Week 3: Polynomial rings I

Let R be a commutative ring with identity. Let K be a field.

Practice Problems

1. Factor the polynomial $x^4 + 1$ in the rings $\mathbb{C}[x]$, $\mathbb{R}[x]$, $\mathbb{Q}[x]$.

2. Construct a surjective ring homomorphism $K[x,y] \to K$ with kernel (x,y). Construct a surjective ring homomorphism $K[x,y] \to K[y]$ with kernel (x). Deduce that (x,y) is maximal and that (x) is prime.

3. Show that $\mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/(x^2+5)$.

Presentation Problems

1. Show that $K[x]$ contains infinitely many primes. *Hint*: Look at Euclid's proof that there are infinitely many primes in $\mathbb{Z}$.

2. Let $I = (xy, (x-y)z) \subseteq K[x,y,z]$. Show that $\sqrt{I} = (xy, xz, yz)$.

3. (a) Show that $K[x,y]/(y^2 - x) \cong K[y]$.

   (b) Show that $K[x,y]/(y^2 - x) \not\cong K[x,y]/(y^2 - x^2)$.

4. (a) Construct an injective ring homomorphism $K[x,y]/(xy) \to K[x] \times K[y]$.

   (b) Show that $K[x,y]/(xy) \not\cong K[x] \times K[y]$.

Module Theory Problem

1. (a) Show that if $f\colon M \to N$ is an R-module homomorphism then the sequence of R-modules

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N \longrightarrow \operatorname{coker} f \longrightarrow 0$$

   is exact, where $\operatorname{coker} f = N/\operatorname{im} f$.

   (b) Show that the blue commutative diagram of R-modules induces the red R-module homomorphisms

$$\begin{tikzcd}
0 \arrow[r] & \ker\alpha \arrow[r] \arrow[d] & A \arrow[r, "\alpha"] \arrow[d, "f"] & A' \arrow[r] \arrow[d, "f'"] & \operatorname{coker}\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\beta \arrow[r] & B \arrow[r, "\beta"] & B' \arrow[r] & \operatorname{coker}\beta \arrow[r] & 0.
\end{tikzcd}$$

   (c) Show that the blue commutative diagrams of R-modules with exact rows induce the red exact sequences of R-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] \\
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
0 \arrow[r, "g"] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "g"] \arrow[d, "\gamma"] & 0 \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] \\
\operatorname{coker}\alpha \arrow[r] \arrow[d] & \operatorname{coker}\beta \arrow[r] \arrow[d] & \operatorname{coker}\gamma \arrow[d] \\
0 & 0 & 0.
\end{tikzcd}$$

(d) Show that the blue commutative diagram of R-modules with exact rows induces the red exact sequence of R-modules

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \ker f \arrow[r] & \ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] \arrow[dddll, "\delta"'] & & \\
 & & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 & \\
 & 0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & & \\
 & & \operatorname{coker}\alpha \arrow[r] \arrow[d] & \operatorname{coker}\beta \arrow[r] \arrow[d] & \operatorname{coker}\gamma \arrow[r] \arrow[d] & \operatorname{coker} g' \arrow[r] & 0 \\
 & & 0 & 0 & 0 & &
\end{tikzcd}$$

This is known as the snake lemma.

Tricky Problems

1. Suppose that R is an integral domain. Let a and b be positive integers with $\gcd(a, b) = 1$. Consider the ring homomorphism $\varphi\colon R[x, y] \to R[t]$ defined by $\varphi(x) = t^b$ and $\varphi(y) = t^a$.

   (a) Show that $(x^a - y^b) \subseteq \ker\varphi$.

   (b) Let $f(x, y) \in \ker\varphi$. Show that we can write $f(x, y) = g(x, y) + h(x, y)$ with $g(x, y) \in (x^a - y^b)$ and $\deg_y h(x, y) \leq b - 1$.

   (c) Show that $h(x, y) \in \ker\varphi$.

   (d) Show that the exponents of $\varphi(x^i y^j)$ are distinct for $0 \leq j \leq b - 1$ and deduce that $h(x, y) = 0$.

   (e) Show that $\ker\varphi = (x^a - y^b)$.

   (f) Show that $(x^a - y^b)$ is a prime ideal of $R[x, y]$.

2. (a) Show that if R is a Noetherian ring then every quotient of R is Noetherian.

   (b) Show that if R and S are Noetherian rings then $R \times S$ is a Noetherian ring.

   (c) Show that R is Noetherian if and only if every ideal of R is finitely generated.

   The remaining parts of this problem will show that if R is Noetherian then so is $R[x]$.

   (d) Let I be an ideal of $R[x]$. Let I' denote the set of leading coefficients of polynomials in I. Prove that I' is an ideal of R, and deduce that $I' = (a_1, a_2, \dots, a_n)$ for some $a_1, a_2, \dots, a_n \in R$.
   By definition of I', for each a_j there is some $f_j \in I$ whose leading coefficient is a_j.

   (e) Let d be a positive integer. Let I_d be the set of leading coefficients of polynomials of degree d in I, as well as 0. Prove that that I_d is an ideal. Deduce that $I_d = (a_{d,1}, a_{d,2}, \dots, a_{d,n_d})$ for each d, and let $f_{d,j} \in I$ be a polynomial of degree d with leading coefficient $a_{d,j}$.

   (f) Let $N = \max_{1 \leq i \leq n} \deg f_i$. Prove that

   $$I = (f_1, f_2, \dots, f_n) + \sum_{d=1}^{N-1} (f_{d,1}, f_{d,2}, \dots, f_{d,n}).$$

   Hint: If they're not equal, then there is some $f \in I$ not in the ideal in the right of minimal degree.

(g) Deduce that if R is a Noetherian ring then $R[x]$ is Noetherian.

This is known as Hilbert's Basis Theorem.