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# Jet fragmentation in $e^+e^-$ annihilation

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## Abstract

A short review of theoretical and experimental results on fragmentation in  $e^+e^-$  annihilation is presented. Starting with an introduction of the concept of fragmentation functions in  $e^+e^-$  annihilation, aspects of scaling violation, multiplicities, small and large  $x$ , longitudinal, gluon, light and heavy quark fragmentation are summarized.

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# 1 Introduction

Fragmentation functions are dimensionless functions that describe the final-state single-particle energy distributions in hard scattering processes. The total  $e^+e^-$  fragmentation function for hadrons of type  $h$  in annihilation at c.m. energy  $\sqrt{s}$ , via an intermediate vector boson  $V = \gamma/Z^0$ , is defined as

$$F^h(x, s) = \frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{dx} (e^+e^- \rightarrow V \rightarrow hX) \quad (1)$$

where  $x = 2E_h/\sqrt{s} \leq 1$  is the scaled hadron energy (in practice, the approximation  $x = x_p = 2p_h/\sqrt{s}$  is often used). Since each hadron of type  $h$  contributes to the fragmentation function (1), its integral with respect to  $x$  gives the average multiplicity of those hadrons:

$$\langle n_h(s) \rangle = \int_0^1 dx F^h(x, s) . \quad (2)$$

Similarly, the integral of  $x F^h(x, s)$  gives the total scaled energy carried by hadrons of that type, which implies the sum rule

$$\sum_h \int_0^1 dx x F^h(x, s) = 2 . \quad (3)$$

Neglecting contributions suppressed by inverse powers of  $s$ , the fragmentation function (1) can be represented as a sum of contributions from the different parton types  $i = u, \bar{u}, d, \bar{d}, \dots, g$ :

$$F^h(x, s) = \sum_i \int_x^1 \frac{dz}{z} C_i(s; z, \alpha_s) D_i^h(x/z, s) . \quad (4)$$

where  $D_i^h$  are the parton fragmentation functions. In lowest order the coefficient function  $C_g$  for gluons is zero, while for quarks  $C_i = g_i(s)\delta(1-z)$  where  $g_i(s)$  is the appropriate electroweak coupling. In particular,  $g_i(s)$  is proportional to the charge-squared of parton  $i$  at  $s \ll M_Z^2$ , when weak effects can be neglected. In higher orders the coefficient functions and parton fragmentation functions are factorization-scheme dependent.

Parton fragmentation functions are analogous to the parton distributions in deep inelastic scattering (Section 9 of [1]). In the fragmentation function  $x$  represents the fraction of a parton's momentum carried by a produced hadron, whereas in a parton distribution it represents the fraction of a hadron's momentum carried by a constituent parton. In both cases, the simplest parton-model approach would predict a scale-independent  $x$  distribution. Furthermore we obtain similar violations of this scaling behaviour when QCD corrections are taken into account.

## 2 Scaling violation

The evolution of the parton fragmentation function  $D_i(x, t)$  with increasing scale  $t = s$ , like that of the parton distribution function  $f_i(x, t)$  with  $t = s$  (see Section 35 of [1]), is

governed by the DGLAP equation [2]

$$t \frac{\partial}{\partial t} D_i(x, t) = \sum_j \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_{ji}(z, \alpha_s) D_j(x/z, t). \quad (5)$$

In analogy to DIS, in some cases an evolution equation for the fragmentation function  $F$  itself (Eq. (4)) can be derived from Eq. (5) [4]. Notice that the splitting function is now  $P_{ji}$  rather than  $P_{ij}$  since here  $D_j$  represents the fragmentation of the final parton, whereas  $f_j$  in Eqs. (35.7) - (37.10) of [1] represented the distribution of the initial parton. The splitting functions again have perturbative expansions of the form

$$P_{ji}(z, \alpha_s) = P_{ji}^{(0)}(z) + \frac{\alpha_s}{2\pi} P_{ji}^{(1)}(z) + \dots \quad (6)$$

where the lowest-order functions  $P_{ji}^{(0)}(z)$  are the same as those in deep inelastic scattering but the higher-order terms [3]<sup>1</sup> are different. The effect of evolution is, however, the same in both cases: as the scale increases, one observes a scaling violation in which the  $x$  distribution is shifted towards lower values. This can be seen from Fig. 1.

As in the case of the parton distribution functions, the most common strategy for solving the evolution equations (5) is to take moments (Mellin transforms) with respect to  $x$ :

$$\tilde{D}(j, t) = \int_0^1 dx x^{j-1} D(x, t), \quad (7)$$

the inverse Mellin transformation being

$$D(x, t) = \frac{1}{2\pi i} \int_C dj x^{-j} \tilde{D}(j, t), \quad (8)$$

where the integration contour  $C$  in the complex  $j$  plane is parallel to the imaginary axis and to the right of all singularities of the integrand. Again we can consider fragmentation function combinations which are non-singlet (in flavour space) such as  $D_V = D_{q_i} - D_{\bar{q}_i}$  or  $D_{q_i} - D_{q_j}$ . In these combinations the mixing with the flavour singlet gluons drops out and for a fixed value of  $\alpha_s$  the solution is simply

$$\tilde{D}_V(j, t) = \tilde{D}_V(j, t_0) \left( \frac{t}{t_0} \right)^{\gamma_{qq}(j, \alpha_s)}, \quad (9)$$

where the quantity  $\gamma_{qq}$  is the Mellin transform of  $\alpha_s P_{qq}/2\pi$ . The resulting dependence on  $t$  violates scaling rules based on naive dimensional analysis, and therefore the power  $\gamma_{qq}$  is called an *anomalous dimension*. For a running coupling  $\alpha_s(t)$ , however, the scaling violation is no longer power-behaved. Inserting the lowest-order term in Eq. (9.5a) of [1] for the running coupling, we find that the solution varies like a power of  $\ln t$ :

$$\begin{aligned} \tilde{D}_V(j, t) &= \tilde{D}_V(j, t_0) \left( \frac{\alpha_s(t_0)}{\alpha_s(t)} \right)^{d_{qq}(j)}, \\ d_{qq}(j) &= \frac{6C_F}{11C_A - 2n_f} \left[ -\frac{1}{2} + \frac{1}{j(j+1)} - 2 \sum_{k=2}^j \frac{1}{k} \right], \end{aligned} \quad (10)$$

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<sup>1</sup>There are misprints in the formulae in the published article. The correct expressions can be found in the preprint version or in [5].

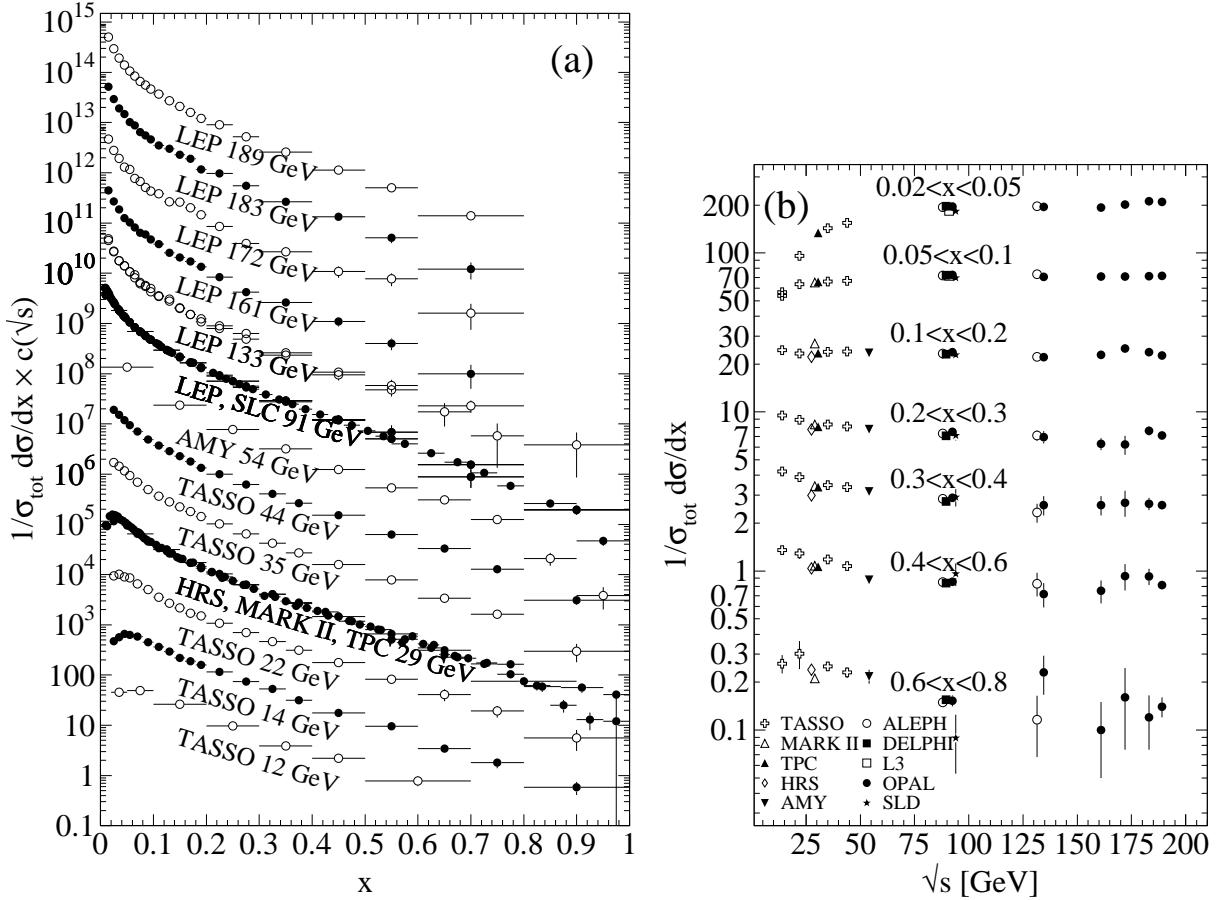


Figure 1: The  $e^+e^-$  fragmentation function for all charged particles is shown (a) for different c.m. energies,  $\sqrt{s}$ , versus  $x$  and (b) for various ranges of  $x$  versus  $\sqrt{s}$ . For the purpose of plotting (a), a scale factor  $c(\sqrt{s}) = 10^i$  was applied to the distributions where  $i$  is ranging from  $i = 0$  ( $\sqrt{s} = 12$  GeV) to  $i = 12$  ( $\sqrt{s} = 189$  GeV). Data are compiled from references [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

where  $C_F = 4/3$ ,  $C_A = 3$ , and  $n_f$  is the number of quark flavours as in Section 9 of [1].

For the singlet fragmentation function

$$D_S = \sum_i (D_{q_i} + D_{\bar{q}_i}) \quad (11)$$

we have mixing with the fragmentation of the gluon and the evolution equation becomes a matrix relation as in the deep inelastic scattering case:

$$t \frac{\partial}{\partial t} \begin{pmatrix} \tilde{D}_S \\ \tilde{D}_g \end{pmatrix} = \begin{pmatrix} \gamma_{qq} & 2n_f\gamma_{gg} \\ \gamma_{gg} & \gamma_{gg} \end{pmatrix} \begin{pmatrix} \tilde{D}_S \\ \tilde{D}_g \end{pmatrix}. \quad (12)$$

The anomalous dimension matrix in this equation has two real eigenvalues,

$$\gamma_{\pm} = \frac{1}{2} [\gamma_{gg} + \gamma_{qq} \pm \sqrt{(\gamma_{gg} - \gamma_{qq})^2 + 8n_f\gamma_{gg}\gamma_{qq}}]. \quad (13)$$

Expressing  $D_S$  and  $D_g$  as linear combinations of the corresponding eigenvectors  $D_+$  and  $D_-$ , we find that they evolve as superpositions of terms of the form (10) with  $\gamma_+$  and  $\gamma_-$  in the place of  $\gamma_{qq}$ .

At small  $x$ , corresponding to  $j \rightarrow 1$ , the  $g \rightarrow g$  anomalous dimension becomes dominant and we find  $\gamma_+ \rightarrow \gamma_{gg} \rightarrow \infty$ ,  $\gamma_- \rightarrow \gamma_{qq} \rightarrow 0$ . This region requires special treatment, which will be presented in Section 4.

There are several complications in the experimental study of scaling violation in jet fragmentation functions [4]. First, the energy dependence of the electroweak couplings  $g_i(s)$  that enter into Eq. (4) is especially strong in the energy region presently under study ( $\sqrt{s} = 20 - 200$  GeV). In particular, the  $b$ -quark contribution varies by more than a factor of 2 in this range. The fragmentation of the  $b$  quark into charged hadrons, including the weak decay products of the  $b$ -flavoured hadron, is expected to be substantially softer than that of the other quarks, so its varying contribution can give rise to a ‘fake’ scaling violation. A smaller, partially compensating effect is expected in charm fragmentation. These effects can be eliminated by extracting the  $b$  and  $c$  fragmentation functions from tagged heavy quark events. Fig. 2 shows the flavour-dependent  $e^+e^-$  fragmentation functions for quarks determined at  $\sqrt{s} = 91$  GeV.

Secondly, one requires the gluon fragmentation function  $D_g(x, s)$  in addition to those of the quarks. Although the gluon does not couple directly to the electroweak current, it contributes in higher order, and mixes with the quarks through evolution. Its fragmentation can be studied in tagged heavy-quark three-jet ( $Q\bar{Q}g$ ) events, or via the longitudinal fragmentation function (to be discussed below). Results for both methods are shown in Fig. 2.

A final complication is that power corrections to fragmentation functions, of the form  $f(x)/Q^p$ , are not so well understood as those in deep inelastic scattering. Theoretical arguments [26, 27] suggest that hadronization can lead to  $1/s$  corrections. Therefore, possible contributions of this form should be included in the parametrization when fitting

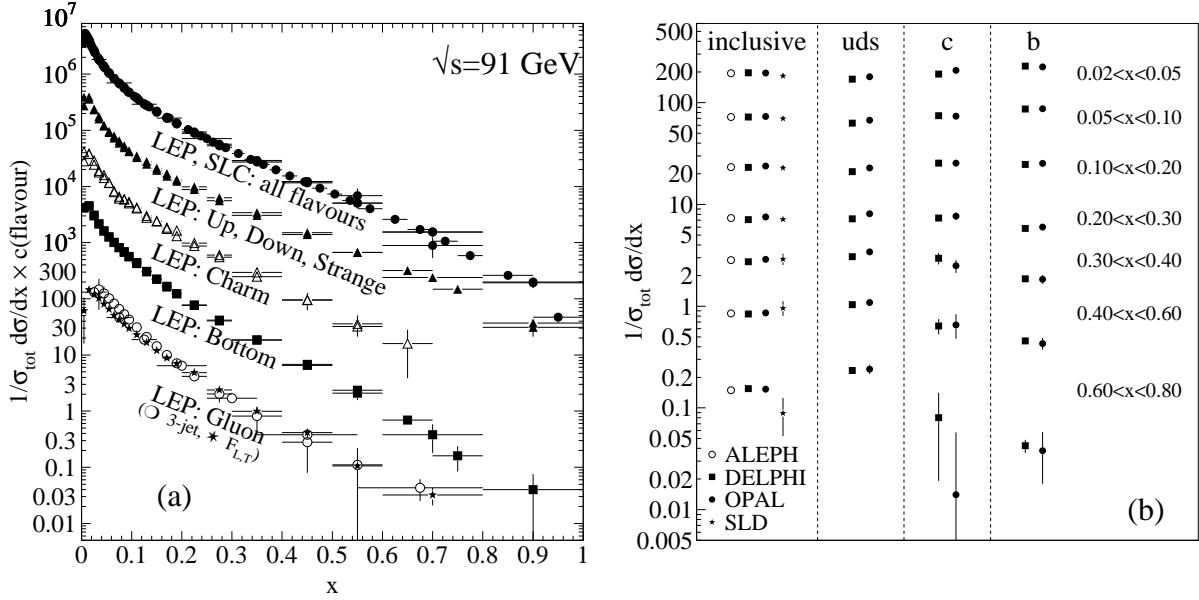


Figure 2: Comparison of the charged-particle and the flavour-dependent  $e^+e^-$  fragmentation functions obtained at  $\sqrt{s} = 91 \text{ GeV}$ . The data are shown (a) for the inclusive, light (up, down, strange) quarks, charm quark, bottom quark, and the gluon versus  $x$ , and (b) in various ranges of  $x$ . For the purpose of plotting (a), a scale factor  $c(\text{flavour}) = 10^i$  was applied to the distributions where  $i$  is ranging from  $i = 0$  (Gluon) to  $i = 4$  (all flavours). Data are compiled from references [12, 13, 14, 15, 16, 21, 22, 23, 24, 25].

the scaling violation. More conservatively, one could also try to fit  $1/Q$  corrections to check for their presence.

Quantitative results of studies of scaling violation in  $e^+e^-$  fragmentation are reported in refs. [22, 28, 29, 30, 31]. The values of  $\alpha_s$  obtained are consistent with the world average in Section 9 of [1].

### 3 Average Multiplicities

The average number of hadrons of type  $h$  in the fragmentation of a parton of type  $i$  at scale  $t$ ,  $\langle n_h(t) \rangle_i$ , is just the integral of the fragmentation function, which is the  $|j| = 1$  moment in the notation of Eq. (7):

$$\langle n_h(t) \rangle_i = \int_0^1 dx D_i^h(x, t) = \tilde{D}_i^h(1, t) . \quad (14)$$

If we try to compute the  $t$  dependence of this quantity, we immediately encounter the problem that the lowest-order expressions for the anomalous dimensions  $\gamma_{gq}$  and  $\gamma_{gg}$  in Eq. (12) are divergent at  $j = 1$ . The reason is that for  $j \leq 1$  the moments of the splitting functions in Eq. (6) are dominated by the region of small  $z$ , where  $P_{gi}(z)$  has a divergence associated with soft gluon emission.

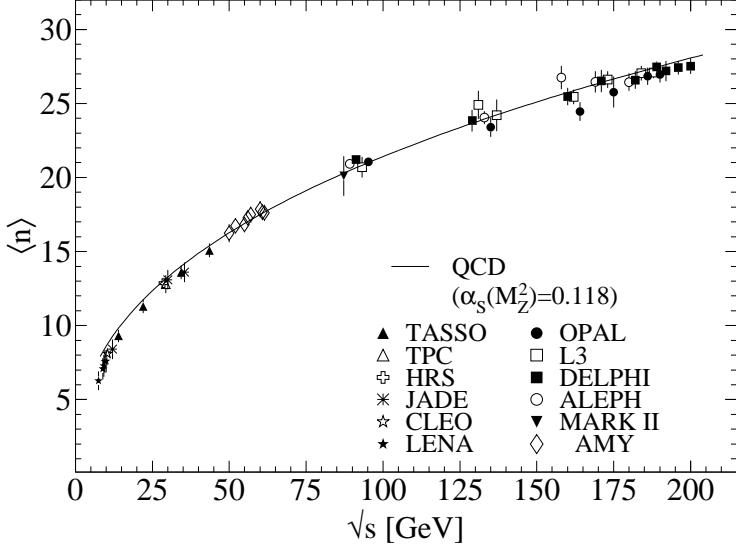


Figure 3: The mean multiplicity of charged particles measured in  $e^+e^-$  annihilation by various experiments [12, 13, 14, 16, 17, 18, 19, 20, 24, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. The measurements include contributions from  $K_S^0$  and  $\Lambda$  decays. Overlaid is the prediction Eq. (16) using  $\alpha_s(M_Z^2) = 0.118$ .

In fact, however, we can still solve the evolution equation for the average multiplicity provided we take into account the suppression of soft gluon emission due to coherence [32, 33]. The leading effect of coherence is that the scale on the right-hand side of the DGLAP equation Eq. (5) is reduced by a factor of  $z^2$ :

$$t \frac{\partial}{\partial t} D_i(x, t) = \sum_j \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_{ji}(z, \alpha_s) D_j(x/z, z^2 t) . \quad (15)$$

This change is not important for most values of  $x$  but it is crucial at small  $x$ . The anomalous dimensions are now found to be finite at  $j = 1$  and Eq. (14) gives to next-to-leading order [34]:

$$\langle n(s) \rangle = a \cdot \exp \left[ \frac{4}{\beta_0} \sqrt{\frac{6\pi}{\alpha_s(s)}} + \left( \frac{1}{4} + \frac{10n_f}{27\beta_0} \right) \ln \alpha_s(s) \right] + c . \quad (16)$$

where  $a$  and  $c$  are constants, and  $\beta_0$  is defined in Eq. (9.4b) of [1]. The resulting prediction, shown by the curve in Fig. 3 after fitting the parameters  $a$  and  $c$ , is in very good agreement with experiment. In this comparison the scale parameter  $\Lambda_{\text{mult}}$  in  $\alpha_s(s)$  was held fixed.  $\Lambda_{\text{mult}}$  is not necessarily equal to  $\Lambda_{\overline{\text{MS}}}$ , because the renormalization scheme dependence of  $\langle n(s) \rangle$  only appears at next-to-next-to-leading order, and corrections to Eq. (16) of this order have not yet been calculated. In fact, however, the value used in Fig. 3,  $\Lambda_{\text{mult}} = 226$  MeV, corresponding to  $\alpha_s(M_Z^2) = 0.118$ , is close to  $\Lambda_{\overline{\text{MS}}}$ , indicating that further higher-order corrections should be small. Higher order corrections to Eq. (16) have been considered in [48, 49, 50].

## 4 Small- $x$ Fragmentation

The behaviour of  $\tilde{D}(j, s)$ , Eq. (7), near  $j = 1$  determines the form of small- $x$  fragmentation functions via the inverse Mellin transformation (8). Keeping the first three terms in a Taylor expansion of the anomalous dimension  $\gamma_{gg}$  around  $j = 1$  gives a simple Gaussian function of  $j$  which transforms into a Gaussian in the variable  $\xi \equiv \ln(1/x)$ :

$$xD(x, s) \propto \exp \left[ -\frac{1}{2\sigma^2}(\xi - \xi_p)^2 \right], \quad (17)$$

where the peak position is

$$\xi_p = \frac{1}{4b\alpha_s(s)} \simeq \frac{1}{4} \ln \left( \frac{s}{\Lambda^2} \right) \quad (18)$$

and the width of the distribution of  $\xi$  is

$$\sigma = \left( \frac{1}{24b} \sqrt{\frac{2\pi}{C_A \alpha_s^3(s)}} \right)^{\frac{1}{2}} \propto \left[ \ln \left( \frac{s}{\Lambda^2} \right) \right]^{\frac{3}{4}}. \quad (19)$$

Again, one can compute next-to-leading corrections to these predictions. In the method of [33], the corrections are included in an analytical form known as the ‘modified leading logarithmic approximation’ (MLLA). Alternatively [51] they can be used to compute the higher moment corrections to the Gaussian form (17). Fig. 4 shows the  $\xi$  distribution at various c.m. energies.

The predicted energy dependence (18) of the peak in the  $\xi$  distribution is a striking illustration of soft gluon coherence, which is the origin of the suppression of hadron production at small  $x$ . Of course, a decrease at very small  $x$  is expected on purely kinematical grounds, but this would occur at particle energies proportional to their masses, i.e. at  $x \propto m/\sqrt{s}$  and hence  $\xi \sim \frac{1}{2} \ln s$ . Thus if the suppression were purely kinematic the peak position  $\xi_p$  would vary twice as rapidly with energy, which is ruled out by the data (see Fig. 5).

## 5 Large- $x$ Fragmentation

At large values of the energy fraction  $x$ , there are enhancements in the coefficient functions  $C_i$  in (4) and the splitting functions  $P_{ji}$  in Eq. (5). These are associated with the emission of soft and/or collinear gluons, which can lead to factors of  $\alpha_s^n \ln^m(1-x)/(1-x)$  with  $m \leq 2n-1$  in the  $n$ th order of perturbation theory. It turns out that in the conventional  $\overline{\text{MS}}$  factorization scheme the largest terms, with  $m > n$ , all occur in the coefficient functions. After the Mellin transformation (7), they are resummed by the following expression [60, 61]

$$\ln \tilde{C}_q(j, s) = - \int_0^1 dz \frac{z^{j-1} - 1}{1-z} \left\{ \int_{(1-z)s}^s \frac{dt}{t} A[\alpha_s(t)] + B[\alpha_s((1-z)s)] \right\} \quad (20)$$

where  $A(\alpha_s)$  and  $B(\alpha_s)$  have perturbative expansions

$$A(\alpha_s) = \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n A^{(n)}, \quad B(\alpha_s) = \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n B^{(n)} \quad (21)$$

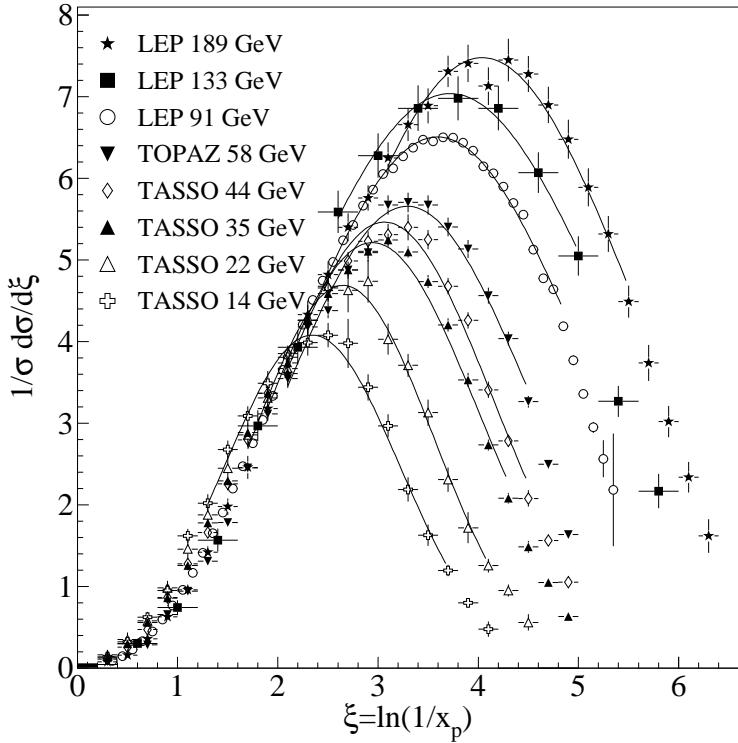


Figure 4: Distribution of  $\xi = \ln(1/x_p)$  at several c.m. energies [7, 12, 14, 15, 17, 18, 19, 20, 52, 53, 54]. At each energy only one representative measurement is shown and measurements between 133 and 189 GeV are not shown due to their small statistical significance. Overlaid are fits of a simple Gaussian function for illustration.

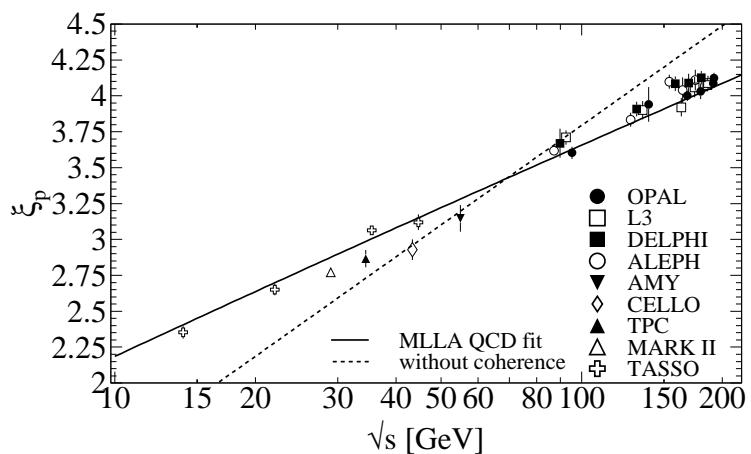


Figure 5: Evolution of the peak position,  $\xi_p$ , of the  $\xi$  distribution with the c.m. energy  $\sqrt{s}$ . The MLLA QCD prediction (solid) and the expectation without gluon coherence (dashed) were fitted to the data [7, 9, 11, 14, 17, 18, 19, 20, 41, 46, 47, 54, 55, 56, 57, 58, 59].

with

$$\begin{aligned} A^{(1)} &= C_F = \frac{4}{3}, \\ A^{(2)} &= C_F \left[ C_A \left( \frac{67}{36} - \frac{\pi^2}{12} \right) - \frac{5}{18} n_f \right] = \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{27} n_f, \\ B^{(1)} &= \frac{3}{4} C_F = 1. \end{aligned} \quad (22)$$

## 6 Longitudinal Fragmentation

In the process  $e^+e^- \rightarrow V \rightarrow hX$ , the joint distribution in the energy fraction  $x$  and the angle  $\theta$  between the observed hadron  $h$  and the incoming electron beam has the general form

$$\frac{1}{\sigma_{\text{tot}}} \frac{d^2\sigma}{dx d\cos\theta} = \frac{3}{8} (1 + \cos^2\theta) F_T(x) + \frac{3}{4} \sin^2\theta F_L(x) + \frac{3}{4} \cos\theta F_A(x), \quad (23)$$

where  $F_T$ ,  $F_L$  and  $F_A$  are respectively the transverse, longitudinal and asymmetric fragmentation functions. All these functions also depend on the c.m. energy  $\sqrt{s}$ . Eq. (23) is the most general form of the inclusive single particle production from the decay of a massive vector boson [4]. As their names imply,  $F_T$  and  $F_L$  represent the contributions from virtual bosons polarized transversely or longitudinally with respect to the direction of motion of the hadron  $h$ .  $F_A$  is a parity-violating contribution which comes from the interference between the vector and axial vector contributions. Integrating over all angles, we obtain the total fragmentation function,  $F = F_T + F_L$ . Each of these functions can be represented as a convolution of the parton fragmentation functions  $D_i$  with appropriate coefficient functions  $C_i^{\text{T,L,A}}$  as in Eq. (4). This representation works in the high energy limit. Furthermore, when  $x \cdot \sqrt{s}$  is of the order of a few hundred MeV, the  $p_\perp$  smearing of the parton can no longer be neglected, and the fragmentation function formalism no longer account correctly for the separation of  $F_T$ ,  $F_L$ , and  $F_A$ . The transverse and longitudinal coefficient functions are [62, 63]

$$\begin{aligned} C_q^{\text{T}}(z) &= \delta(1-z) + O(\alpha_s) \\ C_g^{\text{T}}(z) &= O(\alpha_s) \\ C_q^{\text{L}}(z) &= C_F \frac{\alpha_s}{2\pi} + O(\alpha_s^2) \\ C_g^{\text{L}}(z) &= 4C_F \frac{\alpha_s}{2\pi} \left( \frac{1}{z} - 1 \right) + O(\alpha_s^2), \end{aligned} \quad (24)$$

which implies that

$$F_L(x) = C_F \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} \left[ F_T(z) + 4 \left( \frac{z}{x} - 1 \right) D_g(z) \right] + O(\alpha_s^2). \quad (25)$$

Thus the gluon fragmentation function  $D_g$  can be extracted from measurements of  $F_T$  and  $F_L$ . The next-to-leading order corrections to Eq. (25) are also known [64, 65]. In Fig. 6  $F_T$ ,  $F_L$ , and  $F_A$  measured at  $\sqrt{s} = 91$  GeV are shown. The gluon fragmentation function derived from  $F_T$  and  $F_L$  in [24] is shown in Fig. 2(a).

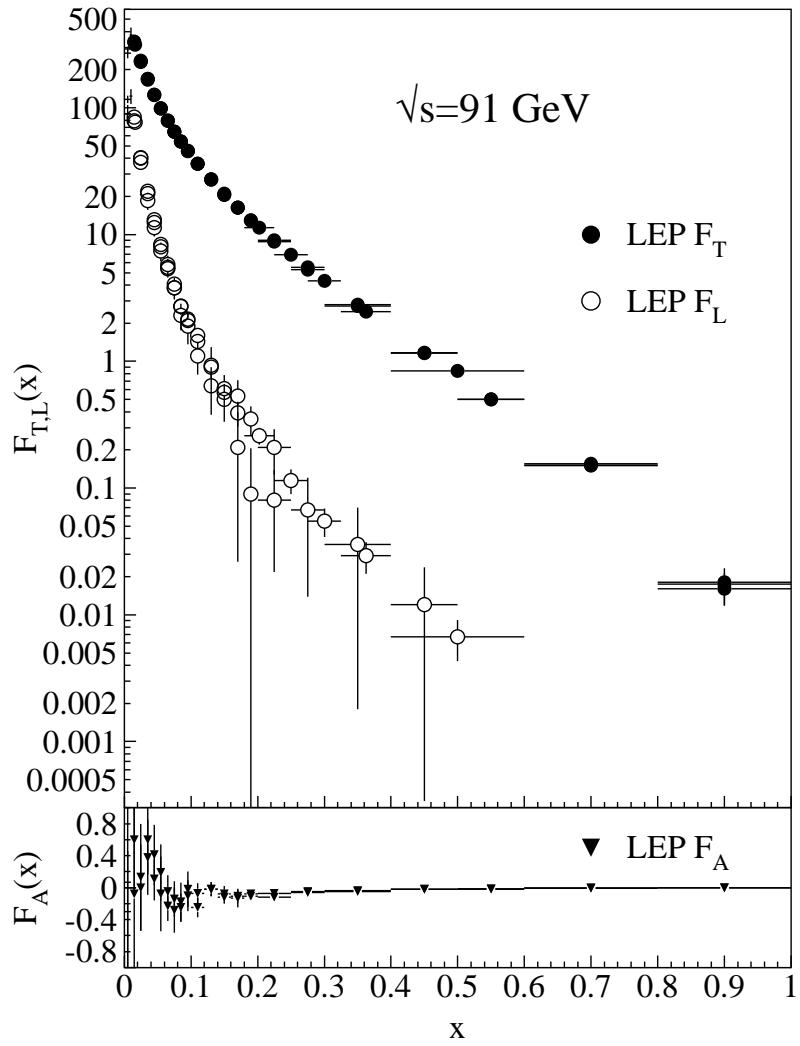


Figure 6: Transverse ( $F_T$ ), longitudinal ( $F_L$ ), and asymmetric ( $F_A$ ) fragmentation functions are shown [13, 24, 29]. Data points with relative errors greater than 100% are not shown.

Summed over all particle types, the total fragmentation function satisfies the energy sum rule (3), which we may write as

$$\frac{1}{2} \int_0^1 dx x F(x) = 1 . \quad (26)$$

Similarly the integrals

$$\frac{1}{2} \int_0^1 dx x F_{T,L}(x) \equiv \frac{\sigma_{T,L}}{\sigma_{\text{tot}}} \quad (27)$$

give the transverse and longitudinal fractions of the total cross section. The perturbative prediction is [64]

$$\begin{aligned} \frac{\sigma_L}{\sigma_{\text{tot}}} &= \frac{\alpha_s}{\pi} + \left( \frac{601}{40} - \frac{6}{5} \zeta(3) - \frac{37}{36} n_f \right) \left( \frac{\alpha_s}{\pi} \right)^2 + O(\alpha_s^3) \\ &\simeq \frac{\alpha_s}{\pi} + (13.583 - 1.028 n_f) \left( \frac{\alpha_s}{\pi} \right)^2 \end{aligned} \quad (28)$$

where  $\zeta(3) = 1.202 \dots$ . Comparing with Eq. (9.17) of [1], we see that the whole of the  $O(\alpha_s)$  correction to  $\sigma_{\text{tot}}$  comes from the longitudinal part, while the  $O(\alpha_s^2)$  correction receives both longitudinal and transverse contributions.

## 7 Gluon fragmentation

As we saw in the previous Section, the gluon fragmentation function can be extracted from the longitudinal fragmentation function using Eq. (25). It can also be deduced from the fragmentation of three-jet events in which the gluon jet is identified, for example by tagging the other two jets with heavy quark decays. The trouble with the latter method is that the relevant coefficient function has not been computed to next-to-leading order, and so the scale and scheme dependence of the extracted fragmentation function is ambiguous. The experimentally measured gluon fragmentation functions are shown in Fig. 2(a).

## 8 Fragmentation models

Although the scaling violation can be calculated perturbatively, the actual form of the parton fragmentation functions is non-perturbative. Perturbative evolution gives rise to a shower of quarks and gluons (partons). Phenomenological schemes are then used to model the carry-over of parton momenta and flavour to the hadrons. Two of the very popular models are the *string fragmentation* [66, 67], implemented in the JETSET [68] and UCLA [69] Monte Carlo event generation programs, and the *cluster fragmentation* of the HERWIG Monte Carlo event generator [70].

## 8.1 String fragmentation

The string-fragmentation scheme considers the colour field between the partons, i.e., quarks and gluons, to be the fragmenting entity rather than the partons themselves. The string can be viewed as a colour flux tube formed by gluon self-interaction as two coloured partons move apart. Energetic gluon emission is regarded as energy-momentum carrying “kinks” on the string. When the energy stored in the string is sufficient, a  $q\bar{q}$  pair may be created from the vacuum. Thus the string breaks up repeatedly into colour singlet systems as long as the invariant mass of the string pieces exceeds the on-shell mass of a hadron. The  $q\bar{q}$  pairs are created according to the probability of a tunneling process  $\exp(-\pi m_{q,\perp}^2/\kappa)$  which depends on the transverse mass squared  $m_{q,\perp}^2 \equiv m_q^2 + p_{q,\perp}^2$  and the string tension  $\kappa \approx 1 \text{ GeV/fm}$ . The transverse momentum  $p_{q,\perp}$  is locally compensated between quark and antiquark. Due to the dependence on the parton mass  $m_q$  and/or the hadron mass  $m_h$ , the production of strange and, in particular, heavy-quark hadrons is suppressed. The light-cone momentum fraction  $z = (E + p_{\parallel})_h / (E + p)_q$ , where  $p_{\parallel}$  is the momentum of the formed hadron  $h$  along the direction of the quark  $q$ , is given by the string-fragmentation function

$$f(z) \sim \frac{1}{z} (1 - z)^a \exp\left(-\frac{bm_{\perp}^2}{z}\right) \quad (29)$$

where  $a$  and  $b$  are free parameters. These parameters need to be adjusted to bring the fragmentation into accordance with measured data, e.g.  $a = 0.11$  and  $b = 0.52 \text{ GeV}^{-2}$  as determined in [71] (for an overview see [72]).

## 8.2 Cluster fragmentation

Assuming a local compensation of colour based on the *pre-confinement* property of perturbative QCD [73], the remaining gluons at the end of the parton shower evolution are split non-perturbatively into quark-antiquark pairs. Colour singlet clusters of typical mass of a couple of GeV are then formed from quark and antiquark of colour-connected splittings. These clusters decay directly into two hadrons unless they are either too heavy (relative to an adjustable parameter `CLMAX`, default value 3.35 GeV), when they decay into two clusters, or too light, in which case a cluster decays into a single hadron, requiring a small rearrangement of energy and momentum with neighbouring clusters. The decay of a cluster into two hadrons is assumed to be isotropic in the rest frame of the cluster except if a perturbative-formed quark is involved. A decay channel is chosen based on the phase-space probability, the density of states, and the spin degeneracy of the hadrons. Cluster fragmentation has a compact description with few parameters, due to the phase-space dominance in the hadron formation.

## 9 Experimental studies

A great wealth of measurements of  $e^+e^-$  fragmentation into identified particles exists. A collection of references, where data on the fragmentation into identified particles can be found, is given for Tab. 37.1 of [1]. As representatives of all the data, Fig. 7 shows fragmentation functions as the scaled momentum spectra of charged particles at several c.m. energies,  $\sqrt{s}$ . In Fig. 8  $x_p$  spectra at  $\sqrt{s} = 91$  GeV are shown for all charged particles, for several identified charged and neutral particles. Heavy flavour particles are dealt with separately in Sect. 10.

The measured fragmentation functions are solutions to the DGLAP equation (5) but need to be parametrized at some initial scale  $t_0$  (usually 2 GeV<sup>2</sup> for light quarks and gluons). A general parametrization is [79]

$$D_{p \rightarrow h}(x, t_0) = Nx^\alpha(1-x)^\beta \left(1 + \frac{\gamma}{x}\right) \quad (30)$$

where the normalization  $N$ , and the parameters  $\alpha$ ,  $\beta$ , and  $\gamma$  in general depend on the energy scale  $t_0$  and also on the type of the parton,  $p$ , and the hadron,  $h$ . Frequently the term involving  $\gamma$  is left out [65, 80, 81, 82, 83]. In the above quoted references the parameters of Eq. (30) are tabulated for many different combinations of partons and hadrons in  $p \rightarrow h$ . The parameters were obtained by fitting data for many different hadron types over a vast range of c.m. energies ( $\sqrt{s} \approx 5 - 200$  GeV).

## 10 Heavy quark fragmentation

It was recognized very early [84] that a heavy flavoured meson should retain a large fraction of the momentum of the primordial heavy quark, and therefore its fragmentation function should be much harder than that of a light hadron. In the limit of a very heavy quark, one expects the fragmentation function for a heavy quark to go into any heavy hadron to be peaked near 1.

When the heavy quark is produced at a momentum much larger than its mass, one expects important perturbative effects, enhanced by powers of the logarithm of the transverse momentum over the heavy quark mass, to intervene and modify the shape of the fragmentation function. In leading logarithmic order (*i.e.*, including all powers of  $\alpha_s \log m_Q/p_T$ ) the total (*i.e.*, summed over all hadron types) perturbative fragmentation function is simply obtained by solving the leading evolution equation for fragmentation functions, Eq. (5), with the initial condition at a scale  $\mu^2 = m_Q^2$  given by

$$D_Q(z, m_Q^2) = \delta(1-z), \quad D_q = 0, \quad D_{\bar{q}} = 0, \quad D_{\bar{Q}} = 0, \quad D_g = 0 \quad (31)$$

where the notation  $D_i(z)$  stands now for the probability to produce a heavy quark  $Q$  from parton  $i$  with a fraction  $z$  of the parton momentum. If the scale  $\mu^2$  is not too large, and one looks at relatively large values of  $z$ , one can assume that the non-singlet evolution

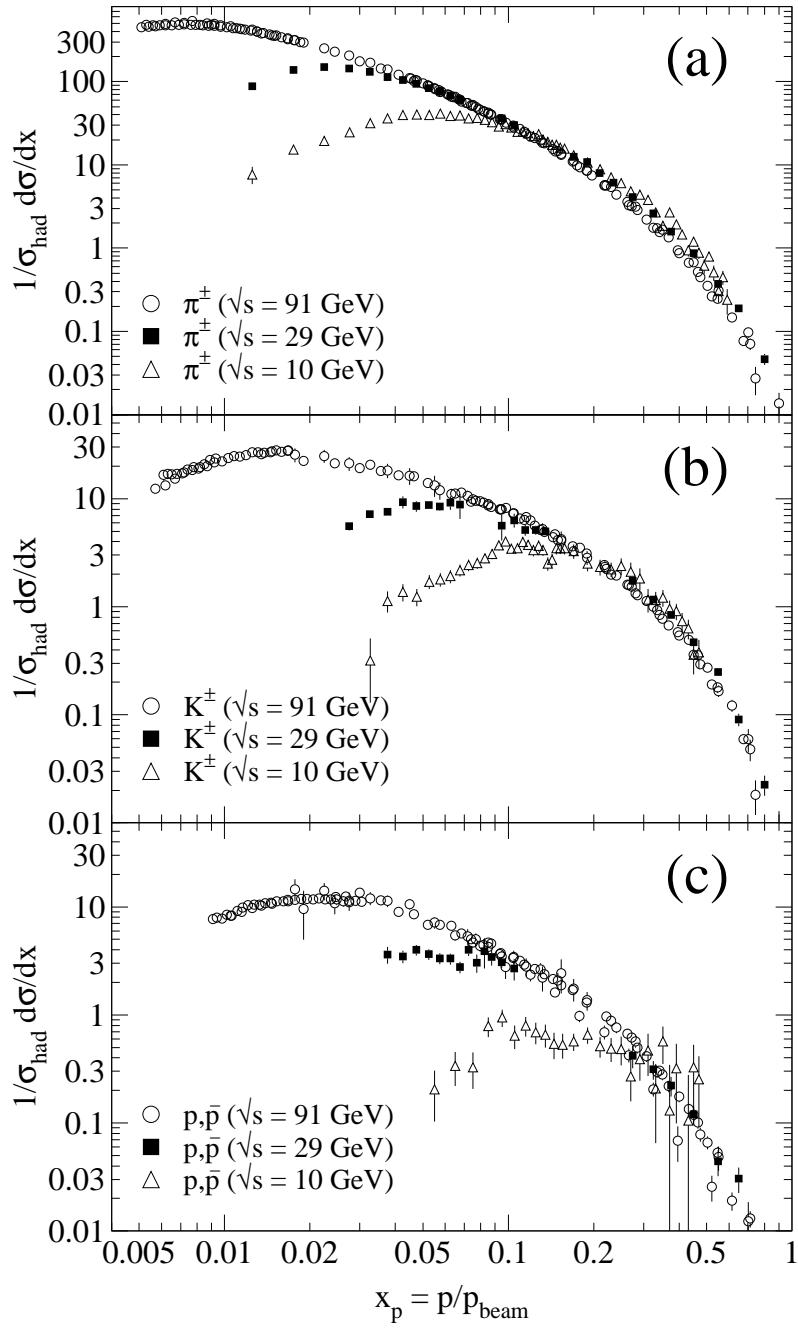


Figure 7: Scaled momentum,  $x_p \equiv 2p/\sqrt{s} = p/p_{\text{beam}}$  spectra of (a)  $\pi^\pm$ , (b)  $K^\pm$ , and (c)  $p/\bar{p}$  at  $\sqrt{s} = 10, 29$ , and  $91 \text{ GeV}$  are shown [10, 74, 75, 76, 77, 78].

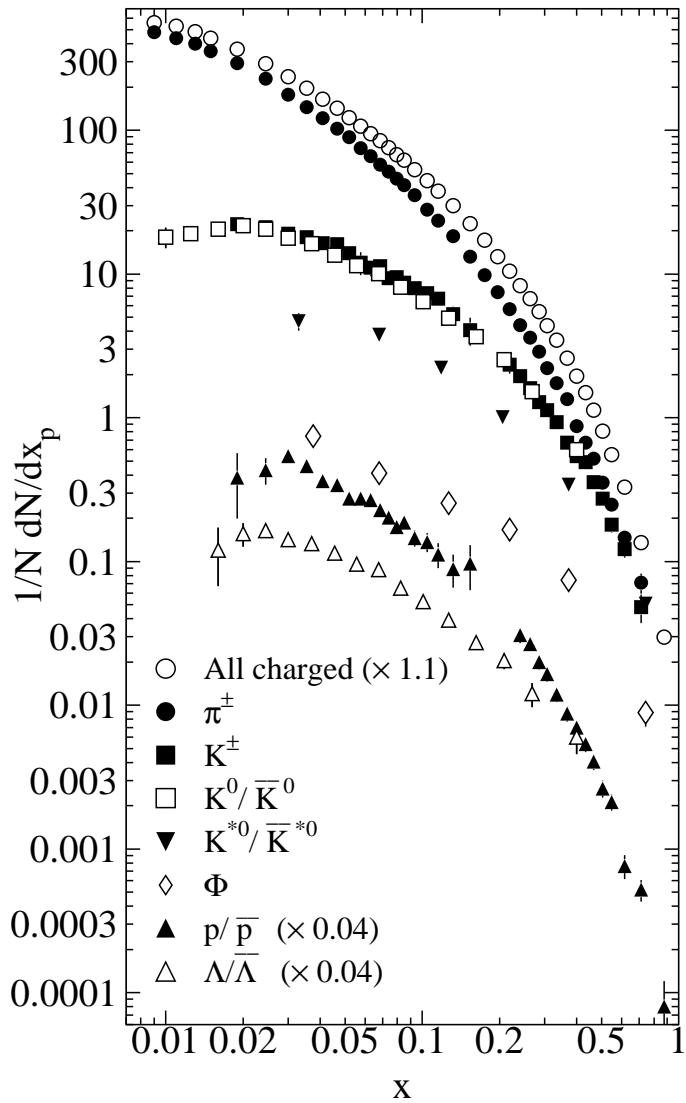


Figure 8: Scaled momentum spectra  $\sqrt{s} = 91$  GeV for all charged particles, for identified charged ( $\pi^\pm$ ,  $K^\pm$ ,  $p/\bar{p}$ ) and for identified neutral particles ( $K_S^0/\bar{K}_S^0$ ,  $K^{*0}/\bar{K}^{*0}$ ,  $\phi$ ,  $\Lambda/\bar{\Lambda}$ ) are shown [78].

embodies most of the physics. Following this assumption, for example, the average value of  $z$  at a scale  $\mu^2$  is easily obtained by solving the evolution equation in the moment representation (cf. Eq. (10))

$$\langle z \rangle = \tilde{D}_Q(2, \mu^2) = \left( \frac{\alpha_s(\mu^2)}{\alpha_s(m_Q^2)} \right)^{\frac{8C_F}{11C_A - 2n_f}} \quad (32)$$

Several extensions of the leading logarithmic result have appeared in the literature:

- Next-to-leading-log (NLL) order results for the perturbative heavy quark fragmentation function have been obtained in [85].
- At large  $z$ , phase space for gluon radiation is suppressed. This exposes large perturbative corrections due to the incomplete cancellation of real gluon radiation and virtual gluon exchange (Sudakov effects), which should be resummed in order to get accurate results. A leading-log (LL) resummation formula has been obtained in [85, 86]. Next-to-leading-log resummation has been performed in [61].
- Fixed order calculations of the fragmentation function at order  $\alpha_s^2$  in  $e^+e^-$  annihilation have appeared in [87]. This result does not include terms of order  $(\alpha_s \log s/m^2)^k$  and  $\alpha_s(\alpha_s \log s/m^2)^k$ , but it does include correctly all terms up to the order  $\alpha_s^2$ , including terms without any logarithmic enhancements.

Inclusion of non-perturbative effects in the calculation of the heavy quark fragmentation function is done in practice by convolving the perturbative result with a phenomenological non-perturbative form. Among the most popular parametrizations we have the following:

$$\text{Peterson et al. [88]: } D_{\text{np}}(z) \propto \frac{1}{z} \left( 1 - \frac{1}{z} - \frac{\epsilon}{1-z} \right)^{-2}, \quad (33)$$

$$\text{Kartvelishvili et al. [89]: } D_{\text{np}}(z) \propto z^\alpha (1-z), \quad (34)$$

$$\begin{aligned} \text{Collins\&Spiller [90]: } D_{\text{np}}(z) \propto & \left( \frac{1-z}{z} + \frac{(2-z)\epsilon_C}{1-z} \right) \times \\ & (1+z^2) \left( 1 - \frac{1}{z} - \frac{\epsilon_C}{1-z} \right)^{-2} \end{aligned} \quad (35)$$

where  $\epsilon$ ,  $\alpha$ , and  $\epsilon_C$  are non-perturbative parameters, depending upon the heavy hadron considered. In general, the non-perturbative parameters do not have an absolute meaning. They are fitted together with some model of hard radiation, which can be either a shower Monte Carlo, a leading-log or NLL calculation (which may or may not include Sudakov resummation), or a fixed order calculation. In [87], for example, the  $\epsilon$  parameter for charm and bottom production is fitted from the measured distributions of refs. [91, 92] for charm, and of [93] for bottom. If the leading-logarithmic approximation (LLA) is used for the perturbative part, one finds  $\epsilon_c \approx 0.05$  and  $\epsilon_b \approx 0.006$ ; if a second order calculation is used one finds  $\epsilon_c \approx 0.035$  and  $\epsilon_b \approx 0.0033$ ; if a NLLO calculation is used instead one finds  $\epsilon_c \approx 0.022$  and  $\epsilon_b \approx 0.0023$ . The larger values found in the LL approximation are consistent with what is obtained in the context of parton shower models [94], as expected.

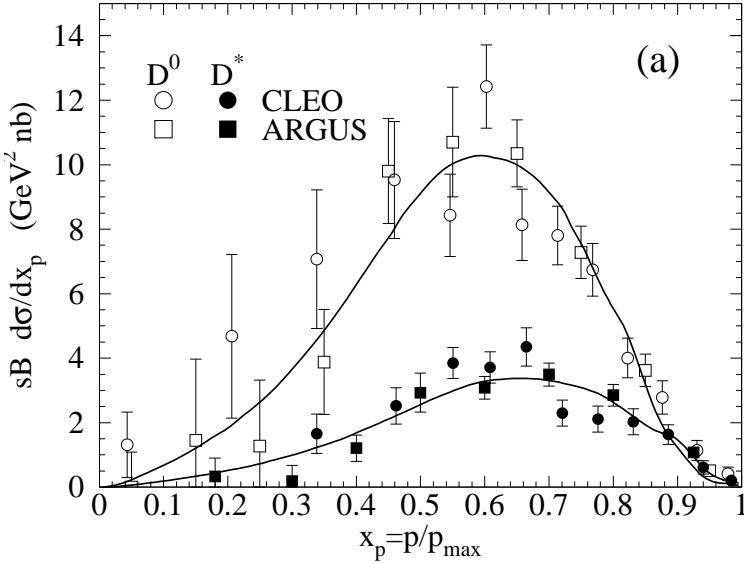


Figure 9: Efficiency-corrected inclusive cross-section measurements for the production of  $D^0$  and  $D^{*+}$  in  $e^+e^-$  measurements at  $\sqrt{s} \approx 10$  GeV. The variable  $x_p$  approximates the Peterson variable  $z$ , but is not identical to it.

The  $\epsilon$  parameter for charm and bottom scales roughly with the inverse square of the heavy flavour mass. This behaviour can be justified by several arguments [84, 95, 96]. It can be used to relate the non-perturbative parts of the fragmentation functions of charm and bottom quarks [87, 97, 98].

The bulk of the available fragmentation function data on charmed mesons (excluding  $J/\Psi(1S)$ ) is from measurements at  $\sqrt{s} = 10$  GeV.<sup>2</sup> Shown in Fig. 9 are the efficiency-corrected (but not branching ratio corrected) CLEO [99] and ARGUS [92] inclusive cross-sections ( $s \cdot \mathcal{B}d\sigma/dx_p$  in units of  $\text{GeV}^2 \text{nb}$ , with  $x_p = p/p_{\max}$ ) for the production of pseudoscalar  $D^0$  and vector  $D^{*+}$  in  $e^+e^-$  annihilation at  $\sqrt{s} \approx 10$  GeV. For the  $D^0$ ,  $\mathcal{B}$  represents the product branching fraction:  $D^{*+} \rightarrow D^0\pi^+$ ,  $D^0 \rightarrow K^-\pi^+$ . These inclusive spectra have not been corrected for cascades from higher states, nor for radiative effects. Note that since the momentum spectra are sensitive to QED and QCD radiative corrections, charm spectra at  $\sqrt{s} = 10$  GeV cannot be compared directly with spectra at higher c.m. energies, and must be appropriately evolved.

Tuning  $\epsilon$  of the function (33) in the JETSET Monte Carlo generator [68] using the parameter set of [71] and including radiative corrections to describe the combined CLEO and ARGUS  $D^0$  and  $D^{*+}$  data gives  $\epsilon_c = 0.043 \pm 0.004$  ( $\chi^2/\text{d.o.f.} = 50.5/41$ ); this is indicated in the solid curves. Measurements of the fragmentation functions for a variety of particles has allowed comparisons between mesons and baryons, and particles of different spin structure, as shown in Table 1. The  $\epsilon$  values listed in this table were obtained from directly fitting the Peterson function (33) to the measured differential cross-section  $\sigma \cdot \mathcal{B}d\sigma/dx_p$ .

<sup>2</sup>This part on charm quark fragmentation including Fig. 9 and Table 1, which have been updated, are taken from the review of D. Besson contributed to C. Caso *et al.*, Eur. Phys. J. **C3**, 1 (1998).

Table 1: The Peterson momentum hardness parameter  $\epsilon$  as obtained from fits of Eq. (33) to  $s \cdot \mathcal{B}d\sigma/dx_p$  of  $e^+e^- \rightarrow \text{particle} + X$  measurements at  $\sqrt{s} \approx 10$  GeV.

Particle	$L$	$\epsilon$	Reference
$D^0$	0	$0.260 \pm 0.024$	[99, 100]
$D^+$	0	$0.156 \pm 0.022$	[99, 100]
$D^{*+}$	0	$0.198 \pm 0.022$	[99, 100]
$D_s$	0	$0.10 \pm 0.02$	[101, 100]
$D_s^*$	0	$0.057 \pm 0.008$	[101, 100]
$D_1(2420)^0$	1	$0.015 \pm 0.004$	[102, 100]
$D_2^*(2460)^0$	1	$0.039 \pm 0.013$	[102, 100]
$D_1(2420)^+$	1	$0.013 \pm 0.006$	[103, 100]
$D_2^*(2460)^+$	1	$0.023 \pm 0.009$	[103, 100]
$D_{s1}(2536)^+$	1	$0.018 \pm 0.008$	[104, 100]
$D_{sJ}(2573)^+$	1	$0.027^{+0.043}_{-0.015}$	[105]
$\Lambda_c^+$	0	$0.267 \pm 0.038$	[106, 107]
$\Xi_c^{+,0}$	0	$0.23 \pm 0.05$	[108, 109]
$\Xi_c^{\prime+,0}$	0	$0.20^{+0.24}_{-0.11}$	[110]
$\Sigma_c(2455)$	0	$0.28 \pm 0.05$	[111, 112]
$\Sigma_c(2520)$	0	$0.30^{+0.10}_{-0.07}$	[113]
$\Xi_c(2645)^+$	0	$0.24^{+0.22}_{-0.10}$	[114]
$\Xi_c(2645)^0$	0	$0.22^{+0.15}_{-0.08}$	[115]
$\Lambda_c(2593)^+$	1	$0.058 \pm 0.022$	[116, 117]
$\Lambda_c(2625)^+$	1	$0.053 \pm 0.014$	[116, 118]
$\Xi_c(2815)$	1	$0.07^{+0.03}_{-0.02}$	[119]

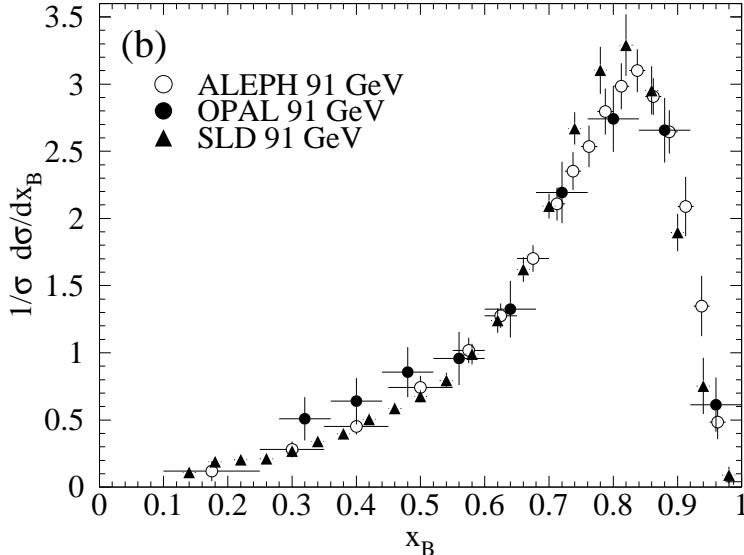


Figure 10: Measured  $e^+e^-$  fragmentation function of  $b$  quarks into  $B$  hadrons [121, 122, 123].

We note from Table 1 that the mass dependence of  $\epsilon$  is less marked than the dependence on the orbital angular momentum structure of the charmed hadron being measured. Orbitally excited  $L = 1$  charmed hadrons ( $D_J$ ,  $D_{sJ}$ , and  $\Lambda_c(2593)$ ,  $\Lambda_c(2625)$ ) show consistently harder spectra (*i.e.*, smaller values of  $\epsilon$ ) than the  $L = 0$  ground states, whereas the data for the ground state charmed baryons  $\Lambda_c$  and  $\Xi_c$  show agreement with the lighter (by  $\approx 400\text{-}600$  MeV) ground-state  $D$  and  $D_s$  charmed mesons. To some extent, the harder spectra of  $L = 1$  hadrons can be attributed to the fact that all the  $L = 1$  charmed hadrons will eventually decay into  $L = 0$  hadrons.

Experimental studies of the fragmentation function for  $b$  quarks have been performed at LEP and SLD [93, 120, 121, 122]. The results are shown in Fig. 10.  $B$  hadrons are commonly identified by their semileptonic decays or by their decay vertex reconstructed from the charged particles emerging from the  $B$  decay. The most recent results of refs. [123] and [122] are in agreement. Both studies fit the  $B$  spectrum using a Monte Carlo shower model supplemented with non-perturbative fragmentation functions to fit their data.

The experiments measure primarily the spectrum of  $B$  meson. This defines a fragmentation function which includes the effect of the decay of higher mass excitations, like the  $B^*$  and  $B^{**}$ . There is an ambiguity in the definition of the fragmentation function, which has to do with what is considered to be a final state hadron. One may prefer to distinguish  $B$  meson produced directly from those arising from decays of  $B^*$  and  $B^{**}$ , and define

$$D_{B|B^*|B^{**}}^{(d)}(z) = D_B^{(d)}(z) + D_{B^*}^{(d)}(z) + D_{B^{**}}^{(d)}(z) \quad (36)$$

where the notation  $D^{(d)}$  stands for directly produced. The fragmentation function which

Table 2: Fraction of events containing  $g \rightarrow c\bar{c}$  and  $g \rightarrow b\bar{b}$  subprocesses in  $Z$  decays, as measured by the various collaborations, compared with theoretical predictions. The central/lower/upper values for the theoretical predictions are obtained with  $m_c = (1.5 \pm 0.3)$  and  $m_b = (4.75 \pm 0.25)$  GeV.

	$\bar{n}_{g \rightarrow c\bar{c}} (\%)$	$\bar{n}_{g \rightarrow b\bar{b}} (\%)$
OPAL	[128] $3.20 \pm 0.21 \pm 0.38$	
ALEPH	[129] $2.65 \pm 0.74 \pm 0.51$	[131] $0.277 \pm 0.042 \pm 0.057$
DELPHI		[130] $0.21 \pm 0.11 \pm 0.09$
SLD		[132] $0.307 \pm 0.071 \pm 0.066$
Theory [126]		
$\Lambda_{\overline{\text{MS}}}^{(5)} = 150 \text{ MeV}$	$1.35^{+0.48}_{-0.30}$	$0.20 \pm 0.02$
$\Lambda_{\overline{\text{MS}}}^{(5)} = 300 \text{ MeV}$	$1.85^{+0.69}_{-0.44}$	$0.26 \pm 0.03$

is directly measured from the  $B$  mesons, irrespective of their origin, is instead given by

$$D_B(x) = D_B^{(d)}(x) + \int F_{B^* \rightarrow B}(y) D_{B^*}^{(d)}(z) \delta(x - yz) dy dz + \int F_{B^{**} \rightarrow B}(y) D_{B^{**}}^{(d)}(z) \delta(x - yz) dy dz . \quad (37)$$

where (for example)  $F_{B^{**} \rightarrow B}(y)$  stands for the probability for a fast  $B^{**}$  to decay to  $B$  with a fraction  $y$  of its energy. Thus, in order to extract  $D_{B|B^*|B^{**}}^{(d)}$  from  $D_B$  a correction factor should be computed, which gives a harder fragmentation function. Thus, for example, ref. [122] obtains the value  $\langle x_B \rangle = 0.714 \pm 0.005(\text{stat.}) \pm 0.007(\text{syst.}) \pm 0.002(\text{model})$ ; postulating a  $B^*$  fraction of 0.75, one gets  $\langle x_{B|B^*} \rangle = 0.718$ ; postulating a  $B^{**}$  fraction of 0.25 yields  $\langle x_{B|B^{**}} \rangle = 0.728$ .

Besides degrading the fragmentation function by gluon radiation, QCD evolution can also generate soft heavy quarks, increasing in the small  $x$  region as  $s$  increases. This effect has been studied both theoretically and experimentally. One important issue is to understand how often  $b\bar{b}$  or  $c\bar{c}$  pairs are produced indirectly, via a gluon splitting mechanism. Several theoretical studies are available on this topic [124, 125, 126, 127]. Experimental studies on charm production via gluon splitting have been presented in refs. [128, 129], and measurements of  $g \rightarrow b\bar{b}$  have been given in [130, 131, 132]. The reported values are given in Table 2. In ref. [126] an explicit calculation of these quantities has been performed. Using these results charm and bottom multiplicities for different values of the masses and of  $\Lambda_{\overline{\text{MS}}}^{(5)}$  were computed in [133]. They are reported in Table 2. As can be seen, charm measurements are somewhat in excess of the predictions. The averaged experimental result for charm,  $(3.10 \pm 0.34)\%$ , is 1 - 2 standard deviations outside the range of the theoretical prediction, preferring lower values of the quark mass and/or a larger value of  $\Lambda_{\overline{\text{MS}}}^{(5)}$ . However, higher-order corrections may well be substantial at the charm quark mass scale. Better agreement is achieved for bottom.

As reported in ref. [126], Monte Carlo models are in qualitative agreement with these results, although the spread of the values they obtain is somewhat larger than the theoretical error estimated by the direct calculation. In particular, for charm one finds that

while HERWIG [70] and JETSET [68] agree quite well with the theoretical calculation, ARIADNE [134] is higher by roughly a factor of 2, and thus is in better agreement with data. For bottom, agreement between theory, models and data is adequate. For a detailed discussion see ref. [135].

The discrepancy with the charm prediction may be due to experimental cuts forcing the final state configuration to be more 3-jet like, which increases the charm multiplicity. Calculations that take this possibility into account are given in [127].

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