# Category Theory: Algebraic Algebra Theory

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**Abstract**

"Indeed, the subject might better have been called abstract function theory, or perhaps even better: archery."

-Steve Awodey (read his book!)

Category theory is a tool that helps organize mathematics. It highlights a sort of "exterior view" of areas of math, allowing one to study mathematical objects in extreme generality without ever "stepping into" those objects. That said, at first sight category theory feels quite daunting. The goal of this talk is to introduce you to the language of category theory, giving concrete examples like the categories **Set**, **Grp**, and **Top**, discuss the ways that categories are used and how they abstract other concepts, then conclude with a discussion of a slightly scary subject, universal properties.

## 1 Definitions and first examples.

Categories have a really abstract definition that seems confusing at first:

**Definition 1.1.** A category $\mathcal{C}$ has objects $A, B, C, ...$ and arrows, called morphisms, between those objects. A morphism $f$ from $A$ to $B$ is denoted $f : A \to B$ or $A \xrightarrow{f} B$. In this case we call $A$ the domain of $f$, denoted $\text{dom} f := A$, and $B$ the codomain, $\text{cod} f := B$. The morphisms of a category must satisfy three axioms:

(1) If $f, g$ are morphisms with $\text{cod} f = \text{dom} g$, then we define the composition[1] of $f$ and $g$ as a map $g \circ f$ from $\text{dom} f$ to $\text{cod} g$.

(2) Composition is associative, that is if composition is defined between $f, g, h$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) Every object $A$ has an identity morphism $id_A : A \to A$ such that for morphisms $f : A \to B$, $g : C \to A$,

$$f \circ id_A = f, \quad id_A \circ g = g.$$

Note that a category isn't always a set; there are some categories we will discuss that cannot be described as a set of objects. The axioms, as with most categorical things, can be visualized well with commutative diagrams. A commutative diagram is a diagram of objects in a category and morphisms between them so that composition is well defined. That is,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

[1] Since there is usually only one operation on the morphisms, sometimes the composition is simply denoted $gf$.

is said to commute, or be commutative, if $h = g \circ f$. There are many different notations in the language of categories, many different arrows denoting different types of morphisms. Among them is the dotted arrow, which denotes an induced morphism, that is, the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dotted, "\varphi"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

is commutative when the existence of $f$ and $g$ guarantee the existence of $\varphi$.
The axioms of a category can then be visualized as three commutative diagrams,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dotted, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd},$$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "f"'] & A \arrow[r, "g"] \arrow[d, "id_A"] & C \\ & A \arrow[ur, "g"'] & \end{tikzcd},$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h \circ g"] & C \arrow[r, "h"] & D \end{tikzcd}.$$

Generally, if you have a collection of objects and can define a transitive and reflexive relation between them, you can think of it as a category. You should be familiar with isomorphisms from linear algebra, which are abstracted in category theory as follows:

**Definition 1.2.** A morphism $f : A \to B$ is called an isomorphism if it has a two-sided inverse, that is, if there exists a morphism $f^{-1} : B \to A$ so that

$$f^{-1} \circ f = id_A \text{ and } f \circ f^{-1} = id_B.$$

In this case, $A$ and $B$ are said to be isomorphic, and we write $A \cong B$. Note also that $id$ is its own inverse, so it is an isomorphism.

Since isomorphism is an equivalence relation, we often discuss objects "up to isomorphism". We say that $A, B$ are equivalent up to isomorphism if $A$ and $B$ are isomorphic. Some categories have special objects that are unique up to iso, called the initial and final objects. The initial object in a category is an object $A$ such that, for any object $B$ in the category, there is a unique morphism $A \to B$. Similarly, the final object is an object $C$ such that for every $B$, there is a unique morphism $B \to C$.

Now we look at a few categories of familiar objects.

**Set**. The category **Set** has as objects sets and functions for morphisms. One should be concerned about Russel's paradox, which is remedied since **Set** need not be a set, it can be a larger object like a proper class.

**Grp**. The category of groups and group homomorphisms is a category that you will be working in all quarter. There is a *subcategory* of **Grp** called **Ab**, the category of abelian groups with group homomorphisms. A subcategory of a category $\mathcal{C}$ is a category $\mathcal{B}$ such that if $A, B$ are objects of $\mathcal{B}$, then they are objects of $\mathcal{C}$, and if $f : A \to B$ is a morphism in $\mathcal{B}$ then it is a morphism $A \to B$ in $\mathcal{C}$.

**Top**. The category of topological spaces has continuous maps as morphisms. If you know some algebraic topology, an interesting construction is that of the homotopy category **hTop**. It has the same objects and morphisms as **Top**, but there are more isomorphisms (in particular, all weak homotopy equivalences are isomorphisms). This category can be constructed from **Top** through a process called localization.

Here we also discuss some objects that can be formalized as categories.

A partially ordered set $(X, \leq)$ is a set $X$ with a relation $\leq$ on it such that $\leq$ is reflexive, transitive, and antisymmetric. We also call these posets for short, and any poset can be realized as a category as follows: let the elements of $X$ be the objects in the category, and define a morphism $a \to b$ iff $a \leq b$. The transitive property is analogous to composition, associativity follows trivially, and identities are provided by reflexivity. Antisymmetry implies that $(X, \leq)$ is a special kind of category, called a skeleton category.

A groupoid is a category where every morphism is iso. See Joke 1.1 in Aluffi to understand why it is called a groupoid...

<u>Exercises!</u>

1. Let $G$ be a directed graph. How can you formalize this as a category?

2. If $G$ is undirected, what kind of category is it?

3. An endomorphism of a set $X$ is a function $f : X \to X$. Formalize the set $\mathrm{End}(X)$ as a category with one object. An automorphism of $X$ is a bijective endomorphism. Show that $\mathrm{Aut}(X)$ is a groupoid with one object.

4. Suppose that $A, B, C, D$ are sets so that $A \subseteq B$ and $C \subseteq D$. We denote the inclusion map $A \to B$, $x \mapsto x$ by $\iota : A \hookrightarrow B$. Let $g : A \to C$ and $f : B \to D$ be functions. Show

that

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook, "\iota"] & C \arrow[d, hook, "\iota"] \\ B \arrow[r, "f"] & D \end{tikzcd}$$

commutes iff $g$ is the restriction of $f$ to $A$.

# 2 Functors.

Functors are the homomorphisms of categories, in the same way that linear maps are vector space homomorphisms, continuous maps are topological homomorphisms, etc.
Thus when defining them, we want functors to preserve the structure of a category. Try to see how the axioms of a functor achieve this.

**Definition 2.1.** A functor $F : \mathcal{C} \to \mathcal{D}$ is a mapping that takes objects $A$ of $\mathcal{C}$ to objects $F(A)$ of $\mathcal{D}$, and morphisms $f$ in $\mathcal{C}$ to morphisms $F(f)$ in $\mathcal{D}$ so that the following three axioms are satisfied:

(1) If $f : A \to B$ is a morphism in $\mathcal{C}$, then $F(f) : F(A) \to F(B)$. This can be thought of as saying "$F$ preserves domains and codomains".

(2) $F(id_A) = id_{F(A)}$; $F$ preserves identities.

(3) $F(g \circ f) = F(g) \circ F(f)$; $F$ respects composition.

As one can see, functors are designed to map one category to another, while preserving the structure of a category. You should look for this exact utility when defining group homomorphisms this quarter, and ring homomorphisms next quarter. Let's go over some examples.

Given a category $\mathcal{C}$, we can define the arrow category $\mathcal{C}^{\to}$ to have the arrows in $\mathcal{C}$, and given objects $f : A \to B$ and $g : C \to D$, an arrow $\varphi : f \to g$ is a pair of arrows $\varphi_1 : A \to C$, $\varphi : B \to D$ so that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ C & \xrightarrow{g} & D \end{array}$$

commutes. In this case, we can regard the domain and codomain operations as functors cod$(-)$,dom$(-) : \mathcal{C}^{\to} \to \mathcal{C}$.

The opposite category $\mathcal{C}^{op}$ is defined to have the same objects as $\mathcal{C}$, but with all the arrows reversed.

# 3 Universal Mapping Properties: The Scary Subject.

"What is a UMP? If no one ask of me, I know, if I wish to explain to him who asks, I know not."
-Math Stack Exchange User

Many times we want to talk about objects that look familiar from mathematics, in the abstract setting of category theory. We want to consider abstractions of things like the cartesian product from set theory, but in the context of category theory we can't define the product by talking about the elements in the objects, since the objects may not be sets. Thus we turn to universal properties, which are special types of diagrams that are satisfied by unique objects (up to iso). We begin with some concrete examples, to get an intuition for these abstract objects.

First, we compare the categorical product to the cartesian product. The categorical product of two objects $A, B$ is denoted $A \times B$. There are "projection maps" $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, and given an object $Z$ with maps $a : Z \to A$, $b : Z \to B$, there is a unique map $u : Z \to A \times B$ making the following diagram commute:

$$\begin{tikzcd} & Z \arrow[dl, "a"'] \arrow[d, dotted, "u"] \arrow[dr, "b"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

This should look really scary. However, consider this universal property in the category of sets. Given sets $A, B$, their cartesian product $A \times B$, and the usual projections $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, we see that for any $Z$ and $a : Z \to A$, $b : Z \to B$, we can uniquely determine the induced map $u$ to be $u(z) = (a(z), b(z))$. This satisfies the diagram and is uniquely determined by the commutativity. Note as well that the condition that $Z$ have a unique map into $A \times B$ is similar to the definition of a final object; this will come up later.

Another universal object is the equalizer. Again we give the diagram first, Then we will consider an example in **Top**. Given objects $A, B$ and two morphisms $f, g : A \to B$, an equalizer of $f, g$ is an object $E$ along with an arrow $e : E \to A$ such that $f \circ e = g \circ e$, that is, $e$ equalizes $f, g$. $E$ is made universal as follows: given an object $Z$ and map $z : Z \to A$ such that $f \circ z = g \circ z$, there is a unique map $u : Z \to E$ so that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dotted, "u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

Note again that the universal property frames $E$ as some kind of final object. Our concrete example will be in the category of topological spaces and continuous maps. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^2$, $g(x, y) = 1$. I claim that the equalizer of these two arrows is the unit circle $S^1 = \{(x, y) : x^2 + y^2 = 1\}$, with the inclusion $\iota : S^1 \hookrightarrow \mathbb{R}^2$. This should make sense, because $S^1$ is defined as

$$S^1 = \{(x,y) : f(x,y) = x^2 + y^2 = 1 = g(x,y)\}.$$

Any set that equalizez $f, g$ will be either a subset of $S^1$, or continuously deformable into a subset of $S^1$, as desired.

# 4 Limits and Calculating Universal Objects.

We want to define limits in a category, which will lead directly to a way of calculating universal objects. First we will need to define indexing categories and cones.

**Indexing Categories**. Let $\mathcal{J}$ be a category. We will name the objects of $\mathcal{J}$ $j, k, l, \ldots$. Let's assume $\mathcal{J}$ is a really basic category, since my tikzcd skills aren't that good and there's a lot of diagrams coming up. Suppose that $\mathcal{J}$ looks something like this:

$$i \xrightarrow{f} j$$

Notice that there are no morphisms between $i, j$. There can be, and these lead to more complicated diagrams going forward, but we will proceed with this basic exampl. If we consider another category $\mathcal{C}$, and a functor $F : \mathcal{J} \to \mathcal{C}$, we get a "copy" of $\mathcal{J}$ sitting inside $C$, which looks something like this:

$$F(i) \xrightarrow{F(f)} F(j)$$

This diagram, and sometimes the functor itself, is called a $\mathcal{J}$-type diagram, and $\mathcal{J}$ is called the indexing category.

**Cones**. Assume we have a category $\mathcal{C}$ and a $\mathcal{J}$-type diagram in $\mathcal{C}$, we want to consider objects of $\mathcal{C}$ that play nice with the diagram. That is, we want to study the objects $A$ with maps $a_i : A \to F(i)$, $a_j : A \to F(j)$, so that

$$\begin{tikzcd} A \arrow[d, "a_i"'] \arrow[dr, "a_j"] & \\ F(i) \arrow[r, "F(f)"'] & F(j) \end{tikzcd}$$

commutes. The collection of these maps $a_i, a_j$ is called the cone at $A$, since in more complicated cases it looks somewhat like a cone with base on the $\mathcal{J}$-diagram and $A$ at the top. We now can define a morphism of cones, which will lead to our definition of limits. Let $A, B$ be objects of $\mathcal{C}$ with cones of $\mathcal{J}$. We will encode the cones as a pair $(A, a_i)$ of $A$ and the set of arrows $a_i$ in the cone, and $(B, b_i)$ defined similarly. We will call a map $\theta : A \to B$ a morphism of cones if for every object $j$ of $\mathcal{J}$,

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[dr, "a_j"'] & B \arrow[d, "b_j"'] \\ & F(j) \end{tikzcd}$$

commutes. That is, $\theta$ preserves the cone structure. One can check that morphisms of cones can be composed to obtain another morphism of cones, so that we get a category of cones $(A, a_j)$ and cone morphisms. This is called the cone category of $F$, and is denoted $\mathrm{Cone}(F)$. If it exists, the limit of the $\mathcal{J}$-type diagram (or equivalently, the limit of the functor $F$) is defined to be the final object in $\mathrm{Cone}(F)$, and is denoted

$$\varprojlim_j F(j).$$

So, why do we care about this abstract nonsense??

Let'a start with an example. Let $\mathcal{J}$ be the category of two objects, $\mathcal{J} = \{i, j\}$, and onle identity morphisms, and let $\mathcal{C}$ be an arbitrary category with objects $A, B$. Take a functor $F : \mathcal{J} \to \mathcal{C}$ so that $F(i) = A$ and $F(j) = B$. For an object $X$, the cone at $X$ will look something like this:

$$A \longleftarrow X \longrightarrow B$$

If the limit exists, we know that it itself is a cone, so that we have the diagram

$$A \longleftarrow \varprojlim_j F(j) \longrightarrow B$$

We might even call these maps $\pi_1, \pi_2$. The condition that $\varprojlim_j F(j)$ be final in the cone category implies that any other cone

$$A \xleftarrow{a} Z \xrightarrow{b} B$$

has a unique morphism into $\varprojlim_j F(j)$, that is, there is a unique arrow $u$ in $\mathcal{C}$ so that

$$\begin{tikzcd}
 & Z \arrow[dl, "a"'] \arrow[d, dotted, "u"] \arrow[dr, "b"] & \\
A & \varprojlim_j F(j) \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

Wait.. this looks familiar! Since final objects are unique up to isomorphism, we see that $\varprojlim_j F(j)$ is (up to isomorphism) the product $A \times B$! Wow! So choosing the right indexing category and functor, you can compute a universal object as the limit of that functor. And since limits are terminal objects, we know that universal objects are always unique up to a unique isomorphism. As an exercise, try finding finding the functor whose limit is the equalizer of two morphisms.

# 5 Extra Topics.

Here I list some interesting subjects in category theory, interesting applications in mathematics, and some stuff that I'm working on. They're listed in order of increasing abstraction and difficulty, but if you're interested I encourage you to ask or read about the more difficult topics. If you want to know about any of these, just let me know, or email me at aquist99@uw.edu.

1. The shape of the category of fields
2. Sheaves
3. The Duality Principle
4. Homology and Cohomology
5. Abelian Categories