Problem 1

Show that the following distributions belong to the exponential family. Find the natural parameter θ, scale parameter ϕ and convex function $b(\theta)$. Also find the $E(Y)$ and $Var(Y)$ as functions of the natural parameter. Specify the canonical link functions.

1. Exponential distribution $Exp(\lambda)$, $f(y;\lambda) = \lambda e^{-\lambda y}$;
2. Binomial distribution $Bin(n,\pi)$, $f(y;\pi) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$, where n is known;
3. Poisson distribution $Pois(\lambda)$, $f(y;\lambda) = \frac{1}{y!}\lambda^y e^{-\lambda}$;
4. Chi-squared distribution $\chi^2_{(k)}$, $f(y;k) = \frac{1}{\Gamma(\frac{k}{2})2^{\frac{k}{2}}} y^{\frac{k}{2}-1} e^{-\frac{y}{2}}$;
5. Negative binomial distribution $NB(m,\beta)$, $f(y;\beta) = \binom{y+m-1}{m-1}\beta^m(1-\beta)^y$, where m is known;
6. The Gamma distribution $Gamma(\alpha,\beta)$, $f(y;\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$, where the shape parameter α is known.

Problem 2

Assume $Y_1, Y_2, ..., Y_n$ are independent and follow a binomial distribution where $Y_i \sim Bin(m,\pi_i)$ and m is known. Furthermore, assume $log\frac{\pi_i}{1-\pi_i} = X_i\beta$. What are the expressions of deviance residuals and Pearson residuals respectively (use $\hat{\beta}$ to represent the MLE)? What are the expressions of the deviance and Pearson's χ^2 statistic?

Problem 3

Consider the binary response variable $Y \sim Bernoulli$ with $P(Y=1) = \pi$ and $P(Y=0) = 1-\pi$. Observations Y_i, $i = 1,\ldots,n$, are independent and identically distributed as Y.

1. Find the Wald test statistic, the score test statistic, and the likelihood ratio test statistic to test hypotheses on $\pi = \pi_0$.
2. With large samples, the Wald test statistic, score test statistic and the likelihood ratio test statistic approximately have the $\chi^2(1)$ distribution. For $n = 10$ and data (0, 1, 0, 0, 1, 0, 0, 0, 1, 0), use these statistics to test null hypotheses on for (i) $\pi = 0.1$, (ii) $\pi = 0.3$, (iii) $\pi = 0.5$.
3. Do the test statistics lead to the same conclusions?

Problem 4

(Optional; PhD required) $Y_i \sim Pois(\lambda), i = 1, ..., n$. We are interested in testing $H_0 : log\lambda = log\lambda_0$. What are the Wald test statistic, the score test statistic, and the likelihood ratio test statistic? How are they different from the test statistics for testing $H_0 : \lambda = \lambda_0$?