Simplicial Sets

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1 CW Complexes

The objects of study in classical homotopy theory are the *homotopy types*. This is not the same thing as a topological space, or even a CW complex, but "CW complex up to homotopy". CW complexes are spaces that admit a construction in stages, starting with some point, then gluing on intervals via their boundary, then gluing on disks via their boundary, and so on, then taking the union of all finite stages. In stage n the "gluing" of n-disks onto the $(n-1)$-skeleton X_{n-1} can be understood categorically as taking a pushout of X_{n-1} with your family of disks $\coprod_{\lambda\in\Lambda} D^n$ along a family of arbitrary continuous maps $\{f_\lambda : S^n \to X\}_{\lambda\in\Lambda}$ ("attaching maps") and standard inclusions $S^n \hookrightarrow D^n$. We could just have easily defined this using (topological) simplex inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, for Δ^n and D^n are convex bodies of the same dimension and so canonically (after picking a basepoint) homeomorphic. So CW complexes are *exactly* the topological spaces that can be obtained from a sequential colimit of pushouts of (coproducts of) the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$. In other words, they're spaces obtained by gluing simplices together with the restriction that one may only glue along the boundary, but the flexibility that arbitrary continuous gluings of that boundary are allowed. But combining the "Simplicial Approximation Theorem" with the following lemma allows us to assume a CW complex is obtained from a very, very structured kind of gluing.

Lemma 1. *Let X be a topological space and $f, g : S^{n-1} \to X$ two homotopic maps. Then the pushouts (or "amalgamation spaces") $D^n \amalg_f X$ and $D^n \amalg_g X$ are homotopy equivalent.*

Proof. Let $H : S^{n-1} \times I \to X$ be a homotopy. The key idea is that we may use the deformation retraction of the "cylinder" $D^n \times I$ onto its boundary minus the top $(D^n\times\{0\})\cup(S^{n-1}\times I)$ to get a deformation retraction of $(D^n\times I)\amalg_H X$ onto $((D^n \times \{0\}) \cup (S^{n-1} \times I)) \amalg_H X$. We have a morphism $J : (D^n \times I) \amalg_H X \to ((D^n \times \{0\}) \cup (S^{n-1} \times I)) \amalg_H X$ induced by the morphism of spans

$$\begin{tikzcd}
D^n \arrow[d, hook] & & S^{n-1} \arrow[ll, hook'] \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, equal] \\
(D^n \times \{0\}) \cup (S^{n-1} \times I) & & S^{n-1} \times I \arrow[ll, hook'] \arrow[r, "H"] & X.
\end{tikzcd}$$

And in fact J is surjective, because every point in the extra bit $S^{n-1} \times (0, 1]$ is glued onto X by H. But it's actually a split monomorphism as well, because morphism of spans above has a left inverse

$$\begin{tikzcd}
D^n & & S^{n-1} \arrow[ll, hook'] \arrow[r, "f"] & X \arrow[d, equal] \\
(D^n \times \{0\}) \cup (S^{n-1} \times I) \arrow[u, two heads] & & S^{n-1} \times I \arrow[ll, hook'] \arrow[r, "H"] \arrow[u, two heads] & X.
\end{tikzcd}$$

This means J is actually a homeomorphism, because it is a surjection with a continuous left inverse. The punchline is that $D^n \amalg_f X$, and by symmetry $D^n \amalg_g X$, are both homeomorphic to deformation retracts of the same space (and hence are homotopy equivalent). □

> Exercise: Reprove Lemma 1 in terms of the simplicial inclusions, using the fact that Δ^n deformation retracts onto any of its "horns" Λ^n_i (those spaces formed by removing the ith face from $\partial\Delta^n$).

2 The simplex category, gluing, and presheaves

Simplicial sets are a more "algebraic" or "combinatorial" way of modelling homotopy types. This has the advantage that it transports more easily to algebraic contexts. E.g., the (1-)category of topological abelian groups is not abelian but the (1-)category of simplicial abelian groups is! We saw above through careful analysis of CW complexes that any homotopy type is built up from gluing together simplices along their boundaries. For CW complexes the gluing was fairly geometric, an actual pushout in the category of topological spaces. Simplicial sets take the opposite approach: they are formal gluings of (formal!) simplices. Before we can define simplicial sets we must discuss the (category of) simplices from which they are glued.

Definition 2. The simplex category Δ has objects the finite nonempty ordinals $[n] = \{0, 1, \ldots, n\}$ and a morphism $[n] \to [m]$ is simply an order preserving function. The augmented simplex category Δ_a is defined in the same way, but the empty ordinal $[-1] = \varnothing$ is included.

Note that Δ is equivalent to the category of all finite totally ordered sets. What does this have to do with actual geometric simplices? The object $[n]$ should be understood as a representation of the geometric n-simplex Δ^n, and its elements $0, \ldots, n$ representing the $(n+1)$-vertices of that simplex. As demonstrated by simplicial or singular homology, it's often more convenient to work with simplices that have a chosen order on their vertices (for manageably and consistently tracking orientation); this is why we're looking at ordered finite sets and not just finite sets[1]. The geometric simplex Δ^n is the convex hull of its vertices $e_0, \ldots, e_n$, and this means that every function of finite sets $\{e_0, \ldots, e_n\} \mapsto \{e_0, \ldots, e_m\}$ has a unique extension to an affine transformation $\Delta^n \to \Delta^m$ sending vertices to vertices. Thus Δ could just as truthfully be described as the category of geometric simplices $\Delta^n \subseteq \mathbb{R}^{n+1}$ with morphisms the affine transformations sending vertices to vertices and preserving the standard order on those vertices.

Definition 3. Let

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \, : \, x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}$$

be the n-dimensional "geometric simplex". The vertices of Δ^n are the standard basis vectors $e_0, \ldots, e_n$ of $\mathbb{R}^{n+1}$ and any point in Δ^n can be uniquely represented as a convex combination $t_0 e_0 + \ldots + t_n e_n$ of them. Given an order-preserving map $f : [n] \to [m]$ there is an induced continuous map $\widetilde{f} : \Delta^n \to \Delta^m$ defined by

$$\widetilde{f}\left(\sum_{i=0}^{n} t_i e_i \right) = \sum_{i=0}^{n} t_i e_{f(i)}.$$

> Exercise: The assignments $[n] \mapsto \Delta^n$ and $f \mapsto \widetilde{f}$ define a faithful functor $\Delta \to \mathsf{Top}$.

There are two important families of maps within Δ, the coface and codegeneracy maps. Geometrically these correspond to the inclusions of a face of a simplex and the projections of a simplex onto one of its faces.

Definition 4. Let n be a positive integer. For $0 \leq i \leq n$ denote by $\delta_i^n : [n-1] \to [n]$ the unique monotone injection which omits i from its range. This is the ith coface map. Concretely,

$$\delta_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

Also define $\sigma_i^n : [n+1] \to [n]$ to be the unique monotone surjection with $\sigma_i^n(i) = \sigma_i^n(i+1)$. This is the ith codegeneracy map. Concretely,

$$\sigma_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

[1] But for those who are interested, there is a theory of unoriented "symmetric simplicial sets"

Geometrically, δ_i^n is the inclusion of the ith face of Δ^n (meaning the face opposite the ith vertex) and σ_n^i is the projection of Δ^{n+1} onto Δ^n where we collapse the edge $[e_i\ e_{i+1}]$ down to a point. Any monotone map $f\ :\ [n] \to [m]$ has a decomposition into a surjection $[n] \twoheadrightarrow [k]$ and an injecton $[k] \hookrightarrow [m]$; this may be easiest to see if we think of $[k]$ as the image f with the order inherited from $[m]$ (passing to the category of all finite nonempty totally ordered sets). Furthermore the injection $[k] \hookrightarrow [m]$ can be decomposed into a composition of coface maps, omitting elements of $[m]$ one at a time, and the surjection $[n] \twoheadrightarrow [k]$ may be decomposed into a composition of codegeneracy maps, squishing together elements $i, i+1$ such that $f(i) = f(i+1)$ one at a time until none remain. This tells us that every morphism in Δ is a composition of face and degeneracy maps. In fact there is a normal form associated to this decomposition, obtained by repeatedly applying the "cosimplicial identites".

Theorem 5. *The simplex category* Δ *is the free category* C *on a sequence of objects* $[0], [1], \dots$ *and families of morphisms* $\{\delta_i^n \in \mathrm{Hom}_{\mathsf{C}}(n-1, n)\}_{n\geq 1, 0\leq i\leq n}$ *and* $\{\sigma_i^n \in \mathrm{Hom}_{\mathsf{C}}(n+1, n)\}_{n\geq 0, 0\leq i\leq n}$*, subject to the relations (for all n)*

$$\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n \quad (\text{if } i < j) \tag{1}$$

$$\sigma_j^{n+1} \circ \delta_i^{n+2} = \delta_i^{n+2} \circ \sigma_{j-1}^n \quad (\text{if } i < j) \tag{2}$$

$$\sigma_j^n \circ \delta_j^{n+1} = \mathrm{id}_{[n]} \tag{3}$$

$$\sigma_j^n \circ \delta_{j+1}^{n+1} = \mathrm{id}_{[n]} \tag{4}$$

$$\sigma_j^{n+1} \circ \delta_i^{n+2} = \delta_{i-1}^{n+1} \circ \sigma_j^n \quad (\text{if } i > j+1) \tag{5}$$

$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \quad (\text{if } i \leq j) \tag{6}$$

We will not prove this theorem in these notes, but we will attempt to explain what these identites say in the simplex category and explain what it means for a category to be presented by generators and relations. The equations (1) and (2) are a commutativity condition, they express (with index shifts appropriate to the δ's and σ's) that omitting a vertex i and then omitting/collapsing a later vertex j is the same as first omitting/collapsing $j - 1 = \delta_i^{-1}(j)$ and then omitting i. The equations (3) and (4) are perhaps the most important identities, because their categorical interpretation is that each δ is a *split monomorphism* and each σ is a *split epimorphism*; explicitly they say that if we omit a vertex and then collapse it with the next/previous vertex, it's the same as doing nothing. The equation (5) can be understood as saying "far away" omissions/collapses do not affect eachother (up to reindexing!). And finally equation (6) expresses that if you collapse twice in a row, the order of collapses matters only in that it shifts up the indexing.

The "free category" part of the theorem is more directly relevant, because it gives an explicit description of functors $\Delta \to \mathsf{C}$ for any category C (like how a presentation of a group G tells you what group homomorphisms $G \to H$ are). One interpretation of a "free structure" is exactly this kind of universal property, i.e. a free thing ("group" or "category equipped with a sequence of objects and families of maps satisfying the cosimplicial identities") is an initial object in the category of things. A free group G on generators $x_1, \dots, x_n$ subject to relations $r_1, \dots, r_m$ is an initial object in the category of tuples $(H, y_1, \dots, y_n)$ of groups H and $\mathbf{y} \in H^n$ such that for each j, interpreting x_i as y_i in w_j gives the identity element of H; a morphism $(H, \mathbf{y}) \to (H', \mathbf{z})$ in this category is of course a group homomorphism $f\ :\ H \to H'$ such that $f(y_i) = z_i$ for each i. Hence a free category on objects $\{X_s\}_{s\in S}$ and morphisms $\{f_\lambda : X_s \to X_t\}_{s,t\in S,\lambda\in\Lambda_{s,t}}$ subject to some equations of morphisms $\{E_j\}_{j\in J}$ is an initial object in the category[2] of categories that are equipped with a chosen family of objects labelled by S and a chosen family of morphisms labelled by the $\Lambda_{s,s'}$ satisfying all equations E_j. There is also a "by hand" construction of a free category on a directed graph/quiver G, e.g. the graph with vertices $\mathbb{N}$ and edges labelled by the coface/codegeneracy maps. This construction is fairly simply, if v, w are vertices in G then a morphism $v \to w$ in the free category is just a path ("formal composition of edges") from v to w in G. One can then quotient the set of arrows of this category by the smallest equivalence relation which contains the equations and "respects composition" (like how a normal subgroup gives an equivalence relation which multiplication).

We might stop and ask at this point why we need the codegeneracies at all. If we're interested in gluing together simplices along their boundaries, surely we just need the face inclusions? It turns out that the category of simplicial sets is much nicer when degeneracies; for example, the geometric realization of the semisimplicial set $\Delta^1 \times \Delta^1$ is an interval union two points, not a square! The theory without degeneracies isn't useless, though, we obtain what are called "semi-simplicial sets". These are called "Δ-complexes" in Hatcher's algebraic topology textbook.

[2]Here I mean the locally small category of small categories such that etc. But in fact an initial object of this category will have the right mapping out property with respect to locally small categories too, as any functor $F\ :\ \mathsf{C} \to \mathsf{D}$ with C factors through a small subcategory D' of D; specifically D' is the full subcategory of D on the objects in the image of F.

We now return to simplicial sets, having gained an understanding of what kind of "formal simplices" we're gluing together. The categorical understanding of "gluing" is that it is a colimit. And vice versa, in many concrete categories a colimit does performing some kind of concrete "gluing". This is all there is to the definition of a simplicial set.

Definition 6. The category of simplicial sets sSet is the free[3] cocompletion of Δ. That is to say sSet has all (small) colimits, comes equipped with a functor $Y : \Delta \to \mathsf{sSet}$, and for any other category D with all (small) colimits and functors $F : \Delta \to \mathsf{D}$ there exists a colimit preserving functor $G : \mathsf{sSet} \to \mathsf{D}$ equipped with an isomorphism $\tau : F \to G \circ Y$. Furthermore (G, τ) is unique in that if we have another colimit-preserving functor $G' : \mathsf{sSet} \to \mathsf{C}$ equipped with an isomorphism $\tau' : F \to G' \circ Y$ then there is a unique isomorphism $\zeta : G \to G'$ making the diagram

$$\begin{tikzcd} & G \circ Y \arrow[d, "\zeta Y"] \\ F \arrow[ur, "\tau'"] \arrow[r, "\tau"'] & G' \circ Y \end{tikzcd}$$

commute.

Intuitively this says that an object of sSet is a formal colimit of some diagram in Δ. One can construct a free cocompletion in this way, but I tried to write it down once and lost two weeks working out technical details. Luckily the free cocompletion of a small category is a recognizable, fairly simple, and extremely well behaved category. The rest of this section will be devoted to proving the following theorem.

Theorem 7. *Let* C *be a small category and* $\mathrm{Psh}(\mathsf{C}) = \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ *be the category of presheaves on* C*. The Yoneda embedding* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *exhibits* $\mathrm{Psh}(\mathsf{C})$ *as the free cocompletion of* C*.*

Most people would find my initial definition of sSet a little silly. The true definition is just $\mathsf{sSet} = \mathrm{Psh}(\Delta)$. Our presentation of Δ tells us that a simplicial set can also be understood as sequence of sets $\{X_n\}_{n \in \mathbb{N}}$ equipped with morphisms s_i^n, d_i^n satisfying the *simplicial identities*, the categorical dual of the cosimplicial identities (because presheaves are contravariant functors $\Delta \to \mathsf{Set}$). We will expand on this later.

One caveat with Theorem7 is that, because the presheaf category is a free construction, already existing colimits in C will almost never be preserved under y. The proof of Theorem7 boils down to the fact that any presheaf on C can be canonically written as a colimit of representable presheaves (those in the image of the Yoneda embedding). This may sound strange, but it's actually just another point of view on the celebrated Yoneda lemma, which we recall below.

Lemma 8. *Let* C *be a small category and* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *the functor* $y(x) = \mathrm{Hom}_{\mathsf{C}}(-, x)$*. For any object* x *of* C *and presheaf* S *on* C *the function* $\varphi : \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \to S(x)$ *defined by* $\varphi(\eta) = \eta_x(\mathrm{id}_x)$ *is a bijection. Furthermore,* φ *defines a natural isomorphism of functors* $\mathsf{C}^{\mathrm{op}} \times \mathrm{Psh}(\mathsf{C}) \to \mathsf{Set}$*.*

For the rest of this document we use the notation φ as in this lemma and set $\psi = \varphi^{-1}$.

Proof. We define an inverse $\psi : S(x) \to \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S)$ by $\psi(s)_z(f) = S(f)(s)$. Unwrapping this a bit, for $s \in S(x)$ we define a natural transformation $\psi(s) : y(x) \to S$ by setting its component on an object z to be the function $\mathrm{Hom}_{\mathsf{C}}(z, x) \to S(z)$ sending $f : z \to x$ to its action on s under S, i.e. $S(f)(s)$. We must verify that $\psi(s)$ is in fact natural for each s. So suppose we have a map $g : z \to w$ in C, we must check that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(w, x) \arrow[r, "g^*"] \arrow[d, "\psi(s)_w"] & \mathrm{Hom}_{\mathsf{C}}(z, x) \arrow[d, "\psi(s)_z"] \\ S(w) \arrow[r, "S(g)"] & S(z) \end{tikzcd}$$

commutes. By unravelling definitions and applying functorality of S we calculate for any $f \in \mathrm{Hom}_{\mathsf{C}}(w, x)$ that

$$\psi(s)_z(g^*(f)) = \psi(s)_z(f \circ g) = S(f \circ g)(s) = S(g)(S(f)(s)) = S(g)(\psi(s)_w(f)).$$

So ψ is natural. And for arbitrary $s \in S(x)$, $\eta \in \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S)$, $z \in \mathrm{Obj}(\mathsf{C})$ and $f \in \mathrm{Hom}_{\mathsf{C}}(z, x)$ we have

$$\varphi(\psi(s)) = \psi(s)_x(\mathrm{id}_x) = S(\mathrm{id}_x)(s) = \mathrm{id}_{S(x)}(s) = s$$

$$\psi(\varphi(\eta))_z(f) = S(f)(\varphi(\eta)) = S(f)(\eta_x(\mathrm{id}_x)) \overset{(!)}{=} \eta_z(S(f)(\mathrm{id}_x)) = \eta_z(f^*(\mathrm{id}_x)) = \eta_z(f).$$

[3] This is not quite freeness in the sense discussed above; it is about 2-initiality in an appropriate 2-category!

The equality labelled (!) holds because of the following naturality square of η:

$$\begin{tikzcd} y(x)(x) \arrow[r, "y(x)(f)"] \arrow[d, "\eta_x"] & y(x)(z) \arrow[d, "\eta_z"] \\ S(x) \arrow[r, "S(f)"] & S(z). \end{tikzcd}$$

This proves φ is a bijection. To check φ is natural it suffices to show it is natural in x for fixed S and natural in S for fixed x. Fix S and write φ_x for φ. We must show that for any morphism $a : u \to v$ in C the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(v), S) \arrow[r, "y(a)^*"] \arrow[d, "\varphi_v"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(u), S) \arrow[d, "\varphi_u"] \\ S(v) \arrow[r, "S(a)"] & S(u) \end{tikzcd}$$

commutes. This is once again just unfolding definitions and using naturality, as for any $\eta : y(v) \to S$ we calculate

$$\varphi_u(y(a)^*(\eta)) = \varphi_u(\eta \circ y(a)) = \eta_u(y(a)_u(\mathrm{id}_u)) = \eta_u(a \circ \mathrm{id}_u) = \eta_u(a)$$
$$S(a)(\varphi_v(\eta)) = S(a)(\eta_v(\mathrm{id}_v)) = \eta_u(y(v)(a)(\mathrm{id}_v)) = \eta_u(\mathrm{id}_v \circ a) = \eta_u(a).$$

Now fix x and write φ_S for S. Let $\beta : S \to T$ be an arbitrary natural transformation. The diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \arrow[r, "\beta_*"] \arrow[d, "\varphi_S"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), T) \arrow[d, "\varphi_T"] \\ S(x) \arrow[r, "\beta_x"] & T(x) \end{tikzcd}$$

commutes because for any $\eta : y(x) \to S$ we have

$$\varphi_T(\beta_*(\eta)) = \varphi_T(\beta \circ \eta) = \beta_x(\eta_x(\mathrm{id}_x)) = \beta_x(\varphi_S(\eta)).$$ □

So what does an isomorphism $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \cong S(x)$ have to do with writing S as a colimit? The key point is that naturality in the x argument means that for any map $f : z \to w$ in C and $s \in S(w)$ we have

$$\varphi_{z,S}(\psi_{w,S}(s) \circ y(f)) = \varphi_{z,S}(y(f)^*(\psi_{w,S}(s))) = S(f)(\varphi_{w,S}(\psi_{w,S}(s))) = S(f)(s).$$

Hence for any $t \in S(z)$, $s \in S(w)$ and map $f : z \to w$ satisfying $S(f)(s) = t$ we have a commutative triangle

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\psi_{z,S}(t)"'] & y(w) \arrow[d, "\psi_{w,S}(s)"] \\ & S. \end{tikzcd}$$

These triangles suggest that S is a cocone under a certain diagram with structure maps $\psi_{w,S}(s) : y(w) \to S$. An object of the indexing category must know about both w and $s \in S(w)$ and a morphism has to be constrained by $S(f)(s) = t$.

Definition 9. Let C be a small category and S a presheaf on C. Define a (small) category $\mathrm{el}(S)$[4] by

$$\mathrm{Obj}(\mathrm{el}(S)) = \{(x, s) : x \in \mathrm{Obj}(\mathsf{C}), s \in S(x)\}$$
$$\mathrm{Hom}_{\mathrm{el}(S)}((z, t), (w, s)) = \{f \in \mathrm{Hom}_{\mathsf{C}}(z, w) : S(f)(s) = t\}.$$

We set $\mathrm{id}_{(x,s)} = \mathrm{id}_x$ and perform composition as in C. The identities are well defined because $S(\mathrm{id}_x)(s) = s$. The composition laws are automatic, and this composition is well defined because if $S(f)(s) = t$ and $S(g)(t) = r$ then

$$S(f \circ g)(s) = S(g)(S(f)(s)) = S(g)(t) = r.$$

This category comes with a forgetful functor $P_S : \mathrm{el}(S) \to \mathsf{C}$. The category $\mathrm{el}(S)$ equipped with P_S is referred to as the category of elements of S (in the special case Δ it is sometimes called the category of simplices of S). It is instructive to think about what happens in the case that S is the forgetful functor of some familiar category, e.g. finite groups (or to make C small and not just essentially small, groups whose underlying set is hereditarily finite).

[4] A reader with stacky inclinations may recognize this as the grothendieck construction, specialized to presheaves of discrete groupoids (sets).

Theorem 10. *Let* C *be a small category and* S *a presheaf on* C*. The morphisms* $\psi_{x,S}(s) : y(x) \to S$ *make* S *into a colimit of the diagram* $y \circ P : \mathrm{el}(S) \to \mathrm{Psh}(\mathsf{C})$*.*

Proof. We already saw that these maps assemble into a cocone by naturality of the Yoneda lemma. Suppose we have a presheaf T on C and natural transformations $\sigma^{x,s} : y(x) \to T$ such that for any morphism $f : (z,t) \to (w,s)$ in $\mathrm{el}(S)$,

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\sigma^{z,t}"'] & y(w) \arrow[d, "\sigma^{w,s}"] \\ & T \end{tikzcd}$$

commutes. We are then required to show there is a unique natural transformation $\beta : S \to T$ making each diagram

$$\begin{tikzcd} y(x) \arrow[d, "\psi_{x,S}(s)"'] \arrow[dr, "\sigma^{x,s}"] & \\ S \arrow[r, "\beta"'] & T \end{tikzcd}$$

commute. Uniqueness is immediate, as naturality of the Yoneda lemma in the presheaf argument gives

$$\beta \circ \psi_{x,S}(s) = \beta_*(\psi_{x,S}(s)) = \psi_{x,T}(\beta_x(s))$$

and hence commutativity of the requisite triangles is equivalent to the identity $\beta_x(s) = \varphi_{x,T}(\sigma^{x,s})$. So we just need to check that the maps $\beta_x(s) = \varphi_{x,T}(\sigma^{x,s})$ assemble into a natural transformation $S \to T$. This means that the square

$$\begin{tikzcd} S(w) \arrow[r, "S(f)"] \arrow[d, "\beta_w"] & S(z) \arrow[d, "\beta_z"] \\ T(w) \arrow[r, "T(f)"] & T(z) \end{tikzcd}$$

must commute for any $f : z \to w$ in C, which in turn is true because for $s \in S(w)$, abbreviating $t = S(f)(s)$, we have

$$T(f)(\beta_w(s)) = T(f)(\varphi_{w,T}(\sigma^{w,s})) = \varphi_{z,T}(y(f)^*(\sigma^{w,s})) = \varphi_{z,T}(\sigma^{w,s} \circ y(f)) = \varphi_{z,T}(\sigma^{z,t}) = \beta_z(t) = \beta_z(S(f)(s)). \quad \square$$

With Theorem 2 we have shown that $\mathrm{Psh}(\mathsf{C})$ is generated from $y(\mathsf{C}) \simeq \mathsf{C}$ under "gluing" (colimits). We now have the tools to prove Theorem 7, which states that this method of gluing objects of C together is universal. The reader may already see how to define a colimit-preserving extension of a functor $F : \mathsf{C} \to \mathsf{D}$ using the colimit formula for presheaves: send $S = \mathrm{colim}(y \circ P)$ to $\mathrm{colim}(F \circ P)$. But in set-theoretic foundations the term $\mathrm{colim}(F \circ P)$ isn't really meaningful; "$\mathrm{colim}(F \circ P)$" is only defined up to isomorphism, not equality. We do not have a canonical choice of colimit in D, and choosing an arbitrary one simultaneously across the proper class of presheaves S requires a stronger choice axiom than is in ZFC. In univalent mathematics there is no issue, since equality and isomorphism are the same thing. In ZFC+Grothendieck universes the "class" of presheaves is only a proper class from the point of view of some ambient inaccessible cardinal κ. Our presheaves are valued in V_κ and so there is just a set of them, to which ZFC's axiom of choice applies. But this noncanonical choice is still awkward, so we opt to consider all choices without bias. In ZFC (or really NBG, so we can talk about classes) we should be able to carry this out with anafunctors (but, also, in any particular case of interest we have explicit constructions of colimits/definable choice functions so this is all moot).

Fix a small category C and a locally small category D.

Definition 11. A realization functor is a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ such that for every presheaf S on C, G preserves the colimit of the diagram $y \circ P_S$.

Lemma 12. *In sufficiently strong foundations, if* D *has all small colimits then for every functor* $F : \mathsf{C} \to \mathsf{D}$ *there exists a realization functor* $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ *such that* $F \cong G \circ y$*.*

Proof. We begin by correcting a deficiency from earlier, which was defining the category of elements as just an objectwise construction and not a functor $\mathrm{Psh}(\mathsf{C}) \to \mathsf{Cat}$. Given a morphism of presheaves $\alpha : S \to T$ we can define a functor $\mathrm{el}(\alpha) : \mathrm{el}(S) \to \mathrm{el}(T)$ by $\mathrm{el}(\alpha)(x, s) = (x, \alpha_x(s))$ on objects and $\mathrm{el}(\alpha)(f) = f$ on morphisms. This is well defined on morphisms by naturality of α, i.e. $S(f)(s) = t$ implies

$$T(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t).$$

The functor laws for $\mathrm{el}(-)$ are easily verified, as $\mathrm{el}(\mathrm{id}_S)(x, s) = (x, (\mathrm{id}_S)_x(s)) = (x, s)$ and

$$\mathrm{el}(\beta \circ \alpha)(x, s) = (x, (\beta \circ \alpha)_x(s)) = (x, \beta_x(\alpha_x(s))) = \mathrm{el}(\beta)(x, \alpha_x(s)) = \mathrm{el}(\beta)(\mathrm{el}(\alpha)(x, s)).$$

Also note that for any $\alpha : S \to T$ we have $P_T \circ \mathrm{el}(\alpha) = P_S$ (the functor $\mathrm{el}(\alpha)$ leaves the first coordinate unchanged). By assumption we may choose for each S a colimit A_S of $F \circ P_S$, with structure maps $\kappa^{S,e} : F(P_S(e)) \to A_S$. Define a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ on objects by $G(S) = A_S$. For a natural transformation $\alpha : S \to T$ the equality $P_T \circ \mathrm{el}(\alpha) = P_S$ allows us to "pull back" the $(y \circ P_T)$-cocone structure on T along $\mathrm{el}(\alpha)$ to a $(y \circ P_S)$-cocone structure, the structure maps of which are $\kappa^{T,\mathrm{el}(\alpha)(e)} : F(P_S(e)) \to A_T$ (for e an object of $\mathrm{el}(S)$). Then by one definition of a colimit (initial cocone) there exists a unique morphism $G(\alpha) : A_S \to A_T$ such that for all $e \in \mathrm{Obj}(\mathrm{el}(S))$ the diagram

$$\begin{tikzcd}
F(P_S(e)) \arrow[d, "\kappa^{S,e}"'] \arrow[dr, "\kappa^{T,\mathrm{el}(\alpha)(e)}"] & \\
A_S \arrow[r, dashed, "G(\alpha)"'] & A_T
\end{tikzcd}$$

commutes. It is easy to verify the functor laws for G using this definition and functoriality of $\mathrm{el}(-)$. So we have defined a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$. Before we show it is a realization functor we argue that $F \cong G \circ y$.

To do so, we examine the structure of $\mathrm{el}(y(x))$ for a general object x of C. The key observation is that (x, id_x) is a terminal object of the category $\mathrm{el}(y(x))$. This is because for any other object $(z, f) \in \mathrm{el}(y(x))$, f itself defines a map $(z, f) \to (x, \mathrm{id}_x)$, and any map $g : (z, f) \to (x, \mathrm{id}_x)$ must satisfy $g^*(\mathrm{id}_x) = f$, i.e. $g = f$. In general if the indexing category of a diagram has a terminal object then the image of that terminal object is itself a colimit of the diagram. In this particular case we find that the structure map $\kappa^{y(x),(x,\mathrm{id}_x)} : F(x) \to A_{y(x)}$ must be an isomorphism. Hence we can define a natural isomorphism $\tau : F \to G \circ y$ by $\tau_x = \kappa^{y(x),(x,\mathrm{id}_x)}$, as long as the square

$$\begin{tikzcd}
F(z) \arrow[r, "F(f)"] \arrow[d, "\tau_z"] & F(w) \arrow[d, "\tau_w"] \\
G(y(z)) \arrow[r, "G(y(f))"] & G(y(w))
\end{tikzcd}$$

commutes for each morphism $f : z \to w$ in C. Equivalently, $G(y(f)) = \tau_w \circ F(f) \circ \tau_z^{-1}$. By the definition of the action of G on morphisms and the equality $\mathrm{el}(y(f))(z, \mathrm{id}_z) = (z, f)$, this is equivalent to commutativity of

$$\begin{tikzcd}
F(z) \arrow[rr, "\kappa^{y(w),(z,f)}"] \arrow[d, "\kappa^{y(z),(z,\mathrm{id}_z)}"'] & & A_{y(w)} \\
A_{y(z)} \arrow[r, "\tau_z^{-1}"] & F(z) \arrow[r, "F(f)"] & F(w), \arrow[u, "\tau_w"]
\end{tikzcd}$$

which immediately reduces to the equation $\kappa^{y(z),(z,\mathrm{id}_z)} = \tau_w \circ F(f)$. But this equation is part of the cocone structure on $A_{y(w)}$, specifically the commuting triangle associated to the morphism $f : (z, f) \to (w, \mathrm{id}_w)$ in $\mathrm{el}(y(w))$.

Finally with τ in hand it is easy to show G is a realization functor. Let S be an arbitrary presheaf. We must show that the morphisms $G(\psi_{x,S}(s)) : G(y(x)) \to S$ make $G(S) = A_S$ a colimit of $G \circ y \circ P_S$. It suffices to show that this is true after transporting the cocone structure across the isomorphism $\tau P_S : F \circ P_S \to G \circ y \circ P_S$, i.e. that the morphisms $G(\psi_{x,S}(s)) \circ \tau_x : F(x) \to S$ make A_S a colimit of $F \circ P_S$. By definition of the action of G on morphisms and the equality $\mathrm{el}(\psi_{x,S}(s))(x, \mathrm{id}_x) = \psi_{x,S}(s)_x(x, \mathrm{id}_x) = s$ we have

$$G(\psi_{x,S}(s)) \circ \tau_x = G(\psi_{x,S}(s)) \circ \kappa^{y(x),(x,\mathrm{id}_x)} = \kappa^{S,(x,s)}$$

and so this $(F \circ P_S)$-cocone structure on A_S is originally chosen one, which we know is colimiting. $\square$

So in order to show Psh(C) is the free cocompletion of C it suffices to show that for any $F : \mathsf{C} \to \mathsf{D}$ with D cocomplete, realization functors extending F are unique up to a unique isomorphism and that they preserve all colimits. We accomplish both by recasting the notion of a realization functor in terms of adjointness.

Definition 13. For any $F : \mathsf{C} \to \mathsf{D}$ the nerve of F is the functor $N(F) : \mathsf{D} \to \mathrm{Psh}(\mathsf{C})$ defined by

$$N(F)(d)(x) = \mathrm{Hom}_{\mathsf{D}}(F(x), d).$$

Lemma 14. *For any $F : \mathsf{C} \to \mathsf{D}$, realization functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$, and isomorphism $\tau : F \to G \circ y$ there is an adjunction $G \dashv N(F)$ with unit map $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G$ satisfying, for each presheaf S and $(x, s) \in \mathrm{el}(S)$,*

$$(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x.$$

Proof. For each presheaf S on C define $\eta_S : S \to N(F)(G(S))$ by $(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x$. These assemble into a natural transformation, i.e. for any morphism $f : z \to w$ in C we have a commuting square

$$\begin{tikzcd}
S(w) \arrow[r, "S(f)"] \arrow[d, "(\eta_S)_w"] & S(z) \arrow[d, "(\eta_S)_z"] \\
\mathrm{Hom}_{\mathsf{D}}(F(w), G(S)) \arrow[r, "F(f)^*"] & \mathrm{Hom}_{\mathsf{D}}(F(z), G(S)).
\end{tikzcd}$$

To prove this square commutes, let $s \in S(w)$ be arbitrary and define $t = S(f)(s)$. By naturality of τ,

$$\begin{aligned}
F(f)^*((\eta_S)_w(s)) &= F(f)^*(G(\psi_{w,S}(s)) \circ \tau_w) \\
&= G(\psi_{w,S}(s)) \circ \tau_w \circ F(f) \\
&= G(\psi_{w,S}(s)) \circ G(y(f)) \circ \tau_z \\
&= G(\psi_{w,S}(s) \circ y(f)) \circ \tau_z \\
&= G(\psi_{z,S}(t)) \circ \tau_z \\
&= (\eta_S)_z(t) \\
&= (\eta_S)_z(S(f)(s)).
\end{aligned}$$

The reader might expect us to now verify η_S is natural in S and write down a counit, but if we use the "universal arrow" characterization of adjunctions this is unnecessary. What we do need to do to obtain an adjunction with (necessarily natural) unit $S \mapsto \eta_S$ is argue that for any object d of D and $\alpha : S \to N(F)(d)$ there exists a unique morphism $\beta : G(S) \to d$ in D such that $\alpha = N(F)(f) \circ \eta_S$. By calculating

$$(N(F)(\beta) \circ \eta_S)_x(s) = N(F)(\beta)_x((\eta_S)_x(s)) = \beta \circ (\eta_S)_x(s) = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$$

we find that a morphism $\beta : G(S) \to d$ satisfies $\alpha = N(F)(\beta) \circ \eta_S$ iff $\alpha_x(s) = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$ for every object (x, s) of $\mathrm{el}(S)$. Then we may push forward the cocone structure on S along α to get a $(y \circ P_S)$-cocone structure on $N(F)(d)$, with structure maps $\alpha \circ \psi_{x,S}(s) : y(x) \to N(F)(d)$. Let $a^{x,s} = \varphi_{x,N(F)(d)}(\alpha \circ \psi_{x,S}(s))$, i.e. $a^{x,s} = \alpha_x(s)$. Then $a^{x,s} \in N(F)(d)(x)$, meaning $a^{x,s}$ is a morphism $F(x) \to d$ in D. In fact these morphisms make d into a cocone under $F \circ P_S$, i.e. for any map $f : (z, t) \to (w, s)$ in $\mathrm{el}(S)$ the diagram

$$\begin{tikzcd}
F(z) \arrow[r, "F(f)"] \arrow[dr, "a^{z,t}"'] & F(w) \arrow[d, "a^{w,s}"] \\
 & d
\end{tikzcd}$$

commutes. This is by naturality of α and the equation $S(f)(s) = t$ (baked into the definition of a morphism $\mathrm{el}(S)$), as

$$a^{w,s} \circ F(f) = F(f)^*(a^{w,s}) = N(F)(d)(f)(a^{w,s}) = N(F)(d)(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t) = a^{z,t}.$$

We may transport this to the structure of a cocone under $G \circ y \circ P_S$ on d, with structure maps $a^{x,s} \circ \tau_x^{-1}$. But because G is a realization functor, $G(S)$ is a colimit of $G \circ y \circ P_s$ with structure maps $G(\psi_{x,S}(s))$. Thus there is a unqiue $\beta : G(S) \to d$ such that $a^{x,s} \circ \tau_x^{-1} = \beta \circ G(\psi_{x,S}(s))$, equivalently $a^{x,s} = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$, as desired. ☐

Corollary 15. *If $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ is a realization functor then there is an adjunction $G \dashv N(G \circ y)$ with unit map $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(G \circ y) \circ G$ satisfying $(\eta_S)_x(s) = G(\psi_{x,S}(s))$. In particular realization functors preserve all colimits.*

Proof. Apply Lemma 14 with $\tau = \mathrm{id}_{G \circ y}$. ☐

A funny implication of Corollary 15 is that there is no difference between "realization functors" $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$, colimit preserving functors $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$, and left adjoints $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$. With this we finally prove Theorem 7.

Proof. Let D be a cocomplete category and $F : \mathsf{C} \to \mathsf{D}$ any functor. Existence of a colimit-preserving functor $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ extending F is immediate from Lemmas 12 and our observation that any realization functor preserves all colimits. Suppose $G, G' : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ preserve all colimits and we have $\tau : F \cong G \circ y, \tau' : F \cong G' \circ y$. Then by Lemma 14 we have adjunctions $G \dashv N(F)$ and $G' \dashv N(F)$ with unit maps $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G$ and $\eta' : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G'$ such that, for each presheaf S and $(x, s) \in \mathrm{el}(S)$,

$$(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x$$
$$(\eta'_S)_x(s) = G'(\psi_{x,S}(s)) \circ \tau'_x.$$

By uniqueness of left adjoints there is a unique isomorphism $\zeta : G \to G'$ such that $\eta' = N(F)\zeta \circ \eta$. For any presheaf S and $(x, s) \in \mathrm{el}(S)$ we may calculate

$$((N(F)\zeta \circ \eta)_S)_x(s) = ((N(F)\zeta)_S)_x((\eta_S)_x(s)) = N(F)(\zeta_S)_x((\eta_S)_x(s)) = \zeta_S \circ (\eta_S)_x(s) = \zeta_S \circ G(\psi_{x,S}(s)) \circ \tau_x.$$

So an isomorphism $\zeta : G \to G'$ satisfies $\eta' = N(F)\zeta \circ \eta$ iff $G'(\psi_{x,S}(s)) \circ \tau'_x = \zeta_S \circ G(\psi_{x,S}(s)) \circ \tau_x$ for each presheaf S and $(x, s) \in \mathrm{el}(S)$. Additionally $\zeta_S \circ G(\psi_{x,S}(s)) = G'(\psi_{x,S}(s)) \circ \zeta_{y(x)}$, so ζ is the unique isomorphism satisfying

$$G'(\psi_{x,S}(s)) \circ \tau'_x = G'(\psi_{x,S}(s)) \circ \zeta_{y(x)} \circ \tau_x$$

for every S and $(x, s) \in \mathrm{el}(S)$. Taking $S = y(x)$ and $s = \mathrm{id}_x$ this equation simplifies to $\tau'_x = \zeta_{y(x)} \circ \tau_x$, and this identity clearly implies the general case. Thus there exists a unique isomorphism $\zeta : G \to G'$ such that $\tau' = \zeta y \circ \tau$. $\square$