Week 6: The Sylow theorems

Practice Problems

1. Let G be a finite group and let H be a subgroup of G. Let P be a Sylow p-subgroup of G. Show that if P is contained in H then P is a Sylow p-subgroup of H.
2. Let p be an odd prime. Find all Sylow p-subgroups of the dihedral group of order $2p$.
3. Find all Sylow subgroups of S_4.

Presentation Problems

1. Let G be a finite group and let H be a subgroup of G. Show that the number of Sylow p-subgroups of H is at most the number of Sylow p-subgroups of G.
2. Let G be a finite group of order pqr for primes $p < q < r$.
   (a) Show that G is not simple.
   (b) Show that G has a normal Sylow r-subgroup.

   It is also true that if all Sylow subgroups of G are cyclic and if r is the largest prime dividing the order of G then G has a normal Sylow r-subgroup but don't try to prove this.
3. Let G be a finite group.
   (a) Let H be a normal subgroup of G and let P be a Sylow subgroup of H. Show that $G = N_G(P)H$.
   (b) Let P be a Sylow subgroup of G and let H be a subgroup of G containing $N_G(P)$. Show that $N_G(H) = H$.
4. Let G be a finite group, let φ be a fixed-point-free automorphism of G, and let p be a prime.
   (a) Show that φ permutes the Sylow p-subgroups of G.
   (b) Show that φ fixes a Sylow p-subgroup of G.
   (c) Show that φ fixes a unique Sylow p-subgroup of G.

Tricky Problems

1. Let G be a finite group of order $p^k m$ where p is a prime not dividing m. Suppose that G has exactly $p+1$ Sylow p-subgroups.
   (a) Show that the union of the Sylow p-subgroups of G has order p^{k+1}.
   (b) Show that if $k \geq 2$ then G is not simple.
2. Let G be a nonabelian simple group of order less than or equal to 100. Prove that $|G| = 60$.

   You can use the previous tricky problem without proof.

   Hint: If you're stuck on some order, try using the techinques in the bonus.

Bonus: Simple Group Techniques

Technique 1: Sylow's Theorems

If there is only one Sylow p-subgroup, then that subgroup is normal.

1. Prove that there is no simple group of order 156.
2. Prove that there is no simple group of order 675.

Technique 2: Element Counting I

Any two distinct subgroups of prime order must intersect trivially. When combined with the previous technique, this can often lead to too many elements in a hypothetical simple group.

1. Prove that there is no simple group of order 105.
2. Prove that there is no simple group of order 380.

Technique 3: Element Counting II

After counting the elements of prime order, there might only be room for one Sylow p-subgroup.

1. Prove that there is no simple group of order 56.
2. Prove that there is no simple group of order 351.

Technique 4: Subgroups of Small Index

Suppose that H is a proper subgroup of a simple group G. Problem 1 from Week 4 gives a nontrivial group homomorphism $G \to S_{[G:H]}$. Since G is simple, this group homomorphism must be injective. In particular, the order of G must divide $[G:H]!$ which is often a contradiction. To find subgroups of small index, consider normalizers of Sylow subgroups.

1. Prove that there is no simple group of order 3393.
2. Prove that there is no simple group of order 4125.

Technique 5: Normalizers of Sylow Intersections

Suppose that $|G| = p^2m$ for a prime p not dividing m. If distinct Sylow p-subgroups of G intersect trivially then we can apply element counting. Otherwise, there will be Sylow p-subgroups P and Q of G with $P \cap Q \neq 1$. However, P and Q are abelian so $N_G(P \cap Q)$ will contain both P and Q. This forces $N_G(P \cap Q)$ to be a large subgroup of G and we can apply the previous technique.

1. Prove that there is no simple group of order 90.
2. Prove that there is no simple group of order 144.

More Advanced Techniques

For more advanced techniques, read section 6.2 of Dummit & Foote.