Example: Beetle Mortality

- Study the relation between the mortality of beetles after 5 hours exposure to gaseous disulphide (CS_2) and the concentrations.

Log10(Dose), x_i	# of beetles, m_i	# killed, y_i
1.6907	59	6
1.7242	60	13
1.7552	62	18
1.7842	56	28
1.8113	63	52
1.8369	59	53
1.8610	62	61
1.8839	60	60

Table: Grouped data.

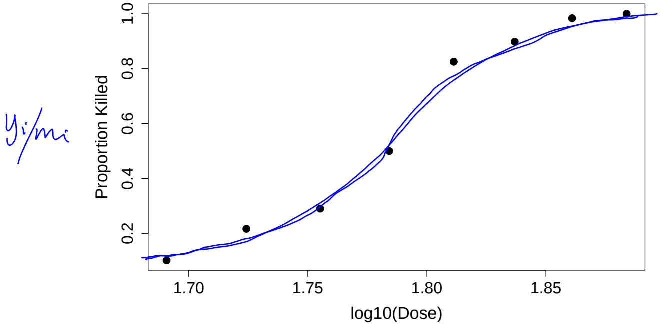

Figure: Beetle mortality data. Proportion killed: y_i/m_i against dose on the $\log_{10}$ scale: x_i.

Example: Median Lethal Dose

- Recall the beetle mortality example, where the primary goal is to study the relation between the mortality rate and the CS2 concentration.
- Suppose we are interested in finding the median lethal dose (LD50), i.e., the dose required to kill half the population.
- In the logistic regression framework, it is equivalent to predicting the dose x_0 that leads to a response rate $\pi_0 = 0.5$.

$$g(\pi) = \beta_0 + \beta_1 x$$

- This type of problem is common in toxicology.

We are interested in x_0 s.t. $\beta_0 + \beta_1 x_0 = g(0.5)$ in LD50 study

- Logit: $x_0 = -\frac{\beta_0}{\beta_1}$ $= f(\beta_0, \beta_1)$ $\quad \log\frac{\pi}{1-\pi} = 0$
- Point Estimate: $\hat{x}_0 = x_0(\hat{\beta}_0, \hat{\beta}_1) = -\hat{\beta}_0/\hat{\beta}_1$

$$\mathrm{var}(f(\hat\beta_0, \hat\beta_1)) = \left(\frac{\partial f}{\partial \beta_0}, \frac{\partial f}{\partial \beta_1}\right) \cdot \underbrace{\begin{pmatrix} \mathrm{var}(\hat\beta_0) & \mathrm{cov}(\hat\beta_0, \hat\beta_1) \\ \mathrm{cov}(\hat\beta_0, \hat\beta_1) & \mathrm{var}(\hat\beta_1) \end{pmatrix}}_{I^{-1}(\beta_0, \beta_1)} \cdot \begin{pmatrix} \frac{\partial f}{\partial \beta_0} \\ \frac{\partial f}{\partial \beta_1} \end{pmatrix}$$

- Asymptotic variance of $\hat{x}_0$:

$$\mathrm{var}(\hat{x}_0) = \left(\frac{\partial x_0}{\partial \beta_0}\right)^2 \mathrm{var}(\hat{\beta}_0) + \left(\frac{\partial x_0}{\partial \beta_1}\right)^2 \mathrm{var}(\hat{\beta}_1)$$
$$+2\left(\frac{\partial x_0}{\partial \beta_0}\right)\left(\frac{\partial x_0}{\partial \beta_1}\right)\mathrm{cov}(\hat{\beta}_0, \hat{\beta}_1)$$

$\mathrm{var}(f(\hat\beta_0, \hat\beta_1)) =$

$$\frac{\partial f}{\partial \beta_0} = -\frac{1}{\beta_1} \qquad \frac{\partial f}{\partial \beta_1} = \frac{\beta_0}{\beta_1^2}$$

- Then the asymptotic CI of x_0 is

$$x_0 \in [x_L, x_R] = \left[\hat{x}_0 - z_{\alpha/2}\sqrt{\mathrm{var}(\hat{x}_0)}, \hat{x}_0 + z_{\alpha/2}\sqrt{\mathrm{var}(\hat{x}_0)}\right]$$

- $(1-\alpha)100\%$ CI of LD50: $[10^{x_L}, 10^{x_R}]$

$-\frac{\beta_0}{\beta_1}$