Appendix: Proof of Theorem 2

Theorem Restatement

The sequences $\{\mathbf{X}_k\}$ and $\{\mathbf{B}_k\}$ generated by Algorithm 2 satisfy

$a)\lim_{k\to\infty}\|\mathbf{X}_{k+1}-\mathbf{B}_{k+1}\|_F=0; b)\lim_{k\to\infty}\|\mathbf{B}_{k+1}-\mathbf{B}_k\|_F=0;$
$c)\lim_{k\to\infty}\|\mathbf{X}_{k+1}-\mathbf{X}_k\|_F=0.$

Proof

We first prove $a)\lim_{k\to\infty}\|\mathbf{X}_{k+1}-\mathbf{B}_{k+1}\|_F=0$. The objective function in the proposed mTNFM model can be transformed as follows

$$\begin{aligned}
&\|\mathbf{C}(\mathbf{Y}-\mathbf{X})\|_F^2+\lambda\|\mathbf{B}\|_{t,*-F}\\
&=\mathcal{L}_{\psi_{k-1}}(\mathbf{X}_k,\mathbf{B}_k,\mathbf{D}_{k-1})-\langle\mathbf{D}_{k-1},\mathbf{X}_k-\mathbf{B}_k\rangle\\
&\quad-\psi_{k-1}/2\|\mathbf{X}_k-\mathbf{B}_k\|_F^2\\
&=\mathcal{L}_{\psi_{k-1}}(\mathbf{X}_k,\mathbf{B}_k,\mathbf{D}_{k-1})-\langle\mathbf{D}_{k-1},(\mathbf{D}_k-\mathbf{D}_{k-1})/\psi_k\rangle\\
&\quad-\psi_{k-1}/2\|(\mathbf{D}_k-\mathbf{D}_{k-1})/\psi_k\|_F^2\\
&=\mathcal{L}_{\psi_{k-1}}(\mathbf{X}_k,\mathbf{B}_k,\mathbf{D}_{k-1})+(2\psi_k)^{-1}(\|\mathbf{D}_{k-1}-\mathbf{D}_k\|_F^2).
\end{aligned}\tag{1}$$

Given $\mathbf{D}\in\mathbb{R}^{3d^2\times N}$, let $T=\min(3d^2,N)$, the sequence of dual variable $\{\mathbf{D}_k\}$ is upper bounded since

$$\begin{aligned}
\|\mathbf{D}_{k+1}\|_F^2&=\|\mathbf{D}_k+\psi(\mathbf{X}_{k+1}-\mathbf{B}_k)\|_F^2\\
&=\psi^2\|(\psi^{-1}\mathbf{D}_k+\mathbf{X}_{k+1})-\mathbf{B}_{k+1}\|_F^2\\
&=\psi^2\|(\mathbf{U}_k(\mathrm{Diag}(\boldsymbol{\sigma}(\mathbf{Z}_k))-\mathbf{P}_{\lambda/\psi_k,t}(\mathbf{Z}_k))\mathbf{V}^\top\|_F^2\\
&=\psi^2\sum_{i=t+1}^{T}(\boldsymbol{\sigma}(\mathbf{Z}_k)-\mathbf{r}^*)^2
\end{aligned}$$

If $\sigma_i(\mathbf{Z}_k)>r_i^*$, $r_i^*=(1+\frac{\lambda/\psi_k}{\|\mathbf{r}\|_2})\cdot(\sigma_i(\mathbf{Z}_k)-\lambda/\psi_k)$. Let $\theta=1+\frac{\lambda/\psi_k}{\|\mathbf{r}\|_2}$. We have $\boldsymbol{\sigma}_i(\mathbf{Z}_k)-r_i^*=(1-\theta)\sigma_i(\mathbf{Z}_k)+\theta\lambda/\psi_k<\lambda/\psi_k$. If $\sigma_i(\mathbf{Z}_k)\le r_i^*$, $r_i^*=0$. We have $\boldsymbol{\sigma}_i(\mathbf{Z}_k)-r_i^*=\boldsymbol{\sigma}_i(\mathbf{Z}_k)\le\lambda/\psi_k$. Overall, we have $\boldsymbol{\sigma}_i(\mathbf{Z}_k)-r_i^*\le\lambda/\psi_k$. Then we can deduce

$$\|\mathbf{D}_{k+1}\|_F^2\le\psi^2\textstyle\sum_{i=t+1}^{T}(\lambda/\psi_k)^2=\lambda^2(T-t).$$

Since $\{\mathbf{D}_k\}$ is upper bounded, $\{\mathbf{D}_{k+1}-\mathbf{D}_k\}$ is upper bounded, i.e., $\exists M>0,\forall k\ge0,\|\mathbf{D}_{k+1}-\mathbf{D}_k\|_F\le M$. According to the squeeze theorem, we have

$$\begin{aligned}
0&\le\lim_{k\to\infty}\|\mathbf{X}_{k+1}-\mathbf{B}_{k+1}\|_F\\
&=\lim_{k\to\infty}\psi_k^{-1}\|\mathbf{D}_{k+1}-\mathbf{D}_k\|_F\le\lim_{k\to\infty}\frac{M}{\psi_k}=0.
\end{aligned}$$

For the augmented Lagrangian term of (1), we have

$$\mathcal{L}_{\psi_{k-1}}(\mathbf{X}_k,\mathbf{B}_k,\mathbf{D}_{k-1})\le\mathcal{L}_{\psi_{k-1}}(\mathbf{X}_{k-1},\mathbf{B}_{k-1},\mathbf{D}_{k-1}),$$

since the global optimum of $\mathbf{X}$ and $\mathbf{B}$ can be got from step 2 and 3 in Algorithm 2. Consequently, we have

$$\begin{aligned}
&\mathcal{L}_{\psi_k}(\mathbf{X}_k,\mathbf{B}_k,\mathbf{D}_k)\\
&=\mathcal{L}_{\psi_{k-1}}(\mathbf{X}_k,\mathbf{B}_k,\mathbf{D}_{k-1})+\langle\mathbf{D}_k-\mathbf{D}_{k-1},\mathbf{X}_k-\mathbf{B}_k\rangle\\
&\quad+(\psi_k-\psi_{k-1})/2\|\mathbf{X}_k-\mathbf{B}_k\|_F^2\\
&=\mathcal{L}_{\psi_{k-1}}(\mathbf{X}_k,\mathbf{B}_k,\mathbf{D}_{k-1})+\langle\mathbf{D}_k-\mathbf{D}_{k-1},\frac{\mathbf{D}_k-\mathbf{D}_{k-1}}{\psi_k}\rangle\\
&\quad+(\psi_k-\psi_{k-1})/(2\psi_{k-1}^2)\|\mathbf{D}_k-\mathbf{D}_{k-1}\|_F^2\\
&\le\mathcal{L}_{\psi_{k-1}}(\mathbf{X}_k,\mathbf{B}_k,\mathbf{D}_{k-1})+\frac{\psi_k+\psi_{k-1}}{2\psi_{k-1}^2}\cdot M^2\\
&\le\mathcal{L}_{\psi_0}(\mathbf{X}_1,\mathbf{B}_1,\mathbf{D}_0)+M^2\textstyle\sum_{k=1}^{\infty}\frac{\psi_k+\psi_{k-1}}{2\psi_{k-1}^2}\\
&=\mathcal{L}_{\psi_0}(\mathbf{X}_1,\mathbf{B}_1,\mathbf{D}_0)+M^2\textstyle\sum_{k=1}^{\infty}\frac{\varphi+1}{2\varphi^{k-1}\psi_0}\\
&\le\mathcal{L}_{\psi_0}(\mathbf{X}_1,\mathbf{B}_1,\mathbf{D}_0)+\frac{M^2}{\psi_0}\textstyle\sum_{k=1}^{\infty}\frac{1}{\varphi^{k-2}},
\end{aligned}$$

where the last inequality is reached using $\varphi+1<2\varphi$. Since $\sum_{k=0}^{\infty}\varphi^{k-2}<\infty$, $\{\mathcal{L}_{\psi_{k-1}}(\mathbf{X}_k,\mathbf{B}_k,\mathbf{D}_{k-1})\}$ is upper bounded. Then we can prove b) and c).

$$\begin{aligned}
&\lim_{k\to\infty}\|\mathbf{B}_{k+1}-\mathbf{B}_k\|_F\\
&=\lim_{k\to\infty}\|\frac{1}{\psi_k}(\mathbf{D}_k-\mathbf{D}_{k+1})+\mathbf{X}_{k+1}-\mathbf{B}_k\|_F\\
&=\lim_{k\to\infty}\|\mathbf{X}_k+\frac{1}{\psi_{k-1}}\mathbf{D}_{k-1}-\mathbf{B}_k+\mathbf{X}_{k+1}-\mathbf{X}_k\\
&\quad-\frac{1}{\psi_{k-1}}\mathbf{D}_{k-1}+\frac{1}{\psi_k}(\mathbf{D}_k-\mathbf{D}_{k+1})\|_F\\
&\le\lim_{k\to\infty}\|\mathbf{X}_k+\frac{1}{\psi_{k-1}}\mathbf{D}_{k-1}-\mathbf{B}_k\|_F+\|\mathbf{X}_{k+1}-\mathbf{X}_k\|_F\\
&\quad+\|\frac{1}{\psi_{k-1}}\mathbf{D}_{k-1}-\frac{1}{\psi_k}\mathbf{D}_k+\frac{1}{\psi_{k-1}}\mathbf{B}_k\|_F\\
&=0.
\end{aligned}$$

$$\begin{aligned}
&\lim_{k\to\infty}\|\mathbf{X}_{k+1}-\mathbf{X}_k\|_F\\
&=\lim_{k\to\infty}\|(\mathbf{C}^\top\mathbf{C}+\frac{\psi_k}{2}\mathbf{I})^{-1}(\mathbf{C}^\top\mathbf{C}\mathbf{Y}+\frac{\psi_k}{2}\mathbf{B}_k-\frac{1}{2}\mathbf{D}_k)\\
&\quad-\frac{1}{\psi_{k-1}}(\mathbf{D}_k-\mathbf{D}_{k-1})-\mathbf{B}_k\|_F\\
&=\lim_{k\to\infty}\|(\mathbf{C}^\top\mathbf{C}+\frac{\psi_k}{2}\mathbf{I})^{-1}(\mathbf{C}^\top\mathbf{C}\mathbf{Y}-\mathbf{C}^\top\mathbf{C}\mathbf{B}_k-\frac{1}{2}\mathbf{D}_k\\
&\quad-\frac{1}{\psi_{k-1}}(\mathbf{D}_k-\mathbf{D}_{k-1})\|_F\\
&\le\lim_{k\to\infty}\|(\mathbf{C}^\top\mathbf{C}+\frac{\psi_k}{2}\mathbf{I})^{-1}(\mathbf{C}^\top\mathbf{C}\mathbf{Y}-\mathbf{C}^\top\mathbf{C}\mathbf{B}_k-\frac{1}{2}\mathbf{D}_k\|_F\\
&\quad+\frac{1}{\psi_{k-1}}\|(\mathbf{D}_k-\mathbf{D}_{k-1})\|_F\\
&=0.
\end{aligned}$$

$\square$