Lecture 9: Value based methods

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References

These slides are almost the exact copy of Practical RL course week 2 slides by Shvechikov Pavel.
Special thanks to YSDA team for making them publicly available.

Original slides link: week02_value_based

Previously in the course

- ## The MDP formalism
  - ### State, Action, Reward, next State

- ## Cross-Entropy Method (CEM)
  - ### easy to implement, good results
  - ### rich theoretical background
  - ### black box
    - #### no knowledge of environment
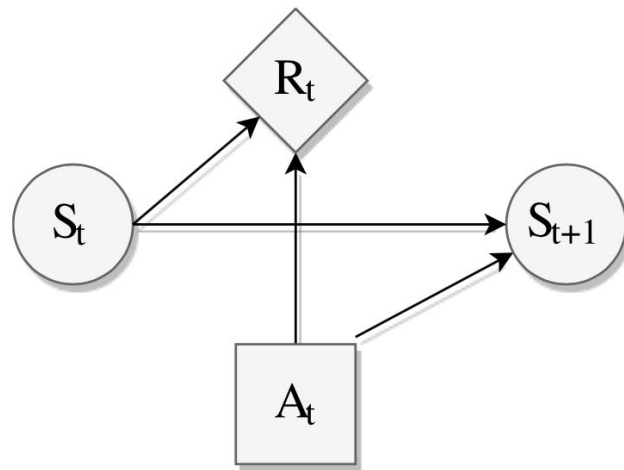   - #### no knowledge of intermediate rewards

elites
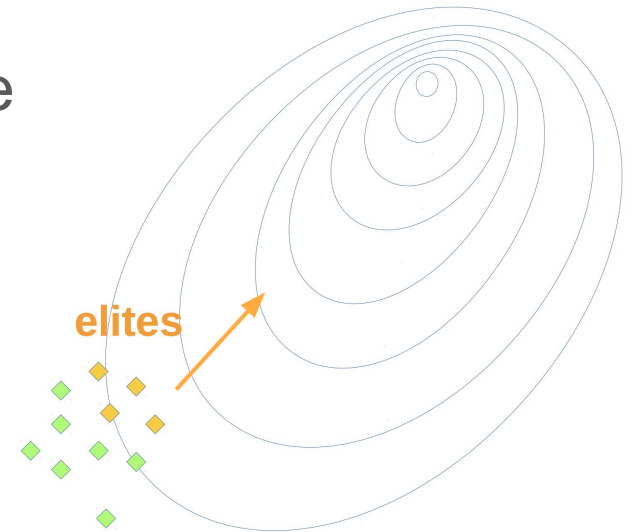
Improve on the CEM → dive into the black box

Given dynamics, how to find an optimal policy?

Definition of Markov Decision Process

MDP is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R} \rangle$, where

1. $\mathcal{S}$ – set of states of the world
2. $\mathcal{A}$ – set of actions
3. $\mathcal{P} : \mathcal{S} \times \mathcal{A} \mapsto \triangle(\mathcal{S})$ – state-transition function, giving us $p(s_{t+1} \mid s_t, a_t)$
4. $\mathcal{R} : \mathcal{S} \times \mathcal{A} \mapsto \mathbb{R}$ – reward function, giving us $\mathbb{E}_R \left[R(s_t, a_t) \mid s_t, a_t \right]$.

Markov property

$$p(r_t, s_{t+1} \mid s_0, a_0, r_0, ..., s_t, a_t) = p(r_t, s_{t+1} \mid s_t, a_t)$$

(next state, expected reward) depend on (previous state, action)

Goal: solve an MDP by finding **an** optimal policy

1. What is the objective?
   a. Reward: discounting and design
   b. Expected objective: state- and action-value function

2. How to evaluate the objective?
   a. Bellman expectation equations

3. How to improve the objective?
   a. Bellman optimality equations

4. Combine evaluation and improvement:
   a. Generalized Policy Iteration

Explaining goals to agent through reward

Reward hypothesis (R.Sutton)

Goals and purposes can be thought of as the maximization of the expected value of the cumulative sum of a received scalar signal

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Cumulative reward is called a return:

$$G_t \triangleq R_t + R_{t+1} + R_{t+2} + ... + R_T$$

E.g.: reward in **chess** – value of taken opponent's piece

Explaining goals to agent through reward

Reward hypothesis (R.Sutton)

Goals and purposes can be thought of as the maximization of the expected value of the cumulative sum of a received scalar signal

Cumulative reward is called a return:

end of an episode

$$G_t \triangleq R_t + R_{t+1} + R_{t+2} + ... + R_T$$

immediate reward

E.g.: reward in **chess** – value of taken opponent's piece

E.g.: data center non-stop cooling system

- **S**tates – temperature measurements
- **A**ctions – different fans speed
- **R = 0**   for exceeding temperature thresholds
- **R = +1** for each second system is cool

What could go wrong with such a design?

E.g.: data center non-stop cooling system

- **S**tates – temperature measurements
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What could go wrong with such a design?

Infinite return for non optimal behaviour!

$$G_t = 1 + 1 + 0 + 1 + 1 + 0 + \ldots = \sum_{t=1}^{\infty} R_t = \infty$$

E.g.: cleaning robot

- **S**tates – dust sensors, air
- **A**ctions – cleaning / rest / conditioning on or off
- **R = 100** for long tedious floor cleaning task done
- **R = 1** for turning air conditioning on-off
- Episode ends each day

What could go wrong with such a design?

E.g.: cleaning robot

- **S**tates – dust sensors, air
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What could go wrong with such a design?

Reward(air) < Reward(cleaning)
Time(air) << Time(cleaning)

Positive feedback loop!

R=1

R=100

Rest

One second

Whole day

SCORE 10500   LAPS —/3   TIME 0:23   TURBO READY
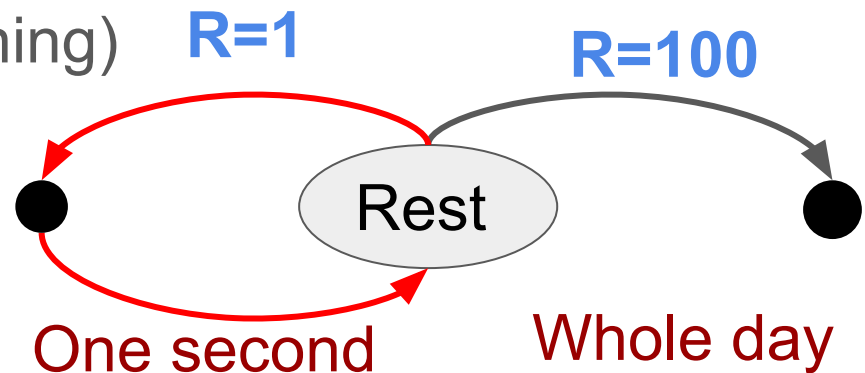
OpenAI blog post about faulty rewards: https://openai.com/blog/faulty-reward-functions/

Reward discounting

Reward discounting

Get rid of infinite sum by discounting $0 \le \gamma < 1$

$$G_t \triangleq R_t + \gamma R_{t+1} + \gamma^2 R_{t+2} + ... = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

discount factor

The same cake compared to today's one worth

- γ times less tomorrow
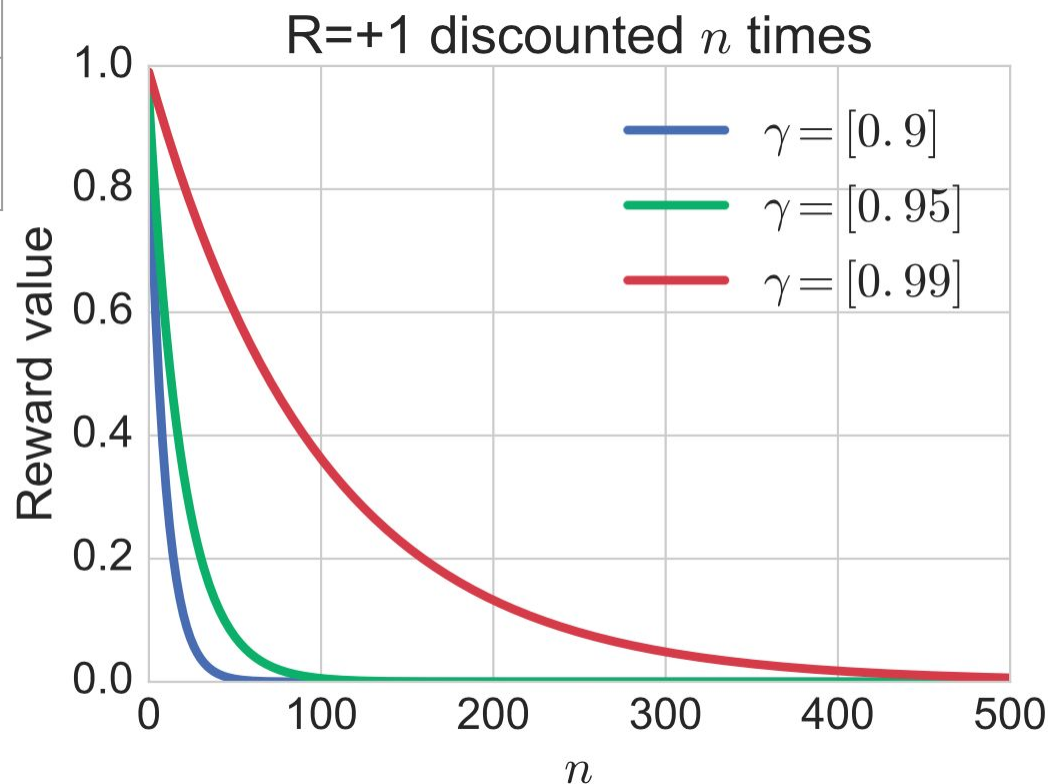- γ^2 times less the day after tomorrow
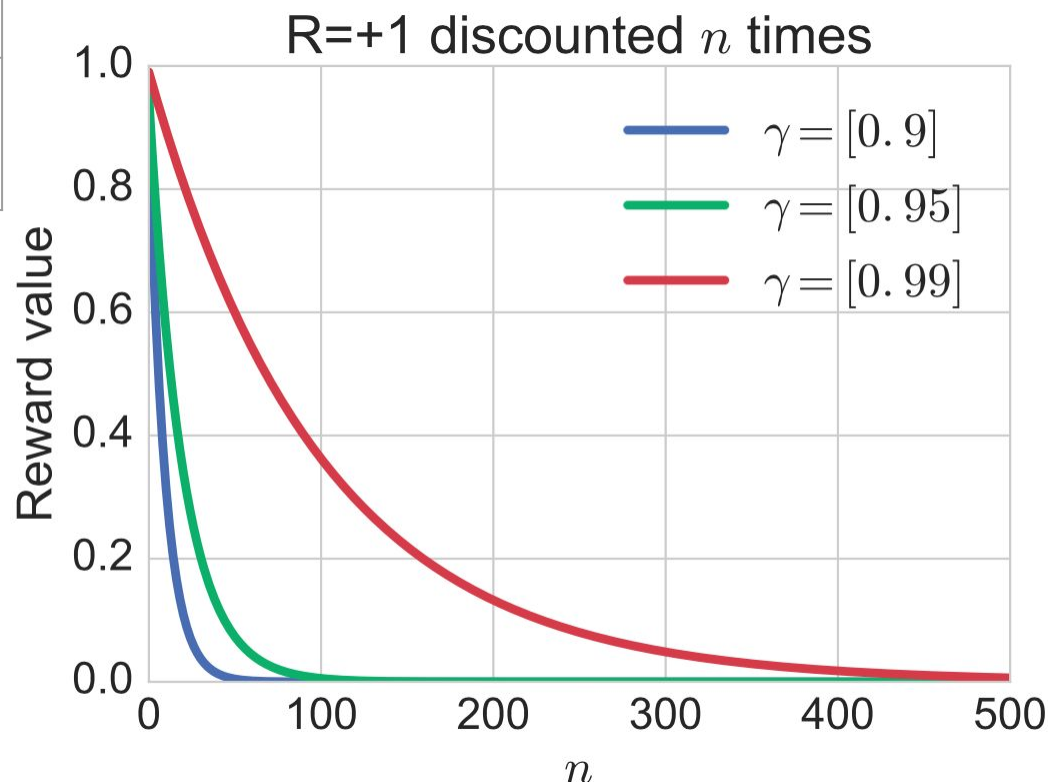

γ will eat it day by day

Discounting makes sums finite

Maximal return for **R = +1**

$$G_0 = \sum_{k=0}^{\infty} \gamma^k = \frac{1}{1 - \gamma}$$

γ	0.9	0.95	0.99
$\frac{1}{1-\gamma}$	10	20	100

R=+1 discounted n times

Discounting makes sums finite

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γ	0.9	0.95	0.99
$\frac{1}{1-\gamma}$	10	20	100

Any discounting changes optimisation task and its solution!

R=+1 discounted n times

Discounting is inherent to humans

- Quasi-hyperbolic $\quad f(t) = \beta \gamma^t$

- Hyperbolic discounting $\quad f(t) = \dfrac{1}{1 + \beta t}$

Laibson, D. (1997). Golden eggs and hyperbolic discounting. The Quarterly Journal of Economics, 112(2), 443-478.

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Mathematical convenience

$$G_t = R_t + \gamma(R_{t+1} + \gamma R_{t+2} + ...)$$
$$= \boxed{R_t + \gamma\, G_{t+1}}$$

Remember this one!
We will need it later

Laibson, D. (1997). Golden eggs and hyperbolic discounting. The Quarterly Journal of Economics, 112(2), 443-478.

Discounting is a stationary end-of-effect model

Any action affects (1) immediate reward (2) next state

Discounting is a stationary end-of-effect model

Any action affects (1) immediate reward (2) next state

Action indirectly affects future rewards

But how long does this effect lasts?

$$G_0 = R_0 + \gamma R_1 + \gamma^2 R_2 + ... + \gamma^T R_T$$
$$= (1 - \gamma)R_0$$
$$+ (1 - \gamma)\gamma(R_0 + R_1)$$
$$+ (1 - \gamma)\gamma^2(R_0 + R_1 + R_2)$$
$$\cdots$$
$$+ \gamma^T \cdot \sum_{t=0}^{T} R_t$$

G is expected return under stationary end-of-effect model

Discounting is a stationary end-of-effect model

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But how long does this effect lasts?

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$$+ (1 - \gamma)\gamma (R_0 + R_1)$$
$$+ (1 - \gamma)\gamma^2 (R_0 + R_1 + R_2)$$
$$...$$
$$+ \gamma^T \cdot \sum_{t=0}^{T} R_t$$

"Effect continuation" probability

"End of effect" probability

G is expected return under stationary end-of-effect model
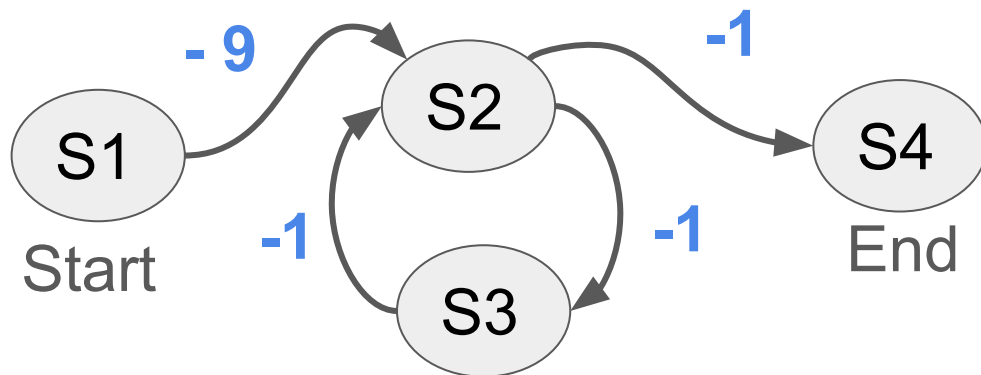
Reward design – don't shift, reward for WHAT

- E.g.: chess – value of taken opponent's piece
  - Problem: agent will not have a desire to win!

- E.g.: cleaning robot, **+100** (cleaning), **+0.1** (on-off)
  - Problem: agent will not bother cleaning the floor!

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Take away: reward only for WHAT, but never for HOW

- 9 **-1**

S1
Start

S2

-1 **-1**

S3

S4
End

Reward design – don't shift, reward for WHAT

- E.g.: chess – value of taken opponent's piece
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Take away: reward only for WHAT, but never for HOW

- 9 S1 (Start) → S2

-1 S1 → S2 (lower path through S3)

-1 S2 → S3

-1 S3 → S2

-1 S2 → S4 (End)
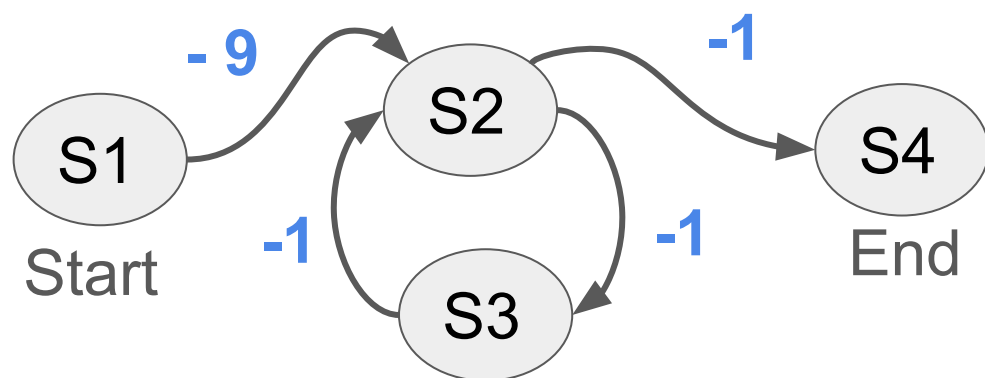
Take away: do not subtract mean from rewards

Reward design – scaling, shaping

What transformations do not change optimal policy?

- Reward **scaling** – division by positive constant
  - May be useful in practise for approximate methods

Ng, A. Y., Harada, D., & Russell, S. (1999, June). Policy invariance under reward transformations: Theory and application to reward shaping. In ICML (Vol. 99, pp. 278-287).

Reward design – scaling, shaping

What transformations do not change optimal policy?

- Reward **scaling** – division by positive constant
  - May be useful in practise for approximate methods
- Reward **shaping** – we could add to all rewards in MDP values of potential-based shaping function F(s, a, s') without changing an optimal policy:

$$F(s, a, s') = \gamma \Phi(s') - \Phi(s)$$

Intuition: when no discounting F adds as much as it subtracts from the total return

Ng, A. Y., Harada, D., & Russell, S. (1999, June). Policy invariance under reward transformations: Theory and application to reward shaping. In ICML (Vol. 99, pp. 278-287).

Expected objective

Optimal policy maximizes expected return

$$\mathbb{E}\left[G_0\right] = \mathbb{E}\left[R_0 + \gamma R_1 + \ldots + \gamma^T R_T\right]$$

$$= \mathbb{E}_{E,\pi_\theta}\left[G_0\right]$$

$$= \mathbb{E}_{\pi_\theta}\left[G_0\right]$$

$$= \mathbb{E}\left[G_0 \mid \pi_\theta\right]$$

$$= \mathbb{E}_{\substack{s_{0:T} \\ a_{0:T}}}\left[G_0\right]$$

$$= \mathbb{E}_{s_0}\left[\mathbb{E}_{a_0|s_0}\left[R_0 + \mathbb{E}_{s_1|s_0,a_0}\left[\mathbb{E}_{a_1|s_1}\left[\gamma R_1 + \ldots\right]\right]\right]\right]$$

$$= \sum_{t=0}^{T}\mathbb{E}_{(s_t,a_t)\sim p_\theta}\left[\gamma^t R_t\right]$$

$$= \mathbb{E}_{\tau \sim p_\theta(\tau)}\left[G(\tau)\right] \qquad \tau \overset{\triangle}{=} (s_0, a_0, s_1, \ldots, a_{T-1}, s_T)$$

$$p_\theta(\tau) = p(s_0)\prod_{t=0}^{T-1}\pi_\theta(a_t|s_t)\, p(s_{t+1}|s_t,a_t)$$

State- and **Action-**value functions

State-value function v(s)

v(s) is <u>expected</u> return conditional on <u>state</u>:

$$v_\pi(s) \triangleq \mathbb{E}_\pi\left[\,G_t\,|\,S_t = s\,\right]$$

$$= \mathbb{E}_\pi\left[\,R_t + \gamma G_{t+1}\,|\,S_t = s\,\right]$$

$$= \sum_a \pi(a\,|\,s) \sum_{r,\,s'} p(r, s'\,|\,s, a)\left[r + \gamma\mathbb{E}_\pi\left[\,G_{t+1}|S_{t+1} = s'\,\right]\right]$$

$$= \sum_a \pi(a\,|\,s) \sum_{r,\,s'} p(r, s'\,|\,s, a)\left[r + \gamma v_\pi(s')\right]$$

Intuition: value of following policy π from state s

State-value function v(s)

v(s) is <u>expected</u> return conditional on <u>state</u>:

stochasticity in policy & environment

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State-value function v(s)

v(s) is <u>expected</u> return conditional on <u>state</u>:

stochasticity in policy & environment

Environment stochasticity

$$v_\pi(s) \triangleq \boxed{\mathbb{E}_\pi} \left[G_t \,|\, S_t = s \right]$$

$$= \mathbb{E}_\pi \left[R_t + \gamma G_{t+1} \,|\, S_t = s \right]$$

$$= \boxed{\sum_a \pi(a \,|\, s)} \boxed{\sum_{r,s'} p(r, s' \,|\, s, a)} \left[r + \gamma \mathbb{E}_\pi \left[G_{t+1} | S_{t+1} = s' \right] \right]$$

Policy stochasticity

$$= \sum_a \pi(a \,|\, s) \sum_{r,s'} p(r, s' \,|\, s, a) \left[r + \gamma v_\pi(s') \right]$$

Intuition: value of following policy π from state s

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v(s) is <u>expected</u> return conditional on <u>state</u>:

stochasticity in policy & environment

Environment
stochasticity

$$v_\pi(s) \doteq \mathbb{E}_\pi [G_t \,|\, S_t = s]$$

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Policy
stochasticity

$$= \sum_a \pi(a \,|\, s) \sum_{r,s'} p(r, s' \,|\, s, a) [r + \gamma v_\pi(s')]$$

By definition

Intuition: value of following policy π from state s

Action-value function q(s, a)

Is <u>expected</u> return conditional on <u>state</u> **and action**:

Intuition: value of following policy π <u>after</u> committing action **a** in state **s**

$$
\begin{aligned}
q_\pi(s, a) &= \mathbb{E}_\pi\left[\,G_t \mid S_t = s, A_t = a\,\right] \\
&= \mathbb{E}_\pi\left[\,R_t + \gamma G_{t+1} \mid S_t = s, A_t = a\,\right] \\
&= \sum_{r, s'} p(r, s' \mid s, a)\left[r + \gamma \mathbb{E}_\pi\left[\,G_{t+1} \mid S_{t+1} = s'\,\right]\right] \\
&= \sum_{r, s'} p(r, s' \mid s, a)\left[r + \gamma v_\pi(s')\right]
\end{aligned}
$$

Action-value function q(s, a)

Is expected return conditional on state **and action**:

Intuition: value of following policy π after committing action **a** in state **s**

No policy stochasticity at first step

$$q_\pi(s, a) = \mathbb{E}_\pi \left[G_t \mid S_t = s, \boxed{A_t = a} \right]$$

$$= \mathbb{E}_\pi \left[R_t + \gamma G_{t+1} \mid S_t = s, A_t = a \right]$$

$$= \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma \mathbb{E}_\pi \left[G_{t+1} \mid S_{t+1} = s' \right] \right]$$

$$= \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma v_\pi(s') \right]$$

Relations between v(s) and q(s,a)

We already know how to write q(s,a) in terms of v(s)

$$q_\pi(s, a) = \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma v_\pi(s') \right]$$

What about v(s) in terms of q(s,a)?

Relations between v(s) and q(s,a)

We already know how to write q(s,a) in terms of v(s)

$$q_\pi(s, a) = \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma v_\pi(s') \right]$$

What about v(s) in terms of q(s,a)?

$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma v_\pi(s') \right]$$

$$= \sum_a \pi(a \mid s) q_\pi(s, a)$$

So, we could now write q(s, a) in terms of q(s,a)!

$$q_\pi(s, a) = \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma \sum_{a'} \pi(a' \mid s') q_\pi(s', a') \right]$$

Bellman expectation equation for **v(s)**

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Recursive definition of v(s) is an important concept in RL

$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma v_\pi(s') \right]$$

$$= \mathbb{E}_\pi \left[R_t + \gamma v_\pi(S_{t+1}) \mid S_t = s \right]$$

Bellman expectation equation for **v(s)**

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Backup diagram

Bellman expectation equation for **v(s)**

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Backup
diagram

Bellman expectation equation for **q(s,a)**

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Backup
diagram
for q(s, a)

Bellman expectation equation for **q(s,a)**

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Backup
diagram
for q(s, a)
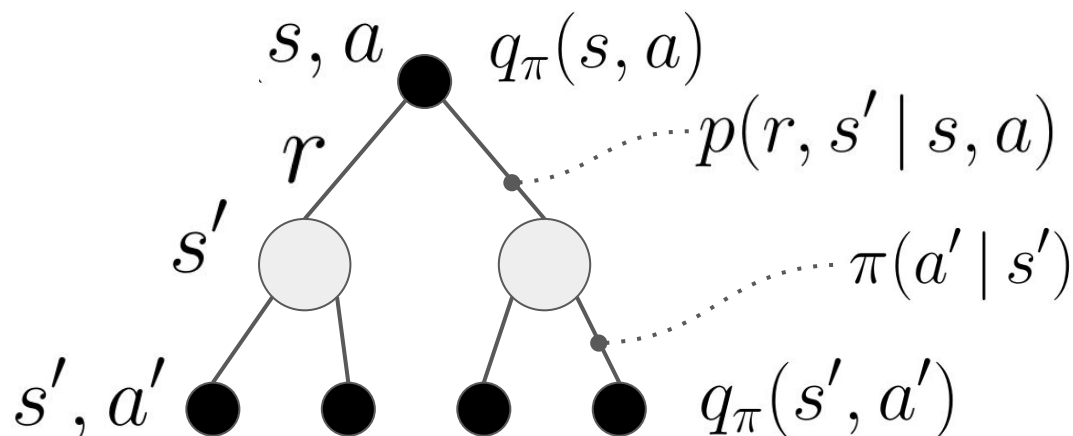
What do we gonna do with value functions?

Already know

- Return, value- and action-value functions
- Bellman equations – assess policy performance

Optimal policy makes

- best actions in each possible state

But how to know which policy is better?

How to compare them?

Bellman optimality equations

Optimal policy is the one with the largest v(s)

We could compare policies on the basis of v(s)

$$\pi \geq \pi' \quad \Leftrightarrow \quad v_\pi(s) \geq v_{\pi'}(s) \quad \forall\, s$$

Best policy π_* is better or equal to any other policy

Use optimal policy from s

In any finite MDP there is always at least one deterministic optimal policy

$$v_*(s) = \max_\pi v_\pi(s)$$

$$q_*(s, a) = \max_\pi q_\pi(s, a)$$

Commit action a, and **afterwards** use optimal policy

Bellman optimality equation for **v(s)**
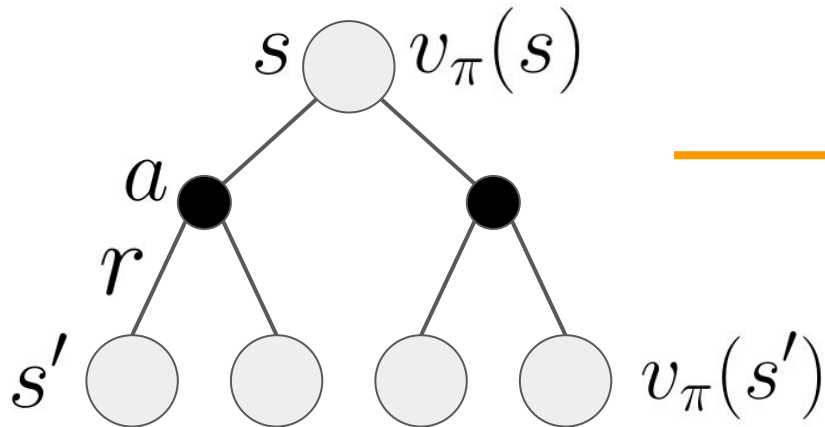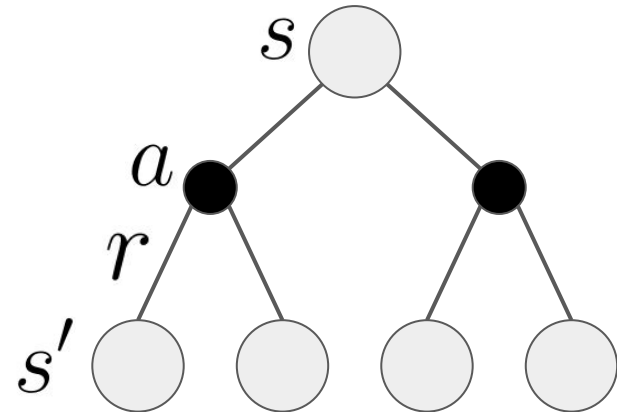
Bellman expectation equation for v(s)

Bellman optimality equation for $v_*(s)$

Bellman optimality equation for **v(s)**
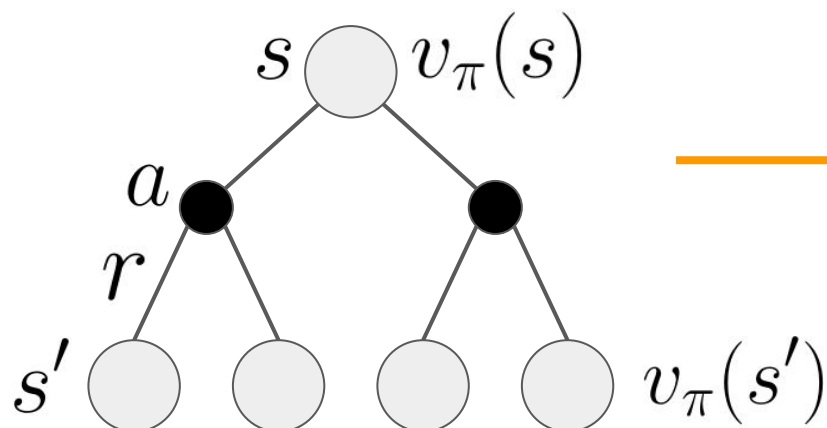
Bellman expectation
equation for v(s)
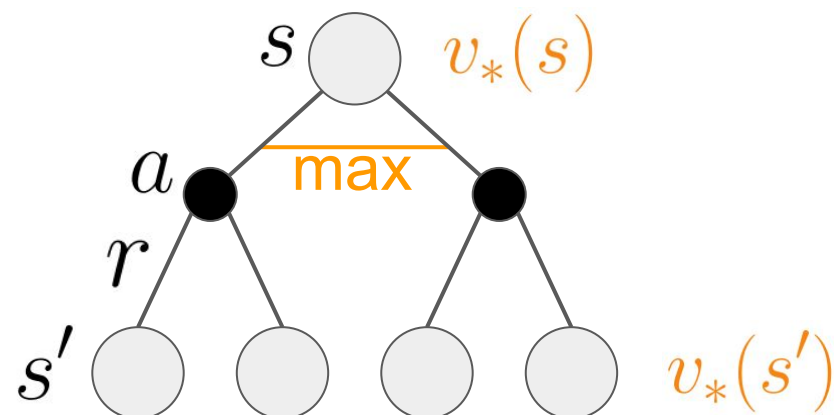
Bellman optimality
equation for v*(s)

Bellman optimality equation for **v(s)**
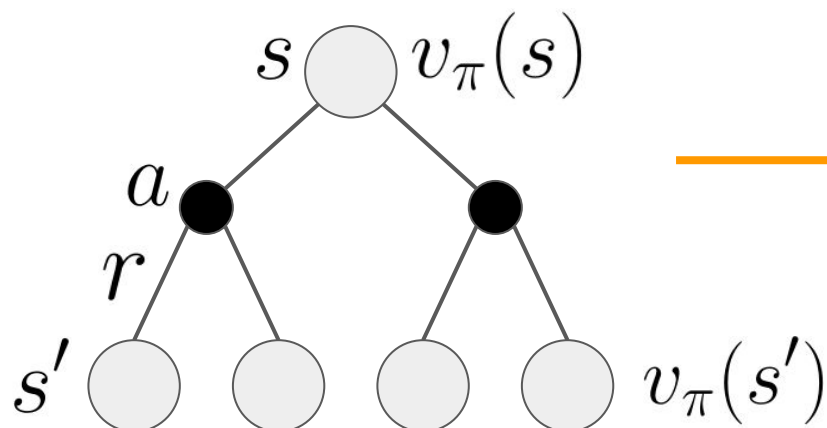
Bellman expectation equation for v(s)
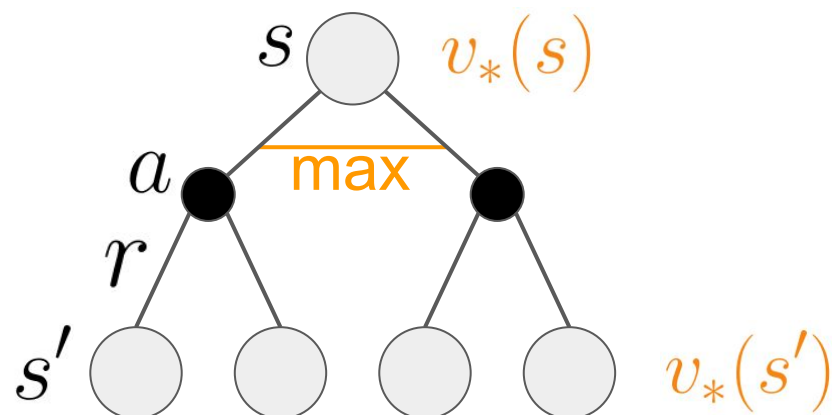
Bellman optimality equation for v*(s)

$$v_*(s) = \max_a \sum_{r,\,s'} p(r, s' \,|\, s, a)\left[r + \gamma v_*(s')\right]$$

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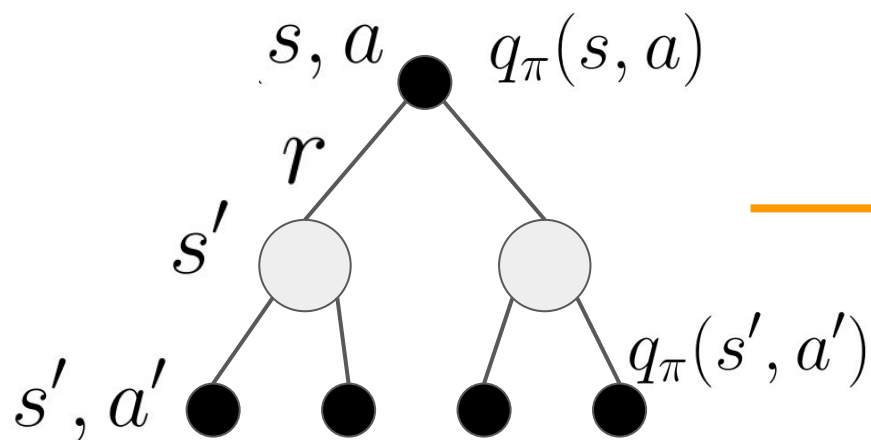
Bellman optimality equation for **q(s,a)**

Bellman expectation equation for q(s,a)

Bellman optimality equation for q*(s, a)

Bellman optimality equation for **q(s,a)**
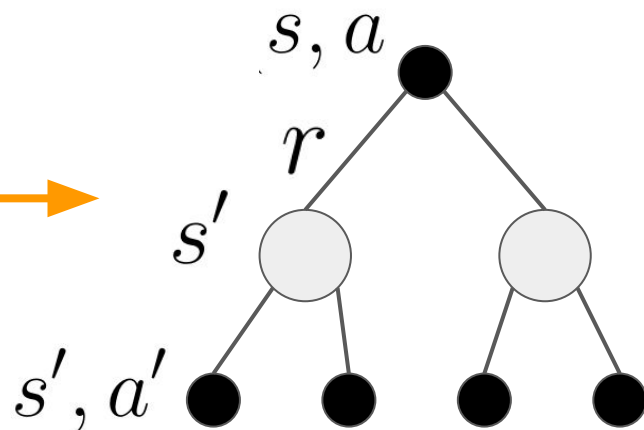
Bellman expectation equation for q(s,a)

Bellman optimality equation for $q_*(s, a)$
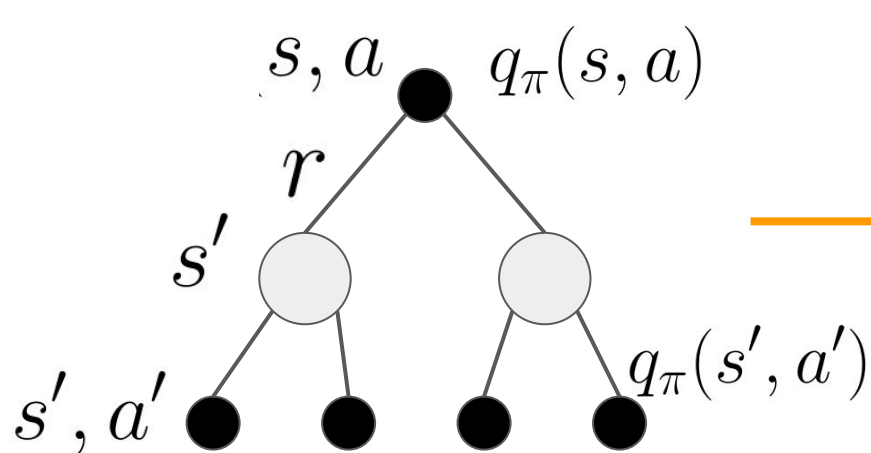
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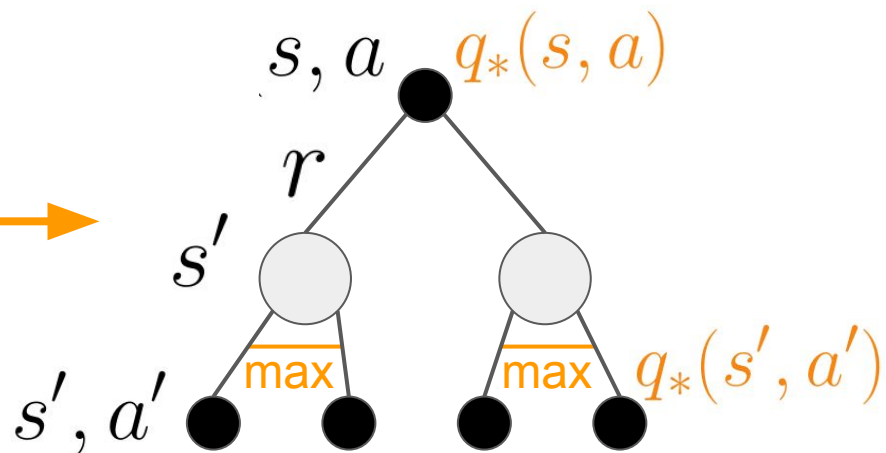
Bellman optimality
equation for q∗(s, a)
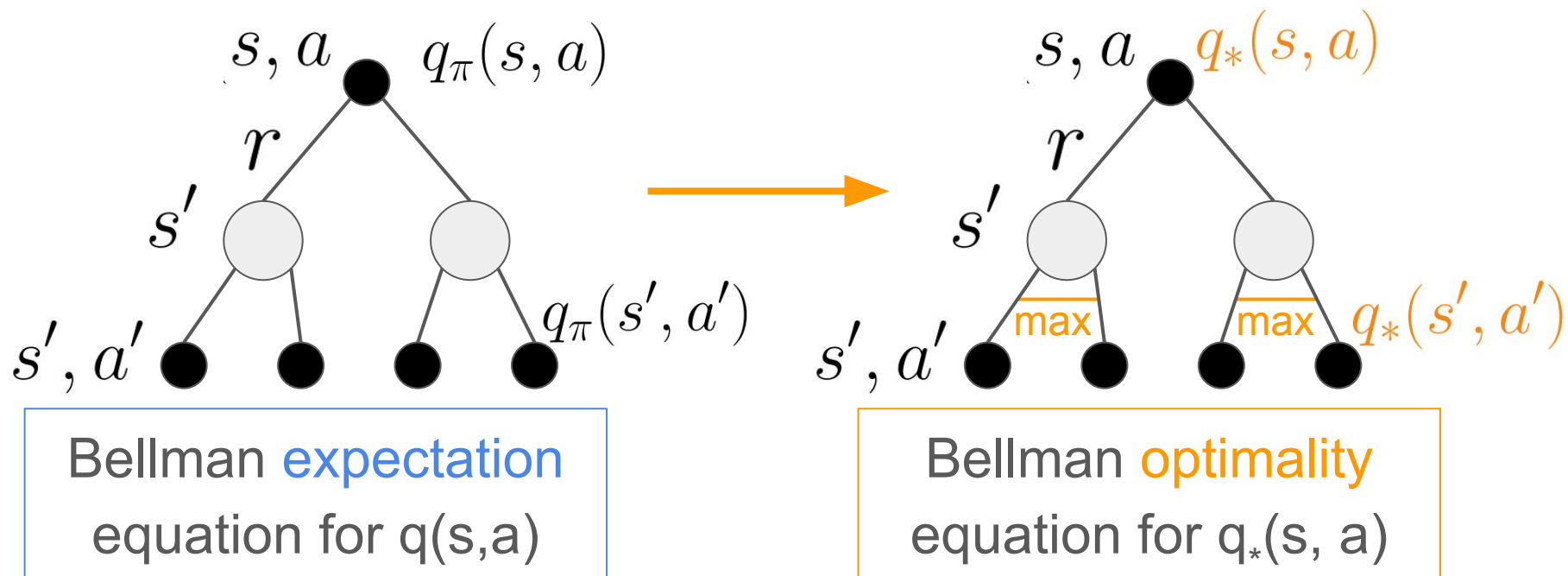
$$q_*(s, a) = \mathbb{E}\left[R_t + \gamma \max_{a'} q_*(S_{t+1}, a') \mid S_t = s, A_t = a \right]$$

$$= \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma \max_{a'} q_*(s', a') \right]$$

Bellman equations: operator view

Bellman equations: operator view

$$[\mathcal{T}^\pi V](s) = \mathbb{E}_{r,s'|s,a=\pi(s)}\big[\, r + \gamma V(s')\,\big]$$

$$[\mathcal{T}^\pi Q](s,a) = \mathbb{E}_{r,s'|s,a}\big[\, r + \gamma \mathbb{E}_{a'\sim\pi(s')}\big[\, Q(s',a')\,\big]\,\big]$$

$$[\mathcal{T}V](s) = \max_a \mathbb{E}_{r,s'|s,a}\big[\, r + \gamma V(s')\,\big]$$

$$[\mathcal{T}Q](s,a) = \mathbb{E}_{r,s'|s,a}\Big[\, r + \gamma \max_{a'} Q(s',a')\,\Big]$$

Bellman equations: operator view

Bellman expectation equation for **v(s)**

$$[\mathcal{T}^\pi V](s) = \mathbb{E}_{r,s'|s,a=\pi(s)} \big[\, r + \gamma V(s') \,\big]$$

Bellman expectation equation for **q(s,a)**

$$[\mathcal{T}^\pi Q](s,a) = \mathbb{E}_{r,s'|s,a} \big[\, r + \gamma \mathbb{E}_{a'\sim\pi(s')} \big[\, Q(s',a') \,\big] \,\big]$$

Bellman optimality equation for **v$_*$(s)**

$$[\mathcal{T} V](s) = \max_a \mathbb{E}_{r,s'|s,a} \big[\, r + \gamma V(s') \,\big]$$

Bellman optimality equation for **q$_*$(s,a)**

$$[\mathcal{T} Q](s,a) = \mathbb{E}_{r,s'|s,a} \big[\, r + \gamma \max_{a'} Q(s',a') \,\big]$$

Bellman equations are **contractions**

Contraction: $d(\mathcal{T}(v), \mathcal{T}(u)) \leq \gamma d(v, u), \quad 0 \leq \gamma < 1$

Operator: $[\mathcal{T}v](s) = \max_a \mathbb{E}_{r,s'|s,a}\left[r + \gamma v(s')\right]$

Denote $a^* = \arg\max_a \mathbb{E}_{r,s'|s,a}\left[r + \gamma v(s')\right]$, then, for any s:

$$[\mathcal{T}v](s) - [\mathcal{T}u](s) \leq r(s, a^*) + \gamma \mathbb{E}_{s'|s,a^*}\left[v(s')\right] - r(s, a^*) - \gamma \mathbb{E}_{s'|s,a^*}\left[u(s')\right]$$

$$= \gamma \mathbb{E}_{s'|s,a^*}\left[v(s') - u(s')\right]$$

$$\leq \gamma \mathbb{E}_{s'|s,a^*}\left[|v(s') - u(s')|\right]$$

$$\leq \gamma \max_{s'}|v(s') - u(s')|$$

$$= \gamma ||v - u||_\infty$$

Hence, taking max over s:

$$||\mathcal{T}v - \mathcal{T}u||_\infty \leq \gamma ||v - u||_\infty$$

Generalized **P**olicy **I**teration:

1. Policy Evaluation

2. Policy Improvement

Policy evaluation

Policy evaluation: motivation

Policy evaluation is also a called **prediction problem:**

- predict value function for a particular policy.

Bellman expectation equation

$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{r,s'} p(r, s' \mid s, a) \left[r + \gamma v_\pi(s') \right]$$

$$= \mathbb{E}_\pi \left[R_t + \gamma v_\pi(S_{t+1}) \mid S_t = s \right]$$

is basically a system of linear equations where

- # of unknowns = # of equations = # of states

Policy evaluation: algorithm

Input π, the policy to be evaluated
Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx v_\pi$

Bellman expectation
equation for v(s)

Policy improvement

Policy improvement: an idea

Once we know what is v(s) for a particular policy

We could improve it by acting greedily w.r.t. v(s)!

$$\pi'(s) \leftarrow \arg\max_{a} \overbrace{\sum_{r,s'} p(r, s' \mid s, a) \left[r + \gamma v_\pi(s') \right]}^{q_\pi(s, a)}$$

This procedure is guaranteed to produce a better policy!

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This procedure is guaranteed to produce a better policy!

if $\qquad q_\pi(s, \pi'(s)) \geq v_\pi(s) \qquad$ for all states

then $\qquad\qquad v_{\pi'}(s) \geq v_\pi(s)$

meaning that $\qquad \pi' \geq \pi$

Policy improvement: convergence

If new policy after improvement

$$\pi'(s) \leftarrow \arg\max_a \overbrace{\sum_{r,s'} p(r,s' \mid s,a)\left[r + \gamma v_\pi(s')\right]}^{q_\pi(s,a)}$$

is the same as the old one

$$\pi' = \pi \qquad \rightarrow \qquad v_{\pi'} = v_\pi$$

then it is optimal, since it satisfies:

$$v_{\pi'}(s) = \max_a \sum_{r,s'} p(r,s' \mid s,a)\left[r + \gamma v_\pi(s')\right]$$

Policy improvement: convergence

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Bellman optimality equation

Determining optimal policy from v$_*$(s), q$_*$(s,a)

If q* is known – how to recover the optimal policy?

$$\pi_*(s) \leftarrow \arg\max_a q_*(s, a)$$

If v* is known – how to recover the optimal policy?

Determining optimal policy from v$_*$(s), q$_*$(s,a)

If q* is known – how to recover the optimal policy?

$$\pi_*(s) \leftarrow \arg\max_a q_*(s, a)$$

If v* is known – how to recover the optimal policy?

$$\pi_*(s) \leftarrow \arg\max_a \overbrace{\sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma v_*(s') \right]}^{q_*(s, a)}$$

Unknown model dynamics → unable to recover optimal policy from v*

Precise evaluation is excessive

Value function

0 iteration

	0.000	0.000	0.000
0.000	0.000	0.000	0.000
0.000	0.000	0.000	

Greedy policy

0 iteration

Value function

0 iteration

	0.000	0.000	0.000
0.000	0.000	0.000	0.000
0.000	0.000	0.000	

5 iteration

	-7.598	-4.986	-3.127
-7.816	-5.834	-2.963	0.543
-6.115	-4.186	0.332	

9999 iteration

	-13.827	-13.289	-11.318
-14.768	-14.193	-10.722	-5.346
-16.111	-13.454	-6.059	

Greedy policy

0 iteration

5 iteration

9999 iteration

Roadmap

Now we know what is

- Policy evaluation (based on Bellman expectation eq)
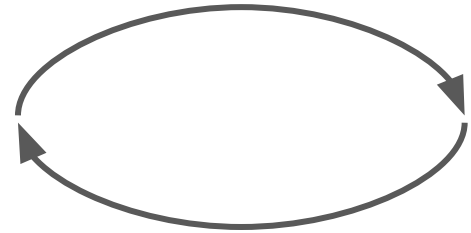
- Policy improvement (based on Bellman optimality eq)

The finishing touches:

how to combine them to obtain optimal policy?

Generalized **P**olicy **I**teration

The idea of policy and value iterations

Policy evaluation

Policy improvement

The idea of policy and value iterations

Generalized policy iteration

1. Evaluate given policy
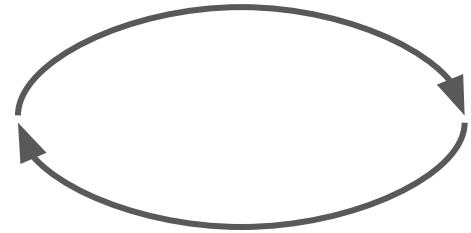2. Improve policy by acting greedily w.r.t. to its value function

Policy evaluation

Policy improvement
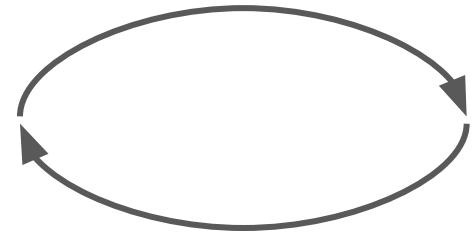
The idea of policy and value iterations

Generalized policy iteration

1. Evaluate given policy
2. Improve policy by acting greedily
    w.r.t. to its value function

Robustness:

● No dependence on initialization
● No need in complete policy evaluation (states / converg.)
● No need in exhaustive update (states)
   ○ Example of update robustness:
      ■ Update only one state at a time
      ■ in a random direction
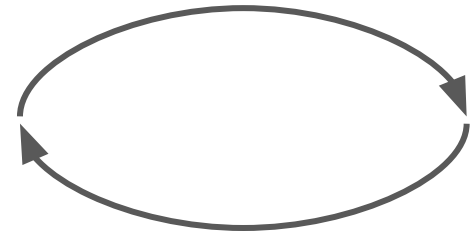   ■ that is correct only in a expectation

Policy evaluation

Policy improvement

The idea of policy and value iterations

Generalized policy iteration

1. Evaluate given policy
2. Improve policy by acting greedily
   w.r.t. to its value function

Policy evaluation

Policy improvement

Policy iteration

1. Evaluate policy until convergence (with some tolerance)
2. Improve policy

Value iteration

1. Evaluate policy only with single iteration
2. Improve policy

Policy iteration

Policy iteration: scheme

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Repeat
   $\quad \Delta \leftarrow 0$
   $\quad$ For each $s \in \mathcal{S}$:
   $\quad\quad v \leftarrow V(s)$
   $\quad\quad V(s) \leftarrow \sum_{s',r} p(s',r|s,\pi(s))[r + \gamma V(s')]$
   $\quad\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number)

   Bellman expectation equation for v(s)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $\quad$ *old-action* $\leftarrow \pi(s)$
   $\quad \pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$
   $\quad$ If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

   q(s,a)

Value iteration

Value iteration

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

Output a deterministic policy, $\pi \approx \pi_*$, such that
    $\pi(s) = \arg\max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$

Bellman optimality equation for v(s)

Value iteration (VI) vs. Policy iteration (PI)

- VI is faster per iteration – $O(|A||S|^2)$
- VI requires many iterations

- PI is slower per iteration – $O(|A||S|^2 + |S|^3)$
- PI requires few iterations

No silver bullet $\rightarrow$ experiment with # of steps spent in policy evaluation phase to find the best