Topology and Groupoids, Chapter 4

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Exercise 4.2.3

Prove that the following are identification maps

(i) The projections $X \times Y \to X$, $X \times Y \to Y$

Let $\pi : X \times Y \to X$ be the projection map. Clearly this is a surjective map, and so by 4.2.4 it suffices to show it is open. Let U be an open set of $X \times Y$. Then $U = \bigcup_{\lambda \in L} U_\lambda \times V_\lambda$ for some family $\{U_\lambda\}_{\lambda \in L}$ of open sets in X and $\{V_\lambda\}_{\lambda \in L}$ of open sets in Y, and thus

$$\pi(U) = \pi\left(\bigcup_{\lambda \in L} U_\lambda \times V_\lambda\right) = \bigcup_{\lambda \in L} U_\lambda$$

Is the union of open sets of X, and thus open. □

Exercise 4.2.5

Let X, Y be topological spaces and $f : X \to Y$ a continuous surjection. Suppose that each point y in Y has a neighbourhood N such that $f|f^{-1}[N], N$ is an identification map. Prove that f is an identification map.

For each $y \in Y$, there is some neghborhood N_y of y such that the restriction $f_y : f^{-1}[N_y] \to N_y$ of f is an identification map. Then if U is such that $f^{-1}[U]$ is open in X, we see first that

$$U = \bigcup_{y \in U} (N_y \cap U)$$

The inclusion $\bigcup_{y \in U}(N_y \cap U) \subseteq U$ holds since $\bigcup_{y \in U}(N_y \cap U) = \left(\bigcup_{y \in U} N_y\right) \cap U \subseteq U$. The other inclusion holds since $U = \bigcup_{y \in U}\{y\}$ and $\{y\} \subseteq N_y \cap U$ for any $y \in U$. Thus

$$f^{-1}[U] = f^{-1}\left[\bigcup_{y \in U}(N_y \cap U)\right] = \bigcup_{y \in U}(f^{-1}[N_y \cap U])$$

Then we see $f^{-1}[N_y \cap U] = f^{-1}[N_y] \cap f^{-1}[U]$, and thus $f^{-1}[N_y \cap U]$ is open in $f^{-1}[N_y]$ for any $y \in U$. But also $f^{-1}[N_y \cap U] \subseteq f^{-1}[N_y]$, so $f^{-1}[N_y \cap U] = f_y^{-1}[N_y \cap U]$, and since f_y is an identification map and $f^{-1}[N_y \cap U]$ is open in $f^{-1}[N_y]$ this implies $N_y \cap U$ is open in N_y for any $y \in U$. Thus for any $y \in U$ there's an open set M_y in Y such that $N_y \cap U = N_y \cap M_y$. But also since N_y is a neighborhood of y, there's some open set O_y containing y contained in N_y, so $O_y \cap M_y \subseteq N_y \cap U$, and also $O_y \cap M_y$ is open in Y for any $y \in U$ since it's the intersection of two open sets. Thus

$$\bigcup_{y \in U}(O_y \cap M_y) \subseteq \bigcup_{y \in U}(N_y \cap U) = U$$

And also for any $y \in U$, we must have $y \in O_y \cap M_y$, and thus

$$U \subseteq \bigcup_{y \in U} (O_y \cap M_y)$$

Which means $U = \bigcup_{y \in U} (O_y \cap M_y)$ is a union of open sets, and thus open.

Thus $f^{-1}[U]$ being open implies U is open, and by assumption f is surjective and continuous, so f is an identification map. $\square$

Exercise 4.2.6

Let A be a subspace of X. A retraction of X onto A is a map $r : X \to A$ such that r is the identity on A. Prove that a retraction of X onto A is an identification map.

Suppose $U \subseteq A$ is some set such that $r^{-1}[U]$ is open. Then $r^{-1}[U] \cap A$ is open in A, and

$$r^{-1}[U] \cap A = \{x \in X : r(x) \in U, x \in A\} = \{x \in A : r(x) \in U\} = \{x \in A : x \in U\} = U$$

Since r is the identity on A. Thus U is open in A, and r is surjective since $r(A) = A$, which means r is an identification map. $\square$

Exercise 4.2.8

Let $f : X \to Y$ be an identification map. For each $A \subseteq X$, let

$$f^\dagger[A] = \{a \in A : f^{-1}[f(a)] \subseteq A\}$$

Prove that the following conditions are equivalent.

(i) f is a closed map.

(ii) If A is closed in X, then also is $f^{-1}[f[A]]$.

(iii) If A is open in X, then so also is $f^\dagger[A]$.

(iv) For each y in Y, every neighbourhood N of $f^{-1}(y)$ contains a saturated neighbourhood of $f^{-1}(y)$.

If A is a closed set and f is a closed and continuous function then $f[A]$ is closed by closedness of f and $f^{-1}[f[A]]$ is closed by continuity of f. Thus (i) implies (ii).

Now assume (ii). We see

$$\begin{aligned} f^{-1}[f[X \setminus A]] &= \{x \in X : f(x) \in f[X \setminus A]\} \\ &= \{x \in X : \exists y \notin A.\ f(x) = f(y)\} \\ &= \{x \in X : f^{-1}[f(x)] \text{ isn't a subset of } A\} \\ &= X \setminus f^\dagger[A] \end{aligned}$$

Thus if A is open, then $X \setminus A$ is closed, so $f^{-1}[f[X \setminus A]]$ is closed, and thus $f^\dagger[A]$ is open.

Now assume (iii). Then for any $y \in Y$ and any neighborhood N of $f^{-1}(y)$ in X, there is some open set O containing $f^{-1}(y)$ contained in N, and thus $f^{\dagger}[O]$ is an open set contained in N as well. Further $f^{-1}(y) \subseteq f^{\dagger}[O]$ since $f^{-1}(y) \subseteq O$ and $f^{-1}[f(b)] = f^{-1}[y] \subseteq O$ for any $b \in f^{-1}(y)$. We show $f^{\dagger}[O]$ is saturated. By definition, this holds if $f^{-1}[f[f^{\dagger}[O]]] = f^{\dagger}[O]$. But But if $x \in f^{-1}[f[f^{\dagger}[O]]]$, then $f(x) \in f[f^{\dagger}[O]]$, so there's some $t \in f^{\dagger}[O]$ such that $f(x) = f(t)$. But $t \in f^{\dagger}[O]$ means $f^{-1}[f(t)] \subseteq O$, and $x \in f^{-1}[f(t)]$ so $x \in O$, and $f^{-1}[f(x)] = f^{-1}[f(t)] \subseteq O$ so $x \in f^{\dagger[O]}$. Then the inclusion $f^{\dagger}[O] \subseteq f^{-1}[f[f^{\dagger}[O]]]$ holds trivially, so $f^{\dagger}[O] = f^{-1}[f[f^{\dagger}[O]]]$ and thus $f^{\dagger}[O]$ is saturated. Thus (iv) holds.

Finally assume (iv) holds. Then assume K is some closed subset of X. We show $f[K]$ is closed. Suppose y is some element of Y such that every neighborhood of y meets $f[K]$. $\square$