Condensed Matter Field Theory

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Contents

The organization follows Altland & Simons's book and I will list supplement materials in the beginning of each section (inspired by Connor Mooney's notes on PDE). The goal is to provide an introduction to condensed matter physics for amateurs.

1 From Particle to Fields

Before delving into the world of condensed matter physics, it might be helpful to get some intuition about what quasi-particles look like and how to write down an effective theory.

My suggestion is to read

1. Alexei Kitaev, Faults-tolerant quantum computation by anyons

first, to get some intuition for condensed matter physics.

1.1 Lagrangian formalism

We begin with the following problem: how to describe the continuum limit of a many-body problem.

It's useful to begin with a concrete example and try to learn rules from it, so let's consider a harmonic chain first:

$$L = \sum_{i=1}^{N} \left[\frac{1}{2} m \dot{\phi}_i^2 - \frac{k_s}{2} (\phi_{i+1} - \phi_i)^2 \right]. \tag{1}$$

To go from discrete variables to continuous variables, we use the Euler-Maclaurin formula

$$\sum_{i=1}^{N} f(i) = \int_1^N f(i) \mathrm{d}i + \frac{1}{2}(f(1) + f(N)) + \dots \tag{2}$$

In the $N \to \infty$ limit, we can simply keep the first integral term and make the change of variable $\mathrm{d}i = \frac{\mathrm{d}i}{\mathrm{d}x} \mathrm{d}x = a^{-1} \mathrm{d}x$, i.e.

$$\sum_{i=1}^{N} = \frac{1}{a} \int_0^L \mathrm{d}x. \tag{3}$$

The renormalization of the field $\phi(x) = a^{-1/2} \phi_I$ is to absorb the additional factors in the mass term. For $\phi_{I+1} - \phi_I$, assume the derivatives of the fields are bounded and continuous enough, then

$$\phi_{I+1} - \phi_I = \phi'(I) + O(\phi'(I + \xi) - \phi'(I)) = a^{3/2} \partial_x \phi(x) + O(a^{5/2} \phi''(x)) \tag{4}$$

Therefore the lagrangian is $O(L||\phi||^2)$, where the norm is the one for C^2 space $||\phi|| := \max_{|\alpha \leq 2} \sup_x |\partial^\alpha \phi(x)|$. A dilation $\phi \mapsto \lambda \phi$ doesn't change the solution of equations of motion, and $||\phi||$ would be determined by other renormalization conditions.

After that, the standard variational method in classical field theory gives the low energy excitations of a many-body problem. In the harmonic chain, the equation of motion is a wave equation and the corresponding excitations are left/right moving waves and hence a collective excitation. Those excitations are building blocks of a condensed matter system and could be quite anti-intuitive, e.g. the topological excitation(anyons) in a topological phase. So in my mind they are first abstract excitations and then it would be lucky if we could find some features to describe them better (e.g. localized in space, described by some algebraic/geometric structure, etc.).

1.2 Hamiltonian formalism

The rules for substituting variables are similar to the Lagrangian formalism.

Notice that there exist massless excitations $\partial_x \phi_\pm \to 0$, which is related to the spontaneously broken translational symmetry.

2 Second Quantization

2.1 Theory

Reading: Chapter 4 of Weinberg's QFT.

The point is that in Fock space any operator can be expressed in annihilation/creation operators

$$\widehat{O} = \sum O_{\alpha\beta} a^\dagger_{\alpha_1} \ldots a^\dagger_{\alpha_N} a_{\beta_1} \ldots a_{\beta_M} \tag{5}$$

where α, β are two multi-indices, $|\alpha| = N, |\beta| = M$. Among which those with the following form

$$\widehat{O}_1 = \sum_{ij} O_{ij} a^\dagger_i a_j \tag{6}$$

are called *one-body operators* and

$$\widehat{O}_2 = \sum_{ijkl} O_{ijkl} a^\dagger_i a^\dagger_j a_k a_l \tag{7}$$

are called *two-body operators*.

One-body operators are diagonalizable, and for H and momentum operators $a^\dagger_k a_k$, we call the coefficient matrix $h(k)$ *Bloch Hamiltonian*.

$$H = \sum a^\dagger_{k,\alpha} h_{\alpha,\beta}(k) a_{k,\beta} \tag{8}$$

Sandwiching the operator with corresponding $a^\dagger |0\rangle$ and $\langle 0| a$, we get the uniqueness of Bloch Hamiltonian (and other coefficients). Similarly, we conclude that $h(k)$ is Hermitian.

We define the *spin operator* $\hat{S}$ as

$$\hat{S} = \sum a^\dagger_{\lambda\alpha} S_{\alpha\alpha'} a_{\lambda\alpha'} \tag{9}$$

where $S = \frac{1}{2}\sigma$, then

$$\begin{aligned}
\hat{S}^z &= \frac{1}{2}\sum(\hat{n}_{\lambda+} - \hat{n}_{\lambda-}) && (10)\\
\hat{S}^+ &= \sum a^\dagger_{\lambda+} a_{\lambda-} && (11)\\
\hat{S}^- &= \sum a^\dagger_{\lambda-} a_{\lambda+} && (12)
\end{aligned}$$

2.1.1 Discrete symmetries on creation/annihilation operators

Let's talk about the most important symmetry operator: time reversal, in this section. Most literature in condensed matter community is unreadable. So my suggestion is to read Appendix C, Chapter 2 of Weinberg's QFT carefully.

The take-home message is that parity operators are NOT physical, and the only meaningful definition is to make them (at least approximately) conserved, i.e. having correct commutating relation with generators of Poincare group or other reasonable groups.

In condensed matter physics we usually don't need to full Poincare symmetry, but we should translate the natural requirements in high energy physics to our new situation: time reversal T should reverse momentum. I know the crystal momentum is not a real momentum, but that's how we make T useful. One immediate corollary is T preserves position index. Another requirement is T reverse angular momentum, and in condensed matter physics we never think about the orbital angular momentum and only care about the spin S, so $TST^{-1} = -S$.

Now the general action of time reversal operator is

$$T\Psi_{p,\sigma,n} = (-1)^{j-\sigma} \sum \mathcal{T}_{nm} \Psi_{-p,-\sigma,m} \tag{13}$$

where $\mathcal{T}$ is unitary, if Ψ are chosen to be orthonormal states. Note that we absorb some phase factor into the definition of Ψ.

In condensed matter physics we stop here, but I strongly recommend you to read the whole appendix mentioned above. It could be proved that in general we cannot set $T^2 = \pm 1$, instead

$$T^2 = (-1)^{2j} e^{\mp i\phi} \tag{14}$$

where ϕ is a scalar matrix. Only when $e^{i\phi} = \pm 1$ we have the simple relation $T^2 \propto 1$ and only when $e^{i\phi} = 1$ could we diagonalize the eigenstates (w.r.t. T).

Now let's return to the second quantization formalism. For spin-0 creation/annihilation operators we have

$$T a_{j,\alpha} T^{-1} = \mathcal{T}_{\alpha\beta} a_{j,\beta} \text{ or } T a_j T^{-1} = \mathcal{T} a_j \tag{15}$$

where we sum over β, and after Fourier transformation

$$T a_{k,\alpha} T^{-1} = \mathcal{T}_{\alpha\beta} a_{-k,\beta} \text{ or } T a_k T^{-1} = \mathcal{T} a_{-k} \tag{16}$$

Taking adjoint we get

$$T a^\dagger_{j,\alpha} T^{-1} = \mathcal{T}_{\alpha\beta}{}^* a^\dagger_{j,\beta} \text{ or } T a^\dagger_j T^{-1} = a^\dagger_j \mathcal{T}^\dagger \tag{17}$$

and

$$T a^\dagger_{k,\alpha} T^{-1} = \mathcal{T}_{\alpha\beta}{}^* a^\dagger_{-k,\beta} \text{ or } T a^\dagger_j T^{-1} = a^\dagger_{-k} \mathcal{T}^\dagger. \tag{18}$$

Notice that in the matrix notation the dagger acts on both the operator and the matrix, or we should think about the algebra $\mathrm{Mat}(N \times N, A)$, where A is the $\mathbb{C}$-algebra of creation/annihilation operators (maybe together with some other symmetry operators) and then define $(a_{ij})^\dagger = (a^\dagger_{ji})$, which goes back to the usual definition of dagger when $A = \mathbb{C}$.

As I said before, a useful definition of a discrete symmetry operator must make it conserved, i.e. $[T, H] = 0$. Let's see the implication on the Bloch Hamiltonian.

$$\begin{aligned}
H = THT^{-1} &= \sum T a^\dagger_k h(k) a_k T^{-1} && (19)\\
&= \sum T a^\dagger_k T^{-1} h^*(k) T a_k T^{-1} && (20)\\
&= \sum a^\dagger_{-k} \mathcal{T}^\dagger h^*(k) \mathcal{T} a_{-k} && (21)\\
&= \sum a^\dagger_k \mathcal{T}^\dagger h^*(-k) \mathcal{T} a_k && (22)
\end{aligned}$$

The uniqueness of Bloch Hamiltonian hence gives

$$\mathcal{T}h(k)\mathcal{T}^{-1} = h^*(-k) = h^T(-k) \tag{23}$$

Another way to deal with the problem is to treat the Bloch Hamiltonian as a module homomorphism. To be more exact, let $A = \mathrm{End}(\mathcal{H})$ be the algebra of endomorphisms of the Hilbert space. Then $A^{\oplus n}$ has a natural (A, A)-bimodule structure and the Bloch Hamiltonian should be treated as a bilinear map, or a A-valued matrix. Then it would be reasonable to write down $Th(k)T^{-1}$ as in literatures. Just remember that $Th(k)T^{-1} = h^*(k)$ in order to go back to our description.

Now let's prove the action on spin operator. The new communication relation is (again use vector notation to hide all the other indices)

$$Ta_{\pm}T^{-1} = \pm\mathcal{T}a_{\mp}, Ta_{\pm}^{\dagger}T^{-1} = \pm a_{\mp}^{\dagger}\mathcal{T}^{\dagger} \tag{24}$$

and then it is easy to show that

$$T\hat{S}^zT^{-1} = -\hat{S}^z, T\hat{S}^{\pm}T^{-1} = -\hat{S}^{\mp} \tag{25}$$

which is just $T\hat{S}T^{-1} = -\hat{S}$.

2.2 Examples

The examples given in the book are usually not self-contained. I am writing to clarify some tricky points and give some good references.

2.2.1 Interacting Electrons

Reading: Chapter 2 of David Tong's note *Applications of Quantum Mechanics*. Section 3.5 of Weinberg's *Lectures on Quantum Mechanics*. Chapter 2 (especially the first section) of Nolting's *Fundamentals of Many-body Physics*.

The Fourier transform of the coulomb potential is evaluated in the distributional sense.

$$V_\epsilon(r) = \frac{e^{-\epsilon r}}{r} \tag{26}$$

$$V_\epsilon(q) = \frac{4\pi}{q^2 + \epsilon^2} \tag{27}$$

We need to keep the ϵ to regularize the $q = 0$ divergence and it can be shown that this divergence is cancelled by the potential created by the ions.

For a derivation of tight-binding approximation, read Phillips, Advanced solid state physics.

2.2.2 Luttinger liquid

This part is pretty tricky. My suggestion is to read the following materials:

1. Fermi liquid and renormalization group

2.2.3 Jordan-Wigner transformation

For anisotropic XY-model, we need to employ Majorana fermions.

Reading: P. D. Sacramento and V. R. Vieira, *Duality and topology*, Appendix. A; Alexei Kitaev, *Unpaired Majorana fermions in quantum wires.*

Remark. This is the first time I cite Kitaev's work. Two great authors are Kitaev and Zinn-Justin. You will learn quite a lot from their book. Maybe I should add Anderson to this short list too.

3 Feynman Path Integral

Reading: Chapter 9 of Weinberg's QFT. Jean Zinn-Justin's *Path Integrals in Quantum Mechanics.*

In QFT, we need to consider the expected value of an operator between ground state, which introduce the ϵ terms in path integral. In condensed matter physics, there is usually a trace operation so we don't need to worry about the vacuum and hence the path integral formula is simplified.

3.1 Construction of Path Integrals

It's important to notice that the formal expression

$$\mathcal{Z} = \int \mathcal{D}q\, e^{iS[q]} \tag{28}$$

does NOT make sense in general. This kind of expressions always come from a limit of a finite dimensional integral. Hence whenever it is not clear how to perform a path integral, we need to go back to the discrete expression and take the limit carefully.

For example, if $A(x_i, x_j) := A_{ij}$ and a good enough interpolation is selected, then an analysis of the discrete expression tells that

$$A^{-1}(x_i, x_j) = \Delta x^{-2} A_{ij}^{-1}. \tag{29}$$

Therefore, to make the path integral well-defined, A_{ij} cannot be arbitrary and must have the above scaling property. Besides, it could be checked that after the Gaussian integral part of the divergent factor

$$\left(\frac{Nm}{it2\pi\hbar}\right)^N = \left(\frac{m}{i\Delta t 2\pi\hbar}\right)^N \tag{30}$$

will be cancelled and we are simply left with

$$\left(\frac{m}{i\hbar}\right)^N. \tag{31}$$

Notice that the partition function itself is not observable and we are only interested in the correlated functions, such a constant factor is irrelevant.

4 Functional Integrals

4.1 Coherent States and SUSY QM

Reading: Chapter 1 of *Complex geometry: an introduction*, Chapter 1 of Etingof's *Representation Theory* or *Tensor Categories*

This formalism makes sense only if we work in the super-spaces (in fact super-modules). Super- is a prefix meaning $\mathbb{Z}_2$-graded and can have nothing to do with supersymmetry in particle physics, just like the prefix quantum could have nothing to do with quanta (e.g. quantum cohomology, quantum group...).

When η is odd, we define $\int d\eta = \partial_\eta$ (hence the integral is just formal notation) and $f(\eta_1, \dots) \in \wedge^*(V)$ where V is the free module over $\{\eta_1, \dots\}$. Remember that they are definitions not properties, otherwise a lot of manipulations do NOT make sense at all.

Also, notice that the bosonic coherent corresponds to the classical limit (recall how QED goes back to electrodynamics) while the fermionic coherent states do NOT correspond to any thing. It's just a formal state, or an element in the super-module over the superalgebra (exterior algebra). Anyway, the η in $|\eta\rangle$ is just a anonymous variable.

4.2 Field Integral

Notice that the entire resolution of identity is

$$\int d(\bar{\psi}, \psi) e^{-\bar{\psi}\psi} |\psi\rangle \langle\psi| \tag{32}$$

while ψ contains indices for all the annihilation/creation operators appears in this problem.

4.3 Bosonization

5 Perturbation Theory

a

6 Broken Symmetry and Collective Phenomena

a

7 Response Functions

b

Sorry I am not familiar with experiments at all, so I don't have much to say at this stage.

8 Renormalization Group

Reading: Jean Zinn-Justin, Phase transition and renormalization group.

9 Topology

For a reference of topological insulators, see B. Andrei Bernevig, Taylor L. Hughes, *Topological Insulators and Topological Superconductors.* For Hopf invariants, see Dale Husemoller, *Fibre Bundles* and his (and some other guys') *Basic Bundle Theory and K-Cohomology Invariants.*

9.1 Algebraic Topology

If you want to do theory, you will need some algebraic topology, seriously. A great introduction to point-set topology is nLab "Introduction to Topology - 1". For algebraic topology, please learn category theory first and then go through May's book.

9.2 Topological Insulators and Topological Superconductors

The main reference is B. Andrei Bernevig, Taylor L. Hughes, *Topological Insulators and Topological Superconductors* and Witten's notes.

I have clarify the meaning of time reversal in previous chapters.

9.3 Quantum Hall Effect

Reading:

1. David Tong, *The Quantum Hall Effect*

2. J. Bellissard, A. van Elst, H. Schulz- Baldes, *The Non-Commutative Geometry of the Quantum Hall Effect*

9.4 An invitation to topological order

I'd like to give a biased review of topological phases.

I hope one day we can see a comprehensive review, like what is done in Stacks Project.

10 Classical Nonequilibrium Statistical Mechanics

Reading: a good book on probability theory

11 Quantum Nonequilibrium Statistical Mechanics

e

12 Last Words

Now you are free to explore the world of condensed matter physics.

I think a great thing is the emergence of something, which looks like the interior definition of manifolds. One important question in quantum gravity is: what is spacetime? Nothing guarantees that it should be a topological space. More general space like topos?