Appendix

To solve the standard nuclear norm minus Frobenius norm minimization (TNFM) problem in closed-form, we propose a proximal operator, defined as follows,

$$\mathbf{P}_\lambda(\mathbf{Z}) = \arg\min_{\mathbf{B}} \frac{1}{2}\|\mathbf{B}-\mathbf{Z}\|_F^2 + \lambda\|\mathbf{B}\|_{t,*-F}, \tag{1}$$

where $\mathbf{Z} \in \mathbb{R}^{m\times n}$ is a given matrix. Assume that $\lambda > 0$ and $\mathbf{Z}\in\mathbb{R}^{m\times n}$ admits SVD as $\mathbf{U_Z}\mathrm{Diag}(\boldsymbol{\sigma}(\mathbf{Z}))\mathbf{V}_\mathbf{Z}^\top$, without loss of generality, let $m \geq n$. The closed-form solution of Equation (1) is given by

$$\mathbf{B}^* = \mathbf{P}_{\lambda,t}(\mathbf{Z}) = \mathbf{U_Z}\mathrm{Diag}(\varrho^*)\mathbf{V}_\mathbf{Z}^\top, \tag{2}$$

where

$$\varrho_i^* = \begin{cases} \sigma_i(\mathbf{Z}), & \text{if } \ 0\leq i < t+1, \\ \big(1+\frac{\lambda}{\|\mathcal{S}_\lambda(\mathbf{r})\|_2}\big)\mathcal{S}_\lambda(\sigma_i(\mathbf{Z})), & \text{if } \ t+1 \leq i \leq n, \end{cases} \tag{3}$$

with $\mathbf{r} = [0,\cdots,0,\sigma_{t+1}(Z),\cdots,\sigma_n(Z)]^\top$ and $\mathcal{S}_\lambda(\mathbf{r})_i = \max(\mathbf{r}_i - \lambda, 0)$ be the soft shrinkage.

Proof:

Consider $\mathbf{B} = \mathbf{U_B}\boldsymbol{\Sigma}_\mathbf{B}\mathbf{V}_\mathbf{B}^\top$ to be its SVD. The first term of problem (1) can be rewritten as

$$\frac{1}{2}\|\mathbf{B}-\mathbf{Z}\|_F^2 = \frac{1}{2}\left(\|\mathbf{B}\|_F^2 - 2\langle\mathbf{B},\mathbf{Z}\rangle + \|\mathbf{Z}\|_F^2\right). \tag{4}$$

According to Von Neumann's trace inequality, the matrix inner product in (4) can be derived as follows:

$$\begin{aligned} \langle\mathbf{B},\mathbf{Z}\rangle &= \mathrm{tr}(\mathbf{B}^\top\mathbf{Z}) \\ &= \mathrm{tr}(\mathbf{V_B}\boldsymbol{\Sigma}_\mathbf{B}\mathbf{U}_\mathbf{B}^\top\mathbf{Z}) \\ &= \mathrm{tr}(\boldsymbol{\Sigma}_\mathbf{B}\mathbf{U}_\mathbf{B}^\top\mathbf{Z}\mathbf{V_B}) \\ &\leq \sum_{i=1}^n \sigma_i(\mathbf{B})\cdot\sigma_i(\mathbf{Z})\cdot\sigma_i(\mathbf{U}_\mathbf{B}^\top\mathbf{V_B}) \quad (5) \\ &= \sum_{i=1}^n \sigma_i(\mathbf{B})\cdot\sigma_i(\mathbf{Z}). \quad (6) \end{aligned}$$

The equality of (5) occurs if and only if

$$\mathbf{U_B} = \mathbf{U_Z}, \mathbf{V_B} = \mathbf{V_Z}, \tag{7}$$

since

$$\begin{aligned} &\mathrm{tr}(\boldsymbol{\Sigma}_\mathbf{B}\mathbf{U}_\mathbf{B}^\top\mathbf{Z}\mathbf{V_B}) \\ =&\mathrm{tr}(\boldsymbol{\Sigma}_\mathbf{B}\mathbf{U}_\mathbf{B}^\top(\mathbf{U_Z}\boldsymbol{\Sigma}_\mathbf{Z}\mathbf{V}_\mathbf{Z}^\top)\mathbf{V_B}) \\ =&\mathrm{tr}(\boldsymbol{\Sigma}_\mathbf{B}(\mathbf{U}_\mathbf{Z}^\top\mathbf{U_Z})\boldsymbol{\Sigma}_\mathbf{Z}(\mathbf{V}_\mathbf{Z}^\top\mathbf{V_Z})) \\ =&\mathrm{tr}(\boldsymbol{\Sigma}_\mathbf{B}\boldsymbol{\Sigma}_\mathbf{Z}) = \sum_{i=1}^n \sigma_i(\mathbf{B})\cdot\sigma_i(\mathbf{Z}). \end{aligned} \tag{8}$$

Therefore, problem (1) can be rewritten as follows:

$$\begin{aligned} &\arg\min_{\mathbf{B}} \frac{1}{2}\|\mathbf{B}-\mathbf{Z}\|_F^2 + \lambda\|\mathbf{B}\|_{t,*-F} \\ =&\arg\min_{\mathbf{B}} \frac{1}{2}\|\mathbf{B}\|_F^2 - \sum_{i=1}^n \sigma_i(\mathbf{B})\sigma_i(\mathbf{Z}) + \frac{1}{2}\|\mathbf{Z}\|_F^2 \\ &+\lambda\bigg(\sum_{i=t+1}^n \sigma_i(\mathbf{B}) - \big(\sum_{i=t+1}^n \sigma_i(\mathbf{B})^2\big)^{\frac{1}{2}}\bigg) \\ =&\arg\min_{\mathbf{B}} \sum_{i=1}^t \bigg(\frac{1}{2}\sigma_i(\mathbf{B})^2 - \sigma_i(\mathbf{B})\sigma_i(\mathbf{Z})\bigg) + \sum_{i=t+1}^n \bigg(\frac{1}{2}\sigma_i(\mathbf{B})^2 \\ &-\sigma_i(\mathbf{B})\sigma_i(\mathbf{Z}) + \lambda\sigma_i(\mathbf{B})\bigg) - \lambda\bigg(\sum_{i=t+1}^n \sigma_i(\mathbf{B})^2\bigg)^{\frac{1}{2}}. \end{aligned} \tag{9}$$

Up to now, original problem (1) has been equivalently transformed into the combination of independent quadratic equations for each $\sigma_i(\mathbf{B})$. Let $F(\boldsymbol{\sigma}(\mathbf{B}))$ denote the objective function of (9). The minimum of F, denoted as $\sigma_i(\mathbf{B})^*$, is given by the first-order optimality condition

$$\frac{\partial F}{\partial \sigma_i(\mathbf{B})} = 0. \tag{10}$$

When $0 \leq i < t+1$, it is trivial to obtain

$$\sigma_i(\mathbf{B})^* = \sigma_i(\mathbf{Z}). \tag{11}$$

When $t+1 \leq i < n+1$, equation (10) is expressed as

$$\bigg(1 - \frac{\lambda}{\|\boldsymbol{\sigma}(\mathbf{B})\|_2}\bigg)\boldsymbol{\sigma}(\mathbf{B}) = \boldsymbol{\sigma}(\mathbf{Z}) - \lambda. \tag{12}$$

Let $\mathbf{r} = [0,\ldots,0,\sigma_{t+1}(\mathbf{Z}),\ldots,\sigma_T(\mathbf{Z})]^\top \in \mathbb{R}^n$ and $\mathcal{S}_\lambda(\mathbf{r}) = \max(\mathbf{r}-\lambda, 0)$, the solution of (12) is

$$\sigma_i(\mathbf{B})^* = \bigg(1 + \frac{\lambda}{\|\mathcal{S}_\lambda(\mathbf{r})\|_2}\bigg)\cdot\mathcal{S}_\lambda(\sigma_i(\mathbf{Z})). \tag{13}$$

Combining (11) and (12), the minimum of F is obtained at

$$\sigma_i(\mathbf{B})^* = \begin{cases} \sigma_i(\mathbf{Z}), & \text{if } \ 0\leq i < t+1, \\ \big(1+\frac{\lambda}{\|\mathcal{S}_\lambda(\mathbf{r})\|_2}\big)\mathcal{S}_\lambda(\sigma_i(\mathbf{Z})), & \text{if } \ t+1 \leq i < n+1. \end{cases} \tag{14}$$

Therefore, the optimal solution of problem (1) is

$$\mathbf{B}^* = \mathbf{U_Z}\mathrm{Diag}(\boldsymbol{\sigma}(\mathbf{B})^*)\mathbf{V}_\mathbf{Z}^\top. \tag{15}$$

$\square$