb{Q}$.
3. Compute the Galois group of $x^4 - 2$ over $\mathbb{Q}$.
4. Compute the Galois group of $x^4 + 8x + 12$ over $\mathbb{Q}$.

Tricky Problems

1. Consider the polynomials $p_1(x) = x^5 - 2x^4 + x^3 + x^2 - x + 1$ and $p_2(x) = x^5 - x - 1$. Let K_j be the splitting field of $p_j(x)$ over $\mathbb{Q}$. Compute $\mathrm{Gal}(K_j/\mathbb{Q})$. Show that there exists a unique quadratic extension $F_j/\mathbb{Q}$ with $F_j \subseteq K_j$. What is this quadratic extension?
2. Let a and n be positive integers. Suppose that a is squarefree and that $a \nmid n$. Show that the Galois group of $x^n - a$ is isomorphic to $\mathbb{Z}/n\mathbb{Z} \rtimes_\varphi (\mathbb{Z}/n\mathbb{Z})^\times$.