Simplicial Sets

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1 CW Complexes

The objects of study in classical homotopy theory are the *homotopy types*. This is not the same thing as a topological space, or even a CW complex, but "CW complex up to homotopy". CW complexes are spaces that admit a construction in stages, starting with some points, then gluing on intervals via their boundary, then gluing on disks via their boundary, and so on, then taking the union of all finite stages. In stage n the "gluing" of n-disks onto the $(n-1)$-skeleton X_{n-1} can be understood categorically as taking a pushout of X_{n-1} with your family of disks $\coprod_{\lambda\in\Lambda} D^n$ along a family of arbitrary continuous maps $\{f_\lambda : S^n \to X\}_{\lambda\in\Lambda}$ ("attaching maps") and standard inclusions $S^n \hookrightarrow D^n$. We could just have easily defined this using (topological) simplex inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, for Δ^n and D^n are convex bodies of the same dimension and so canonically (after picking a basepoint) homeomorphic. So CW complexes are *exactly* the topological spaces that can be obtained from a sequential colimit of pushouts of (coproducts of) the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$. In other words, they're spaces obtained by gluing simplices together with the restriction that one may only glue along the boundary, but the flexibility that arbitrary continuous gluings of that boundary are allowed. But combining the "Simplicial Approximation Theorem" with the following lemma allows us to assume a CW complex is obtained from a very, very structured kind of gluing.

Lemma 1.1. *Let X be a topological space and $f, g : S^{n-1} \to X$ two homotopic maps. Then the pushouts (or "amalgamation spaces") $D^n \amalg_f X$ and $D^n \amalg_g X$ are homotopy equivalent.*

Proof. Let $H : S^{n-1} \times I \to X$ be a homotopy. The key idea is that we may use the deformation retraction of the "cylinder" $D^n \times I$ onto its boundary minus the top $(D^n \times \{0\}) \cup (S^{n-1} \times I)$ to get a deformation retraction of $(D^n \times I) \amalg_H X$ onto $((D^n \times \{0\}) \cup (S^{n-1} \times I)) \amalg_H X$. We have a morphism $J : (D^n \times I) \amalg_H X \to ((D^n \times \{0\}) \cup (S^{n-1} \times I)) \amalg_H X$ induced by the morphism of spans

$$\begin{tikzcd} D^n \arrow[d, hook] & & S^{n-1} \arrow[ll, hook'] \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, equal] \\ (D^n \times \{0\}) \cup (S^{n-1} \times I) & & S^{n-1} \times I \arrow[ll, hook'] \arrow[r, "H"] & X. \end{tikzcd}$$

And in fact J is surjective, because every point in the extra bit $S^{n-1} \times (0, 1]$ is glued onto X by H. But it's actually a split monomorphism as well, because morphism of spans above has a left inverse

$$\begin{tikzcd} D^n & & S^{n-1} \arrow[ll, hook'] \arrow[r, "f"] & X \\ (D^n \times \{0\}) \cup (S^{n-1} \times I) \arrow[u, two heads] & & S^{n-1} \times I \arrow[ll, hook'] \arrow[r, "H"] \arrow[u, two heads] & X. \arrow[u, equal] \end{tikzcd}$$

This means J is actually a homeomorphism, because it is a surjection with a continuous left inverse. The punchline is that $D^n \amalg_f X$, and by symmetry $D^n \amalg_g X$, are both homeomorphic to deformation retracts of the same space (and hence are homotopy equivalent). ☐

Exercise: Reprove Lemma 1.1 in terms of the simplicial inclusions, using the fact that Δ^n deformation retracts onto any of its "horns" Λ^n_i (those spaces formed by removing the ith face from $\partial\Delta^n$).

2 The simplex category, gluing, and presheaves

Simplicial sets are a more "algebraic" or "combinatorial" way of modelling homotopy types. This has the advantage that it transports more easily to algebraic contexts. E.g., the (1-)category of topological abelian groups is not abelian but the (1-)category of simplicial abelian groups is! We saw above through careful analysis of CW complexes that any homotopy type is built up from gluing together simplices along their boundaries. For CW complexes the gluing was fairly geometric, an actual pushout in the category of topological spaces. Simplicial sets take the opposite approach: they are formal gluings of (formal!) simplices. Before we can define simplicial sets we must discuss the (category of) simplices from which they are glued.

Definition 2.1. The simplex category Δ has objects the finite nonempty ordinals $[n] = \{0, 1, \ldots, n\}$ and a morphism $[n] \to [m]$ is simply an order preserving function. The augmented simplex category Δ_a is defined in the same way, but the empty ordinal $[-1] = \varnothing$ is included.

Note that Δ is equivalent to the category of all finite totally ordered sets. What does this have to do with actual geometric simplices? The object $[n]$ should be understood as a representation of the geometric n-simplex Δ^n, and its elements $0, \ldots, n$ representing the $(n+1)$-vertices of that simplex. As demonstrated by simplicial or singular homology, it's often more convenient to work with simplices that have a chosen order on their vertices (for manageably and consistently tracking orientation); this is why we're looking at ordered finite sets and not just finite sets[1]. The geometric simplex Δ^n is the convex hull of its vertices $e_0, \ldots, e_n$, and this means that every function of finite sets $\{e_0, \ldots, e_n\} \mapsto \{e_0, \ldots, e_m\}$ has a unique extension to an affine transformation $\Delta^n \to \Delta^m$ sending vertices to vertices. Thus Δ could just as truthfully be described as the category of geometric simplices $\Delta^n \subseteq \mathbb{R}^{n+1}$ with morphisms the affine transformations sending vertices to vertices and preserving the standard order on those vertices.

Definition 2.2. Let

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}$$

be the n-dimensional "geometric simplex". The vertices of Δ^n are the standard basis vectors $e_0, \ldots, e_n$ of $\mathbb{R}^{n+1}$ and any point in Δ^n can be uniquely represented as a convex combination $t_0 e_0 + \ldots + t_n e_n$ of them. Given an order-preserving map $f : [n] \to [m]$ there is an induced continuous map $\widetilde{f} : \Delta^n \to \Delta^m$ defined by

$$\widetilde{f}\left(\sum_{i=0}^{n} t_i e_i \right) = \sum_{i=0}^{n} t_i e_{f(i)}.$$

Exercise: The assignments $[n] \mapsto \Delta^n$ and $f \mapsto \widetilde{f}$ define a faithful functor $\Delta \to \mathsf{Top}$.

There are two important families of maps within Δ, the coface and codegeneracy maps.

Definition 2.3. Let n be a positive integer. For $0 \leq i \leq n$ denote by $\delta_i^n : [n-1] \to [n]$ the unique monotone injection which omits i from its range. This is the ith coface map. Concretely,

$$\delta_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

Also define $\sigma_i^n : [n+1] \to [n]$ to be the unique monotone surjection with $\sigma_i^n(i) = \sigma_i^n(i+1)$. This is the ith codegeneracy map. Concretely,

$$\sigma_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

Geometrically, δ_i^n is the inclusion of the ith face of Δ^n (meaning the face opposite the ith vertex) and σ_n^i is the projection of Δ^{n+1} onto Δ^n where we collapse the edge $[e_i\ e_{i+1}]$ down to a point.

[1] But for those who are interested, there is a theory of unoriented "symmetric simplicial sets"

Any monotone map $f : [n] \to [m]$ has a decomposition into a surjection $[n] \twoheadrightarrow [k]$ and an injecton $[k] \hookrightarrow [m]$; this may be easiest to see if we think of $[k]$ as the image f with the order inherited from $[m]$ (passing to the category of all finite nonempty totally ordered sets). Furthermore the injection $[k] \hookrightarrow [m]$ can be decomposed into a composition of coface maps, omitting elements of $[m]$ one at a time, and the surjection $[n] \twoheadrightarrow [k]$ may be decomposed into a composition of codegeneracy maps, squishing together elements $i, i+1$ such that $f(i) = f(i+1)$ one at a time until none remain. This tells us that every morphism in Δ is a composition of coface and codegeneracy maps. In fact there is a normal form associated to this decomposition, obtained by repeatedly applying the "cosimplicial identites".

Theorem 2.4. *The simplex category Δ is the free category C on a sequence of objects $[0], [1], \ldots$ and families of morphisms $\{\delta_i^n \in \mathrm{Hom}_{\mathsf{C}}(n-1, n)\}_{n\geq 1, 0\leq i\leq n}$ and $\{\sigma_i^n \in \mathrm{Hom}_{\mathsf{C}}(n+1, n)\}_{n\geq 0, 0\leq i\leq n}$, subject to the relations (for all n)*

$$
\begin{aligned}
\delta_j^{n+1} \circ \delta_i^n &= \delta_i^{n+1} \circ \delta_{j-1}^n \quad (\textit{if } i < j) && (1)\\
\sigma_j^{n+1} \circ \delta_i^{n+2} &= \delta_i^{n+1} \circ \sigma_{j-1}^n \quad (\textit{if } i < j) && (2)\\
\sigma_j^n \circ \delta_j^{n+1} &= \mathrm{id}_{[n]} && (3)\\
\sigma_j^n \circ \delta_{j+1}^{n+1} &= \mathrm{id}_{[n]} && (4)\\
\sigma_j^{n+1} \circ \delta_i^{n+2} &= \delta_{i-1}^{n+1} \circ \sigma_j^n \quad (\textit{if } i > j+1) && (5)\\
\sigma_j^n \circ \sigma_i^{n+1} &= \sigma_i^n \circ \sigma_{j+1}^{n+1} \quad (\textit{if } i \leq j). && (6)
\end{aligned}
$$

We will not prove this theorem in these notes, but we will attempt to explain what these identites say in the simplex category and explain what it means for a category to be presented by generators and relations. The equations (1) and (2) are a commutativity condition, they express (with index shifts appropriate to the δ's and σ's) that omitting a vertex i and then omitting/collapsing a later vertex j is the same as first omitting/collapsing $j - 1 = \delta_i^{-1}(j)$ and then omitting i. The equations (3) and (4) are perhaps the most important identities, because their categorical interpretation is that each δ is a *split monomorphism* and each σ is a *split epimorphism*; explicitly they say that if we omit a vertex and then collapse it with the next/previous vertex, it's the same as doing nothing. The equation (5) can be understood as saying "far away" omissions/collapses do not affect eachother (up to reindexing!). And finally equation (6) expresses that if you collapse twice in a row, the order of collapses matters only in that it shifts up the indexing.

The "free category" part of the theorem is more directly relevant, because it gives an explicit description of functors $\Delta \to \mathsf{C}$ for any category C (like how a presentation of a group G tells you what group homomorphisms $G \to H$ are). One interpretation of a "free structure" is exactly this kind of universal property, i.e. a free thing ("group" or "category equipped with a sequence of objects and families of maps satisfying the cosimplicial identities") is an initial object in the category of things. A free group G on generators $x_1, \ldots, x_n$ subject to relations $r_1, \ldots, r_m$ is an initial object in the category of tuples $(H, y_1, \ldots, y_n)$ of groups H and $\mathbf{y} \in H^n$ such that for each j, interpreting x_i as y_i in w_j gives the identity element of H; a morphism $(H, \mathbf{y}) \to (H', \mathbf{z})$ in this category is of course a group homomorphism $f : H \to H'$ such that $f(y_i) = z_i$ for each i. Hence a free category on objects $\{X_s\}_{s\in S}$ and morphisms $\{f_\lambda : X_s \to X_t\}_{s,t\in S, \lambda\in\Lambda_{s,t}}$ subject to some equations of morphisms $\{E_j\}_{j\in J}$ is an initial object in the category[2] of categories that are equipped with a chosen family of objects labelled by S and a chosen family of morphisms labelled by the $\Lambda_{s,s'}$ satisfying all equations E_j. There is also a "by hand" construction of a free category on a directed graph/quiver G, e.g. the graph with vertices $\mathbb{N}$ and edges labelled by the coface/codegeneracy maps. This construction is fairly simply, if v, w are vertices in G then a morphism $v \to w$ in the free category is just a path ("formal composition of edges") from v to w in G. One can then quotient the set of arrows of this category by the smallest equivalence relation which contains the equations and "respects composition" (like how a normal subgroup gives an equivalence relation which multiplication).

We might stop and ask at this point why we need the codegeneracies at all. If we're interested in gluing together simplices along their boundaries, surely we just need the face inclusions? It turns out that the category of simplicial sets is much nicer when degeneracies; for example, the geometric realization of the semisimplicial set $\Delta^1 \times \Delta^1$ is an interval union two points, not a square! The theory without degeneracies isn't useless, though, we obtain what are called "semi-simplicial sets". These are called "Δ-complexes" in Hatcher's algebraic topology textbook.

We now return to simplicial sets, having gained an understanding of what kind of "formal simplices" we're gluing together. The categorical understanding of "gluing" is that it is a colimit. And vice versa, in many concrete categories a colimit does performing some kind of concrete "gluing". This is all there is to the definition of a simplicial set.

[2]Here I mean the locally small category of small categories such that etc. But in fact an initial object of this category will have the right mapping out property with respect to locally small categories too, as any functor $F : \mathsf{C} \to \mathsf{D}$ with C small factors through a small subcategory D' of D; specifically D' is the full subcategory of D on the objects in the image of F.

Definition 2.5. The category of simplicial sets $s\mathsf{Set}$ is the free[3] cocompletion of Δ. That is to say $s\mathsf{Set}$ has all (small) colimits, comes equipped with a functor $Y : \Delta \to s\mathsf{Set}$, and for any other category D with all (small) colimits and functors $F : \Delta \to \mathsf{D}$ there exists a colimit preserving functor $G : s\mathsf{Set} \to \mathsf{D}$ equipped with an isomorphism $\tau : F \to G \circ Y$. Furthermore (G, τ) is unique in that if we have another colimit-preserving functor $G' : s\mathsf{Set} \to \mathsf{C}$ equipped with an isomorphism $\tau' : F \to G' \circ Y$ then there is a unique isomorphism $\zeta : G \to G'$ making the diagram

$$\begin{tikzcd} & G \circ Y \arrow[d, "\zeta Y"] \\ F \arrow[ur, "\tau'"] \arrow[r, "\tau"] & G' \circ Y \end{tikzcd}$$

commute.

Intuitively this says that an object of $s\mathsf{Set}$ is a formal colimit of some diagram in Δ. One can construct a free cocompletion in this way, but I tried to write it down once and lost two weeks working out technical details. Luckily the free cocompletion of a small category is a recognizable, fairly simple, and extremely well behaved category. The rest of this section will be devoted to proving the following theorem.

Theorem 2.6. *Let* C *be a small category and* $\mathrm{Psh}(\mathsf{C}) = \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ *be the category of presheaves on* C*. The Yoneda embedding* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *exhibits* $\mathrm{Psh}(\mathsf{C})$ *as the free cocompletion of* C*.*

Most people would find my initial definition of $s\mathsf{Set}$ a little silly. The true definition is just $s\mathsf{Set} = \mathrm{Psh}(\Delta)$. Our presentation of Δ tells us that a simplicial set can also be understood as sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ equipped with morphisms s_i^n, d_i^n satisfying the *simplicial identities*, the categorical dual of the cosimplicial identities (because presheaves are contravariant functors $\Delta \to \mathsf{Set}$). We will expand on this later.

One caveat with Theorem 2.6 is that, because the presheaf category is a free construction, already existing colimits in C will almost never be preserved under y. The proof of Theorem 2.6 boils down to the fact that any presheaf on C can be canonically written as a colimit of representable presheaves (those in the image of the Yoneda embedding). This may sound strange, but it's actually just another point of view on the celebrated Yoneda lemma, which we recall below.

Lemma 2.7. *Let* C *be a small category and* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *the functor* $y(x) = \mathrm{Hom}_{\mathsf{C}}(-, x)$*. For any object* x *of* C *and presheaf* S *on* C *the function* $\varphi : \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \to S(x)$ *defined by* $\varphi(\eta) = \eta_x(\mathrm{id}_x)$ *is a bijection. Furthermore,* φ *defines a natural isomorphism of functors* $\mathsf{C}^{\mathrm{op}} \times \mathrm{Psh}(\mathsf{C}) \to \mathsf{Set}$*.*

For the rest of this section we use the notation φ as in this lemma and set $\psi = \varphi^{-1}$.

Proof. We define an inverse $\psi : S(x) \to \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S)$ by $\psi(s)_z(f) = S(f)(s)$. Unwrapping this a bit, for $s \in S(x)$ we define a natural transformation $\psi(s) : y(x) \to S$ by setting its component on an object z to be the function $\mathrm{Hom}_{\mathsf{C}}(z, x) \to S(z)$ sending $f : z \to x$ to its action on s under S, i.e. $S(f)(s)$. We must verify that $\psi(s)$ is in fact natural for each s. So suppose we have a map $g : z \to w$ in C, we must check that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(w, x) \arrow[r, "g^*"] \arrow[d, "\psi(s)_w"] & \mathrm{Hom}_{\mathsf{C}}(z, x) \arrow[d, "\psi(s)_z"] \\ S(w) \arrow[r, "S(g)"] & S(z) \end{tikzcd}$$

commutes. By unravelling definitions and applying functoriality of S we calculate for any $f \in \mathrm{Hom}_{\mathsf{C}}(w, x)$ that

$$\psi(s)_z(g^*(f)) = \psi(s)_z(f \circ g) = S(f \circ g)(s) = S(g)(S(f)(s)) = S(g)(\psi(s)_w(f)).$$

So $\psi(s)$ is natural. And for arbitrary $s \in S(x)$, $\eta \in \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S)$, $z \in \mathrm{Obj}(\mathsf{C})$ and $f \in \mathrm{Hom}_{\mathsf{C}}(z, x)$ we have

$$\varphi(\psi(s)) = \psi(s)_x(\mathrm{id}_x) = S(\mathrm{id}_x)(s) = \mathrm{id}_{S(x)}(s) = s$$
$$\psi(\varphi(\eta))_z(f) = S(f)(\varphi(\eta)) = S(f)(\eta_x(\mathrm{id}_x)) \overset{(!)}{=} \eta_z(y(x)(f)(\mathrm{id}_x)) = \eta_z(f^*(\mathrm{id}_x)) = \eta_z(f).$$

[3]This is not quite freeness in the sense discussed above; it is about 2-initiality in an appropriate 2-category!

The equality labelled (!) holds because of the following naturality square of η:

$$\begin{tikzcd} y(x)(x) \arrow[r, "y(x)(f)"] \arrow[d, "\eta_x"] & y(x)(z) \arrow[d, "\eta_z"] \\ S(x) \arrow[r, "S(f)"] & S(z). \end{tikzcd}$$

This proves φ is a bijection. To check φ is natural it suffices to show it is natural in x for fixed S and natural in S for fixed x. Fix S and write φ_x for φ. We must show that for any morphism $a : u \to v$ in C the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Psh(C)}}(y(v), S) \arrow[r, "y(a)^*"] \arrow[d, "\varphi_v"] & \mathrm{Hom}_{\mathsf{Psh(C)}}(y(u), S) \arrow[d, "\varphi_u"] \\ S(v) \arrow[r, "S(a)"] & S(u) \end{tikzcd}$$

commutes. This is once again just unfolding definitions and using naturality, as for any $\eta : y(v) \to S$ we calculate

$$\varphi_u(y(a)^*(\eta)) = \varphi_u(\eta \circ y(a)) = \eta_u(y(a)_u(\mathrm{id}_u)) = \eta_u(a \circ \mathrm{id}_u) = \eta_u(a)$$
$$S(a)(\varphi_v(\eta)) = S(a)(\eta_v(\mathrm{id}_v)) = \eta_u(y(v)(a)(\mathrm{id}_v)) = \eta_u(\mathrm{id}_v \circ a) = \eta_u(a).$$

Now fix x and write φ_S for S. Let $\beta : S \to T$ be an arbitrary natural transformation. The diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Psh(C)}}(y(x), S) \arrow[r, "\beta_*"] \arrow[d, "\varphi_S"] & \mathrm{Hom}_{\mathsf{Psh(C)}}(y(x), T) \arrow[d, "\varphi_T"] \\ S(x) \arrow[r, "\beta_x"] & T(x) \end{tikzcd}$$

commutes because for any $\eta : y(x) \to S$ we have

$$\varphi_T(\beta_*(\eta)) = \varphi_T(\beta \circ \eta) = \beta_x(\eta_x(\mathrm{id}_x)) = \beta_x(\varphi_S(\eta)). \qquad \square$$

So what does an isomorphism $\mathrm{Hom}_{\mathsf{Psh(C)}}(y(x), S) \cong S(x)$ have to do with writing S as a colimit? The key point is that naturality in the x argument means that for any map $f : z \to w$ in C and $s \in S(w)$ we have

$$\varphi_{z,S}(\psi_{w,S}(s) \circ y(f)) = \varphi_{z,S}(y(f)^*(\psi_{w,S}(s))) = S(f)(\varphi_{w,S}(\psi_{w,S}(s))) = S(f)(s).$$

Hence for any $t \in S(z)$, $s \in S(w)$ and map $f : z \to w$ satisfying $S(f)(s) = t$ there is a commutative triangle

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\psi_{z,S}(t)"'] & y(w) \arrow[d, "\psi_{w,S}(s)"] \\ & S. \end{tikzcd}$$

These triangles suggest that S is a cocone under a certain diagram with structure maps $\psi_{w,S}(s) : y(w) \to S$. An object of the indexing category must know about both w and $s \in S(w)$ and a morphism has to be constrained by $S(f)(s) = t$.

Definition 2.8. Let C be a small category and S a presheaf on C. Define a (small) category $\mathrm{el}(S)$[4] by

$$\mathrm{Obj}(\mathrm{el}(S)) = \{(x, s) : x \in \mathrm{Obj}(\mathsf{C}), s \in S(x)\}$$
$$\mathrm{Hom}_{\mathrm{el}(S)}((z, t), (w, s)) = \{f \in \mathrm{Hom}_{\mathsf{C}}(z, w) : S(f)(s) = t\}.$$

We set $\mathrm{id}_{(x,s)} = \mathrm{id}_x$ and perform composition as in C. The identities are well defined because $S(\mathrm{id}_x)(s) = s$. The composition laws are automatic, and this composition is well defined because if $S(f)(s) = t$ and $S(g)(t) = r$ then

$$S(f \circ g)(s) = S(g)(S(f)(s)) = S(g)(t) = r.$$

This category comes with a forgetful functor $P_S : \mathrm{el}(S) \to \mathsf{C}$. The category $\mathrm{el}(S)$ equipped with P_S is referred to as the category of elements of S (in the special case $\mathsf{C} = \Delta$ it is sometimes called the category of simplices of S). It is instructive to think about what happens in the case that S is the forgetful functor of some familiar category, e.g. finite groups (or to make C small and not just essentially small, groups whose underlying set is hereditarily finite).

[4] A reader with stacky inclinations may recognize this as the grothendieck construction, specialized to presheaves of discrete groupoids (sets).

Theorem 2.9. *Let* C *be a small category and* S *a presheaf on* C*. The morphisms* $\psi_{x,S}(s) : y(x) \to S$ *make* S *into a colimit of the diagram* $y \circ P : \mathrm{el}(S) \to \mathrm{Psh}(\mathsf{C})$.

Proof. We already saw that these maps assemble into a cocone by naturality of the Yoneda lemma. Suppose we have a presheaf T on C and natural transformations $\sigma^{x,s} : y(x) \to T$ such that for any morphism $f : (z,t) \to (w,s)$ in $\mathrm{el}(S)$,

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\sigma^{z,t}"'] & y(w) \arrow[d, "\sigma^{w,s}"] \\ & T \end{tikzcd}$$

commutes. We are then required to show there is a unique natural transformation $\beta : S \to T$ making each diagram

$$\begin{tikzcd} y(x) \arrow[d, "\psi_{x,S}(s)"'] \arrow[dr, "\sigma^{x,s}"] & \\ S \arrow[r, "\beta"'] & T \end{tikzcd}$$

commute. Uniqueness is immediate, as naturality of the Yoneda lemma in the presheaf argument gives

$$\beta \circ \psi_{x,S}(s) = \beta_*(\psi_{x,S}(s)) = \psi_{x,T}(\beta_x(s))$$

and hence commutativity of the requisite triangles is equivalent to the identity $\beta_x(s) = \varphi_{x,T}(\sigma^{x,s})$. So we just need to check that the maps $\beta_x(s) = \varphi_{x,T}(\sigma^{x,s})$ assemble into a natural transformation $S \to T$. This means that the square

$$\begin{tikzcd} S(w) \arrow[r, "S(f)"] \arrow[d, "\beta_w"] & S(z) \arrow[d, "\beta_z"] \\ T(w) \arrow[r, "T(f)"] & T(z) \end{tikzcd}$$

must commute for any $f : z \to w$ in C, which in turn is true because for $s \in S(w)$, abbreviating $t = S(f)(s)$, we have

$$T(f)(\beta_w(s)) = T(f)(\varphi_{w,T}(\sigma^{w,s})) = \varphi_{z,T}(y(f)^*(\sigma^{w,s})) = \varphi_{z,T}(\sigma^{w,s} \circ y(f)) = \varphi_{z,T}(\sigma^{z,t}) = \beta_z(t) = \beta_z(S(f)(s)). \qquad \square$$

With Theorem 2.9 we have shown that Psh(C) is generated from $y(\mathsf{C}) \simeq \mathsf{C}$ under "gluing" (colimits). We now have the tools to prove Theorem 2.6, which states that this method of gluing objects of C together is universal. The reader may already see how to define a colimit-preserving extension of a functor $F : \mathsf{C} \to \mathsf{D}$ using the colimit formula for presheaves: send $S = \mathrm{colim}(y \circ P)$ to $\mathrm{colim}(F \circ P)$. But in set-theoretic foundations the term $\mathrm{colim}(F \circ P)$ isn't really meaningful; "$\mathrm{colim}(F \circ P)$" is only defined up to isomorphism, not equality. We do not have a canonical choice of colimit in D, and choosing an arbitrary one simultaneously across the proper class of presheaves S requires a stronger choice axiom than is in ZFC. In univalent mathematics there is no issue, since equality and isomorphism are the same thing. In ZFC+Grothendieck universes the "class" of presheaves is only a proper class from the point of view of some ambient inacessible cardinal κ. Our presheaves are valued in V_κ and so there is just a set of them, to which ZFC's axiom of choice applies. But this noncanonical choice is still awkward, so we opt to consider all choices without bias. However also in ZFC (or really NBG, so we can talk about classes) we can carry this argument out as long as the target category D has "explicitly defined" or "distinguished" colimits, hich happens in all situations we will see in these notes.

Fix a small category C, a locally small category D, and a functor $F : \mathsf{C} \to \mathsf{D}$.

Definition 2.10. A realization functor is a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ such that for every presheaf S on C, G preserves the colimit of the diagram $y \circ P_S$. Non-rigorously, G is a realization functor if for every S it satisfies the equation

$$G(S) = G\left(\operatorname*{colim}_{e \in \mathrm{el}(S)} y(P_S(e))\right) = \operatorname*{colim}_{e \in \mathrm{el}(S)} G(y(P_S(e))).$$

Say that G extends F if $G \circ y$ is isomorphic to F. In this case G must satisfy the (non-rigorous) identity

$$G(S) = \operatorname*{colim}_{e \in \mathrm{el}(S)} F(P_S(e)).$$

Lemma 2.11. *In sufficiently strong foundations, if* D *has all small colimits then for every functor* $F : \mathsf{C} \to \mathsf{D}$ *there exists a realization functor* $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ *which extends* F.

Proof. We begin by correcting a deficiency from earlier, which was defining the category of elements as a *function* $\mathrm{Obj}(\mathrm{Psh}(\mathsf{C})) \to \mathrm{Obj}(\mathsf{Cat})$ instead of a *functor* $\mathrm{Psh}(\mathsf{C}) \to \mathsf{Cat}$. For a morphism of presheaves $\alpha : S \to T$ define $\mathrm{el}(\alpha) : \mathrm{el}(S) \to \mathrm{el}(T)$ by $\mathrm{el}(\alpha)(x, s) = (x, \alpha_x(s))$ on objects and $\mathrm{el}(\alpha)(f) = f$ on morphisms. This is well defined as

$$T(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t).$$

for any morphism $f : (z, t) \to (w, s)$. The functor laws for $\mathrm{el}(-)$ hold since $\mathrm{el}(\mathrm{id}_S)(x, s) = (x, (\mathrm{id}_S)_x(s)) = (x, s)$ and

$$\mathrm{el}(\beta \circ \alpha)(x, s) = (x, (\beta \circ \alpha)_x(s)) = (x, \beta_x(\alpha_x(s))) = \mathrm{el}(\beta)(x, \alpha_x(s)) = \mathrm{el}(\beta)(\mathrm{el}(\alpha)(x, s)).$$

Also note that for any $\alpha : S \to T$ we have $P_T \circ \mathrm{el}(\alpha) = P_S$ (the functor $\mathrm{el}(\alpha)$ leaves the first coordinate unchanged). By assumption we may choose for each S a colimit A_S of $F \circ P_S$, with structure maps $\kappa^{S,e} : F(P_S(e)) \to A_S$. Define a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ on objects by $G(S) = A_S$. For a natural transformation $\alpha : S \to T$ the equality $P_T \circ \mathrm{el}(\alpha) = P_S$ allows us to "pull back" the $(y \circ P_T)$-cocone structure on T along $\mathrm{el}(\alpha)$ to a $(y \circ P_S)$-cocone structure, the structure maps of which are $\kappa^{T,\mathrm{el}(\alpha)(e)} : F(P_S(e)) \to A_T$ (for e an object of $\mathrm{el}(S)$). Then since A_S is an initial cocone of $F \circ P_S$ there exists a unique morphism $G(\alpha) : A_S \to A_T$ such that for all $e \in \mathrm{Obj}(\mathrm{el}(S))$ the diagram

$$\begin{tikzcd} F(P_S(e)) \arrow[d, "\kappa^{S,e}"'] \arrow[dr, "\kappa^{T,\mathrm{el}(\alpha)(e)}"] & \\ A_S \arrow[r, dashed, "G(\alpha)"'] & A_T \end{tikzcd}$$

commutes. It is easy to verify the functor laws for G using this definition and functoriality of $\mathrm{el}(-)$; we leave this to the reader. So we have defined a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$. We prove $F \cong G \circ y$ and then that G is a realization functor.

To show G extends F we examine the structure of the category $\mathsf{E} = \mathrm{el}(y(x))$ for a general object x of C. The key observation is that (x, id_x) is a terminal object of E. For any other object (z, f) of E we have at least into our proposed terminal object, $f \in \mathrm{Hom}_{\mathsf{E}}((z, f), (x, \mathrm{id}_x))$. And an arbitrary map $g : (z, f) \to (x, \mathrm{id}_x)$ must satisfy $g^*(\mathrm{id}_x) = f$, hence $g = f$. It's a standard result that a diagram with indexing category which admits a terminal object has colimit the image of that terminal object. In parrticular the structure map $\kappa^{y(x),(x,\mathrm{id}_x)} : F(x) \to A_{y(x)}$ must be an isomorphism. Hence we can define a natural isomorphism $\tau : F \to G \circ y$ by $\tau_x = \kappa^{y(x),(x,\mathrm{id}_x)}$, as long as the square

$$\begin{tikzcd} F(z) \arrow[r, "F(f)"] \arrow[d, "\tau_z"] & F(w) \arrow[d, "\tau_w"] \\ G(y(z)) \arrow[r, "G(y(f))"] & G(y(w)) \end{tikzcd}$$

commutes for each morphism $f : z \to w$ in C. Equivalently, $G(y(f)) = \tau_w \circ F(f) \circ \tau_z^{-1}$. By the definition of the action of G on morphisms and the equality $\mathrm{el}(y(f))(z, \mathrm{id}_z) = (z, f)$, this is equivalent to commutativity of

$$\begin{tikzcd} F(z) \arrow[rr, "\kappa^{y(w),(z,f)}"] \arrow[d, "\kappa^{y(z),(z,\mathrm{id}_z)}"'] & & A_{y(w)} \\ A_{y(z)} \arrow[r, "\tau_z^{-1}"] & F(z) \arrow[r, "F(f)"] & F(w), \arrow[u, "\tau_w"] \end{tikzcd}$$

which immediately reduces to the equation $\kappa^{y(z),(z,\mathrm{id}_z)} = \tau_w \circ F(f)$. But this equation is part of the cocone structure on $A_{y(w)}$, specifically the commuting triangle associated to the morphism $f : (z, f) \to (w, \mathrm{id}_w)$ in $\mathrm{el}(y(w))$.

Finally with τ in hand it is easy to show G is a realization functor. Let S be an arbitrary presheaf. We must show that the morphisms $G(\psi_{x,S}(s)) : G(y(x)) \to G(S)$ make $G(S) = A_S$ a colimit of $G \circ y \circ P_S$. It suffices to show that this is true after transporting the cocone structure across the isomorphism $\tau P_S : F \circ P_S \to G \circ y \circ P_S$, i.e. that the morphisms $G(\psi_{x,S}(s)) \circ \tau_x : F(x) \to A_S$ make A_S a colimit of $F \circ P_S$. By definition of the action of G on morphisms and the equality $\mathrm{el}(\psi_{x,S}(s))(x, \mathrm{id}_x) = (x, \psi_{x,S}(s)_x(x, \mathrm{id}_x)) = (x, \varphi_{x,S}(\psi_{x,S}(s))) = (x, s)$ we have

$$G(\psi_{x,S}(s)) \circ \tau_x = G(\psi_{x,S}(s)) \circ \kappa^{y(x),(x,\mathrm{id}_x)} = \kappa^{S,\mathrm{el}(\psi_{x,S}(s))(x,\mathrm{id}_x)} = \kappa^{S,(x,s)}$$

and so this $(F \circ P_S)$-cocone structure on A_S is originally chosen one, which we know is colimiting. $\square$

So in order to show Psh(C) is the free cocompletion of C it suffices to show that realization functors extending F are unique up to a unique isomorphism and that they preserve all colimits. We accomplish both by recasting the notion of a realization functor in terms of adjointness.

Definition 2.12. For any $F : \mathsf{C} \to \mathsf{D}$ the nerve of F is the functor $N(F) : \mathsf{D} \to \mathrm{Psh}(\mathsf{C})$ defined by

$$N(F)(d)(x) = \mathrm{Hom}_{\mathsf{D}}(F(x), d).$$

Lemma 2.13. *For any realization functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ and isomorphism $\tau : F \to G \circ y$ there is an adjunction $G \dashv N(F)$ whose unit map $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G$ satisfies, for each presheaf S and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$,*

$$(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x.$$

Proof. For each presheaf S on C define $\eta_S : S \to N(F)(G(S))$ by $(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x$. These assemble into a natural transformation, i.e. for any morphism $f : z \to w$ in C we have a commuting square

$$\begin{tikzcd} S(w) \arrow[r, "S(f)"] \arrow[d, "(\eta_S)_w"] & S(z) \arrow[d, "(\eta_S)_z"] \\ \mathrm{Hom}_{\mathsf{D}}(F(w), G(S)) \arrow[r, "F(f)^*"] & \mathrm{Hom}_{\mathsf{D}}(F(z), G(S)). \end{tikzcd}$$

To prove this square commutes, let $s \in S(w)$ be arbitrary and define $t = S(f)(s)$. By naturality of τ,

$$\begin{aligned} F(f)^*((\eta_S)_w(s)) &= F(f)^*(G(\psi_{w,S}(s)) \circ \tau_w) \\ &= G(\psi_{w,S}(s)) \circ \tau_w \circ F(f) \\ &= G(\psi_{w,S}(s)) \circ G(y(f)) \circ \tau_z \\ &= G(\psi_{w,S}(s) \circ y(f)) \circ \tau_z \\ &= G(\psi_{z,S}(t)) \circ \tau_z \\ &= (\eta_S)_z(t) \\ &= (\eta_S)_z(S(f)(s)). \end{aligned}$$

The reader might expect us to now verify η_S is natural in S and write down a counit, but if we use the "universal arrow" characterization of adjunctions this is unnecessary. What we do need to do to obtain an adjunction with (necessarily natural) unit $S \mapsto \eta_S$ is argue that for any object d of D and $\alpha : S \to N(F)(d)$ there exists a unique morphism $\beta : G(S) \to d$ in D such that $\alpha = N(F)(\beta) \circ \eta_S$. By calculating

$$(N(F)(\beta) \circ \eta_S)_x(s) = N(F)(\beta)_x((\eta_S)_x(s)) = \beta \circ (\eta_S)_x(s) = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$$

we find that a morphism $\beta : G(S) \to d$ satisfies $\alpha = N(F)(\beta) \circ \eta_S$ iff $\alpha_x(s) = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$ for every object (x, s) of $\mathrm{el}(S)$. We can push forward the $(y \circ P_S)$-cocone structure on S along α to get a $(y \circ P_S)$-cocone structure on $N(F)(d)$, with structure maps $\alpha \circ \psi_{x,S}(s) : y(x) \to N(F)(d)$. Let $a^{x,s} = \varphi_{x,N(F)(d)}(\alpha \circ \psi_{x,S}(s))$, i.e. $a^{x,s} = \alpha_x(s)$. Then $a^{x,s} \in N(F)(d)(x)$, meaning $a^{x,s}$ is a morphism $F(x) \to d$ in D. In fact these morphisms make d into a cocone under $F \circ P_S$, i.e. for any map $f : (z, t) \to (w, s)$ in $\mathrm{el}(S)$ the diagram

$$\begin{tikzcd} F(z) \arrow[r, "F(f)"] \arrow[dr, "a^{z,t}"'] & F(w) \arrow[d, "a^{w,s}"] \\ & d \end{tikzcd}$$

commutes. This is by naturality of α and the equation $S(f)(s) = t$ (baked into the definition of a morphism $\mathrm{el}(S)$), as

$$a^{w,s} \circ F(f) = F(f)^*(a^{w,s}) = N(F)(d)(f)(a^{w,s}) = N(F)(d)(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t) = a^{z,t}.$$

We may transport this along τ to get the structure of a cocone under $G \circ y \circ P_S$ on d, with structure maps $a^{x,s} \circ \tau_x^{-1}$. Because G is a realization functor, $G(S)$ is a colimit of $G \circ y \circ P_s$ with structure maps $G(\psi_{x,S}(s))$. Thus there is a unqiue $\beta : G(S) \to d$ such that $a^{x,s} \circ \tau_x^{-1} = \beta \circ G(\psi_{x,S}(s))$, equivalently $a^{x,s} = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$, as desired. □

Corollary 2.14. *If $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ is a realization functor then there is an adjunction $G \dashv N(G \circ y)$ with unit map $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(G \circ y) \circ G$ satisfying $(\eta_S)_x(s) = G(\psi_{x,S}(s))$. In particular realization functors preserve all colimits.*

Proof. Apply Lemma 2.13 with $\tau = \mathrm{id}_{G \circ y}$. □

A funny implication of Corollary 2.14 is that there is no difference between "realization functors" $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$, colimit preserving functors $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$, and left adjoints $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$. With this we finally prove Theorem 2.6.

Proof. Let D be a cocomplete category and $F : \mathsf{C} \to \mathsf{D}$ any functor. Existence of a colimit-preserving functor $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ extending F is immediate from Lemmas 2.11 and Lemma 2.13. Suppose $G, G' : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ preserve all colimits and we have $\tau : F \cong G \circ y, \tau' : F \cong G' \circ y$. Then by Lemma 2.13 we have adjunctions $G \dashv N(F)$ and $G' \dashv N(F)$ with unit maps $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G$ and $\eta' : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G'$ such that

$$(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x$$
$$(\eta'_S)_x(s) = G'(\psi_{x,S}(s)) \circ \tau'_x$$

for each presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$. By uniqueness of left adjoints there is a unique isomorphism $\zeta : G \to G'$ such that $\eta' = N(F)\zeta \circ \eta$. For any presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$ we may calculate

$$((N(F)\zeta \circ \eta)_S)_x(s) = ((N(F)\zeta)_S)_x((\eta_S)_x(s)) = N(F)(\zeta_S)_x((\eta_S)_x(s)) = \zeta_S \circ (\eta_S)_x(s) = \zeta_S \circ G(\psi_{x,S}(s)) \circ \tau_x.$$

So an isomorphism $\zeta : G \to G'$ satisfies $\eta' = N(F)\zeta \circ \eta$ iff $G'(\psi_{x,S}(s)) \circ \tau'_x = \zeta_S \circ G(\psi_{x,S}(s)) \circ \tau_x$ for each presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$. Additionally $\zeta_S \circ G(\psi_{x,S}(s)) = G'(\psi_{x,S}(s)) \circ \zeta_{y(x)}$, so ζ is unique such that

$$G'(\psi_{x,S}(s)) \circ \tau'_x = G'(\psi_{x,S}(s)) \circ \zeta_{y(x)} \circ \tau_x$$

for every presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$. Taking $S = y(x)$ and $s = \mathrm{id}_x$ this reduces to $\tau'_x = \zeta_{y(x)} \circ \tau_x$, and conversely $\tau'_x = \zeta_{y(x)} \circ \tau_x$ implies the general case. Hence there is a unique iso $\zeta : G \to G'$ with $\tau' = \zeta y \circ \tau$. □

We finish the section by proving that any slice of a presheaf category is still a presheaf category. This enables us to argue by "base change"; the reader who has learned algebraic geometry understands how useful this is. Hopefully this elucidates why we're spending so much time on general presheaf categories if our ultimate interest is simplicial sets.

Lemma 2.15. *Let C be a small category and x an object of C. Then there is an isomorphism of categories between $\mathrm{el}(y(x))$ and the slice category C/x which commutes with their respective projections down to C.*
Furthermore, for any presheaf S on C there is an equivalence of categories $\mathrm{Psh}(\mathrm{el}(S)) \simeq \mathrm{Psh}(\mathsf{C})/S$ such that the composition $\mathrm{el}(S) \hookrightarrow \mathrm{Psh}(\mathrm{el}(S)) \to \mathrm{Psh}(\mathsf{C})/S$ is isomorphic to the cocone structure of S over $P_S \circ y$.[5]

Proof. The isomorphism $\mathrm{el}(y(x)) \cong C/x$ is so simple it might in fact be an equality (depending on how you define the slice category). An object (z, t) of $\mathrm{el}(y(x))$ just corresponds to the object $t : z \to x$ of C/x. Morphisms are the identified under this correspondence because if we have $s : z \to x$ and $t : w \to x$, a map $f : z \to w$ defines a morphism $(z, t) \to (w, s)$ iff $s \circ f = t$ iff it defines a map $s \to t$ in the slice category.

Now let S be a presheaf on C. Let $F : \mathrm{el}(S) \to \mathrm{Psh}(\mathsf{C})/S$ be the cocone structure of S over $y \circ P_S$, so its value on an object (x, s) is the object $\psi(s) : y(x) \to S$ in the slice category and it leaves morphisms unchanged. By the results of this section we have an adjunction $G \dashv N(F)$ such that $F \cong G \circ y$. We prove that G is an equivalence by showing it is fully faithful and essentially surjective (it follows from this that $N(F)$ is a quasi-inverse and the unit/counit of the adjunction $G \dashv N(F)$ are isomorphisms). In fact, we're first going to establish G is essentially surjective assuming it is fully faithful. Let E be the essential image of G and $i : E \hookrightarrow \mathrm{Psh}(\mathsf{C})/S$ the inclusion. Then G factors through i to give an equivalence $\mathrm{Psh}(\mathrm{el}(S)) \to E$, and so in particular E has all colimits and its inclusion i preserves them. Because colimits in slice categories are computed as in the original category, every object of $\mathrm{Psh}(\mathsf{C})/S$ is a colimit of representable presheaves (equipped with structure maps down to S), and hence to show G is essentially surjective it suffices to show E contains all objects of the form $y(x) \xrightarrow{v} S$ for x an object of C. But an object of this form lies in the image of F, which we know is contained in E.

[5] For a diagram $L : \mathsf{J} \to \mathsf{D}$ the structure of a cocone over L on an object d of D is the same thing as a lift of L to D/d.

Now we must show G is fully faithful, i.e. that its action $\mu_{A,B} : \mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(A,B) \to \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(A),G(B))$ is bijective for any presheaves A, B on $\mathrm{el}(S)$. For any morphisms $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : A' \to B$ there is an evident equality $G(\beta\circ\gamma\circ\alpha) = G(\beta)\circ G(\gamma)\circ G(\alpha)$, meaning we have a naturality square for μ

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(A',B) \arrow[r, "\mathrm{Hom}(\alpha,\beta)"] \arrow[d, "\mu_{A',B}"] & \mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(A,B') \arrow[d, "\mu_{A,B'}"] \\
\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(A'),G(B)) \arrow[r, "\mathrm{Hom}(G(\alpha),G(\beta))"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(A),G(B')).
\end{tikzcd}$$

It suffices to show each "partially applied" $\mu_{-,B}$ is an isomorphism $\mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(-,B) \cong \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(-),G(B))$. Because the contravariant Hom-functor preserves limits (with domain the opposite category, i.e. it sends colimits to limits) and G preserves colimits, both the domain and codomain of $\mu_{-,B}$ are limit preserving functors. The full subcategory of $\mathrm{Psh}(\mathrm{el}(S))^{\mathrm{op}}$ on objects A such that $\mu_{A,B}$ is an isomorphism is closed under limits in $\mathrm{Psh}(\mathrm{el}(S))^{\mathrm{op}}$, since naturality of $\mu_{-,B}$ and the fact that the functors it goes between both preserve limits imply that $\mu_{\lim_i A_i,B} = \lim_i \mu_{A_i,B}$, and a limit of isomorphisms is an isomorphism. But $\mathrm{Psh}(\mathrm{el}(S))$ is generated under colimits by representable functors, so $\mathrm{Psh}(\mathrm{el}(S))^{\mathrm{op}}$ is generated under limits by the same, and hence it suffices to show $\mu_{y(e),B}$ is an isomorphism for any object e of $\mathrm{el}(S)$. Now we play the same game and change our goal to proving that each natural transformation $\mu_{y(e),-}$ is an isomorphism, and similarly to before if we can argue that its domain and codomain both preserve colimits we reduce to showing that just the $\mu_{y(e_1),y(e_2)}$ is an isomorphism. This time we can't appeal to a general property of the covariant Hom functor, since it will not always preserve colimits. However the yoneda lemma tells us that $\mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(y(e),-)$ is isomorphic to the functor of evaluation at e, which preserves colimits. But we also know $G(y(e)) \cong F(e)$, and G preserves colimits, so for $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(y(e)),G(-))$ to preserve colimits it suffices for $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e),-)$ to preserve colimits. Let $e = (x,s)$ and define $U : \mathrm{Psh}(\mathsf{C})/S \to \mathrm{Psh}(\mathsf{C})$ to be the forgetful functor. Similarly to how G being a functor makes μ a natural transformation, the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}\left(F(e), T \xrightarrow{\alpha} S\right) \arrow[r, "\alpha_*"] \arrow[d, hook] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e),S) \arrow[d, hook] \\
\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x),T) \arrow[r, "\alpha_*"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x),S).
\end{tikzcd}$$

commutes. In fact it is a cartesian square, since $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e),S) = \{\psi(s)\}$ and $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}\left(F(e), T \xrightarrow{\alpha} S\right)$ is *defined* to be the set of all $\sigma : y(x) \to T$ such that $\alpha_*(\sigma) = \psi(s)$. The conclusion that $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e),-)$ preserves colimits then follows from three facts: the left hand morphism in our cartesian square is the component of a natural transformation $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e),-) \hookrightarrow \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x),U(-))$ at $T \xrightarrow{\alpha} S$, the functor $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x),U(-))$ preserves colimits (since both U and evaluation at x do), and colimits in Set are pullback stable. This last term means that for any morphism $f : X \to Y$ of sets, the pullback functor $- \times_Y X : \mathsf{Set} \to \mathsf{Set}$ preserves colimits. So due to the various universal properties at play, the functor $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}\left(F(e), T \xrightarrow{\alpha} S\right)$ must preserve colimits.

Lastly we must verify $\mu_{y(e),y(e')}$ is an isomorphism for all objects e, e' of $\mathrm{el}(S)$. Because $G \circ y \cong F$ we reduce to checking that F is fully faithful. Let $e = (z,t)$ and $e' = (w,s)$ be objects of $\mathrm{el}(S)$. We have a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{el}(S)}(e,e') \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e),F(e')) \arrow[d, hook] \\
\mathrm{Hom}_{\mathsf{C}}(z,w) \arrow[r] & \mathrm{Hom}_{\mathsf{C}}(y(z),y(w))
\end{tikzcd}$$

where the vertical maps are subset inclusions, the top map is the action of F, and the bottom map is the action of y. We could also view the vertical maps as the actions of P_S and of U, from which perspective commutativity of this diagram holds because $U \circ F = y \circ P_S$. The bottom map is an isomorphism because the yoneda embedding is fully faithful, and so the top map is just the restriction of an isomorphism to certain subsets. Thus in order to show F is fully faithful it suffices to show that a map $f : z \to w$ satisfies $f \in \mathrm{Hom}_{\mathrm{el}(S)}(e,e')$ if and only if $y(f) \in \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e),F(e'))$. Unravelling definitions this means $S(f)(s) = t$ iff $\psi_{w,S}(s) \circ y(f) = \psi_{z,S}(t)$. But this is something we've already used frequently, it's obvious from the definition of $\varphi_{-,S}$ (and the fact φ is an isomorphism). $\square$

3 Examples of simplicial sets

Lemma 2.13 is a powerful tool in understanding simplicial sets. It makes it incredibly easy to define an adjunction between simplicial sets and some other category C, you simply need a functor $\Delta \to \mathsf{C}$. This construction is crucial to understanding simplicial sets, and many important families of simplicial sets arise from adjunctions like this. Perhaps the most important adjunction of this kind, one which the reader has certainly seen before in disguise and which is the source of the term "realization", is the one induced by the functor $\Delta \to \mathsf{Top}$ in Definition 2.2.

Definition 3.1. The singular simplicial set of a topological space X is $\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X)$. This has the structure of a functor as it is the nerve of the geometric simplex functor $\Delta \to \mathsf{Top}$. The geometric realization of a simplicial set S is defined by $|S| = \operatorname*{colim}_{\sigma \in \ell(S)} \Delta^n$. The colimit here is taken over the category $\mathrm{el}(S)$ from definition 2.8, i.e. $|S|$ is a realization functor extending the geometric simplex functor. But note that Top has an explicit choice of colimits, some sort of quotient space of a disjoint union space, so this $|\bullet|$ thing really is a single well defined functor.

The functor Sing occurs implicitly in the definition of singular homology, as the nth chain group of X is just the free abelian group on $\mathrm{Sing}(X)_n$. And furthermore, the differential of the singular chain complex is an alternating sum of the face maps of $\mathrm{Sing}(X)$! Note that the set $\mathrm{Sing}(X)$ is *massive*, for essentially any space X. The only simplicial sets we really know of at this point are the standard simplices $\Delta[n]$, i.e. the representable functors $y([n])$, which are finite in each degree and generated under the face and degneracy maps by finitely many simplices in total. On the other hand if X is a manifold or a CW complex of positive dimension then the set of maps $\Delta^0 \to X$ will be uncountable (it is in bijection with the underlying set of X). This is sort of a reflection of our original frustration with topological spaces: from the perspective of simplicial sets or other combinatorial models of homotopy types they have an enormous amount of redundant data.

The geometric realization functor seems incredibly opaque, but it can be surprisingly simple to calculate in practice. Most simplicial sets, even "small" ones, have infinitely many simplices, so the definition seems somewhat intractible for the purpose of actual computation. But because geometric realization is a left adjoint we know it preserves all colimits, and we know that the geometric realization of $\Delta[n]$ can be canonically identified with Δ^n. So if we have a "presentation" for a simplicial set as a finite colimit (or just simple to undestand colimit) of standard simplices then we get a description of its geometric realization as the same colimit but taken in Top. To illustrate this point we discuss two important families of simplicial sets and calculate their geometric realization. These families are motivated by simple geometric examples, and the presentation we give of them makes it clear their geometric realizations are in fact the topological spaces which model those geometric objects.

Definition 3.2. A simplicial subset of a simplicial set T is a sequence of sets $\{S_n\}_{n=0}^{\infty}$ such that $S_n \subseteq T_n$ for each n and $T(f)(S_n) \subseteq S_m$ for any morphism $f : [m] \to [n]$ in the simplex category (it suffices that this condition hold when f is a coface or codegeneracy map). The inclusions $S_n \subseteq T_n$ assemble into a monomorphism $S \hookrightarrow T$ in $s\mathsf{Set}$.

Definition 3.3. For any n define a simplicial subset $\partial\Delta[n]$ of $\Delta[n]$ by the formula

$$(\partial\Delta[n])_m = \{f \in \Delta[n]_m \mid f \text{ is not surjective}\}.$$

To check that this is well defined we must show that for any $g : [\ell] \to [k]$ we have $g^*((\partial\Delta[n])_k) \subseteq (\partial\Delta[n])_\ell$. But this is clear, because if $f \circ g$ is surjective then f must be as well.

Additionally, for any $0 \leq i \leq n$ define the ith "horn" of $\Delta[n]$ by

$$(\Lambda^n_i)_m = \{\alpha \in \Delta[n]_m \mid [n] \nsubseteq (\alpha([m]) \cup \{i\})\}.$$

The condition $[n] \nsubseteq (\alpha([m]) \cup \{i\})$ says that α must omit some vertex other than the ith, i.e. that α factors through some coface map $\delta_j : [n-1] \to [n]$ for $j \neq i$. So geometrically this is saying Λ^n_i is the union of all faces of $\Delta[n]$ except the ith. This is a simplicial subset for similar reasons to $\partial\Delta[n]$.

Lemma 3.4. *Let C be a small category in which every map factors as a split epimorphism followed by a monomorphism. Suppose S is a presheaf on C which admits a monomorphism into a representable presheaf on C. Define $\mathcal{M}$ to be the full subcategory of* $\mathrm{el}(S)$ *on pairs (w, s) such that $\psi_{w,S}(s)$ is a monomorphism. Then $\mathcal{M}$ is cofinal in* $\mathrm{el}(S)$*. Informally, S is the colimit of its representable subfunctors.*

Proof. Let $i : S \hookrightarrow y(x)$ be a monomorphism. Cofinality says that every object of $\mathrm{el}(S)$ admits a map into some object of $\mathcal{M}$, and that all choices of maps are related in a suitable way. Let $e = (z, t)$ be any object of $\mathrm{el}(S)$ and define $\mathcal{M}_e$ to be the full subcategory of $\mathcal{M}$ on objects $e' = (w, s)$ such that $\psi_{z,S}(t)$ factors through $\psi_{w,S}(s)$. Note that if $e' \in \mathcal{M}_e$ then there is a *unique* morphism $f : z \to w$ such that $\psi_{z,S}(t) = \psi_{w,S}(s) \circ y(f)$, since $\psi_{w,S}(s)$ is a monomorphism (and y is fully faithful). And $\psi_{z,S}(t) = \psi_{w,S}(s) \circ y(f)$ is equivalent to $S(f)(s) = t$, so if $e' \in \mathcal{M}_e$ there exists a unique morphism $e \to e'$. We argue that $\mathcal{M}_e$ has an initial object.

Since y is fully faithful we can write $i \circ \psi_{z,S}(t) = y(f)$ for a unique map $f : z \to x$ in C. By assumption, f admits a factorization $f = j \circ p$ where $p : z \to w$ is split epic and $j : w \to x$ is monic. Let $a : w \to z$ be a section of p and define $s = S(a)(t)$. Then $\psi_{w,S}(s) = \psi_{z,S}(t) \circ y(a)$. This implies

$$i \circ \psi_{w,S}(s) = i \circ \psi_{z,S}(t) \circ y(a) = y(f) \circ y(a) = y(f \circ a) = y(j \circ p \circ a) = y(j)$$

and in particular $i \circ \psi_{w,S}(s)$ is a monomorphism. Hence $\psi_{w,S}(s)$ is a monomorphism, i.e. $(w, s) \in \mathcal{M}$. To show $(w, s) \in \mathcal{M}_e$ we prove $\psi_{z,S}(t) = \psi_{w,S}(s) \circ y(p)$. We can check this after postcomposing with i since i is monic, and

$$i \circ \psi_{w,S}(s) \circ y(p) = y(j) \circ y(p) = y(f) = i \circ \psi_{z,S}(t)$$

so $(w, s) \in \mathcal{M}_e$. We claim that (w, s) is initial. Since every morphism in $\mathcal{M}$ is monic it suffices to show (w, s) admits some map to any other object (v, r) of $\mathcal{M}_e$, uniqueness is immediate. Let $f : z \to w$ be the unique morphism satisfying $S(f)(r) = t$. Then

$$S(f \circ a)(r) = S(a)(S(f)(r)) = S(a)(t) = s$$

which means $f \circ a$ defines a map $(w, s) \to (v, r)$ in $\mathrm{el}(S)$, as desired.

We leave verification of cofinality using this property of $\mathcal{M}_e$ to the reader. $\square$

Lemma 3.4 gives us an easy description of all simplicial subsets of the standard simplices, and of their geometric realizations. If $i : S \hookrightarrow \Delta[n]$ is a monomorphism then representable subfunctors of S are the same thing (under postcomposition with i) as representable subfunctors of $\Delta[n]$ which factor through i. But by the yoneda lemma, representable subfunctors of $\Delta[n]$ are really the same thing as injections into $[n]$ in the simplex category, or even more concretely the data of S is a collection of faces (of any lower dimension) of the n-simplex. Furthermore since S is a functor and compositions of injective maps are injective, any face of a simplex in this collection must still lie in the collection. But this data, a collection of faces closed under taking further faces, is that of an abstract simplicial complex with (ordered) vertices $[n]$! On the other hand, if we have an abstract simplicial complex $L \subseteq 2^{[n]}$ then we can define a simplicial subset L' of $\Delta[n]$ by

$$L'_m = \{\sigma \in \Delta[n]_m : \operatorname{im}\sigma \in L\}.$$

Evidently our simplicial subsets $\partial\Delta[n]$ and Λ^n_i are of this form, where L is the simplicial complex defining the boundary of an n-simplex or the ith horn of an n-simplex. In fact, suppose we have a simplicial complex $L \subseteq 2^{[n]}$ and view it as a poset (category). For any element $\sigma \in L$ we can uniquely write $\sigma = \{i_0, i_1, \dots, i_m\}$ where $0 \le i_0 < i_1 < \dots < i_m \le n$ and the i_j define a morphism $F(\sigma) : [m] \to [n]$ in the simplex category (geometrically, σ is a face of the n-simplex and embedding the m-simplex as that face). Evidently $F(\sigma) \in L'_m$, since $\operatorname{im} F(\sigma) = \sigma$. And since $F(\sigma)$ is an injection we have $([m], F(\sigma)) \in \mathcal{M}$, where $\mathcal{M} \subseteq \mathrm{el}(L')$ is the subcategory from Lemma 3.4. Furthermore if $\sigma \subseteq \tau$ then $F(\tau)$ factors through $F(\sigma)$ (the formula is awkward to write down, but intuitively we're just saying you can include σ into τ and then τ into $\Delta[n]$). Thus F defines a functor $L \to \mathcal{M}$, and in fact an isomorphism (we can write down an inverse $\mathcal{M} \to L$ on objects by sending $([k], \sigma)$ to $\operatorname{im}\sigma$, and if f factors through g we have $\operatorname{im} f \subseteq \operatorname{im} g$, so this assignment is monotone/functorial). Finally within the poset L we have a cofinal poset consisting of the maximal faces and their pairwise intersections (since L is not directed we need the pairwise intersections for cofinality). The point of this is that from a simplicial subset S of $\Delta[n]$ we can explicitly define a simplicial complex L and write down S as a finite colimits of representable functors using L; furthermore, since geometric realization commutes with colimits, $|S|$ is just the geometric simplex complex L! Or even more precisely, the geometric realization of $i : S \hookrightarrow \Delta[n]$ is isomorphic in the arrow category of topological spaces to the closed subset inclusion of L as a geometric subcomplex of Δ^n.

Corollary 3.5. *If $n > 1$ then we have a coequalizer diagram*

$$\coprod_{0 \le i < j \le n} \Delta[n-2] \overset{f}{\underset{g}{\rightrightarrows}} \coprod_{0 \le k \le n} \Delta[n-1] \xrightarrow{p} \partial\Delta[n]$$

in which p is induced by the face maps $y(\delta^n_k) : \Delta[n-1] \to \Delta[n]$ and on the (i, j)th copy of $\Delta[n-2]$ the map f includes into the ith copy of $\Delta[n-1]$ via δ^{n-1}_{j-1} and g includes into the jth copy of $\delta[n-1]$ via δ^{n-1}_i.

Corollary 3.6. *If $0 \leq \ell \leq n$ and $n > 1$ then we have a coequalizer diagram*

$$\coprod_{0 \leq i < j \leq n} \Delta[n-2] \rightrightarrows \coprod_{\substack{0 \leq i \leq n \\ i \neq k}} \Delta[n-1] \longrightarrow \Lambda^n_\ell$$

in which all maps are the same as in Lemma 3.5, just with restricted codomain.

Hopefully this discussion convinces the reader that simplicial sets have some connection to more concrete geometric objects (finite simplicial complexes) and that, although defined very abstractly, the geometric realization of a simplex can often be computed in a reasonable way. We give one more example of a family of simplicial sets and then move on to homotopy theory. This family once again comes from a nerve-realization adjunction: the category Δ is defined as a subcategory of Pos (the category of posets) and Pos embeds in Cat, the category of small categories. So we have a functor $\Delta \to \mathsf{Cat}$ sending $[n]$ to the category (which we also denote $[n]$)

$$[0] \to [1] \to \ldots \to [n].$$

Definition 3.7. The nerve of a small category C is the simplicial set $N(C)_n = \mathrm{Fun}([n], \mathsf{C})$, and taking the nerve is a right adjoint functor.

TODO: Talk about the more concrete description of nerve and talk about $N(BG)$ for a group G. TODO: Work out an example with $\Delta^1 \times \Delta^1$.

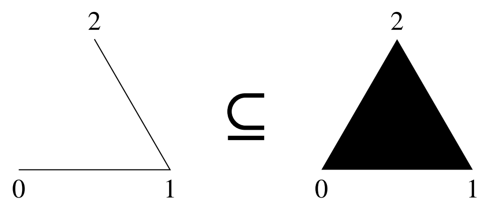
4 Homotopy of simplicial sets

It can sometimes be helpful to think of $\mathrm{Sing}(X)$ as the archetypal simplicial set. That is, we think of a simplicial set X as encoding a space and an element $\sigma \in X_n$ as an n-dimensional simplex within that space (but one which might be highly degenerate). This perspective is enabled by the Yoneda lemma: each object $[n]$ of Δ defines a simplicial set $\Delta[n]$ under the Yoneda embedding, called the standard n-simplex, and $X_n \cong \mathrm{Hom}_{s\mathsf{Set}}(\Delta[n], X)$. Under this perspective, a face map $d_i : X_n \to X_{n-1}$ is literally sending a simplex to its ith face and a degeneracy map $s_i : X_n \to X_{n+1}$ just relabels an n-simplex as a degnerate simplex of higher dimension. But the simplices in an arbitrary simplicial set are not as well behaved as e.g. the simplices that make up a simplicial complex; considering the possibly behavior of singular simplices in $\mathrm{Sing}(X)$ it becomes clear that a simplex is not determined by its faces and that the faces of a simplex do not need to be distinct. In fact due to the presence of degeneracy maps, for any nonempty simplicial there is a simplex whose ith and $(i+1)$st faces are equal for some i.

However thinking of an arbitrary simplicial set as behaving like $\mathrm{Sing}(X)$ can also be dangerous. Continuous maps are very flexible, while morphisms of simplicial sets do not need to be. If we try and think of a simplicial set geometrically, e.g. if we define it by drawing a picture, it may have deceptively few simplices. Compare the "interval" $\Delta[1]$, with its basepoints $\delta^1_1, \delta^1_0 : \Delta[0] \to \Delta[1]$, to the topological interval $I = [0,1] \cong |\Delta[1]|$, with its basepoints $x_0, x_1 : \{*\} \to I$. In Top we have a commutative diagram

$$\begin{tikzcd}
& & \{*\} \arrow[d, "x_1"] \arrow[dd, bend left, "x_1"] \\
& \{*\} \arrow[r, "x_0"] \arrow[d, "x_1"] & I \arrow[d, "f"] \\
\{*\} \arrow[r, "x_0"] \arrow[rr, bend right, "x_0"'] & I \arrow[r, "g"] \arrow[ur, phantom, "\ulcorner", very near end] & I
\end{tikzcd}$$

where the map f compresses I into the right half subinterval and the map g compresses I into the left half subinterval. In essence this is saying that I is a fractal: you can bisect it into two copies of itself. In particular for any topological space X we have a bijection $\mathrm{Hom}_{\mathsf{Top}}(I, X) \to \mathrm{Hom}_{\mathsf{Top}}(I, X) \times_X \mathrm{Hom}_{\mathsf{Top}}(I, X)$, where the maps $\mathrm{Hom}_{\mathsf{Top}}(I, X) \to X$ we're taking a pullback with respect to are evaluation at the endpoint and starting point. The inverse of this map is why we can compose paths in a topological space (and since homotopies are the same as paths in the function space, at least for nice enough spaces, this is also why homotopy of maps is an equivalence relation). This entire story breaks down horribly for the simplicial interval. There are exactly three morphisms $\Delta[1] \to \Delta[1]$, equivalently morphisms $[1] \to [1]$ in Δ: the identity, the map sending everything to the vertex 0, and the map sending everything to the vertex 1. The whole point of our theory of simplicial sets is that they are more combinatorial, more discrete, which is why $\Delta[1]$ has no hope of being a fractal. We do still have a functor $s\mathsf{Set} \to \mathsf{Set}$ sending a simplicial set X to the set of "end to end" paths in X, i.e. $X \mapsto X_1 \times_{X_0} X_1$, and this is corepresented by the pushout of $\Delta[1]$ with itself over $\Delta[1]$, as in Top. It's just that this pushout isn't still $\Delta[1]$. But it is a fairly easy to describe simplicial set, the horn Λ^2_1. The image of the inclusion $\Lambda^2_1 \hookrightarrow \Delta[2]$ under geometric realization is the diagram below.



If we want to have any reasonable way of doing homotopy theory with a simplicial set X, any end-to-end paths in X should have *some* composition.