Supersymmetry (in progress)

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1 Introduction

I found the usual physics-oriented textbooks not clear enough, and I always like starting from the beginning when studying new knowledge. Therefore I decide to summarize how I study supersymmetry in this note. I hope my past self would enjoy it.

2 Preliminaries

Knowledge of abstract algebra and smooth manifolds is required. Other necessary mathematical concepts will be developed in this section, so please be patient. Although seeming to be useless at first glimpse, it finally pays off.

2.1 Central simple algebras

Knowledge of central simple algebras is useful in the discussion of spinors. Of course you can construct spinors by hand, but I prefer a more elegant approach.

In this subsection all modules are right-module, "finite" means "finite dimensional", and $[A : k] \equiv \dim_k A$.

2.1.1 Wedderburn theorem

Let A be a simple ring and M be its nonzero right ideal, with a natural A-module structure.

Structure of bicommutant: Define the commutant and bicommutant of M as

$$A' \equiv \mathrm{End}_A M,\ A'' \equiv \mathrm{End}_{A'} M. \tag{1}$$

Consider the homomorphism $R : A \to A''$, $(m)R(a) = ma$.

1. $A \simeq R(A)$: The kernel $\ker R$ is a two-sided ideal in A and cannot be A itself, so it must be zero and R is monomorphism.

2. $R(M)$ is a right ideal: $n(R(m)a'') = (nm)a'' = n((m)a'') = nR((m)a'')$ (left multiplication belongs to A').

3. $R(A)$ is a right ideal: AM is a two-sided ideal, so $A = AM$ and $R(A) = R(A)R(M)$.

4. $A'' = R(A) \simeq A$: $1_M \in R(A)$.

Lemmas:

1. For a finite dimensional k-algebra A, any nonzero A-module contains a finite dimensional submodule (e.g. xA), whose least dimensional submodule (as k-vector space) must be simple.

2. A simple A-modules M is finite dimensional for it is cyclic.

3. $\mathrm{End}_A M$ is a skew field (Schur's lemma).

Theorem 1. *A finite semisimple algebra is a finite product of matrix algebras over skew fields*

Proof. We only need to prove "a finite simple algebra = a matrix algebra over a skew field". Given a finite simple k-algebra A and consider itself as a A-module.

1. $\Longrightarrow$: Choose a simple submodule M, $K = \mathrm{End}_A M$ is a skew field and $\mathrm{End}_K A \simeq A$. Module over the skew field K is free, so $A \simeq K^{\mathrm{op}}(n \times n)$.

2. $\Longleftarrow$: For any nonzero matrix, multiply by E_{ij} from left and right to extract a nonzero element, then translate to other entries and linearly combine them to generate the whole matrix algebra.

2.1.2 Central algebras

2.1.3 Brauer group

2.1.4 Skolem-Noether theorem

2.2 Graded central simple algebras

in progress

3 Super Linear Algebra

3.1 Categorical approach

3.2 Free modules

4 Spinors

4.1 Clifford algebra

4.2 Spin group

5 Bibliography

1. Stacks Project

2. Pierre Deligne and John W. Morgan, *Notes on supersymmetry*

3. Pierre Deligne, *Notes on spinors*

4. ...