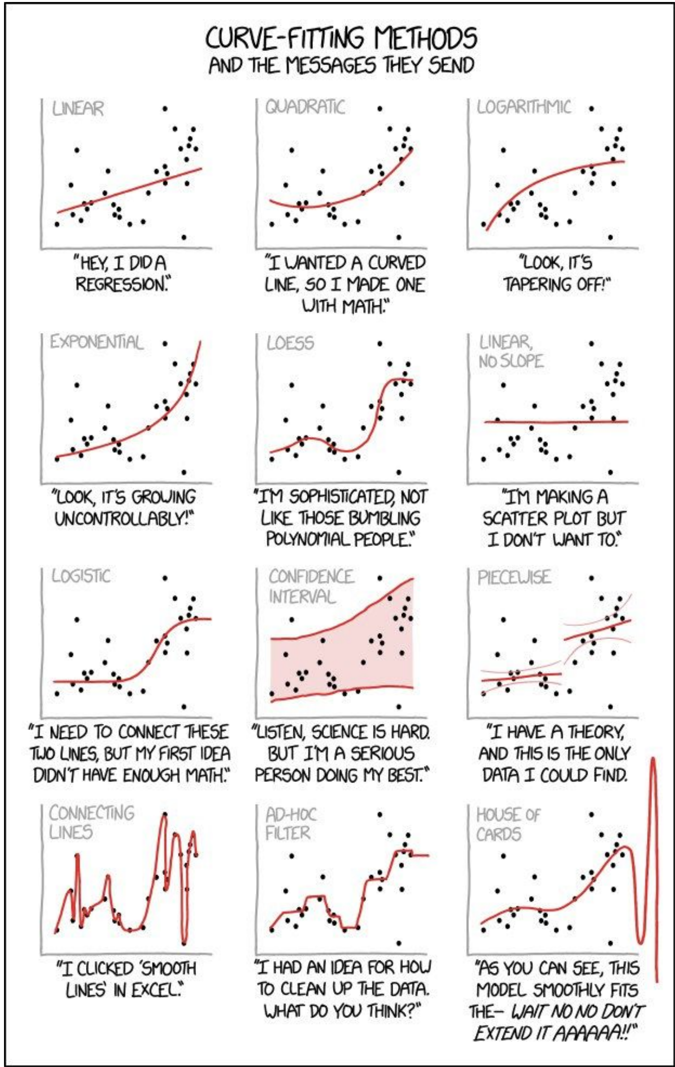
Lecture 2: Linear regression

MIPT, 2019

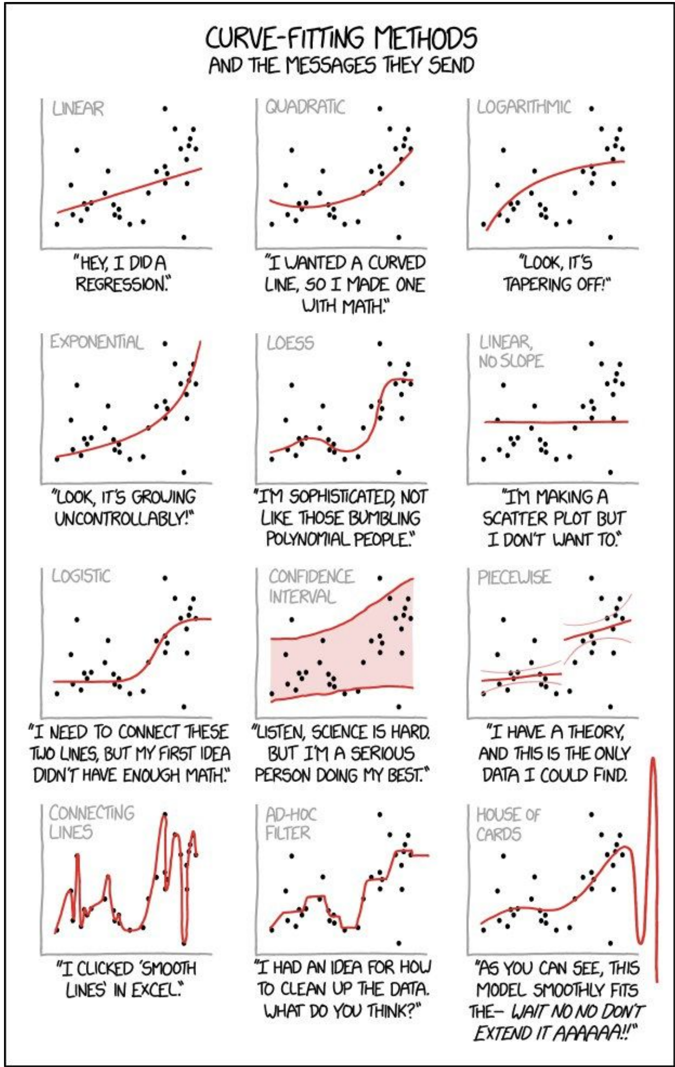
Outline

1. Overview of linear models.
2. Linear regression.
3. Analytical solution.
4. Regularization.
5. Gauss-Markov theorem.
6. Probabilistic interpretation and intuition.


CURVE-FITTING METHODS
AND THE MESSAGES THEY SEND
LINEAR
"HEY, I DID A REGRESSION."
QUADRATIC
"I WANTED A CURVED LINE, SO I MADE ONE WITH MATH."
LOGARITHMIC
"LOOK, IT'S TAPERING OFF!"
EXPONENTIAL
"LOOK, IT'S GROWING UNCONTROLLABLY!"
LOESS
"I'M SOPHISTICATED, NOT LIKE THOSE BUMBLING POLYNOMIAL PEOPLE."
LINEAR, NO SLOPE
"I'M MAKING A SCATTER PLOT BUT I DON'T WANT TO."
LOGISTIC
"I NEED TO CONNECT THESE TWO LINES, BUT MY FIRST IDEA DIDN'T HAVE ENOUGH MATH."
CONFIDENCE INTERVAL
"LISTEN, SCIENCE IS HARD. BUT I'M A SERIOUS PERSON DOING MY BEST."
PIECEWISE
"I HAVE A THEORY, AND THIS IS THE ONLY DATA I COULD FIND."
CONNECTING LINES
"I CLICKED 'SMOOTH LINES' IN EXCEL."
AD-HOC FILTER
"I HAD AN IDEA FOR HOW TO CLEAN UP THE DATA. WHAT DO YOU THINK?"
HOUSE OF CARDS
"AS YOU CAN SEE, THIS MODEL SMOOTHLY FITS THE– *WAIT NO NO DON'T EXTEND IT AAAAAA!!*"

Example questions linear regression can solve (up right picture):

- What will be my monthly spending for the next year?
- Which factor is more important in deciding my monthly spending?
- How monthly income and trips per month are correlated with monthly spending?



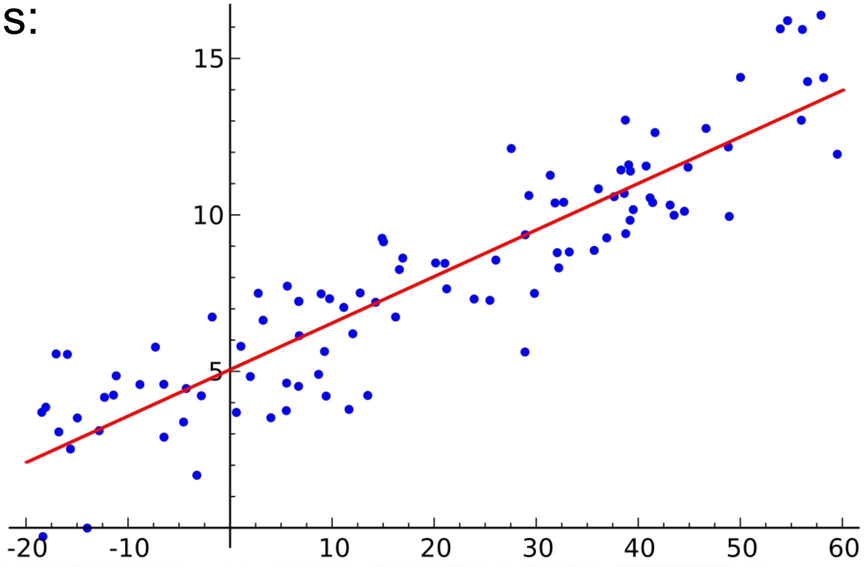
Linear models

$$Y = \underbrace{X_1 + X_2 + X_3}$$

Dependent Variable	Independent Variable
Outcome Variable	Predictor Variable
Response Variable	Explanatory Variable

Linear models

- Predictive models:



Estimated (or predicted) Y value for observation i

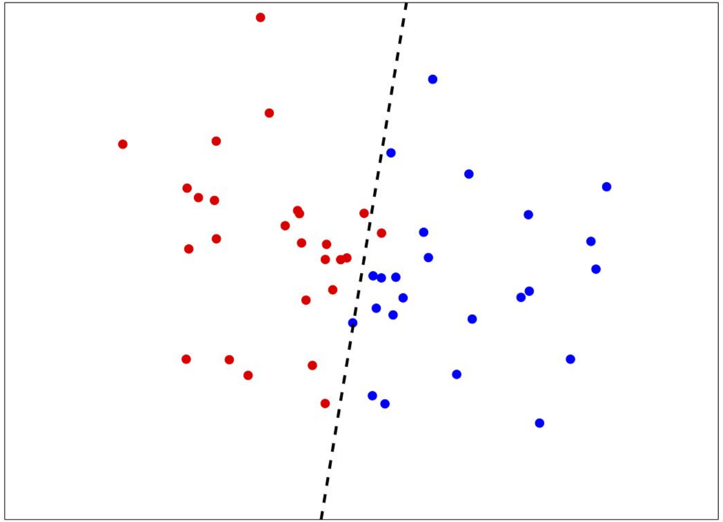
Estimate of the regression intercept

Estimate of the regression slope

Value of X for observation i

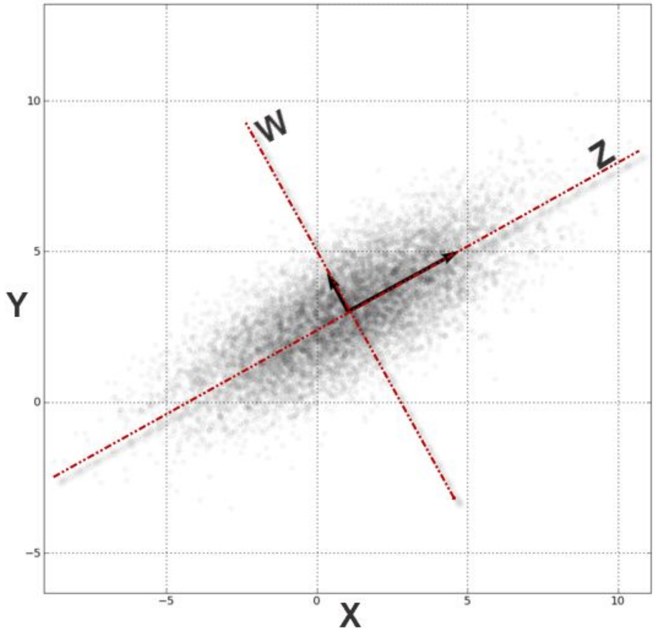
$$\hat{Y}_i = b_0 + b_1 X_i$$

- Predictive models:
- Classification models:



Linear models

- Predictive models:
- Classification models:
- Unsupervised models (e.g. PCA analysis)



Linear models

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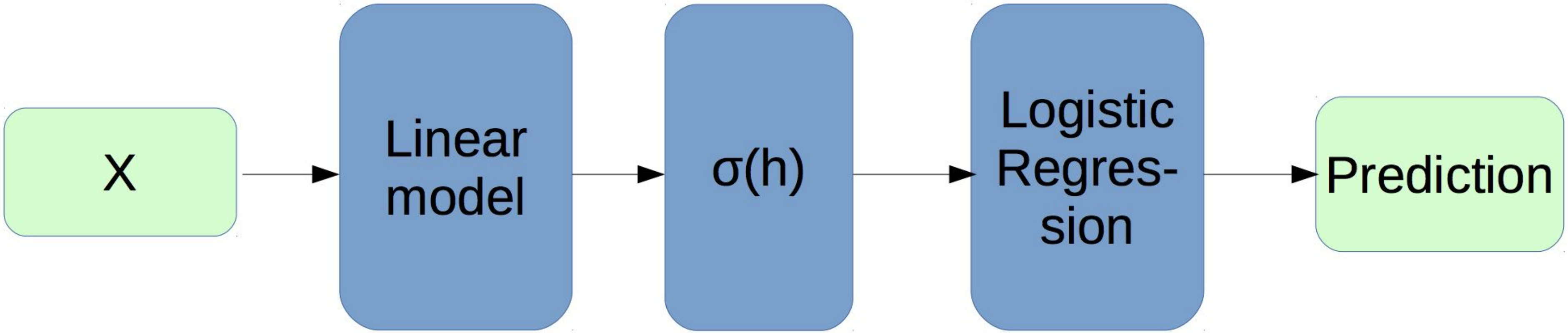
- Building block of other models (ensembles, NNs, etc.)



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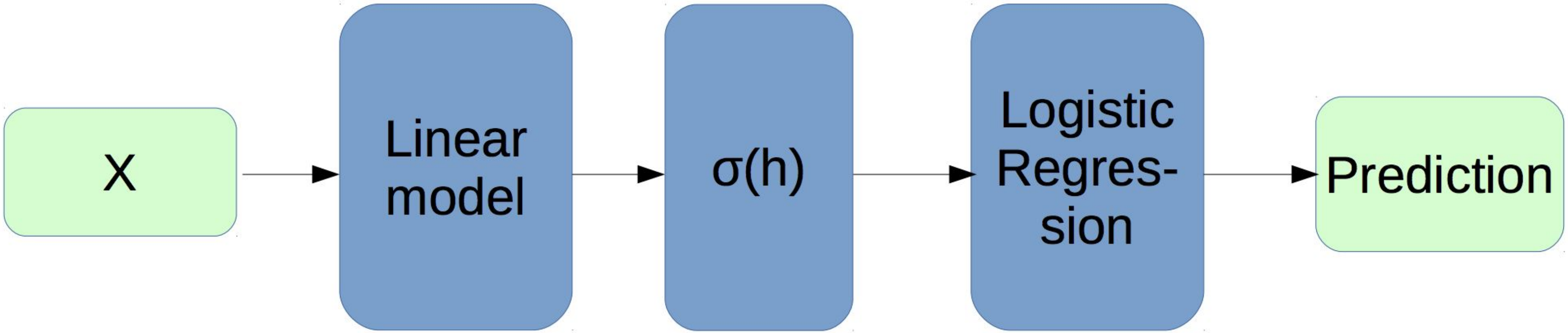

Actually, it's a neural network. We will meet it later.

Linear regression

Linear regression problem statement:

- Training set $\mathcal{L} = \{\mathbf{x}_i, y_i\}_{i=1}^{n}$, where $(\mathbf{x} \in \mathbb{R}^p,\ y \in \mathbb{R})$.

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  Where $\mathbf{w} = (w_1, \ldots w_p)$ is weights vector, w_0 is bias term.

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- Least squares method provides a solution:

$$\hat{\mathbf{w}} = \arg\min_{\mathbf{w}} \|\mathbf{w}_0 + (\mathbf{x}_1 \ldots \mathbf{x}_n)^T \mathbf{w} - (y_1, \ldots, y_n)\|_2^2$$

Analytical solution

Denote quadratic loss function: $Q(\mathbf{w}) = (Y - X\mathbf{w})^T(Y - X\mathbf{w}) = \|Y - X\mathbf{w}\|_2^2$,

where $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n], \quad \mathbf{x}_i \in \mathbb{R}^p$, $Y = [y_1, \ldots, y_n], \quad y_i \in \mathbb{R}$.

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what if this matrix is *singular?*

Unstable solution

In case of multicollinear features the matrix $X^T X$ is almost singular .

It leads to unstable solution:

```
w_true
```

```
array([ 2.68647887, -0.52184084, -1.12776533])
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w_star = np.linalg.inv(X.T.dot(X)).dot(X.T).dot(Y)
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corresponding features are almost collinear

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the coefficients are huge and sum up to almost 0

To make the matrix nonsingular, we can add a diagonal matrix:

$$\hat{\mathbf{w}} = (X^T X + \lambda I)^{-1} X^T Y \,,$$

where $I = \mathrm{diag}[1_1, \ldots, 1_p]$.

Actually, it's a solution for the case $Q(\mathbf{w}) = \|Y - X\mathbf{w}\|_2^2 + \lambda^2 \|\mathbf{w}\|_2^2$.

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exercise: check it by yourself

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Then the solution $\hat{\mathbf{w}} = (X^T X)^{-1} X^T Y$ delivers BLEU: **B**est **L**inear **U**nbiased **E**stimator.

Different norms

Once more: loss functions:

- $MSE = \frac{1}{n}\|\mathbf{x}^T\mathbf{w} - \mathbf{y}\|_2^2$

Regularization terms:

- L_2: $\|\mathbf{w}\|_2^2$

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only works for Gauss-Markov theorem

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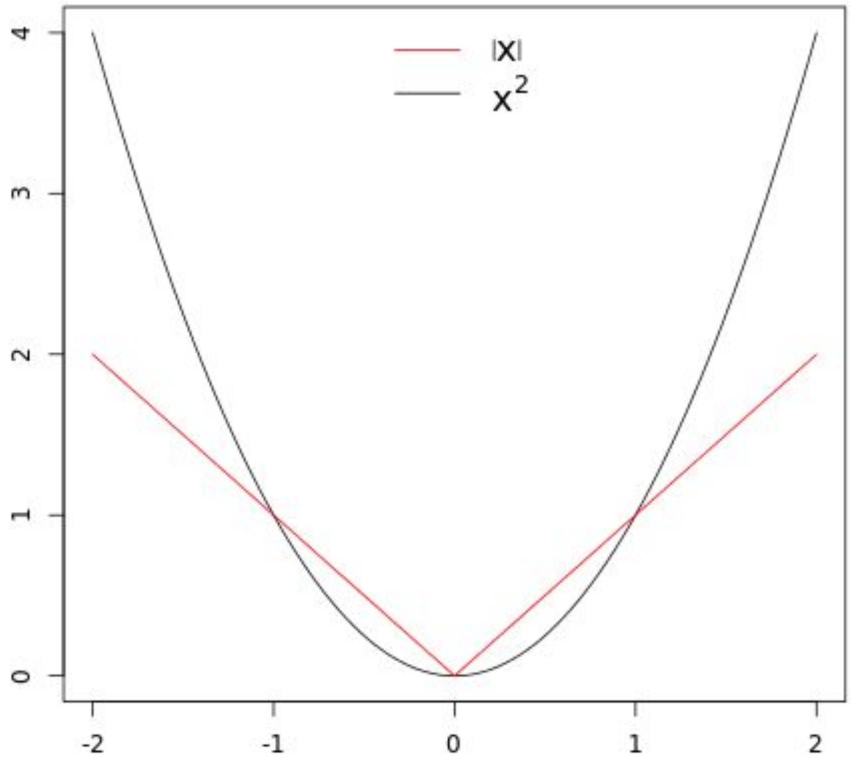
What's the difference?



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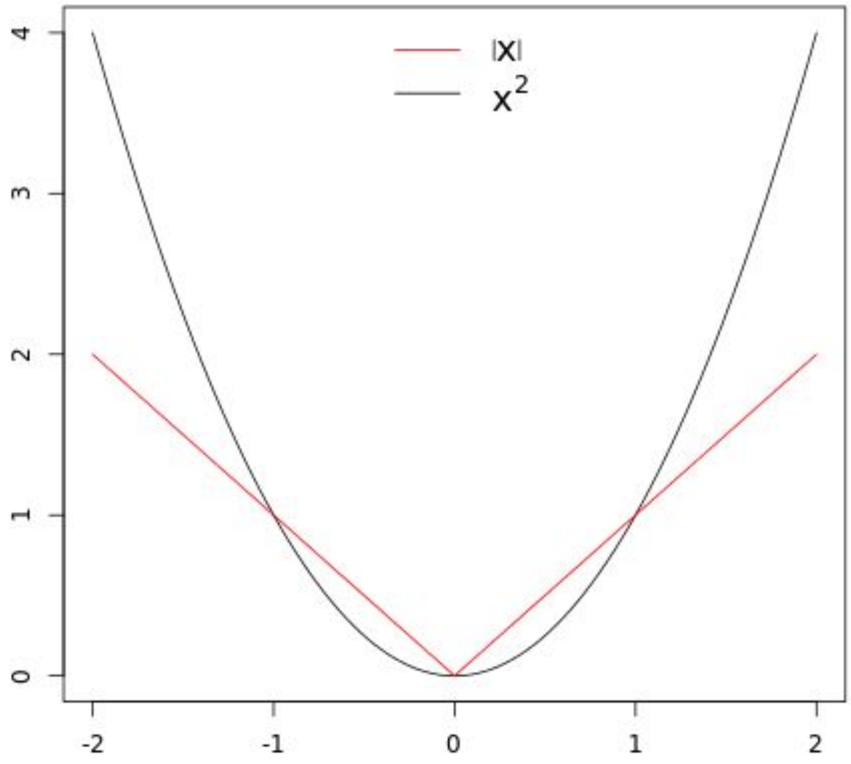

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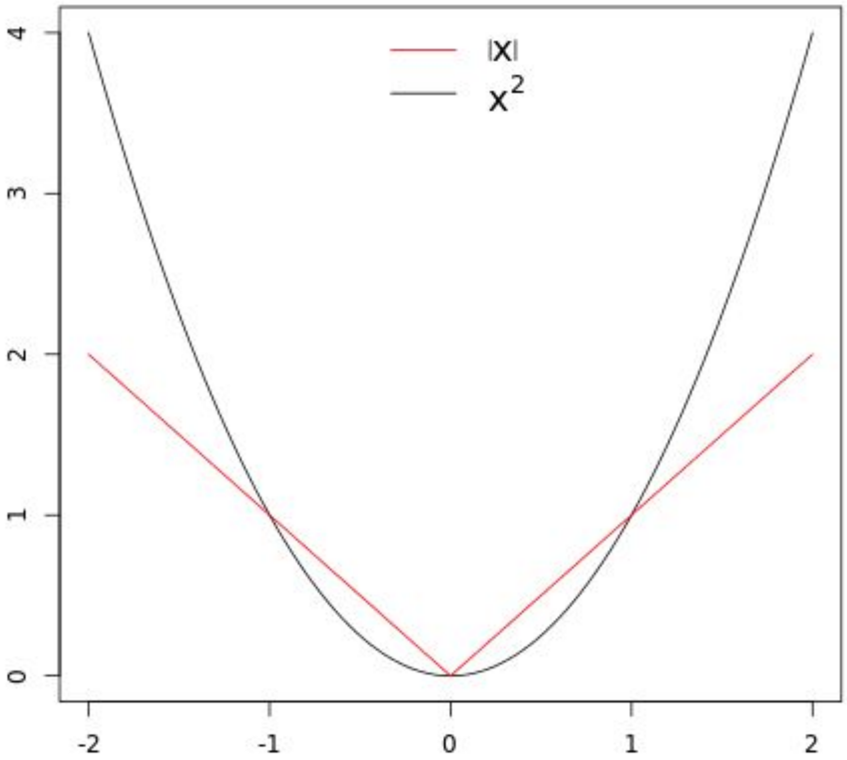

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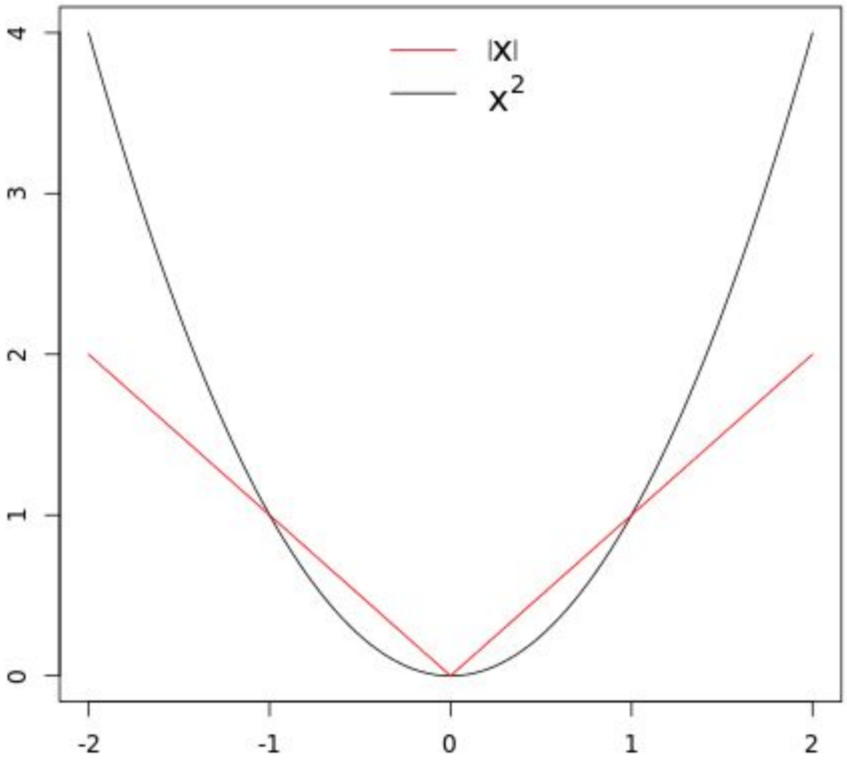

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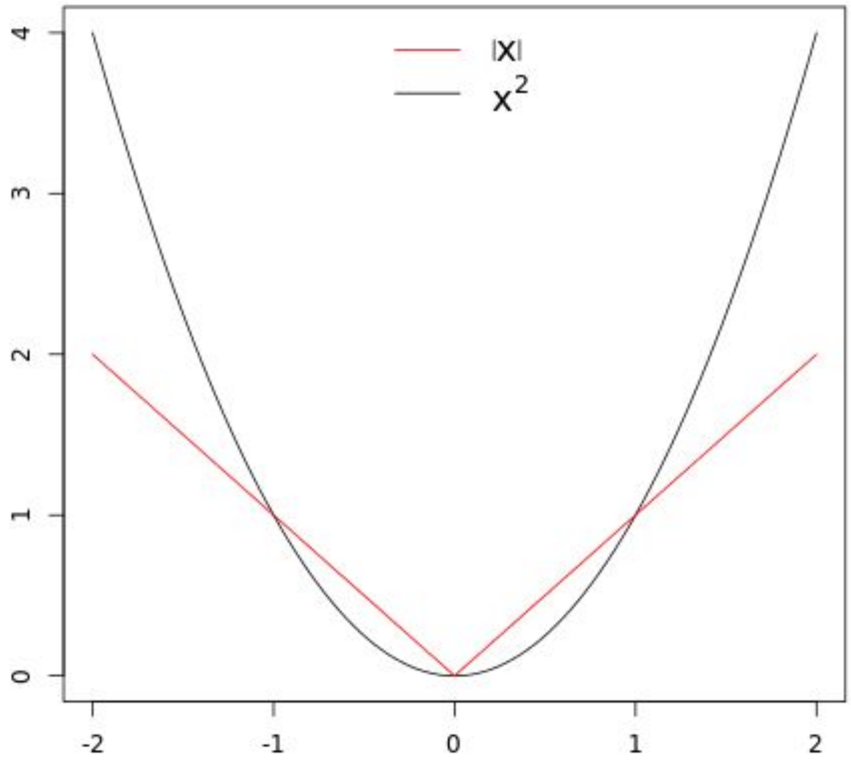

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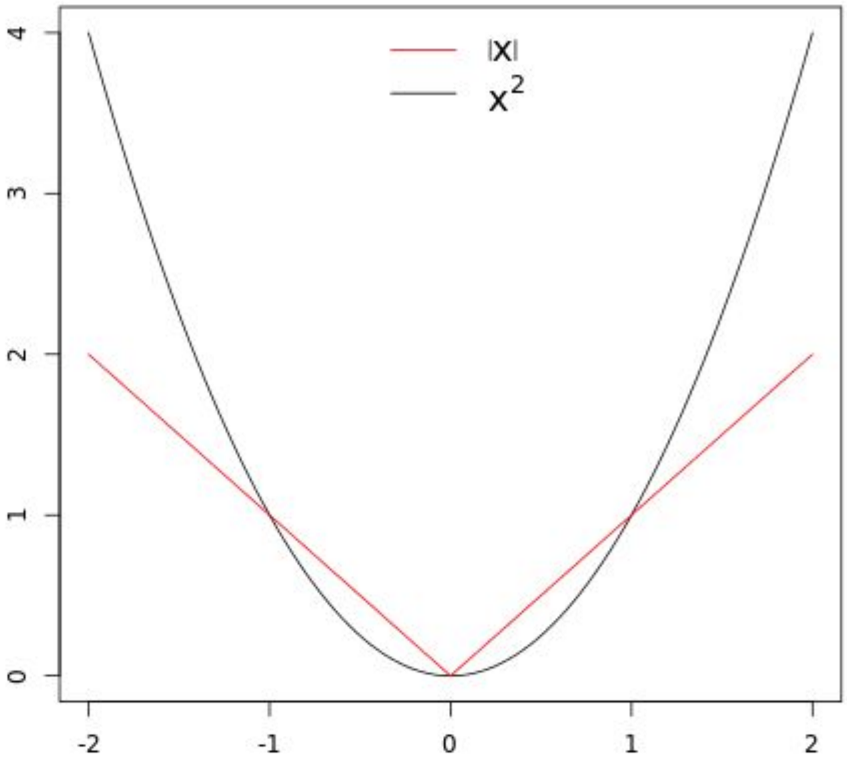
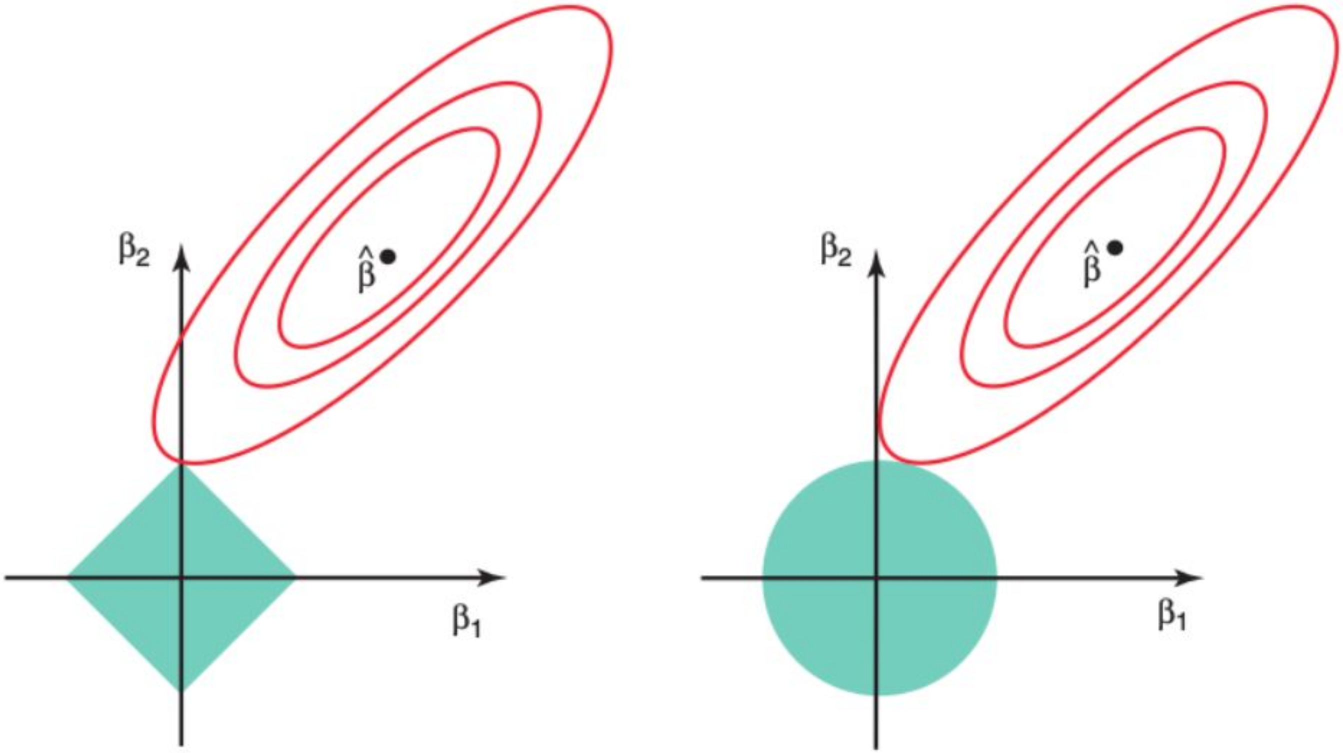
Does what?



Regularization: illustration

$$\mathbf{w} = [\beta_1, \beta_2]$$



source: Modern Multivariate Statistical Techniques

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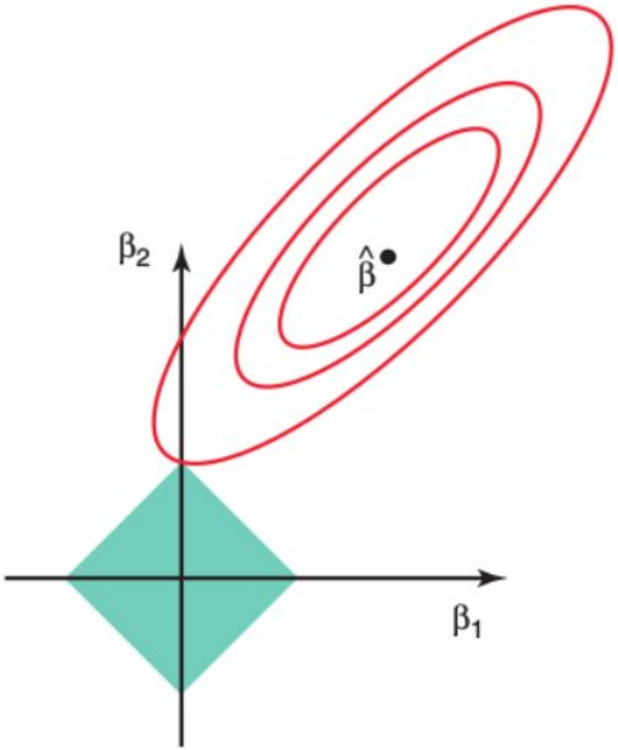
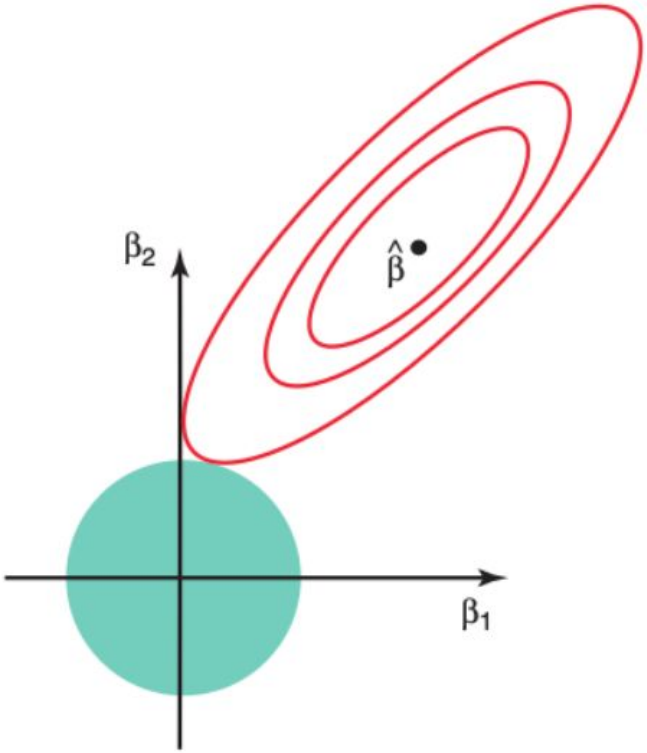


L_1 regularization



L_2 regularization

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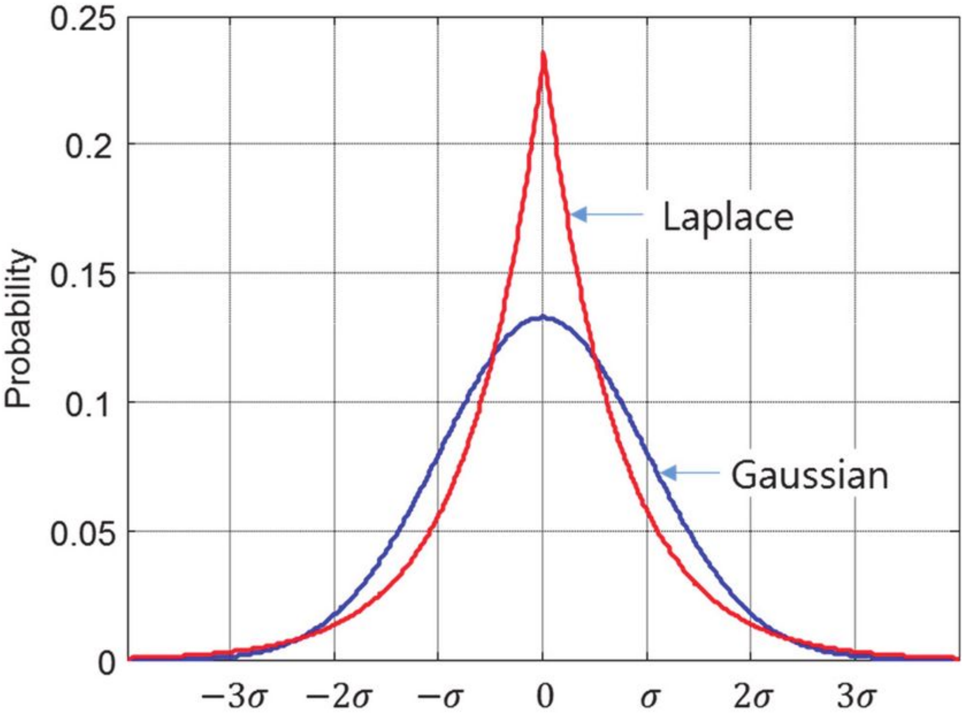
Regularization: probability interpretation

assume **w** elements are sampled from some *specific* distribution (prior distribution for the weights vector)



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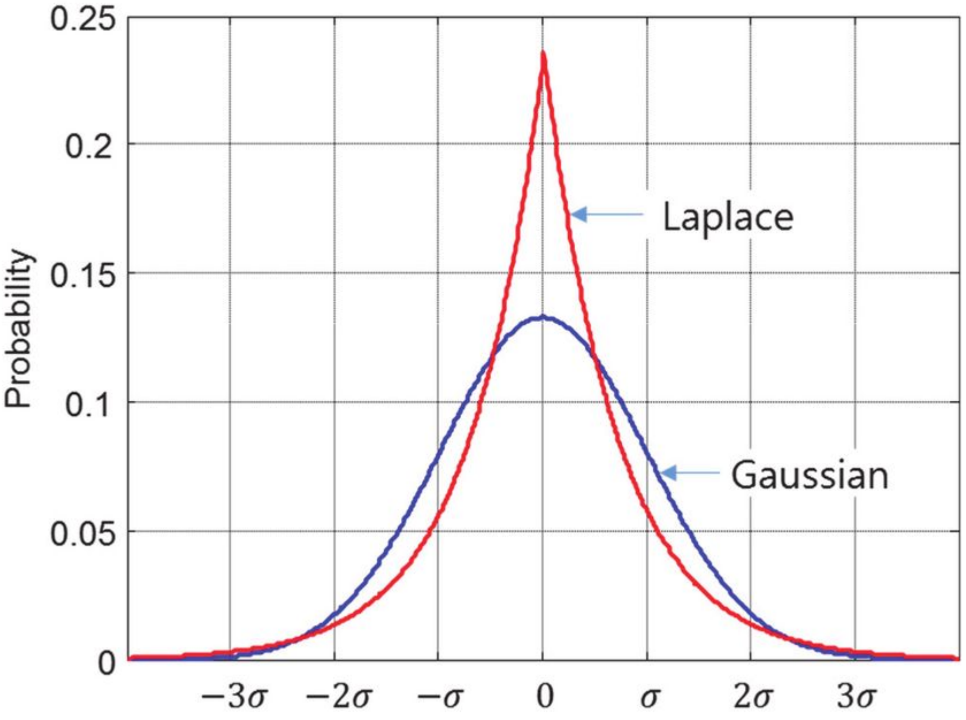

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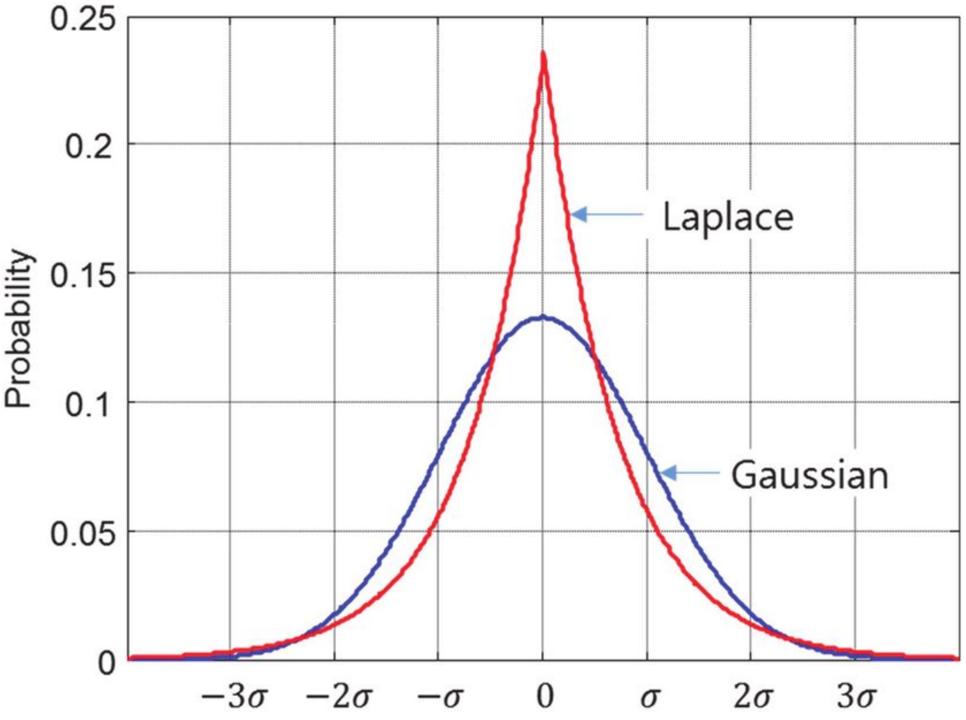

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see seminar extra materials for more

Welcome to the church of Bayes

Дмитрий Ветров - заведующий Международной лабораторией глубинного обучения и байесовских методов

That’s all. Practice coming next.