Topology and Groupoids, Chapter 3

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Lemma 1. *Suppose X can be written as the disjoint union of two connected nonempty subsets C_1, C_2. the only possible disconnection of X is (C_1, C_2).*

Proof. Let (A, B) be a disconnection of X. Let $x \in C_1$; then either $x \in A$ or $x \in B$. W.l.o.g. suppose $x \in A$. We show $C_1 \subseteq A$ by contradiction. Suppose that for some $y \in C_1$, we had $y \in B$. Let $A' = A \cap C_1$ and $B' = B \cap C_1$. Then by assumption $x \in A'$ and $y \in B'$. But also $\overline{A'} \cap B' = \overline{(A \cap C_1)} \cap (B \cap C_1) \subseteq (\overline{A} \cap \overline{C_1}) \cap (B \cap C_1) = (\overline{A} \cap B) \cap \overline{C_1} = \emptyset$ since (A, B) is a disconnection and thus $\overline{A} \cap B = \emptyset$. But also

$$A' \cup B' = (A \cap C_1) \cup (B \cap C_1) = (A \cup B) \cap C_1 = X \cap C_1 = C_1$$

So since $x \in A'$ and $y \in B'$ we have that (A', B') disconnects C_1, a contradiction. Thus $C_1 \subseteq A$. A similar argument shows $C_2 \subseteq B$. But then

$$A = X \setminus B = (C_1 \cup C_2) \setminus B \subseteq (C_1 \cup C_2) \setminus C_2 = C_1 \setminus C_2 = C_1 \setminus (X \setminus C_1) = C_1$$

And since $A \subseteq C_1$ and $C_1 \subseteq A$, this implies $A = C_1$. Then $B = X \setminus A = X \setminus C_1 = C_2$, so (C_1, C_2) is a disconnection of X. $\square$

Exercise 3.5.13

Let $X_n = \{n^{-1}\} \times [-n, n]$ and let $Y = \mathbb{R}^2 \setminus \bigcup_{n \geq 1} X_n$. Prove that Y is connected but not path connected.

We rewrite $Y = Y_1 \cup Y_2$ where $Y_1 =]{\leftarrow}, 0] \times \mathbb{R}$ and $Y_2 = Y \setminus Y_1$. More explicitly, since $\bigcup_{n \geq 1} X_n$ doesn't intersect Y_1, we have

$$Y_2 = (]0, {\rightarrow}[\times \mathbb{R}) \setminus \bigcup_{n \geq 1} X_n$$

Clearly Y_1 is connected, since it's homeomorphic to $]{\leftarrow}, 0]$. We show that also Y_2 is path connected, and thus connected. Let (a, b) be an element on Y_2; we give a path from (a, b) to $(2, 0)$. Since $(a, b) \notin Y_1$, we have $a > 0$, so for some N we have $a > \frac{1}{N}$. Let p be defined by

$$p(t) = \begin{cases} (a, b + 3t(N - b)) & \text{if } t \in [0, \frac{1}{3}] \\ (a + (3t - 1)(2 - a), N) & \text{if } t \in [\frac{1}{3}, \frac{2}{3}] \\ (2, (3 - 3t)N) & \text{if } t \in [\frac{2}{3}, 1] \end{cases}$$

So that p is a path which goes upwards or downwards to a height of N (and since $a > \frac{1}{N}$, any spike X_n at or beyond a has a height lower than N), then goes horizontally from a to 2 (if $a \geq 2$ then none of X_n are in this part of the path; if $a < 2$ then we only go further right and the height of N is higher than any X_n), then goes downwards to from N to 0 (which is safe since all spikes in X_n are to the left of 2). Thus Y_2 is path connected, and so connected.

Now suppose Y were disconnected. Then by Lemma 1 it would be disconnected by (Y_1, Y_2). We show that $(0,0)$ is in the closure of Y_2, and thus $Y_1 \cap \overline{Y_2}$ would be nonempty, so Y would not be disconnected. Let N be a neighborhood of $(0,0)$. Then for some $\varepsilon > 0$ we have $B((0,0),\varepsilon) \subseteq N$. But then for some natural number K we have $\frac{1}{K} < \varepsilon$, so $\frac{1/K+1/(K+1)}{2} < \frac{1}{K} < \varepsilon$ and thus $\left|(0,0) - \left(\frac{1/K+1/(K+1)}{2}, 0\right)\right| = \frac{1/K+1/(K+1)}{2} < \varepsilon$ which means $\left(\frac{1/K+1/(K+1)}{2}, 0\right) \in B((0,0),\varepsilon)$. But we have $0 < \frac{1}{K+1} < \frac{1/K+1/(K+1)}{2} < \frac{1}{K}$, and thus there is no n such that $\left(\frac{1/K+1/(K+1)}{2}, 0\right) \in X_n$, which means $\left(\frac{1/K+1/(K+1)}{2}, 0\right) \in Y_2$. Thus N meets Y_2 for any neighborhood N of $(0,0)$, and thus $(0,0) \in \overline{Y_2}$. Thus Y is connected.

We now show Y is not path connected. If it were, there would be some path $p : [0,1] \to Y$ such that $p(0) = (0,0)$ and $p(1) = (2,0)$. Let $p(t) = (p_1(t), p_2(t))$. Then by the intermediate value theorem, for any $n \in \mathbb{N}$ there is some t_n such that $p_1(t_n) = \frac{1}{n}$. By the extreme value theorem, since $[0,1]$ is compact there is some M such that $|p_2(t)| < M$ for all $t \in [0,1]$. But there must be a natural number K such that $M < K$, and since $p_1(t_K) = \frac{1}{K}$ and $p(t_K) \notin X_K$ we must have $p_2(t) \notin [-K, K]$. But this means $|p_2(t)| \geq K > M$, a contradiction. $\square$