Introduction to Associative Algebras

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1 Introduction

1.1 Convention

R is always a nonzero commutative ring (hence R-Mod and Mod-R are isomorphic to the category of bimodules (R, R)-Mod). All rings have unit.

1.2 Outline

This note is mainly geared towards a upcoming note about supersymmetry. Some basic facts about R-algebras will be discussed:

- Formal construction, including
  - categorical definition
  - tensor product
  - graded structure
- Tensor algebras, symmetric algebras and exterior algebras. As an interesting example, Grassmann varieties will also be discussed.
- Trace and determinant, which generalize the construction in linear algebra.
- Properties of integral elements, which deserves a separate treatment but is usually discussed in the context of field theory in most textbooks.

Although basic knowledge of category theory is assumed, nearly everything is closely related to physics, among which the most important one is supersymmetry.

2 Basic Constructions

2.1 R-algebra

Definition 1. An **R-algebra** is a set A with both ring and R-module structure, where the ring multiplication $(x, y) \mapsto xy$ is a balanced map $(rx)y = x(ry) = r(xy)$.

Definition 2. Homomorphisms are $\mathrm{Hom}_{R-\mathrm{Alg}}(A_1, A_2) = \mathrm{Hom}_{R-\mathrm{Mod}}(A_1, A_2) \bigcap \mathrm{Hom}_{\mathrm{Ring}}(A_1, A_2)$.

Definition 3. $B \subset A$ is a **subalgebra** if B is both a subring and a submodule.

Definition 4. Left/Right/Two-sided **ideals** are submodules closed under left/right/two-sided multiplication by $a \in A$.

Definition 5. Given a two-sided ideal $I \subset A$, the quotient A/I has natural ring and R-module structure, so it is an algebra, called **quotient algebra**.

Definition 6. A is called **division algebra** if it is a division ring.

Definition 7. When A is a free module $A = R^{\oplus X}$, its **dimension** is well-defined $\dim A \equiv |X|$ because of the IBN property of commutative rings.

Since $(r1_A)x = 1_A(rx) = rx = (rx)1_A = x(r1_A)$ and $(rs)1_A = (r1_A)(s1_A)$, $r \mapsto r1_A$ is a ring homomorphism $\sigma : R \to Z_A$ (Z_A is the center of A if viewed as a ring). Conversely, given a ring map $\sigma : R \to A$, it naturally induces an R-module structure on A whose scalar multiplication is $ra \equiv \sigma(r)a$, and A is an algebra iff $\mathrm{im}(\sigma) \subset Z_A$. Therefore, there is a one-to-one correspondence between R-algebra structures on A and $\mathrm{Hom}_{\mathrm{Ring}}(R, Z_A)$, which can be naturally extended to a functor.

Algebras can be viewed as a generalization of rings, which are just $\mathbb{Z}$-algebra (abelian group are just $\mathbb{Z}$-module and $\mathbb{Z}$ is initial in Ring, whose image always lies in the center).

2.2 Matrix algebra

The most important example of non-commutative algebras is the matrix algebra $M_n(R) = \mathrm{End}_R(R^{\oplus n})$. It has dimension n^2 and a basis E_{ij} (the matrix $a_{kl} = \delta_{(k,l),(i,j)}$) satisfying $E_{ij}E_{kl} = \delta_{j,k}E_{il}$. More generally, if an algebra A has a basis $(e_i)_{i\in I}$ as an R-module, then its structure is completely determined by the multiplication of basis $e_ie_j = \sum_{k\in I} a_{i,j}^k e_k$

Definition 8. $(a_{i,j}^k \in R)_{i,j,k\in I}$ is called the **structure constant** of the algebra w.r.t. the basis $(e_i)_{i\in I}$.

Recall that a Lie algebra is defined over a field, it always has the structure constant.

More generally, we can consider the matrix algebra $M_n(A) = \mathrm{End}_{\mathrm{Mod}-A}((A_A)^{\oplus n})$, which contains A as a subring $a \mapsto \mathrm{diag}(a, \dots, a)$. Its R-algebra structure can be constructed from the free left A-module $M_n(R) = \bigoplus_{1\le i,j\le n} AE_{ij}, E_{ij}E_{kl} = \delta_{j,k}E_{il}, aE_{ij} = E_{ij}a$.

2.3 Categorical approach

For future convenience, we redefine R-algebras categorically.

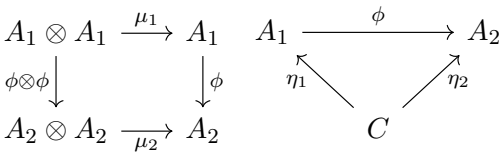
Definition 9. An R-algebra A is an R-module with

1. a multiplication morphism $\mu : A \otimes A \to A$,
2. a unit morphism $\eta : R \to A$,

such that the following diagrams commutes

$$\begin{array}{ccccc} A\otimes(A\otimes A) & & \xrightarrow{\sim} & & (A\otimes A)\otimes A \\ {\scriptstyle \mathrm{id}\otimes\mu}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}\otimes\mu} \\ A\otimes A & \xrightarrow[\mu]{} & A & \xleftarrow[\mu]{} & A\otimes A \end{array} \qquad \begin{array}{ccccc} A\otimes R & \xrightarrow{\mathrm{id}\otimes\eta} & A\otimes A & \xleftarrow{\eta\otimes\mathrm{id}} & R\otimes A \\ & {\scriptstyle\simeq}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\simeq} & \\ & & A & & \end{array}$$

Definition 10. Morphisms between (A_1, μ_1, η_1) and (A_2, μ_2, η_2) are module homomorphisms making the following diagrams commutative



The definition of sub-algebras is already categorical (sub-algebras can be identified with monomorphisms), and left ideals are sub-modules whose left multiplication by A factors through itself $A \otimes I \hookrightarrow A \otimes A \xrightarrow{\mu} I \hookrightarrow A$ (right/two-sided ideals can be defined similarly).

3 Tensor Product

3.1 Finite tensor product

Tensor product in R-Mod is well-known. To make $A \otimes B$ an R-algebra, we need to define

Definition 11 (multiplication morphism)**.** $\mu_{A\otimes B} : (A \otimes B) \otimes (A \otimes B) \xrightarrow{\sim} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\mu_A \otimes \mu_B} A \otimes B$ where the anonymous isomorphism contains only one commutative constraint $c_{B,A}$.

Definition 12 (unit morphism)**.** $\eta_{A\otimes B} : R \xrightarrow{\sim} R \otimes R \xrightarrow{\eta_A \otimes \eta_B} A \otimes B$, where the anonymous isomorphism comes from the fact that R is the unit in monoidal category R-Mod, or more explicitly $r \mapsto r \otimes 1 = 1 \otimes r$.

It can be verified categorically that the commutative constraint is an isomorphism between R-algebras $c(A, B) : A \otimes B \xrightarrow{\sim} B \otimes A$ (using categoricality of $c(A, B)$ and hexagon axiom of braid categories). Now it can be checked that the category R-Alg is actually a symmetric monoidal category.

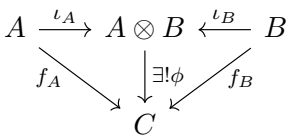
Since R is the unit in monoidal category R-mod, there is a natural isomorphism $A \xrightarrow{\sim} A \otimes R$ and hence a natural module map $\iota_A : A \xrightarrow{\sim} A \otimes R \xrightarrow{\mathrm{id}_A \otimes \eta_B} A \otimes B$. Similarly we have $\iota_B : B \xrightarrow{\sim} R \otimes B \xrightarrow{\eta_A \otimes \mathrm{id}_B} A \otimes B$.

Lemma 1. *It can be checked that these module maps are also homomorphism of algebras. Besides,* $[\mathrm{im}(\iota_A), \mathrm{im}(\iota_B)] = 0$.

The homomorphism ι looks like an embedding; however, it is not a monomorphism in general. For instance, if $\gcd(a, b) = 1$, then $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z}$ is zero algebra ($1 = ua + vb \implies 1 \otimes 1 = vb \otimes ua = 0 \otimes 1 = 1 \otimes 0 = 0 \otimes 0$).

Lemma 2. *When algebra has a basis, the homomorphisms $\iota_{A,B}$ are embeddings, since tensor product preserves direct sum and the image of ι is just a direct factor.*

Proposition 1. *Data $(A \otimes B, \iota_A, \iota_B)$ satisfies the following universal property:*



where $[f_A(A), f_B(B)] = 0$. The unique morphism $\phi : A \otimes B \to C$ is given by the balanced product $(a, b) \mapsto f_A(a) f_B(b)$.

Definition 13. Finite tensor product $(\otimes_{i\in I}A_i, \{f_i : A_i \to \otimes_{i\in I}A_i\}_{i\in I})$ can be defined similarly. However, the commutative constraint might be annoying when working out an explicit construction. In fact, one can write all the possible orderings and define the tensor product as their $\varinjlim$.

For instance, a (S,T)-bimodule structure is equivalent to two ring ($\mathbb{Z}$-algebra) maps $S \to \mathrm{End}(M)$, $T^{\mathrm{op}} \to \mathrm{End}(M)$ with commutative image, which is just a ring map $S \otimes_{\mathbb{Z}} T^{\mathrm{op}} \to \mathrm{End}(M)$.

3.2 Arbitrary tensor product

The universal property of tensor product can be generalized to an arbitrary family of R-algebras. Notice that if $J \subset I$ are two finite sets, there is a natural morphism $f_{JI} : \otimes_{j\in J}A_j \to \otimes_{i\in I}A_i$ (use the morphism ι $|I| - |J|$ times) and $f_{JI}f_{KJ} = f_{KI}$. Finite subsets of I consists of a filtered category, and $J \mapsto \otimes_{j\in J}A_j, [J \to K] \mapsto f_{JK}$ is a functor. Therefore we can define the arbitrary tensor product to be the filtered colimit

Definition 14. $\bigotimes_{i\in I} A_i \equiv \varinjlim_{J\subset I,|J|<\infty} \bigotimes_{j\in J} A_j$ The morphism $\iota_i : A_i \to \otimes_{i\in I}A_i$ is just the composition $A_i \to \otimes_{j\in J}A_j \to \otimes_{i\in I}A_i$ (which is independent of $J \ni i$) and $\phi : \otimes_{i\in I}A_i \to C$ is $\varinjlim$ of $\phi_J : \otimes_{j\in J}A_j \to C$.

Corollary 1. *In the category of commutative R-algebras, the universal property of tensor product is just that of coproduct; besides, the coequilizier* $\mathrm{coker}(f,g) = B/\langle\{f(a) - g(a) : a \in A\}\rangle$ *$(f, g : A \to B)$ exists and hence R-CAlg is co-complete.*

3.3 Base change

Given a ring map $\phi : R \to S$, we can build functors between R-Alg and S-Alg just like modules.

The ring map makes S an R-algebra.

Definition 15 ($P_{R\to S}$ and $\mathcal{F}_{R\to S}$)**.** Morphism $\iota_S : S \to A \otimes_R S$ has S-algebra structure since $\mathrm{im}(\iota_S) \subset Z_{A\otimes_R S}$. Conversely, pullback of $S \to B$ by ϕ gives an R-algebra structure. Therefore we have a pair of functors $R\text{−Mod} \underset{\mathcal{F}_{R\to S}}{\overset{P_{R\to S}}{\rightleftarrows}} S\text{−Mod}$.

Proposition 2. *$P_{R\to S}$ is a monoid functor.*

The isomorphism $P_{R\to S}(A) \otimes_S P_{R\to S}(B) \xrightarrow{\sim} P_{R\to S}(A \otimes_R B)$ is given by $(a \otimes s) \otimes (b \otimes t) \mapsto (a \otimes b) \otimes st$ and $(a \otimes b) \otimes s \mapsto (a \otimes s) \otimes (b \otimes 1)$.

Proposition 3. *$(P_{R\to S}, \mathcal{F}_{R\to S})$ is an adjoint pair.*

The isomorphism $\mathrm{Hom}_R(A, \mathcal{F}_{R\to S}(B)) \xrightarrow{\sim} \mathrm{Hom}_S(A \otimes_R S, B)$ is just $[f : A \to B] \mapsto [f'(a \otimes s) = f(a)s : A \otimes_R S \to B]$, $[\phi : A \to A \otimes_R S] \mapsto [f'\phi : A \to B]$. For $Q \to R \to S$, it can be checked that there is a stronger result than composition of adjoint pairs: $P_{R\to S} \circ P_{Q\to R} \simeq P_{Q\to S}$ and $\mathcal{F}_{Q\to R} \circ \mathcal{F}_{R\to S} = \mathcal{F}_{Q\to S}$.

In the end, let's consider a special base change $P_{R\to R/I}$.

Proposition 4. *Assume $\mathfrak{a}_i \subset A_i, i = 1, 2$ are ideals, then there is a natural isomorphism $(A_1/\mathfrak{a}_1)\otimes_R (A_2/\mathfrak{a}_2) \xrightarrow{\sim} A_1\otimes_R A_2/(\mathfrak{a}_1\otimes_R A_2 + A_1\otimes_R \mathfrak{a}_2) \equiv \mathcal{A}$ since $f_i : A_i/\mathfrak{a}_i \to \mathcal{A}$ satisfies the universal property of tensor product.*

If taking $A_1 = A, \mathfrak{a}_1 = 0, A_2 = R, \mathfrak{a}_2 = I$, we obtain an isomorphism $P_{R\to R/I}(A) \xrightarrow{\sim} A/IA$.

4 Graded Structure

4.1 I-graded structure

Let's begin with graded modules. Suppose $(I,+)$ is a commutative monoid with unit 0.

Definition 16. An **I-graded module** (when $(I,+)\subset(\mathbb{Z},+)$ we often omit the prefix I-) over a commutative ring R is a module $M=\oplus_{i\in I}M_i$. All these modules form a monoid category $(R\text{−Mod}_I,\otimes)$, where **graded morphisms** are module maps preserves grading $\varphi(M_i)\subset N_i$ and tensor product induces a natural graded structure $(M\otimes N)_k=\bigoplus_{i,j\in I,i+j=k}M_i\otimes N_j$ (the existence of such a decomposition is guaranteed by the right exactness of tensor product). $x\in M_i\backslash\{0\}$ is called a **homogeneous element** with **degree** $\deg(x)=i$, and **graded submodules** are those satisfying $N=\oplus_i(N\cap M_i)$.

Definition 17. The definition of I-graded algebra can be directly derived from the categorial definition of R-algebras (we just need to start with the category $R\text{−Mod}_I$). For instance, **graded ideals** are ideals which are graded as submodules.

Lemma 3. *In I-graded algebras (resp. modules), a two-sided ideal (resp. submodule) is graded if and only if it is generated by homogenous elements.*

One direction is obvious; conversely, any element can be written as a R-linear combination of homogenous elements, and the coefficient can be further decomposed into a sum of homogenous elements.

A trivial example of I-graded algebra is the monoid ring $R[I]$. In particular, the polynomial ring is a graded algebra, whose graded structure is known in elementary school.

The first non-trivial example of graded algebra for most students might be the algebra of differential forms (some physicists might be more familiar with the so called Grassmann number). In differential geometry, many constructions share a common $\mathbb{Z}/2\mathbb{Z}$-graded structure, which deserve a separate treatment.

4.2 Sign rule

Definition 18. An I-graded algebra is called **ϵ-commutative** if there exists a homomorphism $\epsilon: I\to\mathbb{Z}/2\mathbb{Z}$ such that $xy=(-1)^{\epsilon(\deg x)\epsilon(\deg y)}yx$ for all homogeneous elements x,y.

When $\epsilon=0$, ϵ-commutativity reduces to the usual commutativity. When $I\subset\mathbb{Z}$ and ϵ is the quotient homomorphism, ϵ-commutativity is also called **anti-commutativity**. We often require **nilsquare** ($x^2=0$) of odd elements $\epsilon(\deg x)=1$ (this is automatically satisfied if $2\in R^\times$).

In general, the above construction is equivalent to assigning a commutative constraint $c(A,B)$. If we require that (i) *the ϵ-commutativity is preserved by tensor product*, (ii) *the natural morphisms $\iota_{A,B}$ is graded*, then it is obvious that $(a_1\otimes 1)(a_2\otimes 1)=a_1a_2\otimes 1$, $(1\otimes b_1)(1\otimes b_2)=1\otimes b_1b_2$. Associativity implies that $(a_1\otimes b_1)(a_2\otimes b_2)=(a_1\otimes 1)[(1\otimes b_1)(a_2\otimes 1)](1\otimes b_2)$, and the fact $A\otimes B$ is ϵ-commutative implies

Definition 19 (Koszul's braided structure)**.** A graded isomorphism $c_\epsilon(M,N):M\otimes N\xrightarrow{\sim}N\otimes M$

$$\begin{aligned}c_\epsilon(M,N):M_i\otimes N_j&\xrightarrow{\sim}N_j\otimes M_i\\ x\otimes y&\mapsto(-1)^{\epsilon(\deg x)\epsilon(\deg y)}y\otimes x\end{aligned}$$

and ϵ-commutative product is just $\mu_A\circ c_\epsilon(A,A)=\mu_A$, where $\mu_A:A\otimes A\to A$ is the multiplication morphism.

Theorem 1 (Koszul sign rule)**.** *Multiplication of homogenous elements is given by*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\epsilon(\deg b_1)\epsilon(\deg a_2)} a_1 a_2 \otimes b_1 b_2$$

and

1. *$A \otimes B$ with this multiplication is an I-graded algebra,*
2. *$\iota_{A,B}$ are homomorphisms between I-graded algebras,*
3. *$A \otimes B$ is ϵ-commutative if so are A, B,*
4. *Koszul braided structure c_ϵ is an isomorphism between I-graded algebras.*

(i)-(iii) lead us to this construction, and (iv) comes from the commutativity of the diagram

$$\begin{array}{ccc}
A \otimes B \otimes A \otimes B & \xrightarrow{c_\epsilon(A,B)\otimes c_\epsilon(A,B)} & B \otimes A \otimes B \otimes A \\
\downarrow{\scriptstyle \mathrm{id}_A \otimes c_\epsilon(B,A) \otimes \mathrm{id}_B} & & \downarrow{\scriptstyle \mathrm{id}_B \otimes c_\epsilon(A,B) \otimes A} \\
A \otimes A \otimes B \otimes B & \xrightarrow{c_\epsilon(A\otimes A, B\otimes B)} & B \otimes B \otimes A \otimes A \\
\downarrow{\scriptstyle \mu_A \otimes \mu_B} & & \downarrow{\scriptstyle \mu_B \otimes \mu_A} \\
A \otimes B & \xrightarrow[c_\epsilon(A,B)]{} & B \otimes A
\end{array}$$

the commutativity of upper part can be seen by playing with braids, and that of lower part is just the functoriality of commutative constraint.

Proposition 5. *Let R-CAlg_I^ϵ be the category of ϵ-commutative I-graded algebras, then tensor product together with Koszul sign rule gives the coproduct in R-CAlg_I^ϵ and hence R-CAlg_I^ϵ is co-complete. If both A and B satisfy $\epsilon(\deg x) = 1 \implies x^2 = 0$, so does $A \otimes B$.*

The universal property of tensor products coincides with the definition of coproduct in $R-\mathrm{CAlg}_I^\epsilon$. The proof of co-completeness is nearly the same as that of commutative R-algebras (Corollary 1), and the nilpotent property comes from direct calculation.

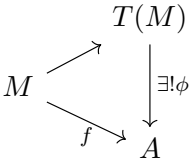
5 Tensor Algebra

5.1 Tensor algebra functor

Tensor product $T^n(M) \equiv \underbrace{M \otimes \cdots \otimes M}_{n}, n \geq 1$, defined by universal property in multilinear maps, is just the $\varinjlim$ of all the possible combinations of parentheses, morphisms between which are just associativity constraints. Notice that $T^0(M) = R$.

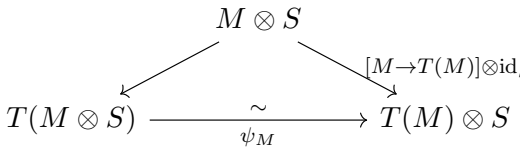
Functoriality of tensor product induces a natural isomorphism $\mu_{i,j} : T^i M \otimes T^j M \xrightarrow{\sim} T^{i+j} M$, which factors through $\underbrace{((M \otimes \dots) \otimes M)}_{i} \otimes \underbrace{((M \otimes \dots) \otimes M)}_{j}$. It can be more explicitly written as $\mu_{i,j} : (x_1 \otimes \cdots \otimes x_n) \otimes (y_1 \otimes \cdots \otimes y_m) \mapsto x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m$.

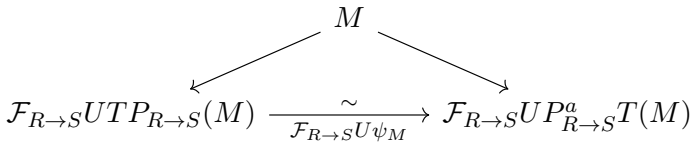
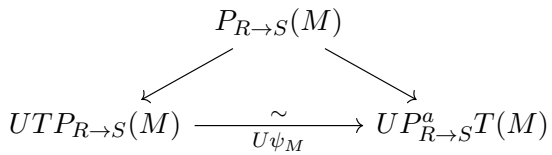
Definition 20. The **tensor algebra** of R-module M is $T(M) \equiv \bigoplus_{n=0}^{\infty} T^i(M)$ with multiplication morphism $\mu_{i,j}$ and unit morphism $R = T^0(M) \hookrightarrow T(M)$. It is a graded algebra and has a natural R-module monomorphism $M = T^1(M) \hookrightarrow T(M)$.

Theorem 2. *For any R-algebra A and R-module map $f : M \to A$, there exists a unique R-module map $\phi : T(M) \to A$ such that the following diagram commutes*  *, where* $\phi(x_1 \otimes \cdots \otimes x_n) = f(x_1) \dots f(x_n)$.

Notice that $\mathrm{Hom}_{R-\mathrm{Alg}}(T(-), -) \xrightarrow{\sim} \mathrm{Hom}_{R-\mathrm{Mod}}(-, U(-))$, i.e. the **tensor algebra functor** $T(-)$ is just the free algebra functor adjoint to the forgetful functor $U(-) : R-\mathrm{Alg} \to R-\mathrm{Mod}$ and the module map $M \to T(M)$ is the unit of this adjoint pair (image of $\mathrm{id}_{T(M)}$).

Corollary 2. *$T(-)$ preserves $\varinjlim$. $M \twoheadrightarrow M/N$ induces $T(M) \to T(M)/\langle N \rangle \xrightarrow{\sim} T(M/N)$.*

Lemma 4. *$T(-)$ commutes with $P_{R\to S}$: there is a unique graded algebra isomorphism $\phi_M : T(M \otimes S) \xrightarrow{\sim} T(M) \otimes S$ such that*  *commutes and hence there exists an isomorphism of functors $T(- \otimes M) \xrightarrow{\sim} T(-) \otimes S$.*

(T, U), $(P_{R\to S}, \mathcal{F}_{R\to S})$ and $(P^a_{R\to S}, \mathcal{F}^a_{R\to S})$ are adjoint pairs (P^a and $\mathcal{F}^a$ are base change functor and its pullback in R-Alg), so $(TP_{R\to S}, \mathcal{F}_{R\to S}U)$ and $(P^a_{R\to S}T, U\mathcal{F}^a_{R\to S})$ are also adjoint pairs. Since $\mathcal{F}_{R\to S}U = U\mathcal{F}^{\dashv}_{\mathcal{R}\to\mathcal{S}}$, the uniqueness of adjoint functor implies $\psi : TP_{R\to S} \xrightarrow{\sim} P^a_{R\to S}T$ and the property of unit implies  and the adjunction $(P_{R\to S}, \mathcal{F}_{R\to S})$ implies  .

5.2 Symmetric algebra and exterior algebra

Definition 21. Symmetric algebra $\mathrm{Sym}(M) \equiv T(M)/I_{\mathrm{Sym}}(M)$ and **exterior algebra** $\bigwedge(M) \equiv T(M)/I_\wedge(M)$, where the two-sided ideals $I_{\mathrm{Sym}}(M) \equiv \langle x \otimes y - y \otimes x \rangle$ and $I_\wedge(M) \equiv \langle x \otimes x \rangle$ are generated by homogenous elements and hence are graded.

Therefore symmetric algebras and exterior algebras are also graded $\mathrm{Sym}(M) = \oplus_{n\geq 0}\mathrm{Sym}^n(M)$, $\bigwedge(M) = \oplus_{n\geq 0}\bigwedge^n(M)$. Multiplication in symmetric algebras and exterior algebras is denoted by xy and $x \wedge y$ respectively.

Since the ideals are generated by homogenous elements of degree 2, $T^0(M)$ and $T^1(M)$ are unaffected and we still have the monomorphisms $M \hookrightarrow \mathrm{Sym}(M)$ and $M \hookrightarrow \bigwedge(M)$. Morphism $M \to N$ induces $I_{\mathrm{Sym}}(M) \to I_{\mathrm{Sym}}(N)$ and $I_\wedge(M) \to I_\wedge(N)$, by the universal property of quotients, $\mathrm{Sym}(-)$ and $\bigwedge(-)$ are functors from $R-\mathrm{Alg}$ to $R-\mathrm{CAlg}_{\mathbb{Z}}$ (**category of commutative graded R-algebras**) and $R-\mathrm{CAlg}^-_{\mathbb{Z}}$ (**category of anti-commutative graded nilsquare R-algebras**).

Lemma 5. *Functors $\mathrm{Sym}(-)$ and $\bigwedge(-)$ commutes with base change functor $P_{R\to S}$.*

Let $\mathrm{Sym}(M^n; A)$ and $\mathrm{Alt}(M^n; A)$ be the module of symmetric and anti-symmetric multilinear maps. The symmetric group S_n acts on the module of multilinear maps by $(\sigma B)(x_1, \dots, x_n) =$

$B(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$, and it is easy to see that $\mathrm{Sym}(M^n; A)$ and $\mathrm{Alt}(M^n; A)$ are two invariant submodules.

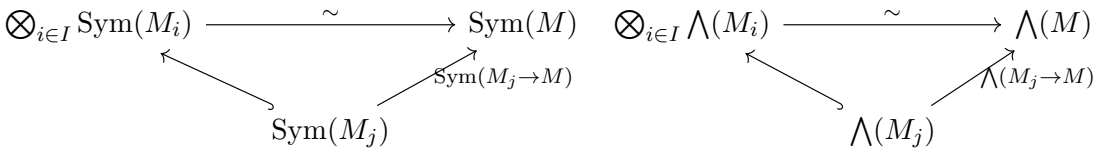
With this understanding, the symmetric algebra and exterior algebra functors have the same categorial property in corresponding categories.

Proposition 6 (universal property)**.** *We have functor isomorphisms*

$$\begin{aligned} \mathrm{Hom}_{R-\mathrm{Mod}}(\mathrm{Sym}^n(M), -) &\xrightarrow{\sim} \quad \mathrm{Sym}(M \times n; -) \\ \phi &\mapsto \quad [(x_1, \dots, x_n) \mapsto \phi(x_1 \dots x_n)] \\ \mathrm{Hom}_{R-\mathrm{Mod}}(\bigwedge^n(M), -) &\xrightarrow{\sim} \quad \mathrm{Alt}(M \times n; -) \\ \phi &\mapsto \quad [(x_1, \dots, x_n) \mapsto \phi(x_1 \wedge \dots \wedge x_n)] \end{aligned}$$

Theorem 3 (adjunction)**.** $(\mathrm{Sym}(-), U : R-\mathrm{CAlg}_{\mathbb{Z}} \to R-\mathrm{Mod})$ *and* $(\bigwedge(-), U : R-\mathrm{CAlg}^-_{\mathbb{Z}} \to R-\mathrm{Mod})$ *are adjoint pairs.*

Corollary 3. $R-\mathrm{CAlg}_{\mathbb{Z}}$ *and* $R-\mathrm{CAlg}^-_{\mathbb{Z}}$ *are co-complete and* $\mathrm{Sym}(-)$ *and* $\bigwedge(-)$ *preserves* $\varinjlim$*. Recall that direct sum is the coproduct in R-Mod and tensor product is the coproduct in* $R-\mathrm{CAlg}_{\mathbb{Z}}$ *and* $R-\mathrm{CAlg}^-_{\mathbb{Z}}$*, if* $M = \oplus_{i\in I} M_i$ *the following diagram commutes*



Corollary 4. $M \twoheadrightarrow M/N$ *induces* $\mathrm{Sym}(M) \to \mathrm{Sym}(M)/\langle N \rangle \xrightarrow{\sim} \mathrm{Sym}(M/N)$ *and* $\bigwedge(M) \to \bigwedge(M)/\langle N \rangle \xrightarrow{\sim} \bigwedge(M/N)$*.*

Corollary 5. *Given free module* $M = \bigoplus_{x\in X} Rx$*, then* $\mathrm{Sym}(M) \simeq R[X]$ *and* $\bigwedge(M) = \bigoplus_{k\geq 0} Rx_1 \wedge \dots \wedge x_k$ *where* $<$ *is a fixed linear order over* X *and* $x_1 < \dots < x_k \in X$*.*

In differential geometry the symmetric algebra and exterior algebra are often identified with submodule of $T(M)$. Let S_n act on $T^n(M)$ by $\sigma(x_1 \otimes \dots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes \dots \otimes \sigma^{-1}(n)$ and $\chi : S_n \to \{\pm 1\}$ be a group character, define $T^n_\chi \equiv \{x \in T^n(M) : \forall \sigma \in S_n, \sigma x = \chi(\sigma)x\}, T_\chi(M) \equiv \bigoplus_{n\geq 1} T^n_\chi(M), T_{\mathrm{Sym}}(M) \equiv T_1(M), T_\wedge \equiv T_{\mathrm{sgn}}(M)$, when $n! \in R^\times$, the endomorphism $e_\chi = e^n_\chi \equiv \frac{1}{n!}\sum_{\sigma\in S_n} \chi(\sigma)^{-1}\sigma$ is idempotent and the Fitting lemma suggests $T^n(M) = \mathrm{im}(e_\chi) \oplus \ker(e_\chi)$ and hence

Theorem 4. *When* $n! \in R^\times$*,* $\ker(e_1) = I^n_{\mathrm{Sym}}, \ker(e_{\mathrm{sgn}}) = I^n_\wedge$*, so we have R-Mod isomorphism* $e_1 : T^n_{\mathrm{Sym}}(M) \xrightarrow{\sim} \mathrm{Sym}^n(M)$ *and* $e_{\mathrm{sgn}} : T^n_\wedge(M) \xrightarrow{\sim} \bigwedge^n(M)$*.*

When $\mathbb{Q} \subset R$, the symmetric algebra and exterior algebra can be embedded into $T(M)$ as R-modules and multiplication of homogenous elements is given by $(x, y) \mapsto e^{\deg x + \deg y}_{1,\mathrm{sgn}}(x \otimes y)$. In particular $T^2(M) = \mathrm{Sym}^2(M) \oplus \bigwedge^2(M)$.

5.3 Grassmann varieties

In this subsection F is a field and $V = F^n$ is a vector space over F.

Definition 22. Grassmann varieties $G(k,V)$ is the collection of k-dimensional subspace of V.

For instance, when $k=1$, $G(1,V)$ is just the **projective space** $\mathbb{P}(V)\equiv(V\backslash\{0\})/F^\times$, which is a smooth manifold. Since given a k-dim subspace W is equivalent to given a $n-k$ subspace in the dual space $V^\vee$, i.e. $W^\perp =\equiv \{f\in V^\perp : f|_W=0\}$, there is a canonical isomorphism $G(k,V)\simeq G(n-k,V^\vee)$. In particular, $G(n-1,V)\simeq\mathbb{P}(V^\vee)$.

Theorem 5 (Plücker embedding)**.** *Given V,k, there is a canonical embedding*

$$\psi: G(k,V)\hookrightarrow \mathbb{P}(\bigwedge^k V)$$
$$W\mapsto F^\times w_1\wedge\cdots\wedge w_k=\bigwedge^k(W)\backslash\{0\}$$

where $w_1,\ldots,w_k$ is a basis of W.

ψ is well-defined since $\bigwedge^k(V)\xrightarrow{\sim}\bigoplus_{a+b=k}(\bigwedge^a(U)\otimes\bigwedge^b(W))$ and $w_1\wedge\cdots\wedge w_k$ is a nonzero element of the direct factor $(a,b)=(0,k)$, spanning $\bigwedge^k(W)$. Conversely if $\psi(W)=F^\times\Lambda$, W can be recovered as $W=\{v\in V: v\wedge\Lambda=0\}$.

Now we turn to the converse problem: what kinds of elements in $\mathbb{P}(\bigwedge^k V)$ is the image of Plücker embedding?

Lemma 6. *There exists a unique F-linear map $\iota: V^\vee\to\mathrm{End}_F(\bigwedge V)$ satisfying*

$$\iota(\tilde v)(v)=\tilde v(v), v\in V=\bigwedge^1 V,$$
$$\iota(\tilde v)(\xi\wedge\eta)=\iota(\tilde v)(\xi)\wedge\eta+(-1)^{\deg\xi}\xi\wedge\iota(\tilde v)(\eta),$$

where $\xi,\eta\in\bigwedge V$ are homogeneous elements and hence by induction $\iota(\tilde v)$ decreases the degree by 1. There is just the familiar inner multiplication in differential geometry. Notice that $\iota(\tilde v)^2=0$, it induces a morphism $\bigwedge^r(V^\vee)\to\mathrm{End}_F(\bigwedge V)$ for all r.

Lemma 7. *Suppose $\Lambda\in\bigwedge^k V$ is nonzero, let $W_\Lambda\subset V$ be the subspace generated by $\{\iota(\Xi)\Lambda:\Xi\in\bigwedge^{k-1}(V^\wedge)\}$, then for any subspace $W_0\subset V$,*

$$\Lambda\in\mathrm{im}\left[\bigwedge^k W_0\hookrightarrow\bigwedge^k V\right]\Leftrightarrow W_0\supset W_\Lambda.$$

In particular, $\dim W_\Lambda\geq k$ *and* $\dim W_\Lambda=k$ *implies* $\psi(W_\Lambda)=F^\times\Lambda$.

Choose $V=U\oplus W_0$ and write the exterior product of direct sums into direct sum of tensor products, the conclusion can be verified by choosing a basis. The last demonstration is obtained by setting $W_0=W_\Lambda$.

By counting dimensions, we have

Lemma 8. *Define $W'_\Lambda\equiv\{w\in W_\Lambda: w\wedge\Lambda=0\}$, then*

$$F^\times\cdot\Lambda\in\mathrm{im}(\psi)\Leftrightarrow W'_\Lambda=W_\Lambda,$$

and $\psi(W_\Lambda)=F^\times\cdot\Lambda$ when the above condition is satisfied.

The condition $W'_\Lambda=W_\Lambda$ is equivalent to $(\iota(\Xi)\Lambda)\wedge\Lambda=0$ for all $\Xi\in\bigwedge^{k-1}(V^\vee)$, so the definition equation for images of Plücker embedding is

Theorem 6 (Plücker relation)**.** *Image of Plücker embedding is the common zero of quadratic homogeneous functions*

$$f_\Xi : \bigwedge^k V \to \bigwedge^{k+1} V$$
$$\Lambda \mapsto (\iota(\Xi)\Lambda) \wedge \Lambda, \Xi \in \bigwedge^{k-1} (V^\vee)$$

Therefore Grassmann variety $G(k, V)$ is embedded as a projective variety.

6 Trace and Determinant

In this section we generalize several linear algebra invariants in the context of modules over a commutative ring. For simplicity we write $\otimes \equiv \otimes_R$.

Assume $E = R^{\oplus n}, n \in \mathbb{Z}_{\geq 0}$. $\bigwedge^n(E)$ is a free R-module of rank 1, so the endomorphisms are of the form $x \mapsto rx$ and can be identified with R.

Definition 23. For $\varphi \in \mathrm{End}_R(E)$, the **determinant** $\det(\varphi)$ is induced by functoriality of exterior product, i.e. $\bigwedge^n(\varphi)$.

After choosing a basis $x_1, \ldots, x_n$ of E and write $\varphi(x_j) = \sum_i x_i a_{ij}$, our definition coincides with the one in linear algebra

$$\det(a_{ij}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{\sigma(1)1} \ldots a_{\sigma(n)n}.$$

Proposition 7. *Given $A \in M_n(R)$, its* **adjugate matrix** *$A^\vee$ is defined just as in linear algebra.*

1. $\det(\varphi\phi) = \det(\varphi)\det(\phi)$ *and* $\det \mathrm{id} = 1$,
2. $\det A^T = \det A$,
3. $AA^\vee = A^\vee A = \det(A) I$, *where I is the identity matrix,*
4. $\varphi \in \mathrm{End}_R(E)$ *is invertible iff* $\det \varphi \in R^\times$ *and* $A^{-1} = (\det A)^{-1} A^\vee$.

(1), (2), (4) are trivial. For (3), we only need to check $A^\vee A = \det(A) I$. Notice that $(\det \varphi)(x_1 \wedge \cdots \wedge x_n) = \sum_i x_i a_{ij} \wedge [\cdots \wedge \widehat{\varphi(x_j)} \wedge \ldots]$, the terms containing x_i in the bracket do not affect the result, so the calculation is reduced to the map $R^{\oplus(n-1)} \hookrightarrow R^{\oplus n} \xrightarrow{\varphi} R^{\oplus n} \twoheadrightarrow R^{\oplus(n-1)}$, where the anonymous maps are just adding or removing the j-component. It can be checked that $(A^\vee A)_{ii} = \det(A)$. The off-diagonal term (i, j) is independent of column i of A, so we can set column i equal to column j and the (i, j) term equals to the (i, i) term $(\det A)$, hence $\bigwedge^n \varphi$ factors through $\bigwedge^n R^{\oplus(n-1)}$, which implies $\det(A) = 0$.

After choosing a dual basis $\check{x}_i$, $[\varphi : x_j \mapsto \sum_j x_i a_{ij}] \in \mathrm{End}_R(E)$ can be identified with $\sum_{i,j} a_{ij} \check{x}_j \otimes x_i \in E^\vee \otimes E$, which can be written in a coordinate-free way $f \otimes x \mapsto f(\cdot)x$.

Definition 24. Trace $\mathrm{Tr}(\varphi)$ is the image of $\mathrm{End}_R(M) \xrightarrow{\sim} E^\vee \otimes E \to R$. In matrix language, $\mathrm{Tr}(A) = \sum_i a_{ii}$.

Now we turn to R-algebra $A = R^{\oplus n}$.

Definition 25. For $a \in A$, the left multiplication $x \mapsto ax$ defines a R-module endomorphism $m_a \in \mathrm{End}_R(A)$. The determinant and trace of free module endomorphism define **norm** $N_{A|R}(a) \equiv \det(m_a)$ and **trace** $\mathrm{Tr}_{A|R}(a) \equiv \mathrm{Tr}(m_a)$.

Lemma 9. *Let A be a commutative R-algebra with basis $(a_j)_j$, E be a free A-module with basis $(e_i)_i$, then E is also a free R-module with basis $(a_j e_i)_{i,j}$ and* $\mathrm{rk}_R(E) = \mathrm{rk}_R(A)\mathrm{rk}_A(E)$.

If $e = \sum u_i e_i$ and $u_i = \sum r_{ij} a_j$, then $e = \sum_{i,j} r_{ij} a_j e_i$.

Theorem 7. *Let A be a commutative R-algebra and finite rank free R-module, E a finite rank free A-module. Notice that $\varphi \in \mathrm{End}_A(E)$ can be embedded into $\mathrm{End}_R(E)$, we have*

$$\mathrm{Tr}_R(\varphi) = \mathrm{Tr}_{A|R}(\mathrm{Tr}_A(\varphi)), \det_R(\varphi) = N_{A|R}(\det_A(\varphi)).$$

Just like the previous lemma, after choosing a basis, this theorem simply reduces to the property of blocked matrices in linear algebra.

Corollary 6. *If A-algebra B is also a finite rank free A-module, replace E with B, for any $b \in B$ we have*

$$\mathrm{Tr}_{B|R}(\varphi) = \mathrm{Tr}_{A|R}(\mathrm{Tr}_{B|A}(\varphi)), N_{B|R}(\varphi) = N_{A|R}(N_{B|A}(\varphi)).$$

Definition 26. Let A be an R-algebra. The symmetric bilinear form $(x, y) \mapsto \mathrm{Tr}_{A|R}(xy)$ is called the **trace form** of A.

Proposition 8. *If A has a basis $x_1, \ldots, x_n$ as an R-module, define $d(x_1, \ldots, x_n) \equiv \det_R(x_i, x_j)$. Basis transformation $y_i = \sum_j t_{ij} x_j$ induces $d(y_1, \ldots, y_n) = \det(T)^2 d(x_1, \ldots, x_n)$, so $d_A \equiv d(x_1, \ldots, x_n)$* mod $R^{\times 2} \in R/R^{\times 2}$ *only depends on the algebra A, called* **discriminant** *of A.*

7 Integral Elements

Assume $A \neq \{0\}$ is an R-algebra. For any pairwise-commutative $x, y, \cdots \in A$, define $R[x, y, \ldots]$ to be the image of evaluation homomorphism

$$\mathrm{ev}_{(x,y,\ldots)} : R[X, Y, \ldots] \twoheadrightarrow R[x, y, \ldots] \subset A$$
$$f(X, Y, \ldots) = \sum (a_k X^{k_X} Y^{k_Y} \ldots) \mapsto \mathrm{ev}_{(x,y,\ldots)}(f) = f(x, y, \ldots) = \sum (a_k x^{k_X} y^{k_Y} \ldots)$$

then $R[x, y, \ldots]$ is always a commutative subalgebra of A.

Definition 27. x is **integral** over R if there is a monic polynomial $f \in R[X]$ vanishing at x, i.e. there exists $n \geq 1$ and $a_0, \ldots, a_{n-1} \in R$ such that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$.

Definition 28. A left S-module N is **faithful** if $sN = \{0\} \Leftrightarrow s = 0$.

Any algebra A is faithful as a $R[x]$-module, and any submodule $N \subset A$ containing 1 is also faithful ($s = s \cdot 1_A \in sN$).

Theorem 8. *For each $x \in A$, the following conditions are equivalent:*

1. *x is integral over R;*
2. *$R[x]$ is a finitely generated R-module;*
3. *$x \in M \subset A$, where M is both a finitely generated R-module and a faithful $R[x]$-module.*

(1) $\implies$ (2) $\implies$ (3) is trivial. For (3) $\implies$ (1), suppose M is generated by $\{b_1, \ldots, b_m\}$, $xb_i = \sum_{j=1}^m b_j t_{ji}$ since M is an $R[x]$-submodule. In the language of matrices, $(b_1 \cdots b_m)(x \cdot \mathrm{id} - T) = 0$. Multiply both side by the adjugate matrix of $(x \cdot \mathrm{id} - T)$, we have $P(x)(b_1 \cdots b_m) = 0$ where $P(x) = \det(x \cdot \mathrm{id} - T)$. Faithfulness implies $P(x) = 0$.

Corollary 7. *$\{x_i\}_{i \in I}$ is a family of pairwise commutative integral elements, then all the elements of $R[(x_i)_{i \in I}]$ are integral. In particular, $x + y$ and xy are integral if x, y are commutative integral elements.*

We only need to consider finite sum, so the problem is reduced to $R[x_1, \ldots, x_n]$. $R_{m+1} \equiv R[x_1, \ldots, x_{m+1}] = R_m[x_{m+1}]$ is a finitely generated R_m-module since x_{m+1} is integral over R_m. By induction R_n is a finitely generated R-module.

When R is a field $\Bbbk$, $\Bbbk[X]$ is a PID and the kernel of evaluation homomorphism $\mathrm{ev}_x : \Bbbk[X] \to A$ can be classified into

Definition 29. $\ker(\mathrm{ev}_x) \equiv (P_x)$

- x is **transcendental** if $P_x = 0$.
- x is **algebraic** if $P_x \neq 0$. The monic polynomial of minimum degree in (P_x) is called the **minimal polynomial** of x.

Lemma 10. *A is a $\Bbbk$-algebra, then for any algebraic element x, it is left invertible iff invertible in $\Bbbk[x]$ iff right invertible.*

The minimal polynomial P_x must have a nonzero constant term (otherwise $P_x(X)/X$ vanishes at x), so $Q(x) = (P_x(x) - P_x(0))/x$ is the inverse of x up to a unit.

Lemma 11. *If $\Bbbk$-algebra A is an integral domain, then $\Bbbk[x] \simeq \Bbbk[X]/(P_x)$ is a field for any algebraic element $x \in A$. Besides, P_x is irreducible and $\dim_\Bbbk \Bbbk[x] = \deg P_x$.*

$\Bbbk[x]$ has no zero factor, so (P_x) is a prime ideal. Prime element is irreducible in PID and $\Bbbk[x]$ is a field. The basis of $\Bbbk[X]/(P_x)$ is obviously $\{x^i : 0 \leq i < \deg P_x\}$.

Lemma 12. *A is an integral domain and $x \in A$. If A is a $\mathbb{C}$-algebra and x is algebraic, then $\mathbb{C}[x] = \mathbb{C}$; if A is a $\mathbb{R}$-algebra and x is algebraic, then $\mathbb{R}[x] = \mathbb{R}$ or $\mathbb{C}$.*

It comes from the fact that irreducible polynomials in $\mathbb{C}[X]$ has degree 1 and those in $\mathbb{R}[X]$ has degree 1 or 2.

Theorem 9 (Frobenius)**.** *If D is a division $\mathbb{R}$-algebra with all elements algebraic, then $D \simeq \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

Assume $\dim_\mathbb{R}(D) \geq 2$ (otherwise D is simply $\mathbb{R}$), the previous lemma implies that $\mathbb{C}$ can be embedded into D. The left multiplication endows D a $\mathbb{C}$-vector space structure, and $x \mapsto ixi^{-1}$ is idempotent. Therefore D can be decomposed into eigen-subspace $D = D^+ \oplus D^-$. D^+ is just $\mathbb{C}$ since it it a $\mathbb{C}$-algebra and every element is algebraic over $\mathbb{R}$. If $D^- = \{0\}$, then $D = \mathbb{C}$; otherwise take $j \in D^- \backslash \{0\}$ and $t \mapsto tj$ is an embedding $D^- \hookrightarrow D^+$. $\dim_\mathbb{C} D^+ = 1$, so $\dim_\mathbb{C} D^-$ is also 1 and $D^- = D^+ j$. The minimal polynomial of j over $\mathbb{R}$ is of degree 2, so $j^2 \in \mathbb{R} \oplus \mathbb{R}j$. The previous embedding implies $j^2 \in D^+$, so $j^2 \in \mathbb{R}_{<0}$. It can be verified that D is just the quaternion $\mathbb{H}$.