

# Simplicial Sets

Brendan Murphy

August 24, 2023

## 1 CW Complexes

The objects of study in classical homotopy theory are the *homotopy types*. This is not the same thing as a topological space, or even a CW complex, but a “CW complex up to homotopy”. CW complexes are spaces that admit a construction in stages, starting with some point, then gluing on intervals via their boundary, then gluing on disks via their boundary, and so on, then taking the union of all finite stages. In stage  $n$  the “gluing” of  $n$ -disks onto the  $(n-1)$ -skeleton  $X_{n-1}$  can be understood categorically as taking a pushout of  $X_{n-1}$  with your family of disks  $\coprod_{\lambda \in \Lambda} D^n$  along a family of arbitrary continuous maps  $\{f_\lambda : S^n \rightarrow X\}_{\lambda \in \Lambda}$  (“attaching maps”) and standard inclusions  $S^n \hookrightarrow D^n$ . We could just have easily defined this using (topological) simplex inclusions  $\partial \Delta^n \hookrightarrow \Delta^n$ , for  $\Delta^n$  and  $D^n$  are convex bodies of the same dimension and so canonically (after picking a basepoint) homeomorphic. So CW complexes are *exactly* the topological spaces that can be obtained from a sequential colimit of pushouts of (coproducts of) the boundary inclusions  $\partial \Delta^n \hookrightarrow \Delta^n$ . In other words, they’re spaces obtained by gluing simplices together with the restriction that one may only glue along the boundary, but the flexibility that arbitrary continuous gluings of that boundary are allowed. But combining the “Simplicial Approximation Theorem” with the following lemma allows us to assume a CW complex is obtained from a very, very structured kind of gluing.

**Lemma 1.** *Let  $X$  be a topological space and  $f, g : S^{n-1} \rightarrow X$  two homotopic maps. Then the pushouts (or “amalgamation spaces”)  $D^n \amalg_f X$  and  $D^n \amalg_g X$  are homotopy equivalent.*

*Proof.* Let  $H : S^{n-1} \times I \rightarrow X$  be a homotopy. The key idea is that we may use the deformation retraction of the “cylinder”  $D^n \times I$  onto its boundary minus the top  $(D^n \times \{0\}) \cup (S^{n-1} \times I)$  to get a deformation retraction of  $(D^n \times I) \amalg_H X$  onto  $((D^n \times \{0\}) \cup (S^{n-1} \times I)) \amalg_H X$ . Observe that we have a morphism  $J : (D^n \times I) \amalg_H X \rightarrow ((D^n \times \{0\}) \cup (S^{n-1} \times I)) \amalg_H X$  induced by the morphism of spans

$$\begin{array}{ccccc} D^n & \longleftarrow & S^{n-1} & \xrightarrow{f} & X \\ \downarrow & & \downarrow & & \parallel \\ (D^n \times \{0\}) \cup (S^{n-1} \times I) & \longleftarrow & S^{n-1} \times I & \xrightarrow{H} & X. \end{array}$$

And in fact this morphism is surjective, because every point in the extra bit  $S^{n-1} \times (0, 1]$  is glued onto  $X$  by  $H$ . But it’s actually a split monomorphism as well, because morphism of spans above has a left inverse

$$\begin{array}{ccccc} D^n & \longleftarrow & S^{n-1} & \xrightarrow{f} & X \\ \uparrow & & \uparrow & & \parallel \\ (D^n \times \{0\}) \cup (S^{n-1} \times I) & \longleftarrow & S^{n-1} \times I & \xrightarrow{H} & X. \end{array}$$

This means  $J$  is actually a homeomorphism, because it is a surjection with a continuous left inverse. The punchline is that  $D^n \amalg_f X$ , and by symmetry  $D^n \amalg_g X$ , are both homeomorphic to deformation retracts of the same space (and hence are homotopy equivalent).  $\square$

Exercise: Reprove Lemma 1 in terms of the simplicial inclusions, using the fact that  $\Delta^n$  deformation retracts onto any of its “horns”  $\Lambda_i^n$  (those spaces formed by removing the  $i$ th face from  $\partial \Delta^n$ ).

## 2 The simplex category, gluing, and presheaves

Simplicial sets are a more “algebraic” or “combinatorial” way of modelling homotopy types. This has the advantage that it transports more easily to algebraic contexts. E.g., the (1-)category of topological abelian groups is not abelian but the (1-)category of simplicial abelian groups is! We saw above through careful analysis of CW complexes that any homotopy type is built up from gluing together simplices along their boundaries. For CW complexes the gluing was fairly geometric, an actual pushout in the category of topological spaces. Simplicial sets take the opposite approach: they are formal gluings of (formal!) simplices. Before we can define simplicial sets we must discuss the (category of) simplices from which they are glued.

**Definition 2.** The simplex category  $\Delta$  has objects the finite nonempty ordinals  $[n] = \{0, 1, \dots, n\}$  and a morphism  $[n] \rightarrow [m]$  is simply an order preserving function. The augmented simplex category  $\Delta_a$  is defined in the same way, but the empty ordinal  $[-1] = \emptyset$  is included.

Note that  $\Delta$  is equivalent to the category of all finite totally ordered sets. What does this have to do with actual geometric simplices? The object  $[n]$  should be understood as a representation of the geometric  $n$ -simplex  $\Delta^n$ , and its elements  $0, \dots, n$  representing the  $(n+1)$ -vertices of that simplex. As demonstrated by simplicial or singular homology, it’s often more convenient to work with simplices that have a chosen order on their vertices (for manageably and consistently tracking orientation); this is why we’re looking at ordered finite sets and not just finite sets<sup>1</sup>. The geometric simplex  $\Delta^n$  is the convex hull of its vertices  $e_0, \dots, e_n$ , and this means that every function of finite sets  $\{e_0, \dots, e_n\} \mapsto \{e_0, \dots, e_m\}$  has a unique extension to an affine transformation  $\Delta^n \rightarrow \Delta^m$  sending vertices to vertices. Thus  $\Delta$  could just as truthfully be described as the category of geometric simplices  $\Delta^n \subseteq \mathbb{R}^{n+1}$  with morphisms the affine transformations sending vertices to vertices and preserving the standard order on those vertices.

**Definition 3.** Let

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^n t_i = 1 \right\}$$

be the  $n$ -dimensional “geometric simplex”. The vertices of  $\Delta^n$  are the standard basis vectors  $e_0, \dots, e_n$  of  $\mathbb{R}^{n+1}$  and any point in  $\Delta^n$  can be uniquely represented as a convex combination  $t_0 e_0 + \dots + t_n e_n$  of them. Given an order-preserving map  $f : [n] \rightarrow [m]$  there is an induced continuous map  $\tilde{f} : \Delta^n \rightarrow \Delta^m$  defined by

$$\tilde{f} \left( \sum_{i=0}^n t_i e_i \right) = \sum_{i=0}^n t_i e_{f(i)}.$$

Exercise: The assignments  $[n] \mapsto \Delta^n$  and  $f \mapsto \tilde{f}$  define a faithful functor  $\Delta \rightarrow \mathbf{Top}$ .

There are two important families of maps within  $\Delta$ , the coface and codegeneracy maps. Geometrically these correspond to the inclusions of a face of a simplex and the projections of a simplex onto one of its faces.

**Definition 4.** Let  $n$  be a positive integer. For  $0 \leq i \leq n$  denote by  $\delta_i^n : [n-1] \rightarrow [n]$  the unique monotone injection which omits  $i$  from its range. This is the  $i$ th coface map. Concretely,

$$\delta_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

Also define  $\sigma_i^n : [n+1] \rightarrow [n]$  to be the unique monotone surjection with  $\sigma_i^n(i) = \sigma_i^n(i+1)$ . This is the  $i$ th codegeneracy map. Concretely,

$$\sigma_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

<sup>1</sup> But for those who are interested, there is a theory of unoriented “symmetric simplicial sets”

Geometrically,  $\delta_i^n$  is the inclusion of the  $i$ th face of  $\Delta^n$  (meaning the face opposite the  $i$ th vertex) and  $\sigma_n^i$  is the projection of  $\Delta^{n+1}$  onto  $\Delta^n$  where we collapse the edge  $[e_i, e_{i+1}]$  down to a point. Any monotone map  $f : [n] \rightarrow [m]$  has a decomposition into a surjection  $[n] \twoheadrightarrow [k]$  and an injection  $[k] \hookrightarrow [m]$ ; this may be easiest to see if we think of  $[k]$  as the image  $f$  with the order inherited from  $[m]$  (passing to the category of all finite nonempty totally ordered sets). Furthermore the injection  $[k] \hookrightarrow [m]$  can be decomposed into a composition of coface maps, omitting elements of  $[m]$  one at a time, and the surjection  $[n] \twoheadrightarrow [k]$  may be decomposed into a composition of codegeneracy maps, squishing together elements  $i, i+1$  such that  $f(i) = f(i+1)$  one at a time until none remain. This tells us that every morphism in  $\Delta$  is a composition of face and degeneracy maps. In fact there is a normal form associated to this decomposition, obtained by repeatedly applying the “cosimplicial identities”.

**Theorem 5.** *The simplex category  $\Delta$  is the free category  $\mathbf{C}$  on a sequence of objects  $[0], [1], \dots$  and families of morphisms  $\{\delta_i^n \in \text{Hom}_{\mathbf{C}}(n-1, n)\}_{n \geq 1, 0 \leq i \leq n}$  and  $\{\sigma_i^n \in \text{Hom}_{\mathbf{C}}(n+1, n)\}_{n \geq 0, 0 \leq i \leq n}$ , subject to the relations (for all  $n$ )*

$$\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n \quad (\text{if } i < j) \quad (1)$$

$$\sigma_j^{n+1} \circ \delta_i^{n+2} = \delta_i^{n+2} \circ \sigma_{j-1}^n \quad (\text{if } i < j) \quad (2)$$

$$\sigma_j^n \circ \delta_j^{n+1} = \text{id}_{[n]} \quad (3)$$

$$\sigma_j^n \circ \delta_{j+1}^{n+1} = \text{id}_{[n]} \quad (4)$$

$$\sigma_j^{n+1} \circ \delta_i^{n+2} = \delta_{i-1}^{n+1} \circ \sigma_j^n \quad (\text{if } i > j+1) \quad (5)$$

$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \quad (\text{if } i \leq j) \quad (6)$$

We will not prove this theorem in these notes, but we will attempt to explain what these identities say in the simplex category and explain what it means for a category to be presented by generators and relations. The equations (1) and (2) are a commutativity condition, they express (with index shifts appropriate to the  $\delta$ 's and  $\sigma$ 's) that omitting a vertex  $i$  and then omitting/collapsing a later vertex  $j$  is the same as first omitting/collapsing  $j-1 = \delta_i^{-1}(j)$  and then omitting  $i$ . The equations (3) and (4) are perhaps the most important identities, because their categorical interpretation is that each  $\delta$  is a *split monomorphism* and each  $\sigma$  is a *split epimorphism*; explicitly they say that if we omit a vertex and then collapse it with the next/previous vertex, it's the same as doing nothing. The equation (5) can be understood as saying “far away” omissions/collapses do not affect each other (up to reindexing!). And finally equation (6) expresses that if you collapse twice in a row, the order of collapses matters only in that it shifts up the indexing.

The “free category” part of the theorem is more directly relevant, because it gives an explicit description of functors  $\Delta \rightarrow \mathbf{C}$  for any category  $\mathbf{C}$  (like how a presentation of a group  $G$  tells you what group homomorphisms  $G \rightarrow H$  are). One interpretation of a “free structure” is exactly this kind of universal property, i.e. a free thing (“group” or “category equipped with a sequence of objects and families of maps satisfying the cosimplicial identities”) is an initial object in the category of things. A free group  $G$  on generators  $x_1, \dots, x_n$  subject to relations  $r_1, \dots, r_m$  is an initial object in the category of tuples  $(H, y_1, \dots, y_n)$  of groups  $H$  and  $\mathbf{y} \in H^n$  such that for each  $j$ , interpreting  $x_i$  as  $y_i$  in  $w_j$  gives the identity element of  $H$ ; a morphism  $(H, \mathbf{y}) \rightarrow (H', \mathbf{z})$  in this category is of course a group homomorphism  $f : H \rightarrow H'$  such that  $f(y_i) = z_i$  for each  $i$ . Hence a free category on objects  $\{X_s\}_{s \in S}$  and morphisms  $\{f_\lambda : X_s \rightarrow X_t\}_{s,t \in S, \lambda \in \Lambda_{s,t}}$  subject to some equations of morphisms  $\{E_j\}_{j \in J}$  is an initial object in the category<sup>2</sup> of categories that are equipped with a chosen family of objects labelled by  $S$  and a chosen family of morphisms labelled by the  $\Lambda_{s,s'}$  satisfying all equations  $E_j$ . There is also a “by hand” construction of a free category on a directed graph/quiver  $G$ , e.g. the graph with vertices  $\mathbb{N}$  and edges labelled by the coface/codegeneracy maps. This construction is fairly simple, if  $v, w$  are vertices in  $G$  then a morphism  $v \rightarrow w$  in the free category is just a path (“formal composition of edges”) from  $v$  to  $w$  in  $G$ . One can then quotient the set of arrows of this category by the smallest equivalence relation which contains the equations and “respects composition” (like how a normal subgroup gives an equivalence relation which multiplication).

We might stop and ask at this point why we need the codegeneracies at all. If we're interested in gluing together simplices along their boundaries, surely we just need the face inclusions? It turns out that the theory of simplicial sets is much nicer when degeneracies are included, especially products and quotients of them, but I do not know an “a priori” motivation for this. The theory without degeneracies isn't useless, though, we obtain what are called “semi-simplicial sets”. These are called “ $\Delta$ -complexes” in Hatcher's algebraic topology textbook.

<sup>2</sup>Here I mean the locally small category of small categories such that etc. But in fact an initial object of this category will have the right mapping out property with respect to locally small categories too, as any functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  with  $\mathbf{C}$  factors through a small subcategory  $\mathbf{D}'$  of  $\mathbf{D}$ ; specifically  $\mathbf{D}'$  is the full subcategory of  $\mathbf{D}$  on the objects in the image of  $F$ .

We now return to simplicial sets, having gained an understanding of what kind of “formal simplices” we’re gluing together. The categorical understanding of “gluing” is that it is a colimit. And vice versa, in many concrete categories a colimit does performing some kind of concrete “gluing”. This is all there is to the definition of a simplicial set.

**Definition 6.** The category of simplicial sets  $\mathbf{sSet}$  is the free<sup>3</sup> cocompletion of  $\Delta$ . That is to say  $\mathbf{sSet}$  has all (small) colimits, comes equipped with a functor  $Y : \Delta \rightarrow \mathbf{sSet}$ , and for any other category  $\mathbf{C}$  with all (small) colimits and a functor  $F : \Delta \rightarrow \mathbf{C}$  there exists a colimit preserving functor  $G : \mathbf{sSet} \rightarrow \mathbf{C}$  equipped with an isomorphism  $\varphi : G \circ Y \rightarrow F$ . Furthermore  $(G, \varphi)$  is unique in that if we have another colimit-preserving functor  $G' : \mathbf{sSet} \rightarrow \mathbf{C}$  equipped with an isomorphism  $\psi : G' \circ Y \rightarrow F$  then there is a unique isomorphism  $\eta : G \rightarrow G'$  making the diagram

$$\begin{array}{ccc} G \circ Y & & \\ \eta Y \downarrow & \searrow \varphi & \\ G' \circ Y & \xrightarrow{\psi} & F \end{array}$$

commute.

Intuitively this says that an object of  $\mathbf{sSet}$  is a formal colimit of some diagram in  $\Delta$ . One can construct a free cocompletion in this way, but I tried to write it down once and lost two weeks working out technical details. Luckily the free cocompletion of a small category is a recognizable, fairly simple category. The rest of this section will be devoted to proving the following theorem.

**Theorem 7.** Let  $\mathbf{C}$  be a small category and  $\mathbf{Psh}(\mathbf{C}) = \mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Set})$  the category of presheaves on  $\mathbf{C}$ . The yoneda embedding  $y : \mathbf{C} \rightarrow \mathbf{Psh}(\mathbf{C})$  exhibits  $\mathbf{Psh}(\mathbf{C})$  as the free cocompletion of  $\mathbf{C}$ .

Because of this, most people would find my original definition of  $\mathbf{sSet}$  a little silly. The true definition is just  $\mathbf{sSet} = \mathbf{Psh}(\Delta) = \mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Set})$ . Our presentation of  $\Delta$  tells us that a simplicial set can also be understood as sequence of sets  $\{X_n\}_{n \in \mathbb{N}}$  equipped with morphisms  $s_i^n, d_i^n$  satisfying the *simplicial identities*, the categorical dual of the cosimplicial identities (because presheaves are contravariant functors  $\Delta \rightarrow \mathbf{Set}$ ). We will expand on this later.

---

<sup>3</sup>This is not quite freeness in the sense discussed above; it is about 2-initiality in an appropriate 2-category!