Categorical Variables and Contingency Tables: Description and Inference

STAT 526
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Reading:
Agresti Ch. 1, 2 and 3
Faraway Ch. 4

Univariate Binomial and Multinomial Measurements

Binomial Distribution

- Probability distribution:
  - $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} Bernouilli(\pi)$
  - $\sum_{i=1}^{n} Y_i \sim Binomial(n, \pi)$
  - $p(y) = \left(\begin{array}{c} n \\ y \end{array} \right) \pi^y (1-\pi)^{n-y}$
  - $\mu = E(Y) = n\pi,\ \sigma^2 = var(Y) = n\pi(1-\pi)$

- Log-ikelihood:
  - $L(\pi) = ylog(\pi) + (n-y)log(1-\pi)$

- Maximum Likelihood Estimator:
  - $\widehat{\pi} = y/n$
  - $E(\widehat{\pi}) = \pi,\ SE(\widehat{\pi}) = \sqrt{\frac{\pi(1-\pi)}{n}}$

Large-sample tests for π

- For a known π_0, test
  $H_0: \ \pi = \pi_0$ vs $H_0: \ \pi \neq \pi_0$

- Wald test:

$$z_W = \frac{\widehat{\pi} - \pi_0}{SE} = \frac{\widehat{\pi} - \pi_0}{\sqrt{\widehat{\pi}(1-\widehat{\pi})/n}} \overset{H_0, approx}{\sim} \mathcal{N}(0,1)$$

- Likelihood ratio Test:

$$\begin{aligned} z_L &= 2(L_1 - L_0) = 2\left(ylog\frac{\widehat{\pi}}{\pi_0} + (n-y)log\frac{1-\widehat{\pi}}{1-\pi_0} \right) \\ &\overset{H_0, approx}{\sim} \chi^2_1 \end{aligned}$$

- Score Test:

$$z_S = \frac{\widehat{\pi} - \pi_0}{SE_0} = \frac{\widehat{\pi} - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}} \overset{H_0, approx}{\sim} \mathcal{N}(0,1)$$

  Closer to $\mathcal{N}(0,1)$ than Wald

Large-sample CI for π

- Based on the Wald test statistic:

$$\widehat{\pi} \pm z_{\alpha/2}\sqrt{\frac{\widehat{\pi}(1-\widehat{\pi})}{n}}$$

Performs poorly unless large n

- Based on the Score Test statistic:

$$\widehat{\pi}\left(\frac{n}{n+z^2_{\alpha/2}}\right)+\frac{1}{2}\left(\frac{z^2_{\alpha/2}}{n+z^2_{\alpha/2}}\right)$$

$$\pm$$

$$z_{\alpha/2}\sqrt{\frac{1}{n+z^2_{\alpha/2}}\left[\widehat{\pi}(1-\widehat{\pi})\left(\frac{n}{n+z^2_{\alpha/2}}\right)+\frac{1}{2}\cdot\frac{1}{2}\left(\frac{z^2_{\alpha/2}}{n+z^2_{\alpha/2}}\right)\right]}$$

Performs better than Wald

Multinomial Distribution

- Probability distribution:
  - $(Y_{i1}, \ldots, Y_{ic}) \sim$
    $\{Y_{ij} = 1$ if in category j, and 0 otherwise $\}$
  - $\sum_{i=1}^{n} Y_{ij} \sim Multinomial(\pi_1, \ldots, \pi_c),\ n = \sum_{j=1}^{c} n_j$
  - $p(n_1, n_2, \ldots, n_{c-1}) = \left(\frac{n!}{n_1! n_2! \ldots n_c!}\right) \pi_1^{n_1} \pi_2^{n_2} \ldots \pi_c^{n_c}$
  - $E(n_j) = n\pi_j$
    $var(n_j) = n\pi_j(1 - \pi_j),\ cov(n_j, n_k) = -n\pi_j\pi_k$

- Log-likelihood:
  - $L(\pi) = \sum_{j=1}^{c} n_j \cdot log\pi_j$

- Maximum Likelihood Estimator:
  - $\widehat{\pi}_j = n_j/n$

Large-Sample Test for $(\pi_1, \ldots, \pi_c)$

- For known $(\pi_{10}, \pi_{20}, \ldots \pi_{c0})$, test
  $H_0: \ \pi_j = \pi_{j0}$ vs $H_0: \ \pi_j \neq \pi_{j0}$

- Pearson test:
  $$X^2 = \sum_{j=1}^{c} \frac{(O_j - E_{j0})^2}{E_{j0}} = \sum_{j=1}^{c} \frac{(n_j - n\pi_{j0})^2}{n\pi_{j0}} \overset{H_0,\ approx}{\sim} \chi^2_{c-1}$$
  E.g. in genetics: test theories of trait inheritance

- Likelihood Ratio test:
  $$G^2 = 2(L_1 - L_0) = 2\sum_{j=1}^{n} log(\frac{n_j}{n\pi_j 0}) \overset{H_0,\ approx}{\sim} \chi^2_{c-1}$$

- Asymptotically equivalent when H_0 is true.

- For $n/c < 5$, X^2 converges faster

Poisson Distribution

- Probability distribution:
  - Y - number of events in a fixed interval of space/time
  - $Y \sim Poisson(\mu)$
  - $p(y) = \frac{e^{-\mu}\mu^y}{y!}$, $y = 0, 1, \ldots;$ $\quad E(Y) = var(Y) = \mu$
  - $Y_1, Y_2, \ldots, Y_c \overset{ind}{\sim} Poisson(\mu_i)$, $\sum_{i=1}^{c} Y_i \sim Poisson(\sum_{i=1}^{c} \mu_i)$

- c indep. Poisson r.v. $|$ total $\sim$ Multinomial

$$
\begin{aligned}
& P(Y_1 = n_1, \ldots, Y_c = n_c \mid \sum_i Y_i = n) \\
= \; & \frac{P(Y_1 = n_1, \ldots, Y_c = n_c)}{P(\sum_i Y_i = n)} \\
= \; & \frac{\prod_i \left[\; exp(-\mu_i)\mu_i^{n_i}/n_i! \;\right]}{exp(-\sum_i \mu_i)\ (\sum_i \mu_i)^n/n!} = \frac{n!}{\prod_i n_i!}\prod_i \pi_i^{n_i}, \;\; \pi_i = \frac{\mu_i}{\sum_i \mu_i}
\end{aligned}
$$

2-Way Contingency Tables

Contingency Tables

- *Contingency Table = Classification Table:* frequency of outcomes

- *Two-Way Table:* frequency outcomes of two categorical variables

- $I \times J$ *table:* a table with I rows and J columns.

- Contingency tables can arise from several sampling schemes
  - Inference depends on the sampling scheme

- Example:

	Lung Cancer		
Smoking	Cases	Controls	Total
Yes	688	650	1338
No	21	59	80
Total	709	709	1418

Joint Distribution and Independence

- Underlying probability distribution of X (smoking) and Y (cancer)
- Joint distribution:
  - π_{ij}, probability of cell (i, j)
- Marginal distribution:
  - $\pi_{i+} = \sum_{j=1}^{J} \pi_{ij}$, probability of row i

    $\pi_{+j} = \sum_{i=1}^{I} \pi_{ij}$, probability of column j
- Conditional distribution:
  - $\pi_{j|i} = \pi_{ij}/\pi_{i+}$, distribution of j given i
- Independence:
  - $\pi_{ij} = \pi_{i+}\pi_{+j}$ for all i and j

Multinomial Sampling

- The total sample size n is fixed, but the row and column totals are not

- X and Y are treated equally
  - $P(X = i, Y = j) = \pi_{ij}, \; i = 1, \ldots, I; \; j = 1, \ldots, J$
  - describe associations with joint distributions.
  - back to the case of the Multinomial distribution

- Likelihood and log-likelihood:

$$Likelihood = \frac{n!}{n_{11}! \cdots n_{IJ}!} \prod_{i}^{I} \prod_{j=1}^{J} \pi_{ij}^{n_{ij}}$$

$$L = \sum_{i=1}^{I} \sum_{j=1}^{J} n_{ij} \log(\pi_{ij}) + constant$$

Multinomial Sampling: Testing for Independence

- Hypotheses:
  - H_0 : reduced model $\pi_{ij} = \pi_{i+}\pi_{+j},$ for all i and j
  - H_a : full model $\pi_{ij} \neq \pi_{i+}\pi_{+j},$ for some i and j

- Pearson $\chi 2$ test:
  - $X^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(O_{ij}-E_{ij})^2}{E_{ij}} \overset{H_0,\ approx.}{\sim} \chi^2_{(I-1)(J-1)}$
  - $O_{ij} = n_{ij},\ E_{ij} = n\widehat{\pi}_{i+}\widehat{\pi}_{+j} = n_{i+}n_{+j}/n$
  - Df $= (I-1)(J-1) = (IJ-1)-(I-1)-(J-1)$

- Likelihood Ratio test:
  - Full model: $\widehat{\pi}_{ij} = n_{ij}/n_{++}$
  - Reduced model: $\widehat{\pi}_{i+} = n_{i+}/n_{++}$; $\widehat{\pi}_{+j} = n_{+j}/n_{++}$.

$$\begin{aligned} G^2 &= 2(L_1 - L_0) \\ &= 2\sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij}\log\frac{n_{ij}n_{++}}{n_{i+}n_{+j}} \overset{H_0,\ approx.}{\sim} \chi^2_{(I-1)(J-1)} \end{aligned}$$

Independent (or Product) Multinomial Sampling

- The row totals n_{i+}, $i = 1, \ldots, I$, are fixed
  - E.g., X is an explanatory variable, and response Y occurs separately at each setting of X.
  - View categorical response as function of categorical predictor
  - Describe associations in terms of conditional distributions

    $$P(Y = j|X = i) = \pi_{j|i}, \quad i = 1, \cdots, I; \quad j = 1, \cdots, J$$

- For a fixed i, $\{n_{ij}, j = 1, \cdots, J\}$ follow a multinomial distribution

$$f(n_{i1} \ldots, n_{iJ}\} = \frac{n_{i+}!}{n_{i1}! \cdots n_{iJ}!} \prod_{j=1}^{J} \pi_{j|i}^{n_{ij}}$$

Compare Proportions

- Independent Multinomial Sampling
- $H_0: \ \pi_1 = \pi_2$ vs $H_a: \ \pi_1 \neq \pi_2$
- ML estimate of the difference:
  - $\widehat{\pi}_1 - \widehat{\pi}_2 = \frac{y_1}{n_1} - \frac{y_2}{n_2}$
  - $SE(\widehat{\pi}_1 - \widehat{\pi}_2) = \left[\frac{\pi_1(1-\pi_1)}{n_1} + \frac{\pi_2(1-\pi_2)}{n_2}\right]^{1/2}$
- Wald Confidence Interval:
  - $\widehat{\pi}_1 - \widehat{\pi}_2 \pm z_{\alpha/2}\widehat{SE}(\widehat{\pi}_1 - \widehat{\pi}_2)$
  - Replace π with $\widehat{\pi}$ to estimate SE
- Usually too narrow
- Better methods (e.g. delta method) exist

Testing for Independence of Rows and Columns

- Independent Multinomial Sampling

- Independence in this context is often called *homogeneity* of the conditional distributions

- X and Y are independent
$\Longleftrightarrow \pi_{j|1} = \cdots = \pi_{j|I}$, for all j

- Can interpret the independence in terms of product of marginal probabilities

- $\pi_{ij} = \pi_{i+}\pi_{+j}$ for all i and j
$\Longleftrightarrow \pi_{j|1} = \cdots = \pi_{j|I}$ for all j

"$\Rightarrow$" $\quad \pi_{j|i} = \pi_{ij}/\pi_{i+} = (\pi_{i+}\pi_{+j})/\pi_{i+} = \pi_{+j}$

"$\Leftarrow$" $\quad$ Let $\pi_{j|i} = a_j$, then $\pi_{+j} = \sum_{i=1}^{I} \pi_{ij} = \sum_{i=1}^{I} \pi_{i+}a_j = a_j$

$\Longrightarrow \pi_{ij} = \pi_{i+}\pi_{+j}$

Testing for Independence of Rows and Columns

- Test the homogeneity of conditional distributions

Row	Column 1	$\cdots$	Column J	Total
1	π_{11} ($\pi_{1\|1}$)	$\cdots$	π_{1J} ($\pi_{J\|1}$)	π_{1+}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
I	π_{I1} ($\pi_{1\|I}$)	$\cdots$	π_{IJ} ($\pi_{J\|I}$)	π_{I+}
Total	π_{+1}	$\cdots$	π_{+J}	π_{++}

- Consider the new notation:
  $\pi_j(x) = P(Y = j|X = x)$

- Although the interpretation is different, use the same Pearson X^2 test and the LR test

Test for Independence: Odds Ratio

- Odds Ratio:

$$\begin{aligned}\theta &= \frac{\pi_{11}/\pi_{12}}{\pi_{21}/\pi_{22}} = \frac{\pi_{11}\pi_{22}}{\pi_{12}\pi_{21}} \\ &= \frac{P(Y=1|X=1)/P(Y=2|X=1)}{P(Y=1|X=2)/P(Y=2|X=2)} \\ &= \frac{P(X=1|Y=1)/P(X=2|Y=1)}{P(X=1|Y=2)/P(X=2|Y=2)}\end{aligned}$$

- Equally valid for prospective (conditional on X), retrospective (conditional on Y) and cross-sectional (multinomial) sampling designs

- MLE: $\widehat{\theta} = \frac{n_{11}/n_{12}}{n_{12}/n_{22}} = \frac{n_{11}n_{22}}{n_{12}n_{21}}$
  - When some $n_{ij} = 0$, $\widehat{\theta}$ is not a good estimator. Is improved by adding 0.5 to each cell count:

$$\tilde{\theta} = \frac{(n_{11}+0.5)(n_{22}+0.5)}{(n_{12}+0.5)(n_{21}+0.5)}$$

Test for Independence: Odds Ratio

- X and Y are independent

$$\Longleftrightarrow \theta = \frac{\pi_{11}/\pi_{12}}{\pi_{21}/\pi_{22}} = \frac{\pi_{11}\pi_{22}}{\pi_{12}\pi_{21}} = 1$$

  - to check, substitute $\pi_{ij} = \pi_{i+}\pi_{+j}$ in the formula above

- Asymptotically, $\log \widehat{\theta} \sim \ N(\log(\theta), \widehat{\sigma}^2)$, where

$$\widehat{\sigma}^2 = \frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}$$

- Large-sample CI for $log\theta$:

$$log\widehat{\theta} \pm z_{\alpha/2}\widehat{SE}(log\widehat{\theta}) = [L, U]$$

- Large-sample CI for θ : $[e^L, e^U]$.
  - Usually too wide

Poisson Sampling

- Observe a process over a period of time, and observe the number of occurrencies
  - No fixed quantities
  - Poisson sampling assumes each $Y_{ij} \overset{ind}{\sim} Poisson(\pi_{ij})$
- Denote Y_{ij} the count of cell (i, j)

$$\sum_{i=1}^{I}\sum_{j=1}^{J} Y_{ij} \sim Poission\left(\sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{ij}\right)$$

- Hypothesis of independence of X and Y has the form $\log(\pi_{ij}) = \lambda + \alpha_i + \beta_j$
  - This is the log-linear model of independence for two-way contingency tables
  - Under independence, $\log(\mu_{ij})$ is an additive function of a row effect α_i and a column effect β_j.
  - Since we don't have a replicate table, the model with the interaction is saturated

Poisson Sampling

- An additive model

$$\log \pi_{ij} = \mu + \alpha_i + \beta_j$$

implies the independence of the margins

$$\begin{aligned} \pi_{ij} &= \frac{E(count)}{\text{sum of all E(count)}} \\ &= \frac{e^{\mu+\alpha_i+\beta_j}}{e^{\mu}(\sum_i e^{\alpha_i})(\sum_j e^{\beta_j})} = \pi_{i+}\pi_{+j}, \end{aligned}$$

where
$\pi_{i+} = e^{\alpha_i}/\sum_i e^{\alpha_i} = \sum_j \pi_{ij}$,
$\pi_{+j} = e^{\beta_j}/\sum_j e^{\beta_j} = \sum_i \pi_{ij}$.

- Test for independence: Pearson X^2 or LR test as before (more on this later)

Hypergeometric Sampling

- Both row and column margins are fixed.

- When X and Y are independent, given the row and column margins, follows hypergeometric distribution

$$\frac{\left(\prod_{i=1}^{I} n_{i+}!\right)\left(\prod_{j=1}^{J} n_{+j}!\right)}{n_{++}!\prod_{i=1}^{I}\prod_{j=1}^{J} n_{ij}!}$$

— the distribution is parameter free

- For a 2×2 table

$$P(n_{11} = k) = \frac{\begin{pmatrix} n_{1+} \\ k \end{pmatrix}\begin{pmatrix} n_{2+} \\ n_{+1} - k \end{pmatrix}}{\begin{pmatrix} n_{++} \\ n_{+1} \end{pmatrix}},$$

$$max(0, n_{1+} + n_{+1} - n) \leq k \leq min(n_{1+}, n_{+1})$$

— *Fisher's exact test:* p-value = total probability of all outcomes more extreme than the one observed.

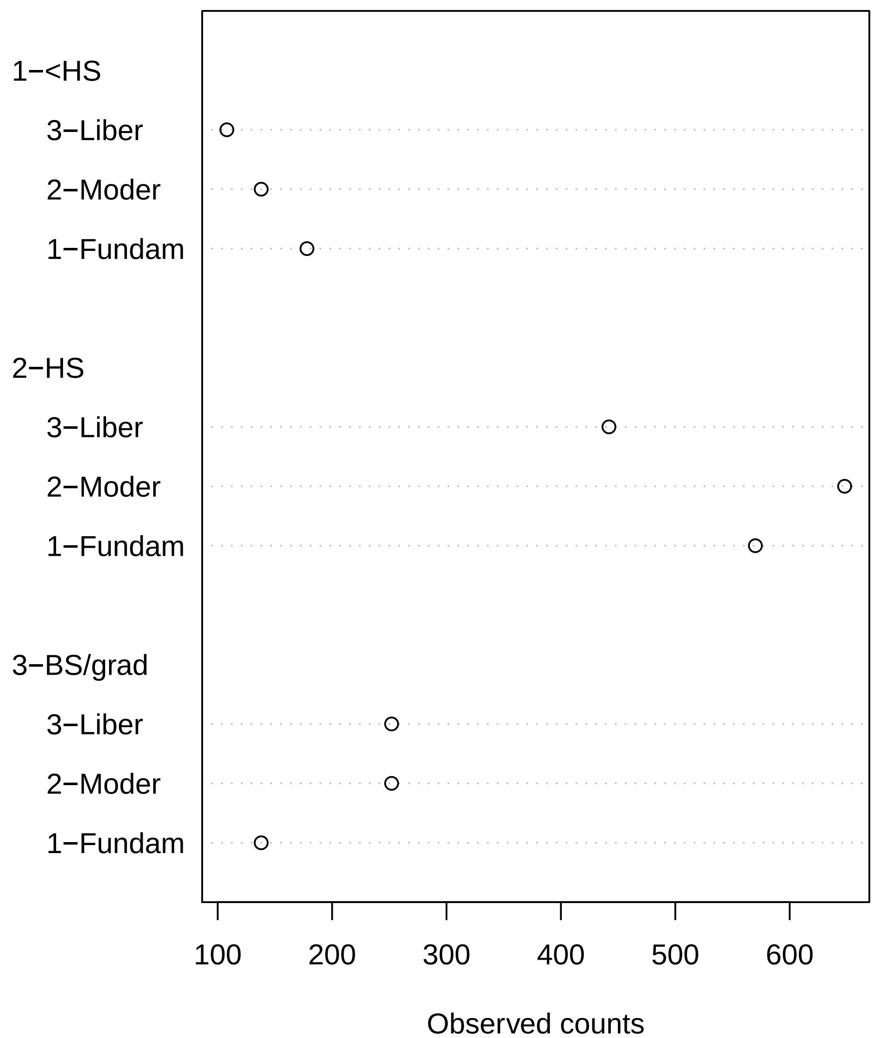
— Takes discrete values for small samples

Case study: Agresti p.80

```
#-------------------read the data---------------------
X <- data.frame(y=c(178, 138, 108, 570, 648,
     442, 138, 252, 252),
     belief=rep(c("1-Fundam", "2-Moder", "3-Liber"), 3),
     degree=rep(c("1-<HS", "2-HS", "3-BS/grad"), 1, each=3)
     )

#----------- a table of observed values (ov)----------
ov <- xtabs(y ~ degree+belief, data=X)
> ov
           belief
degree      1-Fundam 2-Moder 3-Liber
  1-<HS          178     138     108
  2-HS           570     648     442
  3-BS/grad      138     252     252

#-------------export the table into latex--------------
# export the table into latex
library(xtable)
xtable(ov)

\begin{table}[ht]
\begin{center}
\begin{tabular}{rrrr}
  \hline
 & 1-Fundam & 2-Moder & 3-Liber \\
  \hline
1-$<$HS & 178.00 & 138.00 & 108.00 \\
  2-HS & 570.00 & 648.00 & 442.00 \\
.....
```

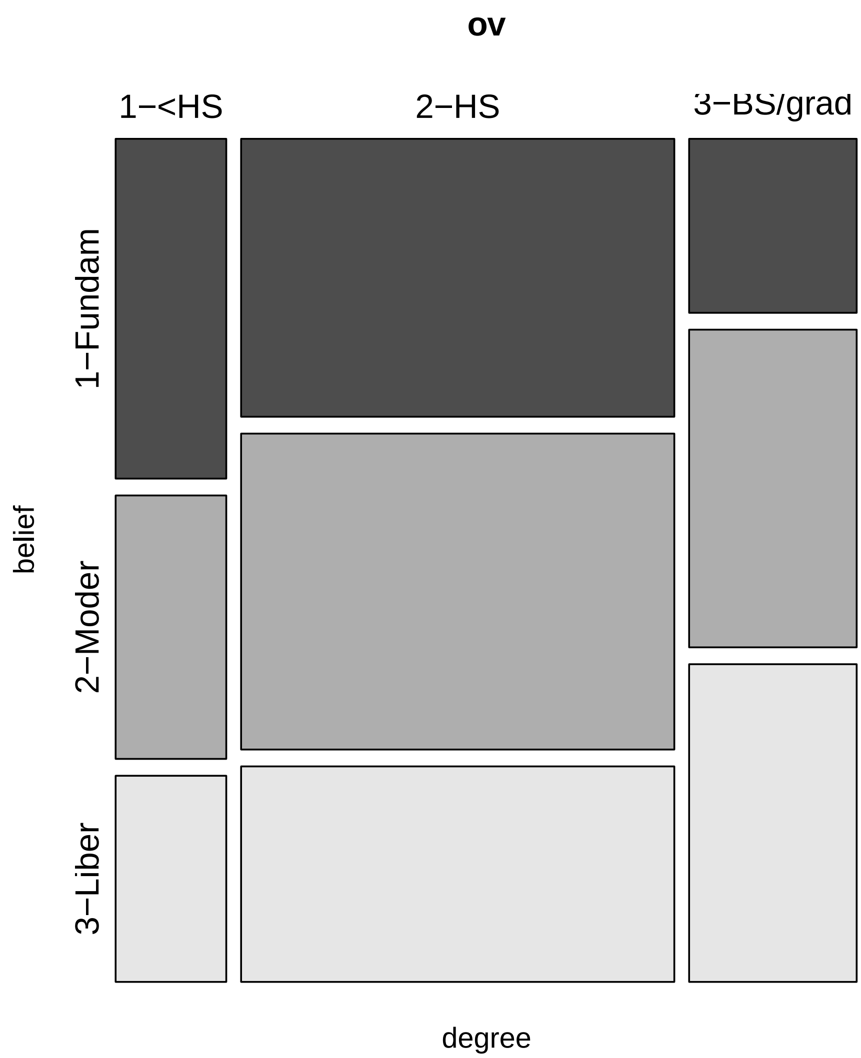
Data visualization

```
#-------------------dotchart---------------------
dotchart(t(ov), xlab="Observed counts")
```



Data visualization

```
#---------------mosaic plot------------------
mosaicplot(ov, color=TRUE)
```



2 x 2 table: Compare Proportions

- Independent multinomial sampling: restrictions on the rows
  - compare proportions of columns, given rows
  - also implements the Pearson X^2 test with Yates correction for small samples (from each O-E, subtract 0.5 if positive, and add 0.5 if negative)

```
> prop.test(ov[1:2,1:2])
2-sample test for equality of proportions
    with continuity correction

data:  ov[1:2, 1:2]
X-squared = 8.7451, df = 1, p-value = 0.003104
alternative hypothesis: two.sided
95 percent confidence interval:
 0.03187153 0.15875016
sample estimates:
   prop 1    prop 2
0.5632911 0.4679803

#-----double-check the proportions------------
> 178/(178+138)
[1] 0.5632911
> 570/(570+648)
[1] 0.4679803
```

2 x 2 table: Hypergeometric

- Sampling conditional on both margins
  - Hypergeometric test
  - compare distributions of counts within the 4 cells
  - H_0 is specified in terms of OR=1
  - produces CI for the OR

```
> fisher.test(ov[1:2,1:2])

Fisher's Exact Test for Count Data

data:  ov[1:2, 1:2]
p-value = 0.002961
alternative hypothesis: true odds ratio is not equal to 1
95 percent confidence interval:
 1.134415 1.897137
sample estimates:
odds ratio
  1.465974
```

I x J table: Pearson X^2

- (Independent) multinomial sampling restrictions on a margin, or on the total
  - H_0 in terms of independence of rows and columns

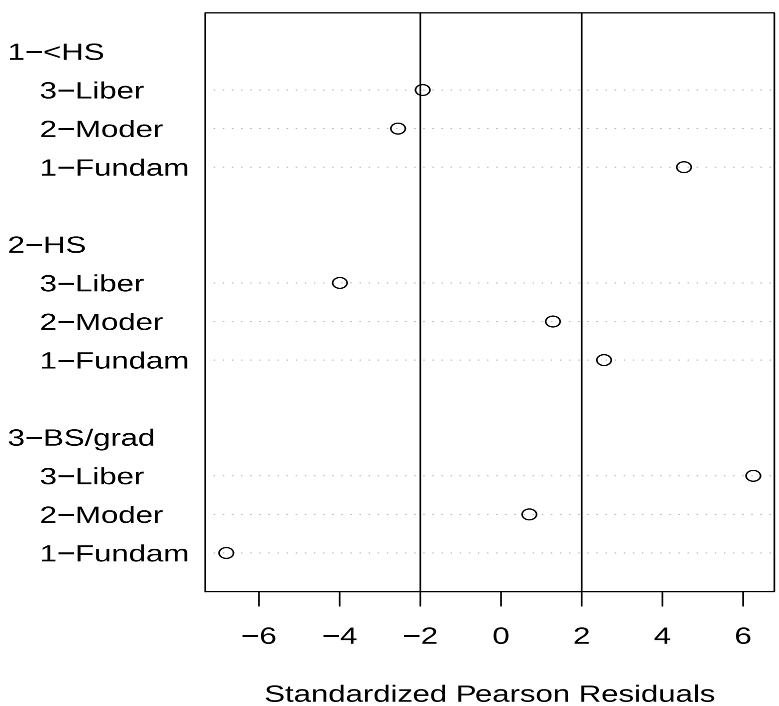
```
> summary(ov)

Call: xtabs(formula = y ~ degree + belief, data = X)
Number of cases in table: 2726
Number of factors: 2
Test for independence of all factors:
Chisq = 69.16, df = 4, p-value = 3.42e-14
```

- Pearson residuals
  - $e_{ij} = \frac{n_{ij} - \widehat{\mu}_{ij}}{\widehat{\mu}_{ij}^{1/2}}$
  - divide residual by $\widehat{SE}(n_{ij})$ in Poisson sampling

- Standardized Pearson residuals
  - $e_{ij} = \frac{n_{ij} - \widehat{\mu}_{ij}}{\sqrt{\widehat{\mu}_{ij} \cdot (1 - p_{i+})(1 - p_{+j})}}$
  - divide residual by $\widehat{SE}(residual)$ in Poisson sampling

Visualizing the association

```
# --Compute Pearson and standardized Pearson residuals ---
e <- apply(ov, 1, sum) %*% t(apply(ov, 2, sum)) / sum(ov)
pearsonResid <- (ov - e)/sqrt(e)

pRow <- 1-apply(ov, 1, sum)  / sum(ov)
pCol <- 1-apply(ov, 2, sum)  / sum(ov)
standPearsonResid <- pearsonResid/ sqrt(pRow %*% t(pCol) )

dotchart( t(standPearsonResid) )
abline(v=c(-2,2))
```



Ordered Categories

- Ordered categories have more info

- Assign scores to categories
  - Rows: $(u_1 \leq, \ldots \leq u_I)$, e.g. $(1, \ldots, I)$
  - Cols: $(v_1 \leq, \ldots \leq v_j)$, e.g. $(1, \ldots, J)$
  - $H_0: \; cor(u,v) = 0$ vs $H_a: \; cor(u,v) \neq 0$
  - produces CI for the OR

- Study the linear trend

$$r = \frac{\sum_{i=1}^{I} \sum_{j=1}^{J} (u_i - \bar{u})(v_j - \bar{v}) n_{ij}}{\sqrt{\left[\sum_{i=1}^{I} \sum_{j=1}^{J} (u_i - \bar{u})^2 n_{ij} \right] \cdot \left[\sum_{i=1}^{I} \sum_{j=1}^{J} (v_i - \bar{v})^2 n_{ij} \right]}}$$

  - $\bar{u} = \sum_{i=1}^{I} \sum_{j=1}^{J} u_i n_{ij}/n; \; \bar{v} = \sum_{i=1}^{I} \sum_{j=1}^{J} v_i n_{ij}/n;$
  - $M^2 = (n-1) r^2 \overset{H_0}{\sim} \chi^2_1$

Case Study: Ordered Categories

```
#---------------existing implementation--------------
> library(coin)
> lbl_test(as.table(ov))

Asymptotic Linear-by-Linear Association Test

data:  belief (ordered) by degree (1-<HS < 2-HS < 3-BS/grad)
chi-squared = 56.0849, df = 1, p-value = 6.939e-14



#-------------------manually-----------------
u <- as.vector(scale(1:3, center=sum(c(1:3)*ov)/sum(ov),
    scale=FALSE))
v <- as.vector(scale(1:3, center=sum(t(ov)*c(1:3))/sum(ov),
    scale=FALSE))

r <- sum(u%*%t(v)*ov) / sqrt(sum(u^2*ov) * sum(t(ov) * v^2))

M2 <- (sum(ov) - 1) * r^2
> 1-pchisq(M2, 1, lower=TRUE)
[1] 6.938894e-14
```

2x2 pairs: Matched Pairs

- Repeated measurements on same subjects
  - ask the same people the same question twice
  - goal: compare proportions
  - absence of association cannot be interpreted as independence

- Example (Agresti Ch. 10.1)
  - Approval of the President's performance, one month apart, for a same sample of Americans.

	Approve	Disapprove
Approve	794.00	150.00
Disapprove	86.00	570.00

- H_0 : Marginal homogeneity. $\pi_{1+} = \pi_{+1}$
  - $\delta = \pi_{1+} - \pi_{+1} = (\pi_{11} + \pi_{12} - (\pi_{11} + \pi_{21}) = \pi_{12} - \pi_{21}$
  - Equivalent to testing table symmetry

Large-sample test and CI

- CI
  - $\widehat{\delta} = p_{+1} - p_{1+} = p_{2+} - p_{+2}$
  - $var(\widehat{\delta}) = [\pi_{1+}(1 - \pi_{+1}) + \pi_{+1}(1 - \pi_{+1})$
    $-2(\pi_{11}\pi_{22} - \pi_{12}\pi_{21})] / n$
  - smaller variance than in independent samples, therefore a more efficient design
  - $\widehat{var}(\widehat{\delta}) = \left[(p_{12} + p_{21}) - (p_{12} - p_{21})^2\right] / n$
  - CI: $\widehat{\delta} \pm z_{\alpha/2}\widehat{SE}(\widehat{\delta})$

- Wald Test
  - $z = \frac{\widehat{\delta}}{\widehat{SE}(\delta)} = \frac{n_{21} - n_{12}}{(n_{21} + n_{12})^{1/2}}$
  - $z^2 \overset{H_0}{\sim} \chi^2_1$ (called McNemar test)
  - Only depends on counts outside of the diagonal

President Approval Example

```
#----------------Read the data--------------
Performance <-
matrix(c(794, 86, 150, 570),
       nrow = 2,  dimnames =
       list("1st Survey" = c("Approve", "Disapprove"),
       "2nd Survey" = c("Approve", "Disapprove"))
       )

> Performance
            2nd Survey
1st Survey   Approve Disapprove
  Approve        794        150
  Disapprove      86        570

#--------------Test---------------------
> mcnemar.test(Performance)

McNemar's Chi-squared test with continuity correction

data:  Performance
McNemar's chi-squared = 16.8178, df = 1, p-value = 4.115e-05
```

- significant change (in fact, drop) in approval ratings