How to Play Perfect Golf Using Math

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30/05/2024

Introduction

- Simple projectile motion
- Projectile with air resistance
- Three dimensional with air resistance
- How to play perfect golf

Simple Projectile

The motion of the projectile in 2D, subject solely to the force of gravity, is our starting point. Later on, we will also examine the role of air resistance. Get ready to understand how objects move in space under the influence of gravity and to explore the variables involved in the trajectory.

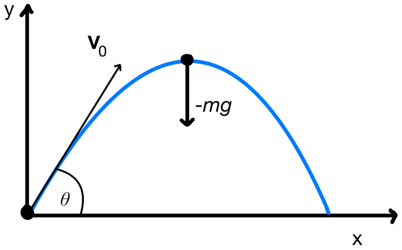


Figure 1: Simple 2D projectile setup

Simple Projectile

In Figure 1, the initial horizontal and vertical velocities are:

$$\dot{x}(0) = v_0 \cos(\theta),$$
$$\dot{y}(0) = v_0 \sin(\theta).$$

Using Newton's Second Law of Motion

$$\begin{aligned} \mathbf{F} &= m\,\mathbf{a} \\ &= m\,(\ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath}) = -mg\,\hat{\jmath}. \end{aligned}$$

Simple Projectile

The horizontal and vertical components are

$$
\begin{aligned}
m\,\ddot{x} &= 0, \\
\ddot{x} &= 0, \\
\dot{x} &= C_1, \\
x &= C_1\,t + C_2.
\end{aligned}
$$

$$
\begin{aligned}
m\,\ddot{y} &= -m\,g, \\
\ddot{y} &= -g, \\
\dot{y} &= -g\,t + C_3, \\
y &= -\frac{1}{2}\,g\,t^2 + C_3\,t + C_4.
\end{aligned}
$$

Simple Projectile

The constants C_1, C_2, C_3, C_4 have been calculated considering the following initial conditions

$$x(0) = 0, \quad y(0) = 0, \quad \dot{x}(0) = v_0 \cos(\theta), \quad \dot{y}(0) = v_0 \sin(\theta).$$

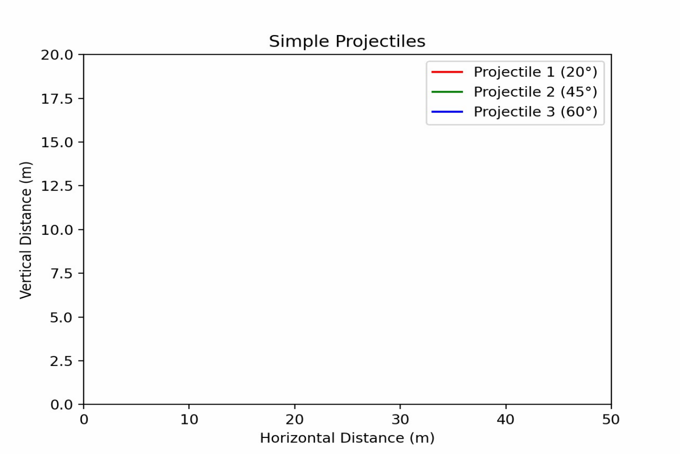
Equation of Motion of Simple Projectile

$$x(t) = v_0 \cos(\theta)\, t,$$
$$y(t) = -\frac{1}{2} g\, t^2 + v_0 \sin(\theta)\, t.$$

The Range of the projectile is given by:

$$R = \frac{v^2 \sin 2\theta}{g}$$

Simple Projectile



Air Resistance

Initially we will assume the air resistance is linearly proportional with velocity $F_{fr} \propto kv$. By Newton's second law:

$$-mg\hat{\jmath} - k(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$$

We split these equation into $\hat{\imath}$ and $\hat{\jmath}$ components to solve for the equations of motion.

$$-k\dot{x} = m\ddot{x}$$

Divide both sides by m and define $\gamma = \frac{k}{m}$. Multiply across by the integrating factor $e^{\gamma t}$ and group.

$$\frac{d}{dt}(\dot{x}e^{\gamma t}) = 0$$

Solving this with the initial conditions yields:

$$x(t) = \frac{v_x}{\gamma}(1 - e^{-\gamma t})$$

Air Resistance

$$-mg - k\dot{y} = m\ddot{y}$$

Following the same process we get:

$$\frac{d}{dt}(\dot{y}e^{\gamma t}) = -ge^{\gamma t}$$

Upon integrating and using $\dot{y}(0) = v_y$

$$\dot{y} = \frac{g}{\gamma} + (v_y + \frac{g}{\gamma})e^{-\gamma t}$$

Repeat this process with $y(0) = 0$

$$y(t) = \frac{1}{\gamma}(v_y + \frac{g}{\gamma})(1 - e^{-\gamma t}) - \frac{gt}{\gamma}$$

Air resistance

To recap

$$x(t) = \frac{v_x}{\gamma}(1 - e^{-\gamma t}) \tag{1}$$

$$y(t) = \frac{1}{\gamma}(v_y + \frac{g}{\gamma})(1 - e^{-\gamma t}) - \frac{gt}{\gamma} \tag{2}$$

From (1) we isolate t:

$$t = -\frac{1}{\gamma}\ln(1 - \frac{x\gamma}{v_x})$$

Substitute into (2):

$$y = \frac{1}{\gamma}(v_y + \frac{g}{\gamma})(1 - e^{\frac{1}{\gamma}\ln(1-\frac{x\gamma}{v_x})\gamma}) - \frac{g}{\gamma}(-\frac{1}{\gamma}\ln(1 - \frac{x\gamma}{v_x}))$$

$$y(x) = \frac{v_y + \frac{g}{\gamma}}{v_x}x + \frac{g}{\gamma^2}\ln(1 - \frac{x\gamma}{v_x})$$

Air Resistance

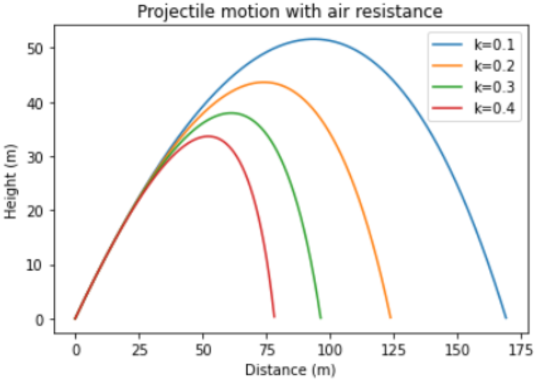


Figure 2: Trajectories with varying air resistance

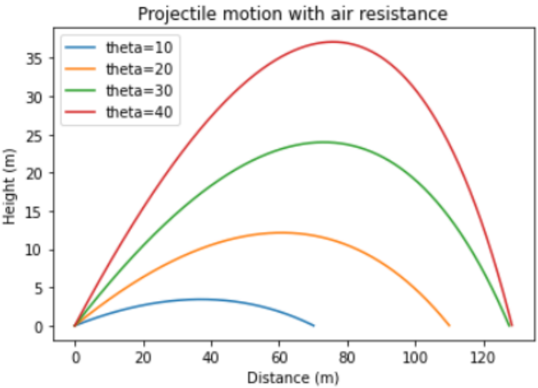


Figure 3: Trajectories with varying angles

Air Resistance

We now consider $F \propto v^2$.

Air Resistance

$$\mathbf{F}_f = -k\,\mathbf{v}\,|\mathbf{v}|,$$

where, k is the constant of proportionality for air drag. We will see later what parameters it takes into account.

Air Resistance

Here,

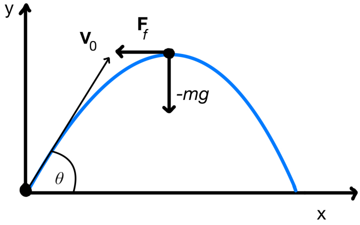


Figure 4: Simple 2D projectile with air resistence

$$\mathbf{F}_f = -k\,\mathbf{v}\,|\mathbf{v}|,$$
$$\mathbf{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath},$$
$$|\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

Air Resistance

As in the simple case, we apply Newton's Second Law to find the equations of motion.

$$
\begin{aligned}
\mathbf{F} &= m\,\mathbf{a} \\
&= m\,(\ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath}) = -mg\,\hat{\jmath} - k\,(\dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath})\,\sqrt{\dot{x}^2 + \dot{y}^2}
\end{aligned}
$$

In component

$$m\,\ddot{x} = -k\,\dot{x}\,\sqrt{\dot{x}^2 + \dot{y}^2},$$

$$m\,\ddot{y} = -g - k\,\dot{y}\,\sqrt{\dot{x}^2 + \dot{y}^2}.$$

Air Resistance

System of ODE's Motion

$$\ddot{x} = -\frac{k}{m}\,\dot{x}\,\sqrt{\dot{x}^2 + \dot{y}^2},$$
$$\ddot{y} = -g - \frac{k}{m}\,\dot{y}\,\sqrt{\dot{x}^2 + \dot{y}^2}$$

$$x(0) = 0, \quad y(0) = 0, \quad \dot{x}(0) = v_0 \cos\theta, \quad \dot{y}(0) = v_0 \sin\theta$$

This is a system of two 2^{nd} - order coupled nonlinear ordinary differential equations for $x(t)$ and $y(t)$.

Air Resistance

To find the solutions we have to do:

- Convert our system of 2^{nd} - order ODE's into a system of 1^{st} - order ODE's.
- Introduce a new variable: velocity components.
- Transition from 2^{nd}-order equations to 1^{st}-order equations.

Air Resistance

Old system:

$$\ddot{x} = -\frac{\mathrm{k}}{\mathrm{m}}\,\dot{x}\,\sqrt{\dot{x}^2 + \dot{y}^2},$$
$$\ddot{y} = -\mathrm{g} - \frac{\mathrm{k}}{\mathrm{m}}\,\dot{y}\,\sqrt{\dot{x}^2 + \dot{y}^2}.$$

New system:

$$v_x = \dot{x},$$
$$v_y = \dot{y},$$
$$\dot{v}_x = -\frac{\mathrm{k}}{\mathrm{m}}\,v_x\,\sqrt{v_x^2 + v_y^2},$$
$$\dot{v}_x = -\mathrm{g} - \frac{\mathrm{k}}{\mathrm{m}}\,v_y\,\sqrt{v_x^2 + v_y^2}.$$

The initial conditions are always the same.

Air Resistance

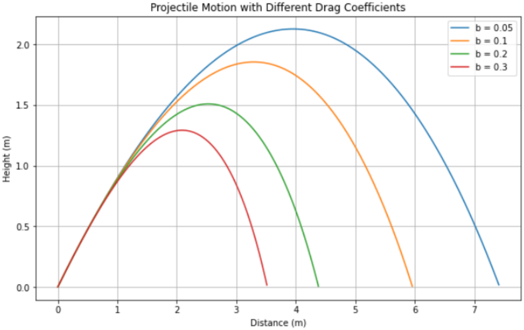


Figure 5: Trajectories for different drag coefficients

Air Drag

Previously, we used k as a constant to describe air resistance. However, the more accurate formula is:

Air Drag

$$F_D = \frac{1}{2} \rho \, C_D \, A \, v^2$$

where

- F_D is the air drag (N),
- ρ is the density of the medium ($kg \, m^{-3}$),
- C_D is the drag coefficient ($kg \, m^{-1}$),
- A is the area of (m^2),
- v is the velocity of the projectile ($m \, s$).

Parameter k

However, k is given by

Parameter k

$$k = \frac{1}{2}\,\rho\,C_D\,A$$

The values ρ, C_D and A are properties of the projectile and depend on the projectile geometry, surface area and weight.

Below are the values of k for 3 types of balls:

Sport	$m(kg)$	$\rho\ (kgm^{-3})$	$C_D(kgm^{-1})$	$A(m^2)$
Golf	0.045	1.22	0.27	0.00014
Rugby	0.44	1.22	0.16	0.0257
Soccer	0.43	1.22	0.17	0.038

Optimal Angle

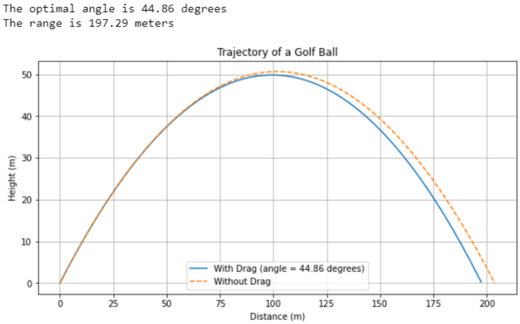


Figure 6: Comparison between air drag and no air drag

Optimal Angle

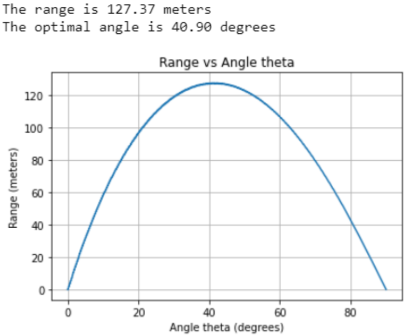


Figure 7: Finding optimal angle

3-Dimensional Projectile

We now consider the generalisation to the three dimensional case.

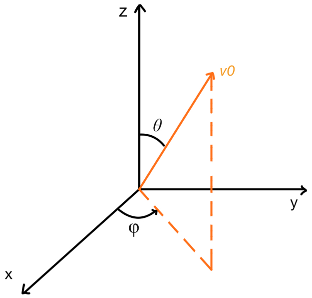


Figure 8: Spherical coordinate system

3-Dimensional Projectile

Similarly to the 2D case, initial velocity in component is:

$$\dot{x}(0) = v_0 \sin\theta \cos\phi,$$
$$\dot{y}(0) = v_0 \sin\theta \sin\phi,$$
$$\dot{z}(0) = v_0 \cos\theta.$$

Newton's Second Law:

$$\mathbf{F} = m\mathbf{a}$$
$$-mg\,\hat{\jmath} - k\left(\dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} + \dot{z}\,\hat{\kappa}\right)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = m\left(\ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath} + \ddot{z}\,\hat{\kappa}\right)$$

3-Dimensional Projectile

The associated ODE system is:

System of ODE's Motion (3D)

$$\ddot{x} = -\frac{k}{m}\,\dot{x}\,\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2},$$

$$\ddot{y} = -g - \frac{k}{m}\,\dot{y}\,\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2},$$

$$\ddot{z} = -\frac{k}{m}\,\dot{z}\,\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}.$$

3-Dimensional Projectile

Convert our system of 2^{nd} - order ODE's into a system of 1^{st} - order ODE's.

New system:

$$
\begin{aligned}
v_x &= \dot{x}, \\
v_y &= \dot{y}, \\
v_z &= \dot{z}, \\
\dot{v}_x &= -\frac{k}{m}\, v_x \sqrt{v_x^2 + v_y^2 + v_z^2}, \\
\dot{v}_y &= -g - \frac{k}{m}\, v_y \sqrt{v_x^2 + v_y^2 + v_z^2}, \\
\dot{v}_z &= -\frac{k}{m}\, v_z \sqrt{v_x^2 + v_y^2 + v_z^2}.
\end{aligned}
$$

3-Dimensional Projectile

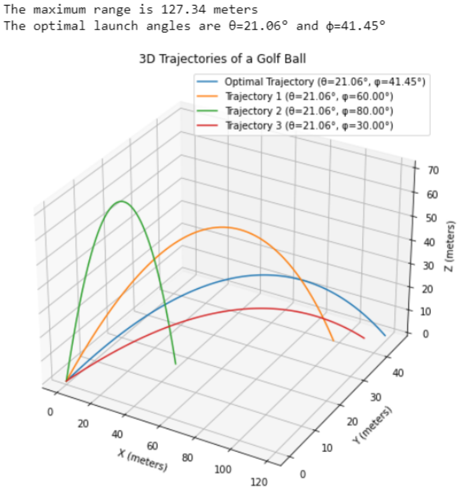


Figure 9: 3d projectile with initial speed 44.7 ms^{-1}

Caddie - input

Inputs	Outputs
Position	Club
Properties of the ball	Speed and angle

```
Input the x-coordinate of the hole: 55
Input the y-coordinate of the hole: 19
Use a pitching wedge
The required initial speed to reach 58.19 meters is 32.51 m/s
```

Figure 10: Inputting desired coordinates

Golf trajectory

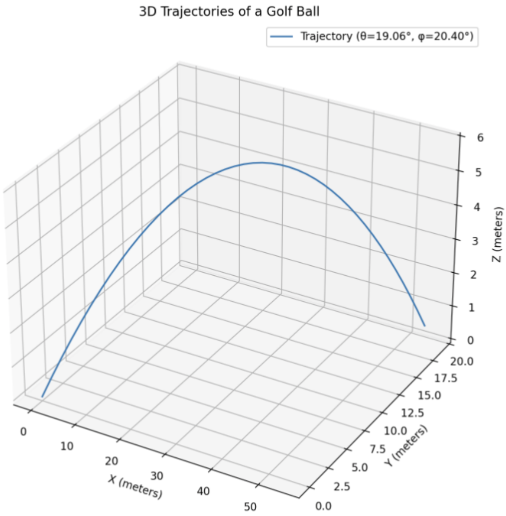


Figure 11: Trajectory to chosen coordinates

Future improvements

- Wind speed
- Ball spin
- Club mechanics
- Height of target