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Problem 1

1. Exponential distribution $Exp(\lambda)$

$f(y;\lambda) = \lambda \cdot e^{-\lambda y} \quad I\{\lambda > 0\}$

$f(y;\lambda) = e^{-\lambda y} \cdot e^{\log\lambda} = e^{[\log\lambda - \lambda y]}$

$= \exp[\log\lambda - \lambda y]$

$= \exp[-\lambda y + \log\lambda]$

$= \exp[-\lambda y 1 - (-\log\lambda)]$

Natural parameter: $-\lambda = \theta \Rightarrow \theta = -\lambda$

scale parameter: $-1 = \phi$

convex function: $b(\theta) = -\log(-\theta)$

$E(Y) = b'(\theta) = -1 \cdot \frac{-1}{-\theta} = -\frac{1}{\theta} = \frac{1}{\lambda}$

$Var(Y) = b''(\theta) = \frac{1}{\theta^2} = \frac{1}{\lambda^2}$

Canonical link function:

we set $\lambda = \eta$

$g(\mu) = \eta = \lambda = b'^{-1}(\mu)$

As inferred above, $\mu = b'(\lambda) = -\frac{1}{\lambda} \Rightarrow b'^{-1}(\mu) = -\frac{1}{\mu}$ $(\lambda > 0)$

so, $g(\mu) = -\frac{1}{\mu}$ is the canonical link function,

since $g^{-1}(\eta)$ is strictly increasing and differentiable.

2. Binomial Distribution: $Bin(n,\pi)$

$f(y;\pi) = \binom{n}{y} \cdot \pi^y \cdot (1-\pi)^{n-y}$, where n is known

with $0 < \pi < 1$, $y = 0, 1, \dots, n$.

$\binom{n}{y} = \frac{n!}{(n-y)!\,y!}$

$f(y;\pi) = \binom{n}{y} \pi^y (1-\pi)^{n-y}$

$= \binom{n}{y} (1-\pi)^n \cdot \left(\frac{\pi}{1-\pi}\right)^y$

$= \binom{n}{y} (1-\pi)^n \cdot \exp\left[\log\left(\frac{\pi}{1-\pi}\right) y\right]$

Define A more general form: $= \exp\left[n \cdot \log(1-\pi) + \log\left(\frac{\pi}{1-\pi}\right) y - \log\binom{n}{y}\right]$

$f(y;\pi) = \exp\left[n\log(1-\pi) + \log\left(\frac{\pi}{1-\pi}\right) y + \log\binom{n}{y}\right]$

Natural parameter: $\theta = \log\left(\frac{\pi}{1-\pi}\right)$

Scale parameter: $\phi = 1$

Convex function: $b(\theta) = -n\log(1-\pi)$, As $\pi = \frac{e^\theta}{1+e^\theta}$, $b(\theta) = n\log(1+e^\theta$

$E(y) = b'(\theta) = n \cdot \frac{e^\theta}{1+e^\theta} = n \cdot \pi$ (Replace $\frac{e^\theta}{1+e^\theta}$ by π)

$Var(y) = b''(\theta) \cdot \phi = n \cdot \frac{e^\theta(1+e^\theta) - e^\theta e^\theta}{[(1+e^\theta)]^2}$

$= n \cdot \frac{e^\theta}{(1+e^\theta)^2} = n \cdot \frac{\frac{\pi}{1-\pi}}{\frac{1}{(1-\pi)^2}} = n\pi(1-\pi)$

For cononical link function:

we set $\theta = \eta$, which is eqivalent to

$\log(\frac{\pi}{1-\pi}) = \eta$ (link) $\pi = \frac{e^\theta}{1+e^\theta}$

As $b'(\theta) = n\pi = \mu = n \cdot \frac{e^\theta}{1+e^\theta} \Rightarrow [b'(\theta)]^{-1}$

$g(\mu) = [b'^{-1}(\mu)] = \log(\frac{\mu}{n-\mu})$ $e^\theta = (\frac{n}{\mu} - 1)^{-1} = \frac{\mu}{n-\mu}$

that is $g(\mu) = \log[\frac{\mu}{n-\mu}]$ $\Rightarrow \theta = -\log(\frac{n}{\mu} - 1)$

3. Poisson Distribution

$f(y;\lambda) = \frac{1}{y!}\lambda^y \cdot e^{-\lambda}$;

$f(y;\lambda) = \exp\{-\log y! + y \cdot \log\lambda - \lambda\}$

$= \exp\{y \cdot \log\lambda - \lambda - \log(y!)\}$

Natural parameter $\theta = \log\lambda$

Scale parameter $\phi = 1$

Convex function: $b(\theta) = \lambda = e^\theta$

Expectation $E(y) = b'(\theta) = e^\theta = e^{\log\lambda} = \lambda$

Variance $Var(y) = b''(\theta) \cdot \phi = e^\theta \cdot 1 = e^\theta = e^{\log\lambda} = \lambda$

Cononical link function:

$b'(\theta) = e^\theta$

$[b'(\theta)]^{-1} = \log\mu$

$g(\mu) = [b'^{-1}(\mu)] = \log\mu$

4. Chi-squared Distribution $\chi^2_{(k)}$

$f(y;k) = \frac{1}{\Gamma(\frac{k}{2}) 2^{\frac{k}{2}}} y^{\frac{k}{2}-1} \cdot e^{-\frac{y}{2}}$

$= \frac{1}{\Gamma(\frac{k}{2}) \cdot 2^{\frac{k}{2}}} \cdot \exp\{(\frac{k}{2} - 1) \cdot \log y - \frac{y}{2}\}$

$= \exp\{(\frac{k}{2} - 1)\log y - \frac{y}{2} - \log(\Gamma(\frac{k}{2})) - \frac{k}{2}\log 2\}$

$= \exp\{(\frac{k}{2}-1)\log y - [\log(\Gamma(\frac{k}{2})) + \frac{k}{2}\log 2] + (-\frac{y}{2})\}$

Natural parameter: $\theta = \frac{k}{2} - 1$

Scale parameter: $\phi = 1$

Convex function: $b(\theta) = \log(\Gamma(\frac{k}{2})) + \frac{k}{2}\log 2$.

Expectation: $b'(\theta) = k$

$=$ (By definition and inference from $\mu = E(X) = k$)

Variance: $Var(y) = b''(\theta) \cdot \phi$

$= 2k$

Canonica link: since $b'(\theta) = k$. $b'(\theta) = 2(\theta+1) = \mu$.

since $\eta = g(\mu)$, we have:

$g^{-1}(\mu) = b'^{-1}(\theta) = \frac{\mu}{2} - 1$

So: $g^{-1}(\mu) = \frac{\mu}{2} - 1$,

which is monotone & increasing on range of domain.

5 Negative binomial Distribution $NB(m, \beta)$

$$f(y;\beta) = \binom{y+m-1}{m-1} \beta^m (1-\beta)^y \quad y = m, m+1, \cdots$$

$$= \binom{y+m-1}{y} \beta^m (1-\beta)^y \quad [y = 0, 1, \cdots$$

$$= \frac{(y+m-1)!}{y!\,(m-1)!} \beta^m \cdot (1-\beta)^y \quad 0 \le \beta \le 1, \ m. \text{ known}]$$

$$= \exp\left\{\log(1-\beta)\cdot y - (-m\log\beta) + \log\left[\frac{(y+m-1)!}{y!\,(m-1)!}\right]\right\}$$

Natural parameter $\theta = \log(1-\beta) \Rightarrow e^\theta = 1-\beta$, $\beta = 1 - e^\theta$

Scale parameter $\phi = 1$

with $b(\theta) = -m\log[(1-e^\theta)]$

Expectation: $b'(\theta) = \frac{-e^\theta}{1-e^\theta} \cdot (-m) = m \cdot \frac{e^\theta}{1-e^\theta} = m \cdot \frac{(1-\beta)}{\beta}$

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Variance: $Var(y) = b''(\theta)\cdot\phi = b''(\theta)$

$$= \left(\frac{me^{\theta}}{(1-e^{\theta})}\right)'$$

$$= \frac{me^{\theta}(1-e^{\theta}) - me^{\theta}(-e^{\theta})}{(1-e^{\theta})^2}$$

$$= \frac{me^{\theta}}{(1-e^{\theta})^2} = \frac{m\cdot(1-\beta)}{\beta^2}$$

Canonical link: $g(E(y)) \to \theta$

As $b'(\theta) = m\cdot\frac{(1-\beta)}{\beta} = \mu$

$$\Rightarrow \mu = \frac{m(\frac{1}{\beta}-1)}{1} \quad (m \text{ is known})$$

$$= m\cdot\left(\frac{1}{1-e^{\theta}} - 1\right)$$

$$= m\cdot\frac{e^{\theta}}{1-e^{\theta}}$$

$$= \frac{m}{e^{-\theta}-1}$$

Get an inverse function of μ,

$$\mu = \frac{m}{e^{-\theta}-1} \qquad \Rightarrow \theta = \log\left(\frac{1}{1+\frac{m}{\mu}}\right)$$

$$e^{-\theta} = \frac{m}{\mu} + 1 \qquad = \log\left(\frac{\mu}{\mu+m}\right)$$

$$-\theta = \log\left(1+\frac{m}{\mu}\right)$$

That is, $\theta = \log\left(\frac{\mu}{\mu+m}\right)$

So, $g(\mu) = \log\left(\frac{\mu}{m+\mu}\right)$

6. Gamma Distribution

Gamma(α, β) with α is known

$$f(y;\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\cdot y^{\alpha-1}\cdot e^{-\beta y}$$

$$f(y;\beta) = \exp\{\alpha\cdot\log\beta - \log\Gamma(\alpha) + (\alpha-1)\log y - \beta y\cdot\log e\}$$

$$= \exp\{-\beta y - \log\Gamma(\alpha) + \alpha\cdot\log\beta + (\alpha-1)\log y\}$$

$$= \exp\left\{\frac{\frac{-\beta y}{\alpha} + \frac{\alpha}{\alpha}\log\beta}{\frac{1}{\alpha}} + (\alpha-1)\log y - \log\Gamma(\alpha)\right\}$$

$$= \exp\left\{\frac{\frac{\beta}{\alpha}y - \log\beta}{-\frac{1}{\alpha}} + (\alpha-1)\log y - \log(\Gamma(\alpha))\right\}$$

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As seen above,

Natural parameter: $\theta = \frac{\beta}{\alpha}$

Scale parameter: $\phi = 1/\alpha$

For Convex function, we need to make a transformation of β.

since $\beta = \alpha\theta = \theta/\phi$, $\log\beta = \log\theta - \log\phi$

$$f(y;\beta) = \exp\left\{\frac{\theta y - \log\theta}{-\phi} + \frac{\log\phi}{\phi} + \left(\frac{1}{\phi} - 1\right)\log y - \log\left(\Gamma\left(\frac{1}{\phi}\right)\right)\right\}$$

$\Rightarrow b(\theta) = \log(\theta)$

Then, we derive:

- Expectation: $E(y) = b'(\theta) = \frac{1}{\theta}$
- Variance $= Var(y) = b''(\theta)\cdot\phi = -\frac{1}{\theta^2}(-\phi) = +\frac{1}{\alpha\theta^2}$

Recall that $\alpha\theta = \beta$, $Var(y) = +\frac{\alpha}{\beta^2}$

To simplify,

$E(y) = \frac{\alpha}{\beta}$

$Var(y) = \frac{\alpha}{\beta^2}$

Canonical Link: $b'(\theta) = \frac{1}{\theta} = \mu \Rightarrow \theta = \frac{1}{\mu}$

$g(\mu) = \frac{1}{\mu}$ is a valid inverse.

Problem 2

$Y_1, Y_2, \cdots, Y_n$ are i.i.d, $Y_i \sim Bin(m, \pi_i)$, m is known.

$$\log\frac{\pi_i}{1-\pi_i} = X_i\beta$$

① Deviance

$$D(y, \hat{\mu}) = 2\{l(y,y) - l(y,\hat{\mu})\}$$

Recall that for binomial distribution

$$f(y,\pi) = \binom{n}{y}\cdot\pi^y\cdot(1-\pi)^{n-y}$$

Likelihood function :

$$L(y|\beta) = \sum_{i=1}^{n} \log\left[\binom{m}{y_i} \pi_i^{y_i} (1-\pi_i)^{m-y_i}\right]$$

$$= \sum_{i=1}^{n} \left[\log\binom{m}{y_i} + y_i \log \pi_i + (m-y_i)\log(1-\pi_i)\right]$$

$$= \sum_{i=1}^{n} \left[y_i \log\left(\frac{\pi_i}{1-\pi_i}\right) + m \log(1-\pi_i) + \log\binom{m}{y_i}\right]$$

Since $Y_i \sim Bin(m, \pi_i)$, $\mu = m\pi_i$, the log-likelihood:

$$l(y,\mu) = y\theta - b(\theta) + c(y)$$

(from Exp. fam) $= \log\left(\frac{\pi}{1-\pi}\right)\cdot y - [-n\log(1-\pi)] + \log\binom{n}{y_i}$

As the canonical link is $g(\mu) = \log\frac{\mu}{m-\mu} = \theta$

so, $l(y,\mu) = \sum_{i=1}^{n}\left[\log\frac{\mu_i}{m-\mu_i} y_i + m\log\left(\frac{m-\mu_i}{m}\right) + \log\binom{m}{y_i}\right] \cdots (1)$

(Derived from $e^{\theta}+1 = \frac{1}{1-\pi}$, $1-\pi = \frac{m-\mu}{m}$)

$$l(y,y) = \sum_{i=1}^{n}\left[\log\frac{y_i}{m-y_i} y_i + m\log\left(\frac{m-y_i}{m}\right) + \log\binom{m}{y_i}\right] \cdots (2)$$

Deviance : $D(y,\hat{\mu}) = 2\{l(y,y) - l(y,\hat{\mu})\}$

$$D(y,\hat{\mu}) = 2\sum_{i=1}^{n}\left[\left(\log\frac{y_i}{m-y_i} - \log\frac{\mu_i}{m-\mu_i}\right) y_i + m\log\left(\frac{m-y_i}{m-\mu_i}\right)\right]$$

$$= 2\sum_{i=1}^{n}\left[\log\left(\frac{y_i}{m-y_i}\cdot\frac{m-\mu_i}{\mu_i}\right) y_i + m\log\left(\frac{m-y_i}{m-\mu_i}\right)\right]$$

Replace with $X_i\hat{\beta}$ $= 2\sum_{i=1}^{n}\left[\log\left(\frac{y_i}{m-y_i}\cdot\frac{m-m\pi_i}{m\pi_i}\right) y_i + m\log\left(\frac{m-y_i}{m-m\pi_i}\right)\right]$

$\pi_i = \frac{1}{e^{X_i\hat{\beta}}+1}$ $\quad = 2\sum_{i=1}^{n}\left\{\log\left[\left(\frac{y_i(1-\pi_i)}{(m-y_i)\pi_i}\right) y_i\right] + \left[\log\frac{m-y_i}{m(1-\pi_i)}\right]\cdot m\right\}$

$\frac{\pi_i}{1-\pi_i} = X_i\hat{\beta}$ $\quad = 2\sum_{i=1}^{n}\left\{y_i\log\left(\frac{1-\pi_i}{\pi_i}\right)\cdot\frac{y_i}{(m-y_i)} + M\log\frac{m-y_i}{M(1-\pi_i)}\right\}$

$$= 2\sum_{i=1}^{n}\left\{y_i\log\frac{y_i}{e^{X_i\hat{\beta}}(m-y_i)} + \log\frac{m-y_i}{m(1-\pi_i)}\cdot m\right\}$$

$$= 2\sum_{i=1}^{n}\left\{y_i\log\frac{y_i}{e^{X_i\hat{\beta}}(m-y_i)} + \log\frac{(m-y_i)(e^{X_i\hat{\beta}}+1)}{m}\cdot m\right\}$$

$$= 2\sum_{i=1}^{n}\left\{y_i\log\frac{y_i(1+e^{X_i\hat{\beta}})}{m e^{X_i\hat{\beta}}} + (m-y_i)\log\frac{(m-y_i)(1+e^{X_i\hat{\beta}})}{m}\right]$$

(2) Pearson Residuals

$$r p_i = \frac{y_i - \hat{\mu}_i}{\sqrt{V(\hat{\mu}_i)}} \quad (y_i - \frac{m e^{x_i \hat{\beta}}}{e^{x_i\hat{\beta}}+1}) = y_i - \hat{\mu}_i$$

$$V(\hat{\mu}_i) = V(\frac{m e^{x_i\hat{\beta}}}{e^{x_i\hat{\beta}}+1})$$

As $V(\hat{\mu}_i) = n P_i (1 - P_i)$.

For $1 - \pi_i = \frac{1}{e^{x_i\hat{\beta}}+1}$,

$$\pi_i = \frac{e^{x_i\hat{\beta}}}{e^{x_i\hat{\beta}}+1}$$

$$r P_i = \frac{y_i - \frac{m e^{x_i\hat{\beta}}}{1+e^{x_i\hat{\beta}}}}{\sqrt{\frac{m e^{x_i\hat{\beta}}}{1+e^{x_i\hat{\beta}}}\left(1 - \frac{e^{x_i\hat{\beta}}}{1+e^{x_i\hat{\beta}}}\right)}}$$

$$r p_i = \frac{y_i(1+e^{x_i\hat{\beta}}) - m e^{x_i\hat{\beta}}}{\sqrt{m e^{x_i\hat{\beta}}}}$$

(3) Deviance Residuals.

$$r D_i = \text{sign}(y_i - \hat{\mu}_i)\sqrt{d_i}$$
$$= \text{sign}(y_i - \frac{m e^{x_i\hat{\beta}}}{1+e^{x_i\hat{\beta}}})\sqrt{d_i}$$

where $d_i = D(y_i, \hat{\mu}_i)$, replace by equation in (1)

$$r D_i = \text{sign}(y_i - \frac{m e^{x_i\hat{\beta}}}{1+e^{x_i\hat{\beta}}})\sqrt{2\sum_{i=1}^{n}\left\{y_i \log\frac{y_i(1+e^{x_i\hat{\beta}})}{m e^{x_i\hat{\beta}}} + (m - y_i)\log\frac{(m-y_i)(1+e^{x_i\hat{\beta}})}{m}\right]}$$

(4) Person χ^2 statistic.

$$G = \sum_{i=1}^{n} (y_i - \hat{\mu}_i)^2 / V(\hat{\mu}_i)$$

$$= \sum_{i=1}^{n} (y_i - \frac{m \cdot e^{x_i\hat{\beta}}}{e^{x_i\hat{\beta}}+1})^2 / V(\hat{\mu}_i)$$

$$= \sum_{i=1}^{n} \frac{(y_i - \frac{m \cdot e^{x_i\hat{\beta}}}{e^{x_i\hat{\beta}}+1})^2}{V(\frac{m e^{x_i\hat{\beta}}}{e^{x_i\hat{\beta}}+1})}$$

$$= \sum_{i=1}^{n} r P_i^2 = \sum_{i=1}^{n} \frac{y_i(1+e^{x_i\hat{\beta}}) - m e^{x_i\hat{\beta}}}{m e^{x_i\hat{\beta}}}.$$

Problem 3.

1. $Y \sim Ber(\pi)$, i.i.d.

$H_0: \pi = \pi_0$. $\quad H_1: \pi \neq \pi_0$

$$l(y,\pi) = \log \left[\prod_{i=1}^{n} (\pi^{y_i} (1-\pi)^{1-y_i} \right]$$
$$= \sum_{i=1}^{n} \log [\pi^{y_i} (1-\pi)^{1-y_i}]$$
$$= \sum_{i=1}^{n} [\log \pi^{y_i} + \log (1-\pi)^{1-y_i}]$$
$$= \sum_{i=1}^{n} [y_i \log \pi + (1-y_i) \log(1-\pi)]$$

$$S(\pi) = \sum_{i=1}^{n} \left[\frac{y_i}{n} + (1-y_i) \cdot \frac{1}{1-\pi} \cdot (-1) \right]$$
$$= \frac{1}{\pi} \sum_{i=1}^{n} y_i - \frac{1}{1-\pi} \sum_{i=1}^{n} (1-y_i)$$
$$= \sum_{i=1}^{n} y_i \left(\frac{1}{\pi} + \frac{1}{1-\pi} \right) - \sum_{i=1}^{n} \frac{1}{1-\pi}$$
$$= \left(\sum_{i=1}^{n} y_i\right) \cdot \frac{1}{\pi(1-\pi)} - n \cdot \frac{1}{1-\pi}$$
$$= \frac{n}{\pi(1-\pi)} (\bar{y} - \pi)$$

Information Matrix:

$$I(\pi) = E\left[\frac{-\partial^2 l(y,\pi)}{\partial \pi^2} \right]$$
$$= E\left[-\frac{\partial}{\partial \pi} \left(\frac{1}{\pi - \pi^2} \sum_{i=1}^{n} y_i - \left(\frac{1}{1-\pi} \right) \cdot n \right] \right.$$
$$= E\left[\frac{1-2\pi}{\pi^2 (1-\pi)^2} \cdot \sum_{i=1}^{n} y_i + \frac{n}{(1-\pi)^2} \right]$$
$$= \frac{n}{(1-\pi)^2} \cdot \left(\frac{1-2\pi+\pi}{\pi} \right)$$
$$= \frac{n}{\pi(1-\pi)} \qquad \hat{\beta} = \bar{y}$$

Wald Test: $T_{sw} = (\hat{\beta} - \beta_0)^T I(\hat{\beta}) (\hat{\beta} - \beta_0)$

$$= \frac{(\bar{y} - \pi_0)^2 n}{\bar{y}(1-\bar{y})}$$

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Score Test: $T_{SS} = S(\pi_0) \times J^{-1}(\pi_0) \times S(\pi_0)$

$$= \frac{n^2(\bar{y} - \pi_0)^2}{\pi_0^2(1-\pi_0)^2} \cdot \frac{\pi_0(1-\pi_0)}{n}$$

$$= \frac{n(\bar{y} - \pi_0)^2}{\pi_0(1-\pi_0)}$$

3. Likelihood Ratio Test statistic:

$$TS_{LR} = 2\,[\,l(y, \hat{\beta}) - l(y, \beta_0)\,]$$

As it can be shown that:

$$2\log\left\{\frac{L(\hat{\pi}\mid H_1)}{L(\hat{\pi}\mid H_0)}\right\} = 2[\log L(\hat{\pi}\mid H_1) - \log L(\pi_0\mid H_0)] \sim \chi^2_1$$

$L(\hat{\pi}\mid H_1)$: under the alternative, likelihood

$L(\hat{\pi}\mid H_0)$: under null — likelihood

$$\log L(\pi) = \log\binom{n}{y} + y \cdot \log \pi + (n-y)\cdot \log(1-\pi)$$

$$\log L(\hat{\pi}\mid H_1) = \log\binom{n}{Y} + y\log\hat{\pi} + (n-y)\log(1-\hat{\pi})$$

$$\log L(\pi_0\mid H_0) = \log\binom{n}{Y} + y\log\pi_0 + (n-y)\cdot\log(1-\pi_0)$$

$$2[\log L(\hat{\pi}\mid H_1) - \log L(\pi_0\mid H_0)]$$

$$= 2\left[\log\binom{n}{y} + y\log\hat{\pi} + (n-y)\log(1-\hat{\pi})\right]$$

$$- 2\left[\log\binom{n}{y} + y\log\pi_0 + (n-y)\log(1-\pi_0)\right]$$

$$= 2\left[\log\left(\frac{\hat{\pi}}{\pi_0}\right) + (n-y)\log\left(\frac{1-\hat{\pi}}{1-\pi_0}\right)\right]$$

$$= 2\left[\log\left(\frac{\hat{\pi}}{\pi_0 n}\right) + (n-y)\cdot\log\left(\frac{1-\hat{\pi}}{(1-\pi_0)n}\right)\right] \sim \chi^2_1$$

The likelihood Ratio Statistic is:

$$T_{LR} = 2\left[y\cdot\log\left(\frac{\hat{\pi}}{\pi_0}\right) + (n-y)\cdot\log\left(\frac{1-\hat{\pi}}{(1-\pi_0)}\right)\right] \sim \chi^2_1$$

where $\hat{\pi} = \frac{y}{n}$ under null

2 see below

$H_0: \pi = 0.1 \quad H_1: \pi \neq 0.1$

Under null: $T_W \sim \chi_1^2$, $\alpha = 5\%$

So our critical value is

$$Z = F^{-1}(1-0.05) = F^{-1}(0.95) = 3.8415$$

(1) For $\pi = 0.1$,

① Wald statistic $= \dfrac{(\bar{y} - \pi_0)^2 n}{\bar{y}(1-\bar{y})}$

$$= \frac{(0.3-0.1)^2 \times 10}{0.3 \times 0.7}$$

$$\approx 1.905 < 3.8415$$

② Score statistic $= \dfrac{10(0.3-0.1)^2}{0.1 \times 0.9} \approx 4.44 > 3.84$

③ log-likelihood statistic:

$$T_L = 2[3 \times \log(\frac{0.3}{0.1}) + 7 \times \log(\frac{0.7}{0.9})]$$

$$= 2 \times [3 \times \log 3 + 7 \times \log(-0.25)]$$

$$\approx 3.073 < 3.8415$$

\# Conclusion:

At the sig. level of 0.05, we shall reject null hypotlesis and accept π is signif. different from 0.1 in score Test.

For wald & log-likelihood test, we cannot reject null at $\alpha = 0.05$

(2) For $\pi = 0.3$

① Wald-stat $= \dfrac{(3 - 10 \times 0.3)}{10 \times 0.3(1-0.3)} = 0 < 3.8415$

② Score stat $= 0$

③ log-likelihood stat.

$$T_L = 2[3 \times \log \frac{0.3}{0.3} + 7 \times \log \frac{0.7}{0.7}]$$

$$= 2 \times 0 = 0$$

Conclu: Under $\alpha = 0.05$, none of them suggests we can Reject Null Hypothe

(3) For $\pi = 0.5$

① Wald-stat $= \dfrac{(3 - 10 \times 0.5)^2}{0.3 \times 0.7}$

$$= \frac{4}{0.21} \approx 1.905 < 3.8415$$

② Score-stat $= 1.60 < 3.8415$

③ log-likelihood.

$$T_L = 2 \times [3 \times \log \frac{0.3}{0.5} + 7 \times \log \frac{0.7}{0.5}]$$

$$\approx 1.65 < 3.8415$$

Conclusion: at sig. level of $\alpha = 0.05$, we cannot reject that Null hypothesis $H_0: \pi = 0.5$ for all three tests

3. Comment: ① the three tests lead to different conclusion, so it's why maybe ~~score and LR test are preferred as it's difficult to~~

Reject. ② Wald, LR, Score tests are asymptotically equivalent $\sim \chi^2_1$ for 1 parameter test.

③ LR is more demanding for H_A, more preferable to Wald