Week 3: Subgroups, normality, and quotient groups

Practice Problems

1. *This problem looks scary but in fact it is easy (and useful).*

   Let G be a group and let H be a subgroup of G. We define the *normalizer* and *centralizer* of H as

   $$N_G(H) = \{g \in G : gHg^{-1} = H\},$$
   $$C_G(H) = \{g \in G : gh = hg \text{ for all } h \in H\}.$$

   (a) Show that $N_G(H)$ is a subgroup of G.

   (b) Show that $C_G(H)$ is a normal subgroup of $N_G(H)$.

   (c) Show that H is a normal subgroup of G if and only if $N_G(H) = G$.

   (d) Let K be a subgroup of G containing H. Show that H is normal in K if and only if K is a subgroup of $N_G(H)$. In other words, $N_G(H)$ is the largest subgroup of G in which H is normal.

2. Let G be the dihedral group of order 8, let $r \in G$ be a reflection, and let $s \in G$ be rotation by 180°. Let $H = \{1, r\}$ and let $N = \{1, r, s, rs\}$. Show that

   - H is a normal subgroup of N,
   - N is a normal subgroup of G,
   - H is not a normal subgroup of G.

   This shows that normality is not transitive.

3. Let G and H be groups. Show that $(G \times H)/H \cong G$.

Presentation Problems

1. Let G be a group and let H be a subgroup of G.

   (a) Construct a homomorphism $N_G(H) \to \mathrm{Aut}(H)$ with kernel $C_G(H)$.

   (b) Construct an injective homomorphism $N_G(H)/C_G(H) \to \mathrm{Aut}(H)$.

   We define the center of G as

   $$Z(G) = C_G(G) = \{g \in G : gh = hg \text{ for all } h \in G\}.$$

   By practice problem 1(b), $Z(G) = C_G(G)$ is a normal subgroup of $G = N_G(G)$.

   (c) Show that $\mathrm{Inn}(G) = \{h \mapsto ghg^{-1} : g \in G\}$ is a normal subgroup of $\mathrm{Aut}(G)$.

   (d) Show that $G/Z(G) \cong \mathrm{Inn}(G)$.

   (e) Show that if $G/Z(G)$ is cyclic then G is abelian.

2. The *quaternion group* is the set

   $$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

   with the relations $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$.

   (a) Determine the subgroup lattice of Q_8.

   (b) Show that every subgroup of Q_8 is normal.

   (c) Determine the center of Q_8.

(d) Let x and y be two random (uniformly distributed and independently chosen) elements of Q_8. Determine the probability that x and y commute.

3. Let G be a group and let A and B be subgroups of G. Show that the elementwise product

$$AB = \{ab : a \in A,\ b \in B\}$$

is a subgroup of G if and only if $AB = BA$.

4. Let G be a group. For $g, h \in G$, the *commutator* of g and h is the element $[g, h] = ghg^{-1}h^{-1}$. Let $[G, G]$ be the subgroup of G generated by commutators $[g, h]$ for $g, h \in G$.

   (a) Show that $[G, G]$ is a normal subgroup of G and that the quotient $G^{\text{ab}} = G/[G, G]$ is abelian.

   (b) Show that if $\varphi\colon G \to A$ is a homomorphism from G to an abelian group A then $[G, G] \subseteq \ker\varphi$ and there exists a unique homomorphism $\widetilde{\varphi}\colon G^{\text{ab}} \to A$ so that the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & A \\ G^{\text{ab}} \arrow[ur, dashed, "\widetilde{\varphi}"'] & \end{tikzcd}$$

   commutes.

   (c) Show that $[G, G]$ is the smallest normal subgroup of G with abelian quotient.

   (d) Construct an interesting homomorphism $\operatorname{Out}(G) \to \operatorname{Aut}(G^{\text{ab}})$ where $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$ is the outer automorphism group of G.

   Warning: $\operatorname{Out}(G)$ is not a subset of $\operatorname{Aut}(G)$.

Tricky Problems

1. Let G be a nonabelian finite group. Let x and y be two random (uniformly distributed and independently chosen) elements of G. Let p be the probability that x and y commute.

   (a) Show that $p \leq 5/8$.

   (b) Show that $p = 5/8$ if and only if $G/Z(G) \cong C_2 \times C_2$.

   You may use these two facts which we will prove next week:

   - If H is a subgroup of G then $|H|$ divides $|G|$.
   - If H is a normal subgroup of G then $|G|/|H| = |G/H|$.

2. Let G be a finite group of order n and let S be a nonempty subset of G. Show that the n-fold product

$$S^n = \underbrace{S \cdots S}_{n} = \{s_1 \cdots s_n : s_1, \ldots, s_n \in S\}$$

   is a subgroup of G. *Hint*: Consider the size of S^k as k increases.