Topology Independent Study, Week 1

Brendan Murphy — 2019

Exercise 1

Is $\mathbb{Q}$ locally compact?

$\mathbb{Q}$ is not locally compact. In fact, no neighborhood in $\mathbb{Q}$ can be compact. Let x be a rational and K a compact neighborhood of x. Then since K is a neighborhood of x, it contains some rational ball centered about x, i.e. $(x-\varepsilon, x+\varepsilon) \cap \mathbb{Q} \subseteq K$ for some ε. By density of the irrationals, we must have $[a,b] \subseteq (x-\varepsilon, x+\varepsilon)$ for some irrational a, b. Define $V = K \setminus [a,b]$ and

$$U_n = \left(a + \frac{1}{k} \cdot \left(\frac{a+b}{2} - a\right), b - \frac{1}{k} \cdot \left(b - \frac{a+b}{2}\right)\right)$$

So that U_n expands out from the midpoint of $[a,b]$ and fills out all of $[a,b]$ except the endpoints. Then $\{V\} \cup \{U_n \cap \mathbb{Q} : n \in \mathbb{N}\}$ is an open cover of $K \setminus \{a,b\}$. Since a and b are irrational and K is a subset of the rationals, $K \setminus \{a,b\} = K$. But any finite subcover of $\{V\} \cup \{U_n \cap \mathbb{Q} : n \in \mathbb{N}\}$ will have a highest N such that it contains $U_N \cap \mathbb{Q}$, and then this subcover will miss $a + \frac{1}{N}\left(\frac{a+b}{2} - a\right)$, so there is no finite subcover of $\{V\} \cup \{U_n \cap \mathbb{Q} : n \in \mathbb{N}\}$, contradicting our assumption K was compact. $\square$

Exercise 3.6.4

Let X be the unit interval with the following topology: the neighborhoods of any nonzero points are as usual, and the neighborhoods of 0 are the usual ones as well as $N \setminus \{x_0, x_1, \ldots\}$ for any sequence of nonzero points $\{x_n\}$ converging to 0. Show X is not a k-space.

We prove that if K is any nonempty compact subset of X, then $\inf\{x \in K : x \neq 0\} > 0$. Let K be a nonempty set such that $\inf\{x \in K : x \neq 0\} = 0$. Then there is a sequence $x_n \neq 0$ in K converging to 0. Let $A_k = \{x_n : n \geq k\}$. Then define $U_k = [0,1] \setminus A_k$. We show that U_k is open for any k. Let a be any point of U. If $a = 0$, then since $[0,1]$ is a neighborhood of 0 in the standard topology and $x_n \to 0$ (no matter how many initial terms you remove), U_k is a neighborhood of 0. If $a \neq 0$, then $x_n \not\to a$, so there's a distance δ such that $|a - x_n| \geq \delta$ for all n. Then $(a-\delta, a+\delta)$ is a neighborhood of a contained in U_k, so U_k is a neighborhood of a. Thus all U_k are open, so $\{U_k : k \in \mathbb{N}\}$ is an open cover of K (since it is an open cover of $[0,1]$), and any finite subcover will miss infinitely many points in the sequence x_k (there's no issue with having infinitely many duplicate terms that have been added in, since $x_n \to 0$ but $x_n \neq 0$ for all n).

We now show that for any such K, $K \setminus \{0\}$ is compact. Let ε be such that $\varepsilon < x$ for all $x \in K \setminus \{0\}$, which exists by the above. Let $U = [0, \varepsilon)$, so that U is an open set containing 0 but disjoint from $K \setminus \{0\}$. Then for any open cover $\mathcal{U}$ of $K \setminus \{0\}$, we can add in U to get an open cover of K, then refine this to a finite open subcover $\mathcal{F}$ of K, and since U is disjoint from $K \setminus \{0\}$, the set $\mathcal{F} \setminus \{U\}$ forms a cover of $K \setminus \{0\}$ and is clearly a subcover of $\mathcal{U}$. Thus the result holds.

But any compact subset is also a compact subspace, and the subspace topology on $K\setminus\{0\}$ inhereted from X coincides with the subspace topology that it would normally inherent from $[0,1]$, since the neighborhoods of points away from 0 are as they are usually. Thus $K\setminus\{0\}$ is a compact subspace of $[0,1]$ under the usual topology, and thus is closed under that usual topology. But by the same remark about neighborhoods away from 0 be as they usually are, this shows $K\setminus\{0\}$ is a closed subset of X.

We now show X is not a k-space. Let $A=(0,1]$. Then A is not closed, since its complement $\{0\}$ is not open. If $\{0\}$ were open, it would be a neighborhood of 0, but all neighborhoods of 0 are uncountable (since they contain an interval $[0,\delta)$ possibly minus a countable number of points). Thus A is a nonclosed space. But for any compact subset K of X, either $K=\emptyset$, and so $A\cap K=\emptyset$ is trivially closed in K, or $A\cap K=K\setminus\{0\}=(K\setminus\{0\})\cap K$ is a closed subset of K as remarked above. □

Exercise 4.3.2

Let $f:\mathbb{R}\to[-1,1]$ be

$$f(x)=\begin{cases}\sin(1/x) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

and give $[-1,1]$ the identification topology with respect to f. Prove that $[-1,1]\setminus\{0\}$ under the subspace topology is as usual, but that the only neighborhood of 0 is the entire space.

By 4.3.1, to show that $[-1,1]\setminus\{0\}$ is as usual under the subspace topology it suffices to show that $[-1,1]\setminus\{0\}$ is as usual under the identification topology with respect to the restricton $g(x)=\sin(1/x)$ of f to $f^{-1}([-1,1]\setminus\{0\})$. But g is continuous, so automatically the implication "U is open in the standard topology on $[-1,1]\setminus\{0\}$" implies "$g^{-1}(U)$ is open in $f^{-1}([-1,1]\setminus\{0\})$". Now suppose U is not open; we show that its preimage is also not open. If U is not open, there's some point $x\in U$ such that U doesn't contain any neighborhood of x, i.e. for all $\varepsilon>0$ there's some $y\in[-1,1]\setminus\{0\}$ such that $y\notin U$ and $|x-y|<\varepsilon$. UNFINISHED

Now suppose U is an open set containing 0. Then $f^{-1}(U)$ is open in $\mathbb{R}$, and so for all $x\in\mathbb{R}$ such that $f(x)=0$, there is some $\varepsilon>0$ such that $(x-\varepsilon,x+\varepsilon)$ is in $f^{-1}(U)$. In particular, for $x=0$ we have that $f^{-1}(U)$ contains $(-\varepsilon,\varepsilon)$. But by making K big enough, the reciprocal of all points in $[2\pi K,2\pi(K+1)]$ lies within that small interval, and so $[-1,1]=f([2\pi K,2\pi(K+1)])\subseteq f(f^{-1}(U))\subseteq U\subseteq[-1,1]$, which shows $U=[-1,1]$. Any neighborhood of 0 must contain an open set containing 0, so this shows any neighborhood of 0 is the whole space. □

Lemma 1. *Suppose A is closed. Then the projection map $\pi:X\to X/A$ is closed. The same holds when "closed" is replaced with open, and the proof is identical.*

Proof. Let C be a closed subset of X. Then if C is disjoint from A, we have $\pi(C)=C$ and $\pi^{-1}(\pi(C))=\pi^{-1}(C)=C$, so the preimage of $\pi(C)$ is closed in X, and thus $\pi(C)$ is closed in X/A. Otherwise $C \cap A$ is nonempty, and so

$$\pi(C) = \pi((C \setminus A) \cup (C \cap A)) = \pi(C \setminus A) \cup \pi(C \cap A) = (C \setminus A) \cup \{A\}$$

Then $\pi(C)$ is closed in X/A, as

$$\pi^{-1}(\pi(C)) = \pi^{-1}((C \setminus A) \cup \{A\}) = \pi^{-1}(C \setminus A) \cup \pi^{-1}(\{A\}) = (C \setminus A) \cup A = C \cup A$$

and so the preimage of $\pi(C)$ is the union of two closed sets, and is thus closed. □

Lemma 2. *Let A be a subset of X and suppose B is a subset of X containing A. Then for any $Q \subseteq A$, if $\pi : X \to X/A$ is the projection map then*

$$\pi(Q \cap B) = \pi(Q) \cap \pi(B)$$

Proof. Either A and Q are disjoint or they are not. If they are, then $Q \cap B = Q \cap (B \setminus A)$, and π is injective when restricted to $X \setminus A$ so

$$\pi(Q \cap B) = \pi(Q \cap (B \setminus A)) = \pi(Q) \cap \pi(B \setminus A) = Q \cap (B \setminus A)$$

But also

$$\pi(Q) \cap \pi(B) = Q \cap (B \setminus A \cup \{A\}) = (Q \cap (B \setminus A)) \cup (Q \cap \{A\}) = Q \cap (B \setminus A)$$

So $\pi(Q \cap B) = \pi(Q) \cap \pi(B)$. If Q and A are not disjoint, then $Q \cap B = ((Q \setminus A) \cap B) \cup (Q \cap A)$, and so

$$\pi(Q \cap B) = \pi((Q \setminus A) \cap B) \cup \pi(Q \cap A) = ((Q \setminus A) \cap B) \cup \{A\}$$

But also

$$\pi(Q) \cap \pi(B) = ((Q \setminus A) \cup \{A\}) \cap ((B \setminus A) \cup \{A\}) = (Q \setminus A \cap B \setminus A) \cup \{A\} = ((Q \setminus A) \cap B) \cup \{A\}$$

Thus the result holds. □

Exercise 4.3.3

Let A, B be subsets of X and suppose $A \subseteq B$ and A is closed. Prove B/A is a subspace of X/A.

Let $\pi : X \to X/A$ be the quotient map. It suffices to show that for any C, C is closed in the subspace topology on B/A iff it is closed in the identification topology. Suppose C is closed in the subspace topology on B/A. Then $C = C' \cap (B/A)$ for some C' closed in X/A. This means $\pi^{-1}(C')$ is closed in X. But this means

$$\pi^{-1}(C) = \pi^{-1}(C' \cap (B/A)) = \pi^{-1}(C') \cap \pi^{-1}(B/A) = \pi^{-1}(C') \cap B$$

Is closed in B, as it is the intersection of a closed set of X and B.

Now suppose C is closed in the identification topology on B/A. Then $\pi^{-1}(C)$ is closed in B, and so $\pi^{-1}(C) = C' \cap B$ for some C' closed in X. Then by Lemma 2,

$$\pi(\pi^{-1}(C)) = \pi(C') \cap \pi(B) = \pi(C') \cap (B/A)$$

And by Lemma 1, $\pi(C')$ is a closed map, which means $\pi(C')$ is closed, and so $\pi(\pi^{-1}(C))$ is closed in the subspace topology on B/A. But also π is surjective, and so $\pi(\pi^{-1}(C)) = C$, so C is closed in the subspace topology on B/A.

Thus the identification and subspace topologies on B/A have the same closed sets, and are thus the same. $\square$

Exercise 4.3.4

Let A be a subset of X. Prove that $X \setminus A$ is a subspace of X/A if and only if the following condition holds: a subset U of $X \setminus A$ is open in $X \setminus A$ if and only if $U = V \cap (X \setminus A)$ where V is open in X and V either contains or is disjoint from A.

A set V is either disjoint from A or is contained in A iff V is π-satured. If V is disjoint from A, then immediately $\pi^{-1}(\pi(V)) = V$ since π is a the identity on $X \setminus A$ and $(X/A) \setminus \{A\}$. If $A \subseteq V$, then $\pi(V) = (V \setminus A) \cup \{A\}$ and so

$$\pi^{-1}(\pi(V)) = \pi^{-1}((V \setminus A) \cup \{A\}) = \pi^{-1}(V \setminus A) \cup \pi^{-1}(\{A\}) = (V \setminus A) \cup A = V$$

Where this last equality holds since $A \subseteq V$. Now if $\pi^{-1}(\pi(V))$, either V and A are disjoint or they share some point a. In this second case,

$$\pi(V) = \pi((V \setminus \{a\}) \cup \{a\}) = \pi(V \setminus \{a\}) \cup \{A\}$$

And so

$$A = \pi^{-1}(\{A\}) \subseteq \pi^{-1}(\pi(V \setminus \{a\})) \cup \pi^{-1}(\{A\}) = \pi^{-1}(\pi(V \setminus \{a\}) \cup \{A\}) = \pi^{-1}(\pi(V)) = V$$

Thus it suffices to show $X \setminus A$ is a subspace of X/A if and only if the condition "a subset U of $X \setminus A$ is open in $X \setminus A$ iff $U = V \cap (X \setminus A)$ where V is open in X and π-satured" holds. Now let $\pi' = \pi \restriction_{X \setminus A, X \setminus A}$ be the restriction of π to our subset of interest. Since π' is the identity, every open subset of $X \setminus A$ is π'-saturated, and thus by 4.3.1 we have that π' is an identification map iff every open set of $X \setminus A$ is the intersection of $X \setminus A$ and an open, π-satured set of X. Since it is tautological that the intersection of an open, π-satured set of X and $X \setminus A$ is open in $X \setminus A$, this means that the condition "a subset U of $X \setminus A$ is open in $X \setminus A$ iff $U = V \cap (X \setminus A)$ where V is open in X and π-satured" is in fact equivalent to just saying π' is an identification map.

Thus we just need to prove that $X \setminus A$ is a subspace of X/A iff $\pi' = id$ is an identification map from the topology induced by X to the topology induced by X/A. id is an identification map if and only if for any set U of $X \setminus A$ in the topology inherited from X/A, U is open iff $id^{-1}(U) = U$ is open in the topology inherited from X. But this says that the topology on $X \setminus A$ is exactly that inherited from X/A, and so the result holds.

Give an example of X, A for which this condition holds, yet A is neither open nor closed in X. Give an example of X, A for which this does not hold.

For the first part, let X be a two point indiscrete space and A a set of one of the points. For the second, let $X = \mathbb{R}$ and $A = \mathbb{Q}$. Then the set $U = \{x \in \,]0,1[\, : x \text{ is irrational}\}$ is open in $\mathbb{R} \setminus \mathbb{Q}$, but cannot be written as $V \cap (\mathbb{R} \setminus \mathbb{Q})$ for any open V which is either disjoint from or contains $\mathbb{Q}$, since the only open set containing $\mathbb{Q}$ is $\mathbb{R}$, and $U \neq \mathbb{R} \setminus \mathbb{Q}$, and any open set V must have some rational point by density. □

Exercise 4.3.5

Let A be a subspace of X. Prove that X/A is Hausdorff if (i) $X \setminus A$ is Hausdorff, (ii) $X \setminus A$ is a subspace of X/A, and (iii) if $x \in X \setminus A$ then x and A have disjoint neighbourhoods in X.

Prove that if $X = [0,2]$, $A = \,]1,2[$, then X/A is not Hausdorff.

□

Exercise 4.3.6

Let A be a closed subset of X and let B be a proper subset of A. Let $X' = X\ B$, $A' = A \setminus B$. Prove that X'/A' is homeomorphic to X/A.

We □