Simplicial Sets

Brendan Murphy

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1 CW Complexes

The objects of study in classical homotopy theory are the *homotopy types*. This is not the same thing as a topological space, or even a CW complex, but "CW complex up to homotopy". CW complexes are spaces that admit a construction in stages, starting with some points, then gluing on intervals via their boundary, then gluing on disks via their boundary, and so on, then taking the union of all finite stages. In stage n the "gluing" of n-disks onto the $(n-1)$-skeleton X_{n-1} can be understood categorically as taking a pushout of X_{n-1} with your family of disks $\coprod_{\lambda\in\Lambda} D^n$ along a family of arbitrary continuous maps $\{f_\lambda : S^n \to X\}_{\lambda\in\Lambda}$ ("attaching maps") and standard inclusions $S^n \hookrightarrow D^n$. We could just have easily defined this using (topological) simplex inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, for Δ^n and D^n are convex bodies of the same dimension and so canonically (after picking a basepoint) homeomorphic. So CW complexes are *exactly* the topological spaces that can be obtained from a sequential colimit of pushouts of (coproducts of) the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$. In other words, they're spaces obtained by gluing simplices together with the restriction that one may only glue along the boundary, but the flexibility that arbitrary continuous gluings of that boundary are allowed. But combining the "Simplicial Approximation Theorem" with the following lemma allows us to assume a CW complex is obtained from a very, very structured kind of gluing.

Lemma 1.1. *Let X be a topological space and $f, g : S^{n-1} \to X$ two homotopic maps. Then the pushouts (or "amalgamation spaces") $D^n \amalg_f X$ and $D^n \amalg_g X$ are homotopy equivalent.*

Proof. Let $H : S^{n-1}\times I \to X$ be a homotopy. The key idea is that we may use the deformation retraction of the "cylinder" $D^n\times I$ onto its boundary minus the top $(D^n\times\{0\})\cup(S^{n-1}\times I)$ to get a deformation retraction of $(D^n\times I)\amalg_H X$ onto $((D^n\times\{0\})\cup(S^{n-1}\times I))\amalg_H X$. We have a morphism $J : (D^n\times I)\amalg_H X \to ((D^n\times\{0\})\cup(S^{n-1}\times I))\amalg_H X$ induced by the morphism of spans

$$\begin{tikzcd}
D^n \arrow[d, hook] & S^{n-1} \arrow[l, hook'] \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, equal] \\
(D^n\times\{0\})\cup(S^{n-1}\times I) & S^{n-1}\times I \arrow[l, hook'] \arrow[r, "H"] & X.
\end{tikzcd}$$

And in fact J is surjective, because every point in the extra bit $S^{n-1}\times(0,1]$ is glued onto X by H. But it's actually a split monomorphism as well, because morphism of spans above has a left inverse

$$\begin{tikzcd}
D^n & S^{n-1} \arrow[l, hook'] \arrow[r, "f"] & X \\
(D^n\times\{0\})\cup(S^{n-1}\times I) \arrow[u, two heads] & S^{n-1}\times I \arrow[l, hook'] \arrow[r, "H"] \arrow[u, two heads] & X. \arrow[u, equal]
\end{tikzcd}$$

This means J is actually a homeomorphism, because it is a surjection with a continuous left inverse. The punchline is that $D^n \amalg_f X$, and by symmetry $D^n \amalg_g X$, are both homeomorphic to deformation retracts of the same space (and hence are homotopy equivalent). $\square$

> Exercise: Reprove Lemma 1.1 in terms of the simplicial inclusions, using the fact that Δ^n deformation retracts onto any of its "horns" Λ^n_i (those spaces formed by removing the ith face from $\partial\Delta^n$).

2 The simplex category, gluing, and presheaves

Simplicial sets are a more "algebraic" or "combinatorial" way of modelling homotopy types. This has the advantage that it transports more easily to algebraic contexts. E.g., the (1-)category of topological abelian groups is not abelian but the (1-)category of simplicial abelian groups is! We saw above through careful analysis of CW complexes that any homotopy type is built up from gluing together simplices along their boundaries. For CW complexes the gluing was fairly geometric, an actual pushout in the category of topological spaces. Simplicial sets take the opposite approach: they are formal gluings of (formal!) simplices. Before we can define simplicial sets we must discuss the (category of) simplices from which they are glued.

Definition 2.1. The simplex category Δ has objects the finite nonempty ordinals $[n] = \{0, 1, \dots, n\}$ and a morphism $[n] \to [m]$ is simply an order preserving function. The augmented simplex category Δ_a is defined in the same way, but the empty ordinal $[-1] = \varnothing$ is included.

Note that Δ is equivalent to the category of all finite totally ordered sets. What does this have to do with actual geometric simplices? The object $[n]$ should be understood as a representation of the geometric n-simplex Δ^n, and its elements $0, \dots, n$ representing the $(n+1)$-vertices of that simplex. As demonstrated by simplicial or singular homology, it's often more convenient to work with simplices that have a chosen order on their vertices (for manageably and consistently tracking orientation); this is why we're looking at ordered finite sets and not just finite sets[1]. The geometric simplex Δ^n is the convex hull of its vertices $e_0, \dots, e_n$, and this means that every function of finite sets $\{e_0, \dots, e_n\} \mapsto \{e_0, \dots, e_m\}$ has a unique extension to an affine transformation $\Delta^n \to \Delta^m$ sending vertices to vertices. Thus Δ could just as truthfully be described as the category of geometric simplices $\Delta^n \subseteq \mathbb{R}^{n+1}$ with morphisms the affine transformations sending vertices to vertices and preserving the standard order on those vertices.

Definition 2.2. Let

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}$$

be the n-dimensional "geometric simplex". The vertices of Δ^n are the standard basis vectors $e_0, \dots, e_n$ of $\mathbb{R}^{n+1}$ and any point in Δ^n can be uniquely represented as a convex combination $t_0 e_0 + \dots + t_n e_n$ of them. Given an order-preserving map $f : [n] \to [m]$ there is an induced continuous map $\widetilde{f} : \Delta^n \to \Delta^m$ defined by

$$\widetilde{f}\left(\sum_{i=0}^{n} t_i e_i \right) = \sum_{i=0}^{n} t_i e_{f(i)}.$$

> Exercise: The assignments $[n] \mapsto \Delta^n$ and $f \mapsto \widetilde{f}$ define a faithful functor $\Delta \to \mathsf{Top}$.

There are two important families of maps within Δ, the coface and codegeneracy maps.

Definition 2.3. Let n be a positive integer. For $0 \leq i \leq n$ denote by $\delta_i^n : [n-1] \to [n]$ the unique monotone injection which omits i from its range. This is the ith coface map. Concretely,

$$\delta_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

Also define $\sigma_i^n : [n+1] \to [n]$ to be the unique monotone surjection with $\sigma_i^n(i) = \sigma_i^n(i+1)$. This is the ith codegeneracy map. Concretely,

$$\sigma_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

Geometrically, δ_i^n is the inclusion of the ith face of Δ^n (meaning the face opposite the ith vertex) and σ_n^i is the projection of Δ^{n+1} onto Δ^n where we collapse the edge $[e_i \; e_{i+1}]$ down to a point.

[1] But for those who are interested, there is a theory of unoriented "symmetric simplicial sets"

Any monotone map $f : [n] \to [m]$ has a decomposition into a surjection $[n] \twoheadrightarrow [k]$ and an injecton $[k] \hookrightarrow [m]$; this may be easiest to see if we think of $[k]$ as the image f with the order inherited from $[m]$ (passing to the category of all finite nonempty totally ordered sets). Furthermore the injection $[k] \hookrightarrow [m]$ can be decomposed into a composition of coface maps, omitting elements of $[m]$ one at a time, and the surjection $[n] \twoheadrightarrow [k]$ may be decomposed into a composition of codegeneracy maps, squishing together elements $i, i+1$ such that $f(i) = f(i+1)$ one at a time until none remain. This tells us that every morphism in Δ is a composition of coface and codegeneracy maps. In fact there is a normal form associated to this decomposition, obtained by repeatedly applying the "cosimplicial identites".

Theorem 2.4. *The simplex category Δ is the free category C on a sequence of objects $[0], [1], \dots$ and families of morphisms $\{\delta^n_i \in \mathrm{Hom}_{\mathsf{C}}(n-1, n)\}_{n\geq 1, 0\leq i\leq n}$ and $\{\sigma^n_i \in \mathrm{Hom}_{\mathsf{C}}(n+1, n)\}_{n\geq 0, 0\leq i\leq n}$, subject to the relations (for all n)*

$$\delta^{n+1}_j \circ \delta^n_i = \delta^{n+1}_i \circ \delta^n_{j-1} \quad (\textit{if } i < j) \tag{1}$$

$$\sigma^{n+1}_j \circ \delta^{n+2}_i = \delta^{n+1}_i \circ \sigma^n_{j-1} \quad (\textit{if } i < j) \tag{2}$$

$$\sigma^n_j \circ \delta^{n+1}_j = \mathrm{id}_{[n]} \tag{3}$$

$$\sigma^n_j \circ \delta^{n+1}_{j+1} = \mathrm{id}_{[n]} \tag{4}$$

$$\sigma^{n+1}_j \circ \delta^{n+2}_i = \delta^{n+1}_{i-1} \circ \sigma^n_j \quad (\textit{if } i > j+1) \tag{5}$$

$$\sigma^n_j \circ \sigma^{n+1}_i = \sigma^n_i \circ \sigma^{n+1}_{j+1} \quad (\textit{if } i \leq j). \tag{6}$$

We will not prove this theorem in these notes, but we will attempt to explain what these identites say in the simplex category and explain what it means for a category to be presented by generators and relations. The equations (1) and (2) are a commutativity condition, they express (with index shifts appropriate to the δ's and σ's) that omitting a vertex i and then omitting/collapsing a later vertex j is the same as first omitting/collapsing $j - 1 = \delta_i^{-1}(j)$ and then omitting i. The equations (3) and (4) are perhaps the most important identities, because their categorical interpretation is that each δ is a *split monomorphism* and each σ is a *split epimorphism*; explicitly they say that if we omit a vertex and then collapse it with the next/previous vertex, it's the same as doing nothing. The equation (5) can be understood as saying "far away" omissions/collapses do not affect eachother (up to reindexing!). And finally equation (6) expresses that if you collapse twice in a row, the order of collapses matters only in that it shifts up the indexing.

The "free category" part of the theorem is more directly relevant, because it gives an explicit description of functors $\Delta \to \mathsf{C}$ for any category C (like how a presentation of a group G tells you what group homomorphisms $G \to H$ are). One interpretation of a "free structure" is exactly this kind of universal property, i.e. a free thing ("group" or "category equipped with a sequence of objects and families of maps satisfying the cosimplicial identities") is an initial object in the category of things. A free group G on generators $x_1, \dots, x_n$ subject to relations $r_1, \dots, r_m$ is an initial object in the category of tuples $(H, y_1, \dots, y_n)$ of groups H and $\mathbf{y} \in H^n$ such that for each j, interpreting x_i as y_i in w_j gives the identity element of H; a morphism $(H, \mathbf{y}) \to (H', \mathbf{z})$ in this category is of course a group homomorphism $f : H \to H'$ such that $f(y_i) = z_i$ for each i. Hence a free category on objects $\{X_s\}_{s\in S}$ and morphisms $\{f_\lambda : X_s \to X_t\}_{s,t\in S, \lambda\in\Lambda_{s,t}}$ subject to some equations of morphisms $\{E_j\}_{j\in J}$ is an initial object in the category[2] of categories that are equipped with a chosen family of objects labelled by S and a chosen family of morphisms labelled by the $\Lambda_{s,s'}$ satisfying all equations E_j. There is also a "by hand" construction of a free category on a directed graph/quiver G, e.g. the graph with vertices $\mathbb{N}$ and edges labelled by the coface/codegeneracy maps. This construction is fairly simply, if v, w are vertices in G then a morphism $v \to w$ in the free category is just a path ("formal composition of edges") from v to w in G. One can then quotient the set of arrows of this category by the smallest equivalence relation which contains the equations and "respects composition" (like how a normal subgroup gives an equivalence relation which multiplication).

We might stop and ask at this point why we need the codegeneracies at all. If we're interested in gluing together simplices along their boundaries, surely we just need the face inclusions? It turns out that the category of simplicial sets is much nicer when degeneracies; for example, the geometric realization of the semisimplicial set $\Delta^1 \times \Delta^1$ is an interval union two points, not a square! The theory without degeneracies isn't useless, though, we obtain what are called "semi-simplicial sets". These are called "Δ-complexes" in Hatcher's algebraic topology textbook.

We now return to simplicial sets, having gained an understanding of what kind of "formal simplices" we're gluing together. The categorical understanding of "gluing" is that it is a colimit. And vice versa, in many concrete categories a colimit does performing some kind of concrete "gluing". This is all there is to the definition of a simplicial set.

[2]Here I mean the locally small category of small categories such that etc. But in fact an initial object of this category will have the right mapping out property with respect to locally small categories too, as any functor $F : \mathsf{C} \to \mathsf{D}$ with C small factors through a small subcategory D' of D; specifically D' is the full subcategory of D on the objects in the image of F.

Definition 2.5. The category of simplicial sets $s\mathsf{Set}$ is the free[3] cocompletion of Δ. That is to say $s\mathsf{Set}$ has all (small) colimits, comes equipped with a functor $Y : \Delta \to s\mathsf{Set}$, and for any other category D with all (small) colimits and functors $F : \Delta \to \mathsf{D}$ there exists a colimit preserving functor $G : s\mathsf{Set} \to \mathsf{D}$ equipped with an isomorphism $\tau : F \to G \circ Y$. Furthermore (G, τ) is unique in that if we have another colimit-preserving functor $G' : s\mathsf{Set} \to \mathsf{C}$ equipped with an isomorphism $\tau' : F \to G' \circ Y$ then there is a unique isomorphism $\zeta : G \to G'$ making the diagram

$$\begin{tikzcd} & G \circ Y \arrow[d, "\zeta Y"] \\ F \arrow[ur, "\tau'"] \arrow[r, "\tau"'] & G' \circ Y \end{tikzcd}$$

commute.

Intuitively this says that an object of $s\mathsf{Set}$ is a formal colimit of some diagram in Δ. One can construct a free cocompletion in this way, but I tried to write it down once and lost two weeks working out technical details. Luckily the free cocompletion of a small category is a recognizable, fairly simple, and extremely well behaved category. The rest of this section will be devoted to proving the following theorem.

Theorem 2.6. *Let* C *be a small category and* $\mathrm{Psh}(\mathsf{C}) = \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ *be the category of presheaves on* C*. The Yoneda embedding* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *exhibits* $\mathrm{Psh}(\mathsf{C})$ *as the free cocompletion of* C*.*

Most people would find my initial definition of $s\mathsf{Set}$ a little silly. The true definition is just $s\mathsf{Set} = \mathrm{Psh}(\Delta)$. Our presentation of Δ tells us that a simplicial set can also be understood as sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ equipped with morphisms s_i^n, d_i^n satisfying the *simplicial identities*, the categorical dual of the cosimplicial identities (because presheaves are contravariant functors $\Delta \to \mathsf{Set}$). We will expand on this later.

One caveat with Theorem 2.6 is that, because the presheaf category is a free construction, already existing colimits in C will almost never be preserved under y. The proof of Theorem 2.6 boils down to the fact that any presheaf on C can be canonically written as a colimit of representable presheaves (those in the image of the Yoneda embedding). This may sound strange, but it's actually just another point of view on the celebrated Yoneda lemma, which we recall below.

Lemma 2.7. *Let* C *be a small category and* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *the functor* $y(x) = \mathrm{Hom}_{\mathsf{C}}(-, x)$*. For any object* x *of* C *and presheaf* S *on* C *the function* $\varphi : \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \to S(x)$ *defined by* $\varphi(\eta) = \eta_x(\mathrm{id}_x)$ *is a bijection. Furthermore,* φ *defines a natural isomorphism of functors* $\mathsf{C}^{\mathrm{op}} \times \mathrm{Psh}(\mathsf{C}) \to \mathsf{Set}$*.*

For the rest of this section we use the notation φ as in this lemma and set $\psi = \varphi^{-1}$.

Proof. We define an inverse $\psi : S(x) \to \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S)$ by $\psi(s)_z(f) = S(f)(s)$. Unwrapping this a bit, for $s \in S(x)$ we define a natural transformation $\psi(s) : y(x) \to S$ by setting its component on an object z to be the function $\mathrm{Hom}_{\mathsf{C}}(z, x) \to S(z)$ sending $f : z \to x$ to its action on s under S, i.e. $S(f)(s)$. We must verify that $\psi(s)$ is in fact natural for each s. So suppose we have a map $g : z \to w$ in C, we must check that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(w, x) \arrow[r, "g^*"] \arrow[d, "\psi(s)_w"] & \mathrm{Hom}_{\mathsf{C}}(z, x) \arrow[d, "\psi(s)_z"] \\ S(w) \arrow[r, "S(g)"] & S(z) \end{tikzcd}$$

commutes. By unravelling definitions and applying functorality of S we calculate for any $f \in \mathrm{Hom}_{\mathsf{C}}(w, x)$ that

$$\psi(s)_z(g^*(f)) = \psi(s)_z(f \circ g) = S(f \circ g)(s) = S(g)(S(f)(s)) = S(g)(\psi(s)_w(f)).$$

So $\psi(s)$ is natural. And for arbitrary $s \in S(x)$, $\eta \in \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S)$, $z \in \mathrm{Obj}(\mathsf{C})$ and $f \in \mathrm{Hom}_{\mathsf{C}}(z, x)$ we have

$$\varphi(\psi(s)) = \psi(s)_x(\mathrm{id}_x) = S(\mathrm{id}_x)(s) = \mathrm{id}_{S(x)}(s) = s$$
$$\psi(\varphi(\eta))_z(f) = S(f)(\varphi(\eta)) = S(f)(\eta_x(\mathrm{id}_x)) \overset{(!)}{=} \eta_z(y(x)(f)(\mathrm{id}_x)) = \eta_z(f^*(\mathrm{id}_x)) = \eta_z(f).$$

[3]This is not quite freeness in the sense discussed above; it is about 2-initiality in an appropriate 2-category!

The equality labelled (!) holds because of the following naturality square of η:

$$\begin{tikzcd} y(x)(x) \arrow[r, "y(x)(f)"] \arrow[d, "\eta_x"] & y(x)(z) \arrow[d, "\eta_z"] \\ S(x) \arrow[r, "S(f)"] & S(z). \end{tikzcd}$$

This proves φ is a bijection. To check φ is natural it suffices to show it is natural in x for fixed S and natural in S for fixed x. Fix S and write φ_x for φ. We must show that for any morphism $a : u \to v$ in C the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(v), S) \arrow[r, "y(a)^*"] \arrow[d, "\varphi_v"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(u), S) \arrow[d, "\varphi_u"] \\ S(v) \arrow[r, "S(a)"] & S(u) \end{tikzcd}$$

commutes. This is once again just unfolding definitions and using naturality, as for any $\eta : y(v) \to S$ we calculate

$$\varphi_u(y(a)^*(\eta)) = \varphi_u(\eta \circ y(a)) = \eta_u(y(a)_u(\mathrm{id}_u)) = \eta_u(a \circ \mathrm{id}_u) = \eta_u(a)$$
$$S(a)(\varphi_v(\eta)) = S(a)(\eta_v(\mathrm{id}_v)) = \eta_u(y(v)(a)(\mathrm{id}_v)) = \eta_u(\mathrm{id}_v \circ a) = \eta_u(a).$$

Now fix x and write φ_S for S. Let $\beta : S \to T$ be an arbitrary natural transformation. The diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \arrow[r, "\beta_*"] \arrow[d, "\varphi_S"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), T) \arrow[d, "\varphi_T"] \\ S(x) \arrow[r, "\beta_x"] & T(x) \end{tikzcd}$$

commutes because for any $\eta : y(x) \to S$ we have

$$\varphi_T(\beta_*(\eta)) = \varphi_T(\beta \circ \eta) = \beta_x(\eta_x(\mathrm{id}_x)) = \beta_x(\varphi_S(\eta)).$$

□

So what does an isomorphism $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \cong S(x)$ have to do with writing S as a colimit? The key point is that naturality in the x argument means that for any map $f : z \to w$ in C and $s \in S(w)$ we have

$$\varphi_{z,S}(\psi_{w,S}(s) \circ y(f)) = \varphi_{z,S}(y(f)^*(\psi_{w,S}(s))) = S(f)(\varphi_{w,S}(\psi_{w,S}(s))) = S(f)(s).$$

Hence for any $t \in S(z)$, $s \in S(w)$ and map $f : z \to w$ satisfying $S(f)(s) = t$ there is a commutative triangle

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\psi_{z,S}(t)"'] & y(w) \arrow[d, "\psi_{w,S}(s)"] \\ & S. \end{tikzcd}$$

These triangles suggest that S is a cocone under a certain diagram with structure maps $\psi_{w,S}(s) : y(w) \to S$. An object of the indexing category must know about both w and $s \in S(w)$ and a morphism has to be constrained by $S(f)(s) = t$.

Definition 2.8. Let C be a small category and S a presheaf on C. Define a (small) category $\mathrm{el}(S)$[4] by

$$\mathrm{Obj}(\mathrm{el}(S)) = \{(x, s) : x \in \mathrm{Obj}(\mathsf{C}), s \in S(x)\}$$
$$\mathrm{Hom}_{\mathrm{el}(S)}((z, t), (w, s)) = \{f \in \mathrm{Hom}_{\mathsf{C}}(z, w) : S(f)(s) = t\}.$$

We set $\mathrm{id}_{(x,s)} = \mathrm{id}_x$ and perform composition as in C. The identities are well defined because $S(\mathrm{id}_x)(s) = s$. The composition laws are automatic, and this composition is well defined because if $S(f)(s) = t$ and $S(g)(t) = r$ then

$$S(f \circ g)(s) = S(g)(S(f)(s)) = S(g)(t) = r.$$

This category comes with a forgetful functor $P_S : \mathrm{el}(S) \to \mathsf{C}$. The category $\mathrm{el}(S)$ equipped with P_S is referred to as the category of elements of S (in the special case $\mathsf{C} = \Delta$ it is sometimes called the category of simplices of S). It is instructive to think about what happens in the case that S is the forgetful functor of some familiar category, e.g. finite groups (or to make C small and not just essentially small, groups whose underlying set is hereditarily finite).

[4]A reader with stacky inclinations may recognize this as the grothendieck construction, specialized to presheaves of discrete groupoids (sets).

Theorem 2.9. *Let* C *be a small category and* S *a presheaf on* C. *The morphisms* $\psi_{x,S}(s) : y(x) \to S$ *make* S *into a colimit of the diagram* $y \circ P : \mathrm{el}(S) \to \mathrm{Psh}(\mathsf{C})$.

Proof. We already saw that these maps assemble into a cocone by naturality of the Yoneda lemma. Suppose we have a presheaf T on C and natural transformations $\sigma^{x,s} : y(x) \to T$ such that for any morphism $f : (z,t) \to (w,s)$ in $\mathrm{el}(S)$,

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\sigma^{z,t}"'] & y(w) \arrow[d, "\sigma^{w,s}"] \\ & T \end{tikzcd}$$

commutes. We are then required to show there is a unique natural transformation $\beta : S \to T$ making each diagram

$$\begin{tikzcd} y(x) \arrow[d, "\psi_{x,S}(s)"'] \arrow[dr, "\sigma^{x,s}"] & \\ S \arrow[r, "\beta"'] & T \end{tikzcd}$$

commute. Uniqueness is immediate, as naturality of the Yoneda lemma in the presheaf argument gives

$$\beta \circ \psi_{x,S}(s) = \beta_*(\psi_{x,S}(s)) = \psi_{x,T}(\beta_x(s))$$

and hence commutativity of the requisite triangles is equivalent to the identity $\beta_x(s) = \varphi_{x,T}(\sigma^{x,s})$. So we just need to check that the maps $\beta_x(s) = \varphi_{x,T}(\sigma^{x,s})$ assemble into a natural transformation $S \to T$. This means that the square

$$\begin{tikzcd} S(w) \arrow[r, "S(f)"] \arrow[d, "\beta_w"] & S(z) \arrow[d, "\beta_z"] \\ T(w) \arrow[r, "T(f)"] & T(z) \end{tikzcd}$$

must commute for any $f : z \to w$ in C, which in turn is true because for $s \in S(w)$, abbreviating $t = S(f)(s)$, we have

$$T(f)(\beta_w(s)) = T(f)(\varphi_{w,T}(\sigma^{w,s})) = \varphi_{z,T}(y(f)^*(\sigma^{w,s})) = \varphi_{z,T}(\sigma^{w,s} \circ y(f)) = \varphi_{z,T}(\sigma^{z,t}) = \beta_z(t) = \beta_z(S(f)(s)). \quad \square$$

With Theorem 2.9 we have shown that $\mathrm{Psh}(\mathsf{C})$ is generated from $y(\mathsf{C}) \simeq \mathsf{C}$ under "gluing" (colimits). We now have the tools to prove Theorem 2.6, which states that this method of gluing objects of C together is universal. The reader may already see how to define a colimit-preserving extension of a functor $F : \mathsf{C} \to \mathsf{D}$ using the colimit formula for presheaves: send $S = \mathrm{colim}(y \circ P)$ to $\mathrm{colim}(F \circ P)$. But in set-theoretic foundations the term $\mathrm{colim}(F \circ P)$ isn't really meaningful; "$\mathrm{colim}(F \circ P)$" is only defined up to isomorphism, not equality. We do not have a canonical choice of colimit in D, and choosing an arbitrary one simultaneously across the proper class of presheaves S requires a stronger choice axiom than is in ZFC. In univalent mathematics there is no issue, since equality and isomorphism are the same thing. In ZFC+Grothendieck universes the "class" of presheaves is only a proper class from the point of view of some ambient inacessible cardinal κ. Our presheaves are valued in V_κ and so there is just a set of them, to which ZFC's axiom of choice applies. But this noncanonical choice is still awkward, so we opt to consider all choices without bias. However also in ZFC (or really NBG, so we can talk about classes) we can carry this argument out as long as the target category D has "explicitly defined" or "distinguished" colimits, hich happens in all situations we will see in these notes.

Fix a small category C, a locally small category D, and a functor $F : \mathsf{C} \to \mathsf{D}$.

Definition 2.10. A realization functor is a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ such that for every presheaf S on C, G preserves the colimit of the diagram $y \circ P_S$. Non-rigorously, G is a realization functor if for every S it satisfies the equation

$$G(S) = G\left(\operatorname*{colim}_{e \in \mathrm{el}(S)} y(P_S(e))\right) = \operatorname*{colim}_{e \in \mathrm{el}(S)} G(y(P_S(e))).$$

Say that G extends F if $G \circ y$ is isomorphic to F. In this case G must satisfy the (non-rigorous) identity

$$G(S) = \operatorname*{colim}_{e \in \mathrm{el}(S)} F(P_S(e)).$$

Lemma 2.11. *In sufficiently strong foundations, if* D *has all small colimits then for every functor* $F : \mathsf{C} \to \mathsf{D}$ *there exists a realization functor* $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ *which extends* F.

Proof. We begin by correcting a deficiency from earlier, which was defining the category of elements as a *function* $\mathrm{Obj}(\mathrm{Psh}(\mathsf{C})) \to \mathrm{Obj}(\mathsf{Cat})$ instead of a *functor* $\mathrm{Psh}(\mathsf{C}) \to \mathsf{Cat}$. For a morphism of presheaves $\alpha : S \to T$ define $\mathrm{el}(\alpha) : \mathrm{el}(S) \to \mathrm{el}(T)$ by $\mathrm{el}(\alpha)(x, s) = (x, \alpha_x(s))$ on objects and $\mathrm{el}(\alpha)(f) = f$ on morphisms. This is well defined as

$$T(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t).$$

for any morphism $f : (z, t) \to (w, s)$. The functor laws for $\mathrm{el}(-)$ hold since $\mathrm{el}(\mathrm{id}_S)(x, s) = (x, (\mathrm{id}_S)_x(s)) = (x, s)$ and

$$\mathrm{el}(\beta \circ \alpha)(x, s) = (x, (\beta \circ \alpha)_x(s)) = (x, \beta_x(\alpha_x(s))) = \mathrm{el}(\beta)(x, \alpha_x(s)) = \mathrm{el}(\beta)(\mathrm{el}(\alpha)(x, s)).$$

Also note that for any $\alpha : S \to T$ we have $P_T \circ \mathrm{el}(\alpha) = P_S$ (the functor $\mathrm{el}(\alpha)$ leaves the first coordinate unchanged). By assumption we may choose for each S a colimit A_S of $F \circ P_S$, with structure maps $\kappa^{S,e} : F(P_S(e)) \to A_S$. Define a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ on objects by $G(S) = A_S$. For a natural transformation $\alpha : S \to T$ the equality $P_T \circ \mathrm{el}(\alpha) = P_S$ allows us to "pull back" the $(y \circ P_T)$-cocone structure on T along $\mathrm{el}(\alpha)$ to a $(y \circ P_S)$-cocone structure, the structure maps of which are $\kappa^{T,\mathrm{el}(\alpha)(e)} : F(P_S(e)) \to A_T$ (for e an object of $\mathrm{el}(S)$). Then since A_S is an initial cocone of $F \circ P_S$ there exists a unique morphism $G(\alpha) : A_S \to A_T$ such that for all $e \in \mathrm{Obj}(\mathrm{el}(S))$ the diagram

$$\begin{tikzcd}
F(P_S(e)) \arrow[d, "\kappa^{S,e}"'] \arrow[dr, "\kappa^{T,\mathrm{el}(\alpha)(e)}"] & \\
A_S \arrow[r, dashed, "G(\alpha)"'] & A_T
\end{tikzcd}$$

commutes. It is easy to verify the functor laws for G using this definition and functoriality of $\mathrm{el}(-)$; we leave this to the reader. So we have defined a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$. We prove $F \cong G \circ y$ and then that G is a realization functor.

To show G extends F we examine the structure of the category $\mathsf{E} = \mathrm{el}(y(x))$ for a general object x of C. The key observation is that (x, id_x) is a terminal object of E. For any other object (z, f) of E we have at least into our proposed terminal object, $f \in \mathrm{Hom}_{\mathsf{E}}((z, f), (x, \mathrm{id}_x))$. And an arbitrary map $g : (z, f) \to (x, \mathrm{id}_x)$ must satisfy $g^*(\mathrm{id}_x) = f$, hence $g = f$. It's a standard result that a diagram with indexing category which admits a terminal object has colimit the image of that terminal object. In parrticular the structure map $\kappa^{y(x),(x,\mathrm{id}_x)} : F(x) \to A_{y(x)}$ must be an isomorphism. Hence we can define a natural isomorphism $\tau : F \to G \circ y$ by $\tau_x = \kappa^{y(x),(x,\mathrm{id}_x)}$, as long as the square

$$\begin{tikzcd}
F(z) \arrow[r, "F(f)"] \arrow[d, "\tau_z"] & F(w) \arrow[d, "\tau_w"] \\
G(y(z)) \arrow[r, "G(y(f))"] & G(y(w))
\end{tikzcd}$$

commutes for each morphism $f : z \to w$ in C. Equivalently, $G(y(f)) = \tau_w \circ F(f) \circ \tau_z^{-1}$. By the definition of the action of G on morphisms and the equality $\mathrm{el}(y(f))(z, \mathrm{id}_z) = (z, f)$, this is equivalent to commutativity of

$$\begin{tikzcd}
F(z) \arrow[rr, "\kappa^{y(w),(z,f)}"] \arrow[d, "\kappa^{y(z),(z,\mathrm{id}_z)}"'] & & A_{y(w)} \\
A_{y(z)} \arrow[r, "\tau_z^{-1}"] & F(z) \arrow[r, "F(f)"] & F(w), \arrow[u, "\tau_w"]
\end{tikzcd}$$

which immediately reduces to the equation $\kappa^{y(z),(z,\mathrm{id}_z)} = \tau_w \circ F(f)$. But this equation is part of the cocone structure on $A_{y(w)}$, specifically the commuting triangle associated to the morphism $f : (z, f) \to (w, \mathrm{id}_w)$ in $\mathrm{el}(y(w))$.

Finally with τ in hand it is easy to show G is a realization functor. Let S be an arbitrary presheaf. We must show that the morphisms $G(\psi_{x,S}(s)) : G(y(x)) \to G(S)$ make $G(S) = A_S$ a colimit of $G \circ y \circ P_S$. It suffices to show that this is true after transporting the cocone structure across the isomorphism $\tau P_S : F \circ P_S \to G \circ y \circ P_S$, i.e. that the morphisms $G(\psi_{x,S}(s)) \circ \tau_x : F(x) \to A_S$ make A_S a colimit of $F \circ P_S$. By definition of the action of G on morphisms and the equality $\mathrm{el}(\psi_{x,S}(s))(x, \mathrm{id}_x) = (x, \psi_{x,S}(s)_x(x, \mathrm{id}_x)) = (x, \varphi_{x,S}(\psi_{x,S}(s))) = (x, s)$ we have

$$G(\psi_{x,S}(s)) \circ \tau_x = G(\psi_{x,S}(s)) \circ \kappa^{y(x),(x,\mathrm{id}_x)} = \kappa^{S,\mathrm{el}(\psi_{x,S}(s))(x,\mathrm{id}_x)} = \kappa^{S,(x,s)}$$

and so this $(F \circ P_S)$-cocone structure on A_S is originally chosen one, which we know is colimiting. ☐

So in order to show Psh(C) is the free cocompletion of C it suffices to show that realization functors extending F are unique up to a unique isomorphism and that they preserve all colimits. We accomplish both by recasting the notion of a realization functor in terms of adjointness.

Definition 2.12. For any $F : \mathsf{C} \to \mathsf{D}$ the nerve of F is the functor $N(F) : \mathsf{D} \to \mathrm{Psh}(\mathsf{C})$ defined by

$$N(F)(d)(x) = \mathrm{Hom}_{\mathsf{D}}(F(x), d).$$

Lemma 2.13. *For any realization functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ and isomorphism $\tau : F \to G \circ y$ there is an adjunction $G \dashv N(F)$ whose unit map $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G$ satisfies, for each presheaf S and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$,*

$$(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x.$$

Proof. For each presheaf S on C define $\eta_S : S \to N(F)(G(S))$ by $(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x$. These assemble into a natural transformation, i.e. for any morphism $f : z \to w$ in C we have a commuting square

$$\begin{tikzcd}
S(w) \arrow[r, "S(f)"] \arrow[d, "(\eta_S)_w"] & S(z) \arrow[d, "(\eta_S)_z"] \\
\mathrm{Hom}_{\mathsf{D}}(F(w), G(S)) \arrow[r, "F(f)^*"] & \mathrm{Hom}_{\mathsf{D}}(F(z), G(S)).
\end{tikzcd}$$

To prove this square commutes, let $s \in S(w)$ be arbitrary and define $t = S(f)(s)$. By naturality of τ,

$$\begin{aligned}
F(f)^*((\eta_S)_w(s)) &= F(f)^*(G(\psi_{w,S}(s)) \circ \tau_w) \\
&= G(\psi_{w,S}(s)) \circ \tau_w \circ F(f) \\
&= G(\psi_{w,S}(s)) \circ G(y(f)) \circ \tau_z \\
&= G(\psi_{w,S}(s) \circ y(f)) \circ \tau_z \\
&= G(\psi_{z,S}(t)) \circ \tau_z \\
&= (\eta_S)_z(t) \\
&= (\eta_S)_z(S(f)(s)).
\end{aligned}$$

The reader might expect us to now verify η_S is natural in S and write down a counit, but if we use the "universal arrow" characterization of adjunctions this is unnecessary. What we do need to do to obtain an adjunction with (necessarily natural) unit $S \mapsto \eta_S$ is argue that for any object d of D and $\alpha : S \to N(F)(d)$ there exists a unique morphism $\beta : G(S) \to d$ in D such that $\alpha = N(F)(\beta) \circ \eta_S$. By calculating

$$(N(F)(\beta) \circ \eta_S)_x(s) = N(F)(\beta)_x((\eta_S)_x(s)) = \beta \circ (\eta_S)_x(s) = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$$

we find that a morphism $\beta : G(S) \to d$ satisfies $\alpha = N(F)(\beta) \circ \eta_S$ iff $\alpha_x(s) = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$ for every object (x, s) of $\mathrm{el}(S)$. We can push forward the $(y \circ P_S)$-cocone structure on S along α to get a $(y \circ P_S)$-cocone structure on $N(F)(d)$, with structure maps $\alpha \circ \psi_{x,S}(s) : y(x) \to N(F)(d)$. Let $a^{x,s} = \varphi_{x,N(F)(d)}(\alpha \circ \psi_{x,S}(s))$, i.e. $a^{x,s} = \alpha_x(s)$. Then $a^{x,s} \in N(F)(d)(x)$, meaning $a^{x,s}$ is a morphism $F(x) \to d$ in D. In fact these morphisms make d into a cocone under $F \circ P_S$, i.e. for any map $f : (z, t) \to (w, s)$ in $\mathrm{el}(S)$ the diagram

$$\begin{tikzcd}
F(z) \arrow[r, "F(f)"] \arrow[dr, "a^{z,t}"'] & F(w) \arrow[d, "a^{w,s}"] \\
 & d
\end{tikzcd}$$

commutes. This is by naturality of α and the equation $S(f)(s) = t$ (baked into the definition of a morphism $\mathrm{el}(S)$), as

$$a^{w,s} \circ F(f) = F(f)^*(a^{w,s}) = N(F)(d)(f)(a^{w,s}) = N(F)(d)(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t) = a^{z,t}.$$

We may transport this along τ to get the structure of a cocone under $G \circ y \circ P_S$ on d, with structure maps $a^{x,s} \circ \tau_x^{-1}$. Because G is a realization functor, $G(S)$ is a colimit of $G \circ y \circ P_s$ with structure maps $G(\psi_{x,S}(s))$. Thus there is a unqiue $\beta : G(S) \to d$ such that $a^{x,s} \circ \tau_x^{-1} = \beta \circ G(\psi_{x,S}(s))$, equivalently $a^{x,s} = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$, as desired. ☐

Corollary 2.14. *If $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ is a realization functor then there is an adjunction $G \dashv N(G \circ y)$ with unit map $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(G \circ y) \circ G$ satisfying $(\eta_S)_x(s) = G(\psi_{x,S}(s))$. In particular realization functors preserve all colimits.*

Proof. Apply Lemma 2.13 with $\tau = \mathrm{id}_{G \circ y}$. ☐

A funny implication of Corollary 2.14 is that there is no difference between "realization functors" $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$, colimit preserving functors $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$, and left adjoints $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$. With this we finally prove Theorem 2.6.

Proof. Let D be a cocomplete category and $F : \mathsf{C} \to \mathsf{D}$ any functor. Existence of a colimit-preserving functor $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ extending F is immediate from Lemmas 2.11 and Lemma 2.13. Suppose $G, G' : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ preserve all colimits and we have $\tau : F \cong G \circ y, \tau' : F \cong G' \circ y$. Then by Lemma 2.13 we have adjunctions $G \dashv N(F)$ and $G' \dashv N(F)$ with unit maps $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G$ and $\eta' : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G'$ such that

$$(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x$$
$$(\eta'_S)_x(s) = G'(\psi_{x,S}(s)) \circ \tau'_x$$

for each presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$. By uniqueness of left adjoints there is a unique isomorphism $\zeta : G \to G'$ such that $\eta' = N(F)\zeta \circ \eta$. For any presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$ we may calculate

$$((N(F)\zeta \circ \eta)_S)_x(s) = ((N(F)\zeta)_S)_x((\eta_S)_x(s)) = N(F)(\zeta_S)_x((\eta_S)_x(s)) = \zeta_S \circ (\eta_S)_x(s) = \zeta_S \circ G(\psi_{x,S}(s)) \circ \tau_x.$$

So an isomorphism $\zeta : G \to G'$ satisfies $\eta' = N(F)\zeta \circ \eta$ iff $G'(\psi_{x,S}(s)) \circ \tau'_x = \zeta_S \circ G(\psi_{x,S}(s)) \circ \tau_x$ for each presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$. Additionally $\zeta_S \circ G(\psi_{x,S}(s)) = G'(\psi_{x,S}(s)) \circ \zeta_{y(x)}$, so ζ is unique such that

$$G'(\psi_{x,S}(s)) \circ \tau'_x = G'(\psi_{x,S}(s)) \circ \zeta_{y(x)} \circ \tau_x$$

for every presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$. Taking $S = y(x)$ and $s = \mathrm{id}_x$ this reduces to $\tau'_x = \zeta_{y(x)} \circ \tau_x$, and conversely $\tau'_x = \zeta_{y(x)} \circ \tau_x$ implies the general case. Hence there is a unique iso $\zeta : G \to G'$ with $\tau' = \zeta y \circ \tau$. $\square$

3 Simplicial sets

The definition of a simplicial set as a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$ has an obvious generalization to categories other than Set.

Definition 3.1. Let C be a category. The category of simplicial objects in C is $s\mathsf{C} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{C})$, i.e. a simplicial object in C is a functor $\Delta^{\mathrm{op}} \to \mathsf{C}$. For a simplicial object X we often write X_n to abbreviate $X([n])$.

By applying a simplicial object $X : \Delta^{\mathrm{op}} \to \mathsf{C}$ to the generating morphisms of the simplex category we obtain its *face* and *degeneracy* maps. We denote face maps by $d_i^n : X_n \to X_{n-1}$ and degeneracy maps by $s_i^n : X_n \to X_{n+1}$. Because X is a contravariant functor out of the simplex category, the face and degeneracy maps of X satisfy the dual of the cosimplicial identites, called the simplicial identities. Explicitly for a sequence of objects $X_0, X_1, \ldots,$ in C and families of morphisms $\{d_i^n \in \mathrm{Hom}_{\mathsf{C}}(X_n, X_{n-1})\}_{n\geq 1, 0\leq i\leq n}$ and $\{s_i^n \in \mathrm{Hom}_{\mathsf{C}}(X_n, X_{n+1})\}_{n\geq 0, 0\leq i\leq n}$ the simplicial identities are

$$\begin{aligned}
d_i^n \circ d_j^{n+1} &= d_{j-1}^n \circ d_i^{n+1} && (\text{if } i < j)\\
d_i^{n+2} \circ s_j^{n+1} &= s_{j-1}^n \circ d_i^{n+1} && (\text{if } i < j)\\
d_j^{n+1} \circ s_j^n &= \mathrm{id}_{X_n} &&\\
d_{j+1}^{n+1} \circ s_j^n &= \mathrm{id}_{X_n} &&\\
d_i^{n+2} \circ s_j^{n+1} &= s_j^n \circ d_{i-1}^{n+1} && (\text{if } i > j+1)\\
s_i^{n+1} \circ s_j^n &= s_{j+1}^{n+1} \circ s_i^n && (\text{if } i \leq j).
\end{aligned}$$

These are satisfied if the objects $X_\bullet$ and families of d's and s's come from a simplicial object X, and vice versa any sequence of objects and family of maps satisfying these equations extends to a unique functor $\Delta^{\mathrm{op}} \to \mathsf{C}$ by Lemma 2.4. Also, the 3rd and 4th identity tell us the maps d_j^n are (split) epimorphisms and the s_j^n are (split) monomorphisms. In this point of view on simplicial sets, a morphism $X \to Y$ is a sequence of morphisms $X_k \to Y_k$ which intertwine the face and degeneracy maps of X with the face and degeneracy maps of Y. This is formally similar to the definition of connective chain complex, which is also a sequence of objects and structure maps and where a morphism between such is a degreewise morphism intertwining the structure maps. So one entirely valid way of thinking of a simplicial object X in a category C is as a diagram

$$\cdots \;\mathrel{\substack{\to\\ \leftarrow\\ \to\\ \leftarrow\\ \to\\ \leftarrow\\ \to\\ \leftarrow\\ \to}}\; X_3 \;\mathrel{\substack{\to\\ \leftarrow\\ \to\\ \leftarrow\\ \to\\ \leftarrow\\ \to}}\; X_2 \;\mathrel{\substack{\to\\ \leftarrow\\ \to\\ \leftarrow\\ \to}}\; X_1 \;\mathrel{\substack{\to\\ \leftarrow\\ \to}}\; X_0$$

where there horizontal maps, indexed by their vertical position, satisfy the simplicial identities.

Our motivation to consider simplicial sets was that they are formal gluings of simplices, and we hope there is some reasonable notion of homotopy on them which allows them to model homotopy types. General simplicial objects are a natural to consider because any functor $F : \mathsf{Set} \to \mathsf{C}$ induces a pushforward functor $s\mathsf{Set} \to s\mathsf{C}$. But for general categories C there is no analogue of Theorem 2.9; we can't even formulate an analogous statement because there is no realization of the standard simplices in C. By a "realization of the standard simplices" we really just mean a functor $\Delta \to \mathsf{C}$, or equivalently[5] a simplicial object in C^{op}. But many categories do come equipped with a kind of realization of the standard simplices as objects of that category, and when they do there is some structure we can exploit.

Definition 3.2. Let C be a category. The category of cosimplicial objects in C is $\mathrm{Fun}(\Delta, \mathsf{C})$, i.e. a simplicial object in C is a functor $\Delta \to \mathsf{C}$. For a cosimplicial object X we often write X^n to abbreviate $X([n])$.

Example 3.3. The identity functor $\mathrm{Id}_\Delta : \Delta \to \Delta$ is a cosimplicial object of the simplex category. Rather tautologically, every cosimplicial object in any category is a pushforward of this.

Example 3.4. The functor $\Delta \to \mathsf{Top}$ described in Definition 2.2 is a cosimplicial topological space.

Example 3.5. By definition Δ is a full subcategory of the category Pos of partially ordered sets and monotone functions. The inclusion $\Delta \subseteq \mathsf{Pos}$ is a cosimplicial poset. If we further include Pos into Cat as the small thin categories in which every isomorphism is an identity we obtain a cosimplicial category $\Delta \to \mathsf{Cat}$. The category represented by $[n]$ is

$$0 \to 1 \to \cdots \to n-1 \to n.$$

Example 3.6. The Yoneda embedding $y : \Delta \to s\mathsf{Set}$ is a cosimplicial object in $s\mathsf{Set}$.

In all examples above (except the stupid one about a cosimplicial object in Δ) the category in which our cosimplicial object lives is itself cocomplete. And we just spent a whole section proving $s\mathsf{Set}$ is the free cocompletion of Δ, so we know that a cosimplicial object $F : \Delta \to \mathsf{C}$ in a cocomplete category C immediately gives rise to an adjunction[6] $G : s\mathsf{Set} \rightleftarrows \mathsf{C} : N(F)$ in which G extends F. We call this adjunction the *realization-nerve* adjunction induced by F. The left adjoint gets its name the cosimplicial object in Example 3.4 and the right adjoint from Example 3.5.

Definition 3.7. The cosimplicial object of geometric simplices in Top induces an adjunction $|-| : s\mathsf{Set} \leftrightarrows \mathsf{Top} : \mathrm{Sing}$. For a simplicial set X the space $|X|$ is called the geometric realization of X. For topological space X the simplicial set $\mathrm{Sing}(X)$ is called the singular simplicial set of X.

Elaborating on this definition, for a topological space X we have $\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X)$. The ith face map $\mathrm{Sing}(X)_n \to \mathrm{Sing}(X)_{n-1}$ is given by precomposition with the inclusion of Δ^{n-1} as the ith face of Δ^n, or less rigorously by the restriction of a singular simplex $\Delta^n \to X$ to the ith face. The ith degeneracy map $\mathrm{Sing}(X)_n \to \mathrm{Sing}(X)_{n+1}$ is given by precomposition with the projection $\Delta^{n+1} \to \Delta^n$ which collapses the edge connecting the ith and $(i+1)$st vertices; hence it is "really" not an $(n+1)$-simplex in X at all, but at most an n-simplex (but arbitrary continuous maps are not well behaved enough to make this precise). Note that $\mathrm{Sing}(X)$ occurs implicitly in the definition of integral singular homology, as the singular chain complex has underlying graded abelian group the free abelian group on the underlying graded set of $\mathrm{Sing}(X)$ and its differentials are alternating sums of face maps of $\mathrm{Sing}(X)$.

It can sometimes be helpful to think of $\mathrm{Sing}(X)$ as the archetypal simplicial set. That is, we think of a simplicial set X as encoding a space and an element $\sigma \in X_n$ as an n-dimensional simplex within that space (but one which might be highly degenerate). This perspective is enabled by the Yoneda lemma: each object $[n]$ of Δ defines a simplicial set $\Delta[n]$ under the Yoneda embedding, called the standard n-simplex, and $X_n \cong \mathrm{Hom}_{s\mathsf{Set}}(\Delta[n], X)$. Under this perspective, a face map $d_i : X_n \to X_{n-1}$ is literally sending a simplex to its ith face and a degeneracy map $s_i : X_n \to X_{n+1}$ just relabels an n-simplex as a degnerate simplex of higher dimension. But the simplices in an arbitrary simplicial set are not as well behaved as e.g. the simplices that make up a simplicial complex; considering the possibly behavior of singular simplices in $\mathrm{Sing}(X)$ it becomes clear that a simplex is not determined by its faces and that the faces of a simplex do not need to be distinct. In fact due to the presence of degeneracy maps, for any nonempty simplicial there is a simplex whose ith and $(i+1)$st faces are equal for some i.

[5]This is a little misleading. The categories $\mathrm{Fun}(\Delta, \mathsf{C})$ and $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{C}^{\mathrm{op}})$ have the same objects but they are *not* equivalent. In general $\mathrm{Fun}(\mathsf{C}, \mathsf{D})^{\mathrm{op}} = \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{D}^{\mathrm{op}})$, because a natural transformation will be oriented the same way that morphisms in the target category are. Relatedly, the free completion of a small category C is $\mathrm{Psh}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathrm{Fun}(\mathsf{C}, \mathsf{Set})^{\mathrm{op}}$, not $\mathrm{Psh}(\mathsf{C}^{\mathrm{op}}) = \mathrm{Fun}(\mathsf{C}, \mathsf{Set})$.

[6]The results in section 2 imply there is an essentially unique adjunction like this, so we can harmlessly pretend there is a single well defined one.

However thinking of an arbitrary simplicial set as behaving like $\mathrm{Sing}(X)$ can also be dangerous. Continuous maps are very flexible, while morphisms of simplicial sets do not need to be. If we try and think of a simplicial set geometrically, e.g. if we define it by drawing a picture, it may have deceptively few simplices. Compare the "interval" $\Delta[1]$, with its basepoints $\delta^1_1, \delta^1_0 : \Delta[0] \to \Delta[1]$, to the topological interval $I = [0,1] \cong |\Delta[1]|$, with its basepoints $x_0, x_1 : \{*\} \to I$. In Top we have a commutative diagram

$$\begin{tikzcd}
& & \{*\} \arrow[d, "x_1"] \arrow[dd, bend left, "x_1"] \\
& \{*\} \arrow[r, "x_0"] \arrow[d, "x_1"] & I \arrow[d, "f"] \\
\{*\} \arrow[r, "x_0"] \arrow[rr, bend right, "x_0"'] & I \arrow[r, "g"] & I \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

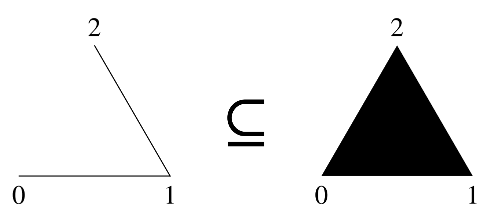
where the map f compresses I into the right half subinterval and the map g compresses I into the left half subinterval. In essence this is saying that I is a fractal: you can bisect it into two copies of itself. In particular for any topological space X we have a bijection $\mathrm{Hom}_{\mathsf{Top}}(I, X) \to \mathrm{Hom}_{\mathsf{Top}}(I, X) \times_X \mathrm{Hom}_{\mathsf{Top}}(I, X)$, where the maps $\mathrm{Hom}_{\mathsf{Top}}(I, X) \to X$ we're taking a pullback with respect to are evaluation at the endpoint and starting point. The inverse of this map is why we can compose paths in a topological space (and since homotopies are the same as paths in the function space, at least for nice enough spaces, this is also why homotopy of maps is an equivalence relation). This entire story breaks down horribly for the simplicial interval. There are exactly three morphisms $\Delta[1] \to \Delta[1]$, equivalently morphisms $[1] \to [1]$ in Δ: the identity, the map sending everything to the vertex 0, and the map sending everything to the vertex 1. The whole point of our theory of simplicial sets is that they are more combinatorial, more discrete, which is why $\Delta[1]$ has no hope of being a fractal. We do still have a functor $s\mathsf{Set} \to \mathsf{Set}$ sending a simplicial set X to the set of "end to end" paths in X, i.e. $X \mapsto X_1 \times_{X_0} X_1$, and this is corepresented by the pushout of $\Delta[1]$ with itself over $\Delta[1]$, as in Top. It's just that this pushout isn't still $\Delta[1]$. But it is a fairly easy to describe simplicial set.

Definition 3.8. For any $n \geq 1$, m and $0 \leq i \leq n$ define the ith "horn" of $\Delta[n]$ by

$$(\Lambda^n_i)_m = \{\alpha \in \Delta[n]_m \mid [n] \nsubseteq (\alpha([m]) \cup \{i\})\}.$$

The condition $[n] \nsubseteq (\alpha([m]) \cup \{i\})$ says that α must omit some vertex other than the ith, i.e. that α factors through some coface map $\delta_j : [n-1] \to [n]$ for $j \neq i$. These sets assemble into a simplicial set Λ^n_i whose action on morphisms is (or whose face and degeneracy maps are) inherited from Δ^n; this is well defined because $\mathrm{im}(f^*\alpha) \subseteq \mathrm{im}\,\alpha$. The degreewise inclusions $(\Lambda^n_i)_m \subseteq \Delta[n]_m$ assemble into a (mono)morphism of simplicial sets $\Lambda^n_i \hookrightarrow \Delta[n]$.

Geometrically Λ^n_i is the union of all faces of $\Delta[n]$ except the ith. For example, the image of the inclusion $\Lambda^2_1 \hookrightarrow \Delta[2]$ under geometric realization is the diagram below.



By construction, the coface maps $y(\delta^n_j) : \Delta[n-1] \to \Delta[n]$ for $j \neq i$ each factor through Λ^n_i and they are mutually surjective onto Λ^n_i. Essentially $\Lambda^n_i = \bigcup_{j \neq i} \mathrm{im}(y(\delta^n_j))$; this is at least true degreewise. We can obtain from this a "presentation" of Λ^n_i, i.e. a way of writing it as a colimit of standard simplices, using the following lemma.

Lemma 3.9. *Let J be a finite set and C a small category. Define $\mathcal{P}$ to be the poset of all subsets of J with size ≤ 2 and suppose we are given a diagram $G : \mathcal{P}^{\mathrm{op}} \to \mathrm{Psh}(\mathsf{C})$ in which each map $G(\varnothing \leq \{j\}) : G(j) \to G(\varnothing)$ is a monomorphism and each square*

$$\begin{tikzcd} G(\{a,b\}) \arrow[r] \arrow[d] & G(\{a\}) \arrow[d] \\ G(\{b\}) \arrow[r] & G(\varnothing) \end{tikzcd}$$

is cartesian. If T is a sub-presheaf of $G(\varnothing)$ (meaning an objectwise subset closed under the functor structure maps) and for any object x of C we have $T(x) = \bigcup_{j \in J} \mathrm{im}(G(\varnothing \leq \{j\}))$. Each morphism $G(U) \to G(\varnothing)$ with U nonempty then uniquely factors through $T \hookrightarrow G(\varnothing)$, giving T the structure of a cocone under $G|_{\mathcal{P}\setminus\{\varnothing\}}$. This structure makes T a colimit of $G|_{\mathcal{P}\setminus\{\varnothing\}}$.

Proof. Colimits and limits are computed objectwise, so we immediately reduce to the case $\mathsf{C} = \{*\}$ and $\mathrm{Psh}(\mathsf{C}) = \mathsf{Set}$. Then we may use the monomorphisms $G(\{j\}) \to G(\varnothing)$ to identify each $G(\{j\})$ with a subset of $G(\varnothing)$, the union of which is T. The squares being cartesian means we can identify $G(\{a,b\})$ with $G(\{a\}) \cap G(\{b\})$, and so being a colimit is just saying that if a set is a union of some subsets then a map defined on each subset which agrees on overlaps can be glued together to a map on the whole set. $\square$