Simplicial Sets

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1 CW Complexes

The objects of study in classical homotopy theory are the *homotopy types*. This is not the same thing as a topological space, or even a CW complex, but "CW complex up to homotopy". CW complexes are spaces that admit a construction in stages, starting with some point, then gluing on intervals via their boundary, then gluing on disks via their boundary, and so on, then taking the union of all finite stages. In stage n the "gluing" of n-disks onto the $(n-1)$-skeleton X_{n-1} can be understood categorically as taking a pushout of X_{n-1} with your family of disks $\coprod_{\lambda\in\Lambda} D^n$ along a family of arbitrary continuous maps $\{f_\lambda : S^n \to X\}_{\lambda\in\Lambda}$ ("attaching maps") and standard inclusions $S^n \hookrightarrow D^n$. We could just have easily defined this using (topological) simplex inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, for Δ^n and D^n are convex bodies of the same dimension and so canonically (after picking a basepoint) homeomorphic. So CW complexes are *exactly* the topological spaces that can be obtained from a sequential colimit of pushouts of (coproducts of) the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$. In other words, they're spaces obtained by gluing simplices together with the restriction that one may only glue along the boundary, but the flexibility that arbitrary continuous gluings of that boundary are allowed. But combining the "Simplicial Approximation Theorem" with the following lemma allows us to assume a CW complex is obtained from a very, very structured kind of gluing.

Lemma 1. *Let X be a topological space and $f, g : S^{n-1} \to X$ two homotopic maps. Then the pushouts (or "amalgamation spaces") $D^n \amalg_f X$ and $D^n \amalg_g X$ are homotopy equivalent.*

Proof. Let $H : S^{n-1} \times I \to X$ be a homotopy. The key idea is that we may use the deformation retraction of the "cylinder" $D^n \times I$ onto its boundary minus the top $(D^n \times \{0\}) \cup (S^{n-1} \times I)$ to get a deformation retraction of $(D^n \times I) \amalg_H X$ onto $((D^n \times \{0\}) \cup (S^{n-1} \times I)) \amalg_H X$. We have a morphism $J : (D^n \times I) \amalg_H X \to ((D^n \times \{0\}) \cup (S^{n-1} \times I)) \amalg_H X$ induced by the morphism of spans

$$\begin{tikzcd}
D^n \arrow[d, hook] & S^{n-1} \arrow[l, hook'] \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, equal] \\
(D^n \times \{0\}) \cup (S^{n-1} \times I) & S^{n-1} \times I \arrow[l, hook'] \arrow[r, "H"] & X.
\end{tikzcd}$$

And in fact J is surjective, because every point in the extra bit $S^{n-1} \times (0, 1]$ is glued onto X by H. But it's actually a split monomorphism as well, because morphism of spans above has a left inverse

$$\begin{tikzcd}
D^n & S^{n-1} \arrow[l, hook'] \arrow[r, "f"] & X \arrow[d, equal] \\
(D^n \times \{0\}) \cup (S^{n-1} \times I) \arrow[u, two heads] & S^{n-1} \times I \arrow[l, hook'] \arrow[r, "H"] \arrow[u, two heads] & X.
\end{tikzcd}$$

This means J is actually a homeomorphism, because it is a surjection with a continuous left inverse. The punchline is that $D^n \amalg_f X$, and by symmetry $D^n \amalg_g X$, are both homeomorphic to deformation retracts of the same space (and hence are homotopy equivalent). $\square$

> Exercise: Reprove Lemma 1 in terms of the simplicial inclusions, using the fact that Δ^n deformation retracts onto any of its "horns" Λ^n_i (those spaces formed by removing the ith face from $\partial\Delta^n$).

2 The simplex category, gluing, and presheaves

Simplicial sets are a more "algebraic" or "combinatorial" way of modelling homotopy types. This has the advantage that it transports more easily to algebraic contexts. E.g., the (1-)category of topological abelian groups is not abelian but the (1-)category of simplicial abelian groups is! We saw above through careful analysis of CW complexes that any homotopy type is built up from gluing together simplices along their boundaries. For CW complexes the gluing was fairly geometric, an actual pushout in the category of topological spaces. Simplicial sets take the opposite approach: they are formal gluings of (formal!) simplices. Before we can define simplicial sets we must discuss the (category of) simplices from which they are glued.

Definition 2. The simplex category Δ has objects the finite nonempty ordinals $[n] = \{0, 1, \ldots, n\}$ and a morphism $[n] \to [m]$ is simply an order preserving function. The augmented simplex category Δ_a is defined in the same way, but the empty ordinal $[-1] = \emptyset$ is included.

Note that Δ is equivalent to the category of all finite totally ordered sets. What does this have to do with actual geometric simplices? The object $[n]$ should be understood as a representation of the geometric n-simplex Δ^n, and its elements $0, \ldots, n$ representing the $(n+1)$-vertices of that simplex. As demonstrated by simplicial or singular homology, it's often more convenient to work with simplices that have a chosen order on their vertices (for manageably and consistently tracking orientation); this is why we're looking at ordered finite sets and not just finite sets[1]. The geometric simplex Δ^n is the convex hull of its vertices $e_0, \ldots, e_n$, and this means that every function of finite sets $\{e_0, \ldots, e_n\} \mapsto \{e_0, \ldots, e_m\}$ has a unique extension to an affine transformation $\Delta^n \to \Delta^m$ sending vertices to vertices. Thus Δ could just as truthfully be described as the category of geometric simplices $\Delta^n \subseteq \mathbb{R}^{n+1}$ with morphisms the affine transformations sending vertices to vertices and preserving the standard order on those vertices.

Definition 3. Let

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,:\, x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}$$

be the n-dimensional "geometric simplex". The vertices of Δ^n are the standard basis vectors $e_0, \ldots, e_n$ of $\mathbb{R}^{n+1}$ and any point in Δ^n can be uniquely represented as a convex combination $t_0 e_0 + \ldots + t_n e_n$ of them. Given an order-preserving map $f \,:\, [n] \to [m]$ there is an induced continuous map $\widetilde{f} \,:\, \Delta^n \to \Delta^m$ defined by

$$\widetilde{f}\left(\sum_{i=0}^{n} t_i e_i \right) = \sum_{i=0}^{n} t_i e_{f(i)}.$$

> Exercise: The assignments $[n] \mapsto \Delta^n$ and $f \mapsto \widetilde{f}$ define a faithful functor $\Delta \to \mathsf{Top}$.

There are two important families of maps within Δ, the coface and codegeneracy maps. Geometrically these correspond to the inclusions of a face of a simplex and the projections of a simplex onto one of its faces.

Definition 4. Let n be a positive integer. For $0 \leq i \leq n$ denote by $\delta_i^n \,:\, [n-1] \to [n]$ the unique monotone injection which omits i from its range. This is the ith coface map. Concretely,

$$\delta_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

Also define $\sigma_i^n \,:\, [n+1] \to [n]$ to be the unique monotone surjection with $\sigma_i^n(i) = \sigma_i^n(i+1)$. This is the ith codegeneracy map. Concretely,

$$\sigma_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

[1] But for those who are interested, there is a theory of unoriented "symmetric simplicial sets"

Geometrically, δ_i^n is the inclusion of the ith face of Δ^n (meaning the face opposite the ith vertex) and σ_n^i is the projection of Δ^{n+1} onto Δ^n where we collapse the edge $[e_i\ e_{i+1}]$ down to a point. Any monotone map $f : [n] \to [m]$ has a decomposition into a surjection $[n] \twoheadrightarrow [k]$ and an injecton $[k] \hookrightarrow [m]$; this may be easiest to see if we think of $[k]$ as the image f with the order inherited from $[m]$ (passing to the category of all finite nonempty totally ordered sets). Furthermore the injection $[k] \hookrightarrow [m]$ can be decomposed into a composition of coface maps, omitting elements of $[m]$ one at a time, and the surjection $[n] \twoheadrightarrow [k]$ may be decomposed into a composition of codegeneracy maps, squishing together elements $i, i+1$ such that $f(i) = f(i+1)$ one at a time until none remain. This tells us that every morphism in Δ is a composition of face and degeneracy maps. In fact there is a normal form associated to this decomposition, obtained by repeatedly applying the "cosimplicial identites".

Theorem 5. *The simplex category* Δ *is the free category* C *on a sequence of objects* $[0], [1], \dots$ *and families of morphisms* $\{\delta_i^n \in \mathrm{Hom}_{\mathsf{C}}(n-1, n)\}_{n\geq 1, 0\leq i\leq n}$ *and* $\{\sigma_i^n \in \mathrm{Hom}_{\mathsf{C}}(n+1, n)\}_{n\geq 0, 0\leq i\leq n}$*, subject to the relations (for all n)*

$$\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n \quad (\textit{if } i < j) \tag{1}$$

$$\sigma_j^{n+1} \circ \delta_i^{n+2} = \delta_i^{n+2} \circ \sigma_{j-1}^n \quad (\textit{if } i < j) \tag{2}$$

$$\sigma_j^n \circ \delta_j^{n+1} = \mathrm{id}_{[n]} \tag{3}$$

$$\sigma_j^n \circ \delta_{j+1}^{n+1} = \mathrm{id}_{[n]} \tag{4}$$

$$\sigma_j^{n+1} \circ \delta_i^{n+2} = \delta_{i-1}^{n+1} \circ \sigma_j^n \quad (\textit{if } i > j+1) \tag{5}$$

$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \quad (\textit{if } i \leq j) \tag{6}$$

We will not prove this theorem in these notes, but we will attempt to explain what these identites say in the simplex category and explain what it means for a category to be presented by generators and relations. The equations (1) and (2) are a commutativity condition, they express (with index shifts appropriate to the δ's and σ's) that omitting a vertex i and then omitting/collapsing a later vertex j is the same as first omitting/collapsing $j-1 = \delta_i^{-1}(j)$ and then omitting i. The equations (3) and (4) are perhaps the most important identities, because their categorical interpretation is that each δ is a *split monomorphism* and each σ is a *split epimorphism*; explicitly they say that if we omit a vertex and then collapse it with the next/previous vertex, it's the same as doing nothing. The equation (5) can be understood as saying "far away" omissions/collapses do not affect eachother (up to reindexing!). And finally equation (6) expresses that if you collapse twice in a row, the order of collapses matters only in that it shifts up the indexing.

The "free category" part of the theorem is more directly relevant, because it gives an explicit description of functors $\Delta \to \mathsf{C}$ for any category C (like how a presentation of a group G tells you what group homomorphisms $G \to H$ are). One interpretation of a "free structure" is exactly this kind of universal property, i.e. a free thing ("group" or "category equipped with a sequence of objects and families of maps satisfying the cosimplicial identities") is an initial object in the category of things. A free group G on generators $x_1, \dots, x_n$ subject to relations $r_1, \dots, r_m$ is an initial object in the category of tuples $(H, y_1, \dots, y_n)$ of groups H and $\mathbf{y} \in H^n$ such that for each j, interpreting x_i as y_i in w_j gives the identity element of H; a morphism $(H, \mathbf{y}) \to (H', \mathbf{z})$ in this category is of course a group homomorphism $f : H \to H'$ such that $f(y_i) = z_i$ for each i. Hence a free category on objects $\{X_s\}_{s\in S}$ and morphisms $\{f_\lambda : X_s \to X_t\}_{s,t\in S, \lambda\in\Lambda_{s,t}}$ subject to some equations of morphisms $\{E_j\}_{j\in J}$ is an initial object in the category[2] of categories that are equipped with a chosen family of objects labelled by S and a chosen family of morphisms labelled by the $\Lambda_{s,s'}$ satisfying all equations E_j. There is also a "by hand" construction of a free category on a directed graph/quiver G, e.g. the graph with vertices $\mathbb{N}$ and edges labelled by the coface/codegeneracy maps. This construction is fairly simply, if v, w are vertices in G then a morphism $v \to w$ in the free category is just a path ("formal composition of edges") from v to w in G. One can then quotient the set of arrows of this category by the smallest equivalence relation which contains the equations and "respects composition" (like how a normal subgroup gives an equivalence relation which multiplication).

We might stop and ask at this point why we need the codegeneracies at all. If we're interested in gluing together simplices along their boundaries, surely we just need the face inclusions? It turns out that the category of simplicial sets is much nicer when degeneracies; for example, the geometric realization of the semisimplicial set $\Delta^1 \times \Delta^1$ is an interval union two points, not a square! The theory without degeneracies isn't useless, though, we obtain what are called "semi-simplicial sets". These are called "Δ-complexes" in Hatcher's algebraic topology textbook.

[2]Here I mean the locally small category of small categories such that etc. But in fact an initial object of this category will have the right mapping out property with respect to locally small categories too, as any functor $F : \mathsf{C} \to \mathsf{D}$ with C factors through a small subcategory D' of D; specifically D' is the full subcategory of D on the objects in the image of F.

We now return to simplicial sets, having gained an understanding of what kind of "formal simplices" we're gluing together. The categorical understanding of "gluing" is that it is a colimit. And vice versa, in many concrete categories a colimit does performing some kind of concrete "gluing". This is all there is to the definition of a simplicial set.

Definition 6. The category of simplicial sets sSet is the free[3] cocompletion of Δ. That is to say sSet has all (small) colimits, comes equipped with a functor $Y : \Delta \to \mathsf{sSet}$, and for any other category C with all (small) colimits and a functor $F : \Delta \to \mathsf{C}$ there exists a colimit preserving functor $G : \mathsf{sSet} \to \mathsf{C}$ equipped with an isomorphism $\varphi : F \to G \circ Y$. Furthermore (G, φ) is unique in that if we have another colimit-preserving functor $G' : \mathsf{sSet} \to \mathsf{C}$ equipped with an isomorphism $\psi : F \to G' \circ Y$ then there is a unique isomorphism $\eta : G \to G'$ making the diagram

$$\begin{tikzcd} & G \circ Y \arrow[d, "\eta Y"] \\ F \arrow[ur, "\psi"] \arrow[r, "\varphi"] & G' \circ Y \end{tikzcd}$$

commute.

Intuitively this says that an object of sSet is a formal colimit of some diagram in Δ. One can construct a free cocompletion in this way, but I tried to write it down once and lost two weeks working out technical details. Luckily the free cocompletion of a small category is a recognizable, fairly simple, and extremely well behaved category. The rest of this section will be devoted to proving the following theorem.

Theorem 7. *Let* C *be a small category and* $\mathrm{Psh}(\mathsf{C}) = \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ *be the category of presheaves on* C*. The Yoneda embedding* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *exhibits* $\mathrm{Psh}(\mathsf{C})$ *as the free cocompletion of* C*.*

Most people would find my initial definition of sSet a little silly. The true definition is just $\mathsf{sSet} = \mathrm{Psh}(\Delta)$. Our presentation of Δ tells us that a simplicial set can also be understood as sequence of sets $\{X_n\}_{n \in \mathbb{N}}$ equipped with morphisms s_i^n, d_i^n satisfying the *simplicial identities*, the categorical dual of the cosimplicial identities (because presheaves are contravariant functors $\Delta \to \mathsf{Set}$). We will expand on this later.

One caveat with Theorem7 is that because the presheaf category is a free construction, already existing colimits in C will almost never be preserved under y. The proof of Theorem7 boils down to the fact that any presheaf on C can be canonically written as a colimit of representable presheaves (those in the image of the Yoneda embedding). This may sound strange, but it's actually just another point of view on the celebrated Yoneda lemma, which we recall below.

Lemma 8. *Let* C *be a small category and* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *the functor* $y(x) = \mathrm{Hom}_{\mathsf{C}}(-, x)$*. For any object* x *of* C *and presheaf* F *on* C *the function*

$$\varphi : \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), F) \to F(x)$$

defined by $\varphi(\eta) = \eta_x(\mathrm{id}_x)$ *is a bijection. Furthermore,* φ *defines a natural isomorphism of functors* $\mathsf{C}^{\mathrm{op}} \times \mathrm{Psh}(\mathsf{C}) \to \mathsf{Set}$*.*

Proof. We define an inverse $\psi : F(x) \to \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), F)$ by $\psi(t)_z(f) = F(f)(t)$. Unwrapping this a bit, for $t \in F(x)$ we define a natural transformation $\psi(t) : y(x) \to F$ by setting its component on an object z to be the function $\mathrm{Hom}_{\mathsf{C}}(z, x) \to F(z)$ sending $f : z \to x$ to its action on t under F, i.e. $F(f)(t)$. We must verify that $\psi(t)$ is in fact natural for each t. So suppose we have a map $g : z \to w$ in C, we must check that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(w, x) \arrow[r, "g^*"] \arrow[d, "\psi(t)_w"] & \mathrm{Hom}_{\mathsf{C}}(z, x) \arrow[d, "\psi(t)_z"] \\ F(w) \arrow[r, "F(g)"] & F(z) \end{tikzcd}$$

commutes. Given any $f \in \mathrm{Hom}_{\mathsf{C}}(w, x)$ by simply unravelling definitions and applying functorality of F we check

$$\psi(t)_z(g^*(f)) = \psi(t)_z(f \circ g) = F(f \circ g)(t) = F(g)(F(f)(t)) = F(g)(\psi(t)_w(f)).$$

So ψ is well defined. And for arbitrary $t \in F(x)$, $\eta \in \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), F)$, $z \in \mathrm{Obj}(\mathsf{C})$ and $f \in \mathrm{Hom}_{\mathsf{C}}(z, x)$ we have

$$\varphi(\psi(t)) = \psi(t)_x(\mathrm{id}_x) = F(\mathrm{id}_x)(t) = \mathrm{id}_{F(x)}(t) = t$$
$$\psi(\varphi(\eta))_z(f) = F(f)(\varphi(\eta)) = F(f)(\eta_x(\mathrm{id}_x)) \overset{(!)}{=} \eta_z(F(f)(\mathrm{id}_x)) = \eta_z(f^*(\mathrm{id}_x)) = \eta_z(f).$$

[3]This is not quite freeness in the sense discussed above; it is about 2-initiality in an appropriate 2-category!

The equality labelled (!) is exactly the following naturality square, applying both paths to $\mathrm{id}_x \in y(x)(x)$

$$\begin{tikzcd} y(x)(x) \arrow[r, "y(x)(f)"] \arrow[d, "\eta_x"] & y(x)(z) \arrow[d, "\eta_z"] \\ F(x) \arrow[r, "F(f)"] & F(z). \end{tikzcd}$$

We may check φ is natural by fixing F and varying x, then fixing x and varying F. Fix F and let $\varphi_x(\eta) = \eta_x(\mathrm{id}_x)$. For any morphism $a : x \to x'$ in C we must verify the commutativity of the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x'), F) \arrow[r, "y(a)^*"] \arrow[d, "\varphi_{x'}"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), F) \arrow[d, "\varphi_x"] \\ F(x') \arrow[r, "F(a)"] & F(x). \end{tikzcd}$$

This is once again just unfolding definitions and using naturality: for any $\eta : y(x') \to F$,

$$\varphi_x(y(a)^*(\eta)) = \varphi_x(\eta \circ y(a)) = \eta_x(y(a)_x(\mathrm{id}_x)) = \eta_x(a \circ \mathrm{id}_x) = \eta_x(a)$$
$$F(a)(\varphi_{x'}(\eta)) = F(a)(\eta_{x'}(\mathrm{id}_{x'})) = \eta_x(y(x')(a)(\mathrm{id}_{x'})) = \eta_x(\mathrm{id}_{x'} \circ a) = \eta_x(a).$$

Now fix x, write $\varphi_F(\eta) = \eta_x(\mathrm{id}_x)$, and take some natural transformation $\beta : F \to G$. The diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), F) \arrow[r, "\beta_*"] \arrow[d, "\varphi_F"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), G) \arrow[d, "\varphi_G"] \\ F(x) \arrow[r, "\beta_x"] & G(x) \end{tikzcd}$$

commutes because for any $\eta : y(x) \to F$ we have

$$\varphi_G(\beta_*(\eta)) = \varphi_G(\beta \circ \eta) = \beta_x(\eta_x(\mathrm{id}_x)) = \beta_x(\varphi_F(\eta)).$$

□

So what does the equation $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), F) \cong F(x)$ have to do with writing F as a colimit? Well we certainly get a lot of maps from representables into F, i.e. for each object x and $s \in F(x)$ we have a natural transformation $\iota^{x,s} : y(x) \to F$ uniquely specified by the constraint $\iota^{x,s}_x(\mathrm{id}_x) = s$. These should give F the structure of a cocone under some diagram of representables, if only we could identify that diagram (and check that the various ι's are compatible). Given any map $f : z \to w$ and $s \in F(w)$ naturality of the Yoneda lemma gives a commutative square

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(w), F) \arrow[r, "y(f)^*"] \arrow[d] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(z), F) \arrow[d] \\ F(w) \arrow[r, "F(f)"] & F(z). \end{tikzcd}$$

The vertical maps above indicate evaluation at id_w and id_z. We may then calculate

$$(\iota^{w,s} \circ y(f))_z(\mathrm{id}_z) = F(f)(\iota^{w,s}_w(\mathrm{id}_w)) = F(f)(s)$$

and deduce $\iota^{w,s} \circ y(f) = \iota^{z,F(f)(s)}$. Changing persepctive slightly, for $z \in \mathrm{Obj}(\mathsf{C})$, $t \in F(z)$ and $w \in \mathrm{Obj}(\mathsf{C})$, $s \in F(w)$, any map $f : z \to w$ satisfying $F(f)(s) = t$ entails a commutative triangle

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\iota^{z,t}"'] & y(w) \arrow[d, "\iota^{w,s}"] \\ & F. \end{tikzcd}$$

So surely F is a cocone under some diagram of representables, specifically one where each representable $y(x)$ is incarnated many time, the instances of it labelled by elements of $F(x)$, and where maps must preserve these labels.

Definition 9. Let C be a small category and F a presheaf on C. Define a (small) category $\mathrm{el}(F)$[4] by

$$\mathrm{Obj}(\mathrm{el}(F)) = \{(x,s) : x \in \mathrm{Obj}(\mathsf{C}), s \in F(x)\}$$
$$\mathrm{Hom}_{\mathrm{el}(F)}((z,t),(w,s)) = \{f \in \mathrm{Hom}_{\mathsf{C}}(z,w) : F(f)(s) = t\}.$$

We perform composition as in C and set $\mathrm{id}_{(x,s)} = \mathrm{id}_x$, the latter of which is is well defined because $F(\mathrm{id}_x)(s) = s$. The composition laws are automatic, and this composition is well defined because if $F(f)(u) = v$ and $F(g)(v) = w$ then

$$F(f \circ g)(s) = F(g)(F(f)(s)) = F(g)(t) = r.$$

This category comes with a forgetful functor $P : \mathrm{el}(F) \to \mathsf{C}$. The category $\mathrm{el}(F)$ equipped with this functor is referred to as the category of elements of F (in the special case Δ it is sometimes called the category of simplices of F).

The preceding discussion establishes that F has the structure of a cocone under the diagram $\mathrm{el}(F) \to \mathsf{C} \xrightarrow{y} \mathrm{Psh}(\mathsf{C})$. Now that we know the specific diagram of which F is a cocone under we may it is an initial cocone (i.e. a colimit).

Theorem 10. *Let C be a small category and F a presheaf on C. For any object x of C and $s \in F(x)$ there is a unique morphism $\iota^{x,s} : y(x) \to F$ satisfying $\iota^{x,s}_x(\mathrm{id}_x) = s$ and these make F a colimit of the diagram $y \circ P : \mathrm{el}(F) \to \mathrm{Psh}(\mathsf{C})$.*

Proof. We have already discussed the cocone structure on F. Suppose we have another cocone, i.e. a presheaf G on C and natural transformations $\sigma^{x,s} : y(x) \to G$ such that for any morphism $f : (z,t) \to (w,s)$ in $\mathrm{el}(F)$ the diagram

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\sigma^{z,t}"'] & y(w) \arrow[d, "\sigma^{w,s}"] \\ & G \end{tikzcd}$$

commutes. We are then required to show there is a unique natural transformation $\beta : F \to G$ making each diagram

$$\begin{tikzcd} y(x) \arrow[d, "\iota^{x,s}"'] \arrow[dr, "\sigma^{x,s}"] & \\ F \arrow[r, "\beta"'] & G \end{tikzcd}$$

commute. Uniqueness is immediate, as the diagram above implies that

$$\beta_x(s) = \beta_x(\iota^{x,s}_x(\mathrm{id}_x)) = \sigma^{x,s}_x(\mathrm{id}_x).$$

So we just need to check that $\beta_x(s) = \sigma^{x,s}_x(\mathrm{id}_x)$ does in fact define a natural transformation $F \to G$. This means that for any map $f : z \to w$ in C the square

$$\begin{tikzcd} F(w) \arrow[r, "F(f)"] \arrow[d, "\beta_w"] & F(z) \arrow[d, "\beta_z"] \\ G(w) \arrow[r, "G(f)"] & G(z) \end{tikzcd}$$

must commute. And this is true because for any $s \in F(w)$, if we let $t = F(f)(s) \in F(x)$ then

$$G(f)(\beta_w(s)) = G(f)(\sigma^{w,s}_w(\mathrm{id}_w)) = \sigma^{w,s}_z(y(w)(f)(\mathrm{id}_w)) = \sigma^{w,s}_z(\mathrm{id}_w \circ f) = \sigma^{w,s}_z(f)$$
$$\beta_z(F(f)(s)) = \beta_z(t) = \sigma^{z,t}_z(\mathrm{id}_z) = (\sigma^{w,s} \circ y(f))_z(\mathrm{id}_z) = \sigma^{w,s}_z(y(f)_z(\mathrm{id}_z)) = \sigma^{w,s}_z(f \circ \mathrm{id}_z) = \sigma^{w,s}_z(f).$$

□

With Theorem 2 we have shown that $\mathrm{Psh}(\mathsf{C})$ is generated from C under "gluing" (colimits). We now have the tools to prove Theorem 7, which states that this method of gluing objects of C together is universal. The reader may already see how to define a colimit-preserving extension of a functor $F : \mathsf{C} \to \mathsf{D}$ using the colimit formula for presheaves: put F inside the colimit. This does in fact work to prove existence, so the main bulk of the proof will be about showing uniqueness. We extract that bulk into the following lemma, which is useful in its own right and characterizes colimit-preserving extensions in terms of adjunctions.

[4] A reader with stacky inclinations may recognize this as the grothendieck construction, specialized to presheaves of discrete groupoids (sets).

Lemma 11. *Let* C *be a small category with Yoneda embedding* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *and* D *some cocomplete category with a functor* $F : \mathsf{C} \to \mathsf{D}$*. Define a functor* $N(F) : \mathsf{D} \to \mathrm{Psh}(\mathsf{C})$ *by* $N(F)(d)(x) = \mathrm{Hom}_{\mathsf{D}}(F(x), d)$*. For any colimit-preserving functor* $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ *there is a bijection between isomorphisms* $\varphi : F \to G \circ y$ *and adjunctions* $G \vdash N(F)$*, where the unit* $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G$ *of the adjunction corresponding to* φ *acts on representables by*

$$(\eta_{y(x)})_x(\mathrm{id}_x) = \varphi_x.$$

Since left adjoints automatically preserve colimits, we can read this lemma as saying (in a more specific/refined way) that colimit preserving extensions of F are the same thing left adjoints to $N(F)$. Uniqueness of extensions will then be an easy corollary, since left adjoints are unique.

Proof. First suppose we are given an isomorphism $\varphi : F \to G \circ y$. Let S be an arbitrary presheaf. Recall that there is a projection $P : \mathrm{el}(S) \to \mathsf{C}$ and structure maps $\iota^{x,s} : y(x) \to S$ as in Theorem 2 making S into a colimit of $y \circ P$. We now define a natural transformation $\eta_S : S \to N(F)(G(S))$ by saying its component at an object x of C is the function

$$(\eta_S)_x(s) = G(\iota^{x,s}) \circ \varphi_x.$$

Why this equation? Well, we are asked for an element of $N(F)(G(S))(x) = \mathrm{Hom}_{\mathsf{D}}(F(x), G(S))$, and we get it by composing the isomorphism $F(x) \cong G(y(x))$ with the action of G on the natural transformation $\iota^{x,s} : y(x) \to S$ corresponding to the given element $s \in S(x)$. As in all good categorical proofs, this is the only definition we could give. The equation $(\eta_{y(x)})_x(\mathrm{id}_x) = \varphi_x$ holds because if $S = y(x)$ then $\iota^{x,\mathrm{id}_x} = \mathrm{id}_{y(x)}$. We must check that the components $\{(\eta_S)_x\}_{x \in \mathrm{Obj}(\mathsf{C})}$ assemble into a natural transformation, i.e. that for any morphism $f : z \to w$ in C the square

$$\begin{tikzcd}
S(w) \arrow[r, "S(f)"] \arrow[d, "(\eta_S)_w"] & S(z) \arrow[d, "(\eta_S)_z"] \\
\mathrm{Hom}_{\mathsf{D}}(F(w), G(S)) \arrow[r, "F(f)^*"] & \mathrm{Hom}_{\mathsf{D}}(F(z), G(S)).
\end{tikzcd}$$

commutes. Let $s \in S(w)$ be arbitrary. Then

$$F(f)^*((\eta_S)_w(s)) = F(f)^*(G(\iota^{w,s}) \circ \varphi_w) = G(\iota^{w,s}) \circ \varphi_w \circ F(f) \overset{(!)}{=} G(\iota^{w,s}) \circ G(y(f)) \circ \varphi_z = G(\iota^{w,s} \circ y(f)) \circ \varphi_z.$$

The (!) equality is by naturality of φ. And if $t = S(f)(s)$ then $\iota^{w,s} \circ y(f) = \iota^{z,t}$ (this is part of S being a cocone). Thus

$$F(f)^*((\eta_S)_w(s)) == G(\iota^{z,t}) \circ \varphi_z = (\eta_S)_z(t) = (\eta_S)_z(S(f)(s))$$

proving naturality of η_S for fixed S. The reader might expect us to now verify η_S is natural in S, then perhaps write down a colimit, but if we use the "universal arrow" characterization of adjunctions this is unnecessary. What we do need to do to obtain an adjunction with (necessarily natural) unit $S \mapsto \eta_S$ is argue that for any object d of D and $\alpha : S \to N(F)(d)$ there is a unique morphism $f : G(S) \to d$ in D such that $\alpha = N(F)(f) \circ \eta_S$. By calculating

$$(N(F)(f) \circ \eta_S)_x(s) = N(F)(f)_x((\eta_S)_x(s)) = f \circ (\eta_S)_x(s) = f \circ G(\iota^{x,s}) \circ \varphi_x.$$

we find that a morphism $f : G(S) \to d$ satisfies $\alpha = N(F)(f) \circ \eta_S$ iff $\alpha_x(s) = f \circ G(\iota^{x,s}) \circ \varphi_x$ for all $(x, s) \in \mathrm{el}(S)$. We can push forward the cocone structure on S along α to obtain the structure of a cocone under $y \circ P$ on $N(F)(d)$, with structure maps $\alpha \circ \iota^{x,s} : y(x) \to N(F)(d)$. By the Yoneda lemma these correspond to elements $a^{x,s} = (\alpha \circ \iota^{X,s})_x(\mathrm{id}_x) = \alpha_x(s)$, or rather morphisms $a^{x,s} : F(x) \to d$ in D. In fact these morphisms make d into a cocone under $F \circ P$, i.e. for any map $f : (z, t) \to (w, s)$ in $\mathrm{el}(S)$ the diagram

$$\begin{tikzcd}
F(z) \arrow[r, "F(f)"] \arrow[dr, "a^{z,t}"'] & F(w) \arrow[d, "a^{w,s}"] \\
& d
\end{tikzcd}$$

commutes. This is by naturality of α and the equation $S(f)(s) = t$ (baked into the definition of a morphism $\mathrm{el}(S)$), as

$$a^{w,s} \circ F(f) = N(F)(d)(f)(a^{w,s}) = N(F)(d)(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t) = a^{z,t}.$$

Now by assumption G preserves colimits and we have isomorphisms $\varphi P : F \circ P \to G \circ y \circ P$, hence $G(S)$ is a colimit $F \circ P$ when equipped with the maps $G(\iota^{x,s}) \circ \varphi_x$. Finally because $G(S)$ is a colimit, there is a unique morphism $f : G(S) \to d$ factoring the colimit structure on d through the universal one, i.e. such that $\alpha_x(s) = f \circ G(\iota^{x,s}) \circ \varphi_x$ for all $(x, s) \in \mathrm{el}(S)$, which is what we were trying to prove.

This establishes a function from the set of natural isomorphisms $F \to G \circ y$ to the set of adjunction structures on the pair $(G, N(F))$. We show this function is a bijection in two steps: first given any adjunction $G \vdash N(F)$ if we denote its unit by $\eta : \mathrm{Id}_{\mathsf{Psh(C)}} \to N(F) \circ G$ then the natural transformations $\eta_{y(x)} : y(x) \to N(F)(G(y(x)))$ correspond under the Yoneda lemma to elements $\varphi_x \in \mathrm{Hom}_{\mathsf{D}}(F(x), G(y(x)))$, defined by $\varphi_x = (\eta_{y(x)})_x(\mathrm{id}_x)$, and these elements assemble into a natural isomorphism $F \cong G \circ y$; second, if we have two adjunctions $G \vdash N(F)$ with units η, η' such that $\eta_{y(x)} = \eta'_{y(x)}$ for all $x \in \mathrm{Obj}(\mathsf{C})$ then they are in fact the same adjunction.

For the first part, fix $x \in \mathrm{Obj}(\mathsf{C})$ and observe that $\mathrm{id}_{F(x)} \in N(F)(F(x))(x)$ corresponds to some natural transformation $\alpha : y(x) \to N(F)(F(x))$; explicitly, for $f : z \to x$ in C we have $\alpha_z(f) = F(f)$. Let $\beta : G(y(x)) \to F(x)$ be the adjunct morphism of α. We argue that β is inverse to φ_x, hence φ is a natural isomorphism. To verify the equation $\varphi_x \circ \beta = \mathrm{id}_{G(y(x))}$ it suffices to verify the adjunct equation $N(F)(\varphi_x) \circ \alpha = \eta_{y(x)}$. By direct computation,

$$N(F)(\varphi_x)(\alpha_x(\mathrm{id}_x)) = N(F)(\varphi_x)(\mathrm{id}_{F(x)}) = (\varphi_x)_*(\mathrm{id}_{F(x)}) = \varphi_x = (\eta_{y(x)})_x(\mathrm{id}_x),$$

and so $N(F)(\varphi_x) \circ \alpha = \eta_{y(x)}$ by the Yoneda lemma. Hence β is right inverse to φ_x. To prove the other equation $\beta \circ \varphi_x = \mathrm{id}_{F(x)}$ we restate it as $N(F)(\beta)_x((\eta_{y(x)})_x(\mathrm{id}_x)) = \alpha_x(\mathrm{id}_x)$, i.e. $N(F)(\beta) \circ \eta_{y(x)} = \alpha$. But $N(F)(\beta) \circ \eta_{y(x)}$ is exactly the formula for the adjunct of β, which we know to be α.

Finally we show that for any natural transformations $\eta, \eta' : \mathrm{Id}_{\mathsf{Psh(C)}} \to N(F) \circ G$, if $\eta_{y(x)} = \eta'_{y(x)}$ for all $x \in \mathrm{Obj}(\mathsf{C})$ then $\eta = \eta'$; as the rest of the data of an adjunction is uniquely specified by the unit, this will conclude our proof. Let S be an arbitrary presheaf. The equality $\eta_S = \eta'_S$ holds iff $(\eta_S)_x(s) = (\eta'_S)_x(s)$ for every object $x \in \mathrm{Obj}(\mathsf{C})$ and $s \in S(x)$. If $\iota^{x,s} : y(x) \to S$ is the map satisfying $\iota^{x,s}_x(\mathrm{id}_x) = s$ then by naturality of η, η' we have

$$(\eta_S)_x(s) = (\eta_S)_x(\iota^{x,s}_x(\mathrm{id}_x)) = N(F)(G(\iota^{x,s}))_x((\eta_{y(x)})_x(\mathrm{id}_x)) = N(F)(G(\iota^{x,s}))_x((\eta'_{y(x)})_x(\mathrm{id}_x)) = (\eta'_S)_x(s).$$

□

We conclude this section by proving Theorem 7.

Proof. Let C be a small category with Yoneda embedding $y : \mathsf{C} \to \mathsf{Psh(C)}$. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor from C to a cocomplete category D. For any presheaf S on C define $P_S : \mathrm{el}(S) \to \mathsf{C}$ to be the canonical projection and let $\iota^{S,e} : y(P_S(e)) \to S$ be the structure maps realizing S as a colimit of $y \circ P_S$. Choose[5] for each S a colimit A_S of $F \circ P_S$, with structure maps $\kappa^{S,e} : F(P_S(e)) \to A_S$. Here it is essential that $\mathrm{el}(S)$ is a small category, as our assumption that D is cocomplete means that it has all *small* colimits. This is a necessary restriction, because a category with all colimits of its own size must be thin! Define $G : \mathsf{Psh(C)} \to \mathsf{D}$ by $G(S) = A_S$ on objects. Given a morphism of presheaves $\alpha : S \to T$ we have an induced functor $\mathrm{el}(\alpha) : \mathrm{el}(S) \to \mathrm{el}(T)$ defined by $\mathrm{el}(\alpha)(x, s) = (x, \alpha_x(s))$ on objects and $\mathrm{el}(\alpha)(f) = f$ on morphisms. This is well defined on morphisms by naturality of α, i.e. $S(f)(s) = t$ implies

$$T(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t).$$

The functor $\mathrm{el}(\alpha)$ obviously satisfies $P_T \circ \mathrm{el}(\alpha) = P_S$, hence we may pull back cocones under $F \circ P_T$ to cocones under $F \circ P_S$. In particular we can pull back the cocone structure of A_T to give it the structure of a cocone under $F \circ P_S$, and finally this allows us to define $G(\alpha)$ as the unique map making the diagram

$$\begin{tikzcd}
F(P_S(e)) \arrow[d, "\kappa^{S,e}"'] \arrow[dr, "\kappa^{T,\mathrm{el}(\alpha)(e)}"] & \\
A_S \arrow[r, dashed, "G(\alpha)"'] & A_T
\end{tikzcd}$$

commute for each $e \in \mathrm{Obj}(\mathrm{el}(S))$. This characterization allows us to easily reduce checking the functor laws for G to checking them for $\mathrm{el} : \mathsf{Psh(C)} \to \mathsf{Cat}$. But in fact both of the identites $\mathrm{el}(\mathrm{id}_S) = \mathrm{id}_{\mathrm{el}(S)}$ and $\mathrm{el}(\beta \circ \alpha) = \mathrm{el}(\beta) \circ \mathrm{el}(\alpha)$ are easily verified from the definition of el. Hence G is a well defined functor.

UNFINISHED □

[5]There is a possible foundational issue here. The axiom of choice in ZFC does not imply "global choice" or choice on classes. But the class of presheaves is large, so we need choice on classes to do this! If we work in ZFC + grothendieck universes then there is no issue, as a "large set" is still a set. In weaker foundations the correct object to consider is the always-extant colimit *anafunctor*.