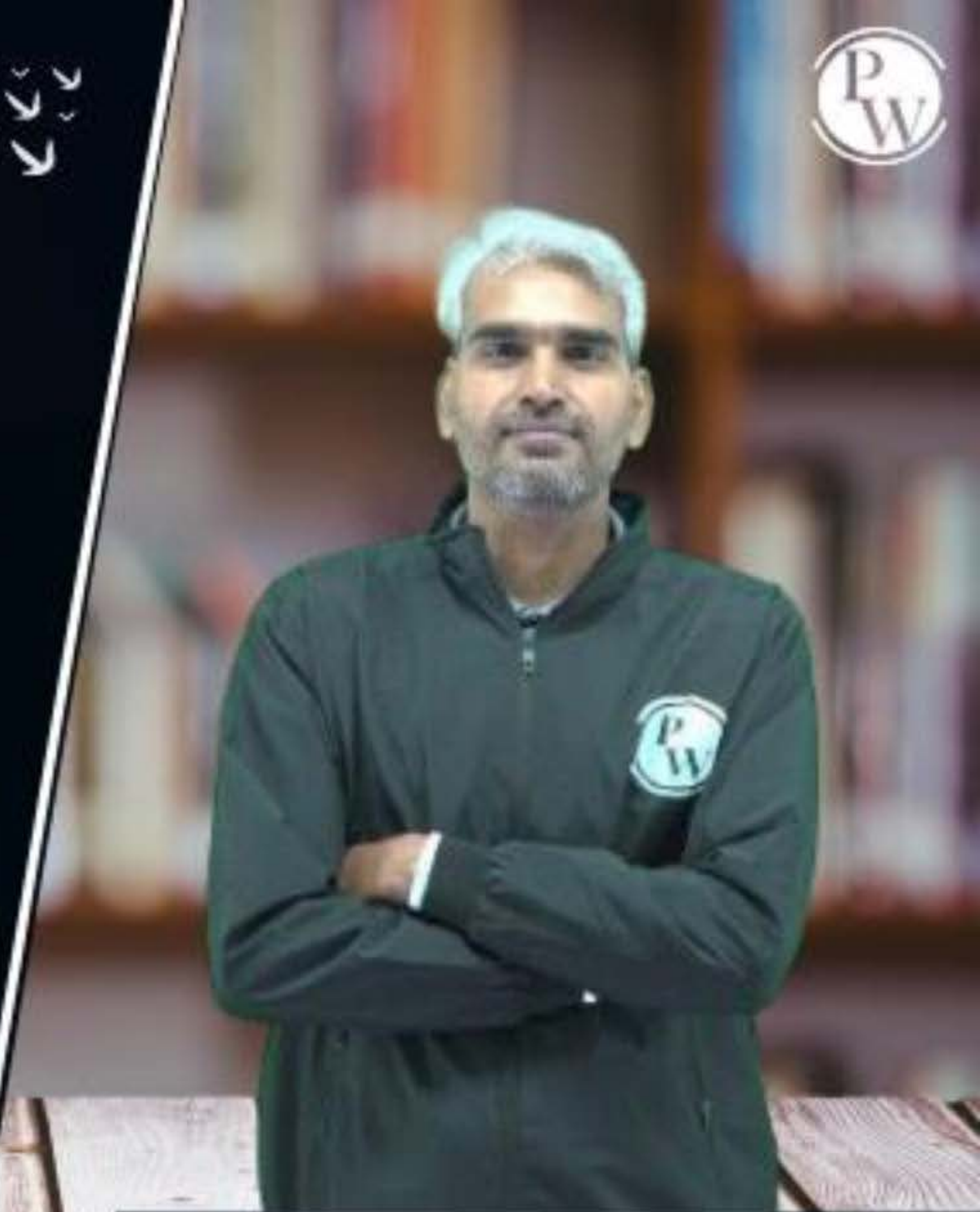
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 06



By- Vishal Sir

Recap of Previous lecture

- **Topic** Composite of two relations
- **Topic** Equivalence relation
- **Topic** Equivalence class
- **Topic** Partition of a set
- **Topic** Number of partitions of a set

Slide

Todays Topic

- **Topic** Number of partitions of a set
- **Topic** Number of equivalence relation
- **Topic** Bell number
- **Topic** Questions

Slide

Topic : Equivalence relation

- A relation R on set A is called an Equivalence relation if and only if relation R is

(1) Reflexive

and (2) Symmetric

and (3) Transitive

$$\boxed{\text{Equivalence Relation} = \text{Reflexive} + \text{Symmetric} + \text{Transitive}}$$

Slide

eg. Let $A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (3,3)\} = \Delta_A$

$\left.\begin{array}{l}\text{Reflexive } \checkmark \\ \text{Symmetric } \checkmark \\ \text{Transitive } \checkmark\end{array}\right\}$ ∴ Equivalence Relation

$\boxed{\Delta_A \text{ is the smallest Equivalence rel}^n \text{ on set } A}$

$R_2 =$ "$A \times A$" $=$

$\left.\begin{array}{l}\text{Reflexive } \checkmark \\ \text{Symmetric } \checkmark \\ \text{Transitive } \checkmark\end{array}\right\}$ ∴ Equivalence Rel^n

$\boxed{\text{'}A \times A\text{' is the largest equivalence rel}^n \text{ on set } A}$

Which of the following are Equivalence relⁿ on set A

eg. Let $A = \{1, 2, 3\}$

$$R_3 = \{\underbrace{(1,1), (2,2), (3,3)}, \underbrace{(1,2), (2,1)}\}$$

Reflexive ✓
Symmetric ✓
Transitive ✓
$\therefore$ Equivalence Rel^n

$$R_4 = \{\underbrace{(1,1)\ (2,2), (3,3)}, \underbrace{(1,3), (3,1)}\}$$

Reflexive ✓
Symmetric ✓
Transitive ✓
$\therefore$ Equivalence Rel^n

$$R_5 = \{\underbrace{(1,1)\ (2,2), (3,3)}, \underbrace{(2,3), (3,2)}\}$$

Reflexive ✓
Symmetric ✓
Transitive ✓
$\therefore$ Equivalence Rel^n

eg. Let $A = \{1, 2, 3\}$

$$R_6 = \{ \overbrace{\underbrace{(1,1), (2,2), (3,3)}_{\text{Reflexive } \checkmark}, (1,2), (2,1), (1,3), (3,1)}^{\text{Symmetric } \checkmark} \}$$

$\therefore$ By transitivity $(2, 3)$ must be present in R_6

But $(2,3) \notin R_6$ $\therefore$ Not transitive

Hence not an Equivalence Rel^n

* Check whether the statement is true or false :

> " Every symmetric as well as transitive relation on set 'A', is also a reflexive relation on set A "

eg let $A = \{1, 2, 3\}$

$R = \{\ \}$ ← it is symmetric as well as transitive but it is not reflexive

Empty relation on set A

eg: let $A = \{1, 2, 3, 4\}$

$R = \{\underbrace{(1,1), (2,2), (1,2), (2,1)}\}$

Symmetric + Transitive

But it is not reflexive

Topic : Equivalence Class

- Let R is an Equivalenc relation on set A.

- Equivalence class of an element $x \in A$ w.r.t. Equivalence relation R Can be denoted by $[x]$ And it is defined as,

$$[x] = \{ y \mid x R y \}$$

(or)

$$[x] = \{ y \mid (x, y) \in R \}$$

Slide

eg: Let $A = \{1, 2, 3, 4, 5\}$
and R is an Equivalence relation on set A.

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (4,5), (5,4)\}$

Find Equivalence Class of every element of set A w.r.t. Equivalence relation R.

Solu^n.

$[1] = \{1, 2\}$

Same Equivalence Class for element '1' & '2'

i.e. $[1] = [2]$

$[2] = \{2, 1\} = \{1, 2\}$

$[3] = \{3\}$

Three distinct Equivalence Classes for elements of set A.

$[4] = \{4, 5\}$

Same Equivalence Class for '4' & '5'

i.e. $[4] = [5]$

$[5] = \{5, 4\} = \{4, 5\}$

- Distinct Equivalence Classes of elements of set A are $\{1,2\}$, $\{3\}$, $\{4,5\}$

- Set of all distinct Equivalence Class of elements of set A w.r.t. Equivalence Rel^n 'R' defines the Partition of set 'A' w.r.t. Equivalence Rel^n R

i.e. $\{\{1,2\}, \{3\}, \{4,5\}\}$ is a Partition of set A w.r.t. Equivalence Rel^n R.

- **Note :→** (1) Equivalence class of two distinct elements of the set may be equal eg:- $[1] = [2]$ even when $1 \neq 2$

Note :- (2) Set of all distinct Equivalence Classes of elements of set A will be a partition of set A with respect to given Equivalence relation

Topic : Partition of a set

- Partition of a "set A" is the set of non-Empty subsets of set A such that each element of set A appears in Exactly one non-Empty subset of that set

(or)

- Let A is any set and $A_1, A_2, A_3, \ldots A_K$ are non-Empty subsets of set A, then

$\{A_1, A_2, A_3, \ldots \ldots A_K\}$ is a partition of set A if and only if (i) $A_i \cap A_j = \phi \;\; \forall i, j \; (i \neq j)$ and (ii) $\bigcup_{i=1}^{K} A_i = A$

Slide

eg:- Let $A = \{1, 2, 3, 4, 5\}$
Which of the following is/are Partitions of set A.

(a) $\{\{1,2,3\}, \{3,4\}, \{5\}\}$ $\cap = \{3\} \neq \emptyset$ $\therefore$ Not a Partition

(b) $\{\{1,2\}, \{3\}, \{4\}\}$ $\Rightarrow$ Union is not equal to 'A' $\therefore$ Not a partition

(c) $\{\{1,2,3\}, \{4\}, \{5\}, \{\ \}\}$ Empty subset is not allowed in a set w.r.t. partition

(d) $\{\{1,2\}, \{3\}, \{4\}, \{5\}\}$ ✓

(e) $\{\{1,2,3,4,5\}\}$ ✓

(f) $\{\{1,2\}, \{3\}, \{4,5\}\}$ ✓

* Find the number of partitions of "set A" when

(1) $|A| = 0$

(2) $|A| = 1$

(3) $|A| = 2$

(4) $|A| = 3$

(5) $|A| = 4$

① Let $A = \{ \ \}$, i.e. $|A| = 0$

Partition of set A = $\{ \qquad \}$

Partition itself may be an Empty set, but an Empty set can never be an element of the set w.r.t. Partition

Partition of set A will be a set

→ and elements of this set will be non Empty subsets of set A

→ If set A itself is Empty, then no non-empty subset of set A.

∴ No element in the Set w.r.t. Partition of set A.

i.e, $\boxed{\text{if } |A| = 0 \text{ then number of partitions of set } A = 1}$

$|A| = 1 \Rightarrow \underbrace{1}$

Only one subset of size = 1

② Let $A = \{a\}$, i.e. $|A| = 1$

Partition of set A = $\{ \{a\} \}$

Partition of set A will be a set

and elements will be non-Empty subsets of set A

w.r.t. set of Cardinality '1', there will be only One non-Empty subset, and it will be set A itself.

i.e., $\boxed{\text{If } |A| = 1 \text{ then number of Partitions of set } A = 1}$

(3) Let $A = \{a, b\}$ i.e. $|A| = 2$

$|A| = 2 \Rightarrow \underbrace{1 + 1}$ Two subsets each of size = 1

$\underbrace{2}$ only one subset of size = 2

Partitions of set $A = \{\{a\}, \{b\}\}$ & $\{\{a, b\}\}$

$\{\{a\}, \{b\}\} \cong \{\{b\}, \{a\}\}$

These are the only two partitions w.r.t. set $A = \{a, b\}$

i.e, $\boxed{\text{If } |A| = 2 \text{ then number of partitions of set } A = 2}$

$|A| = 1 \Rightarrow 1$

$|A| = 2 \Rightarrow 1 + 1$

2

$|A| = 3 \Rightarrow 1 + 1 + 1$

$1 + 2 \equiv 2 + 1$

3

$|A| = 4 \Rightarrow 1 + 1 + 1 + 1$

$1 + 1 + 2$

$1 + 3$

$2 + 2$

4

$|A| = 5 \Rightarrow$

$1 + 1 + 1 + 1 + 1$

$1 + 1 + 1 + 2$

$1 + 1 + 3$

$1 + 2 + 2$

$1 + 4$

$2 + 3$

5

$|A| = 6 \Rightarrow$

$1 + 1 + 1 + 1 + 1 + 1$

$1 + 1 + 1 + 1 + 2$

$1 + 1 + 1 + 3$

$1 + 1 + 2 + 2$

$1 + 1 + 4$

$1 + 2 + 3$

$1 + 5$

$2 + 2 + 2$

$2 + 4$

$3 + 3$

6

④ Let $A = \{a, b, c\}$ i.e. $|A| = 3$

$|A| = 3 \Rightarrow 1 + 1 + 1$

$1 + 2$

3

Partitions of set $A = \{\{a\}, \{b\}, \{c\}\}$ & $\{\{a\}, \{b, c\}\}$ & $\{\{a, b, c\}\}$

&

$\{\{b\}, \{a, c\}\}$

&

$\{\{c\}, \{a, b\}\}$

i.e, $\boxed{\text{If } |A| = 3 \text{ then number of partitions of set } A = 5}$

$|A| = 3 \Rightarrow 1+1+1$

$1+2$

3

④ Let $A = \{a, b, c\}$ i.e. $|A| = 3$

Partitions of set $A = \{\{\ \},\{\ \},\{\ \}\}$ or $\{\{\ \},\{\ ,\ \}\}$ or $\{\{\ ,\ ,\ \}\}$

$\{\{a\},\{b\},\{c\}\}$
$\{\{a\},\{c\},\{b\}\}$
$\{\{b\},\{a\},\{c\}\}$
$\{\{b\},\{c\},\{a\}\}$
$\{\{c\},\{a\},\{b\}\}$
$\{\{c\},\{b\},\{a\}\}$

$= \left(\dfrac{{}^3C_1 * {}^2C_1 * {}^1C_1}{3!}\right)$ or ${}^3C_1 * {}^2C_2$ or ${}^3C_3$

$= 1$ $\qquad = 3$ $\qquad = 1$

i.e., $\boxed{\text{If } |A| = 3 \text{ then number of partitions of set } A = 1 + 3 + 1 = 5}$

(5) Let $A = \{a, b, c, d\}$ i.e. $|A| = 4$, then Find the no. of Partitions of set A

$|A| = 1+1+1+1 \Rightarrow \dfrac{{}^4C_1 * {}^3C_1 * {}^2C_1 * {}^1C_1}{4!} = 1$

$1+1+2 \Rightarrow \dfrac{{}^4C_1 * {}^3C_1}{2!} * {}^2C_2 \Rightarrow 6$

$1+3 \Rightarrow {}^4C_1 * {}^3C_3 \Longrightarrow 4$

$2+2 \Rightarrow \dfrac{{}^4C_2 * {}^2C_2}{2!} \Rightarrow 3$

$4 \Rightarrow {}^4C_4 \Longrightarrow 1$

i.e, $\boxed{\text{If } |A| = 4 \text{ then number of partitions of set } A = 1+6+4+3+1 = 15}$

H.W. 6 Let $A = \{a, b, c, d, e\}$ i.e. $|A| = 5$ then Find the no. of Partitions of set A

i.e, $\boxed{\text{If } |A| = 5 \text{ then number of partitions of set } A = 1+10+10+15+5+10+1} = 52$

$$|A| = 5 \Rightarrow 1+1+1+1+1 = \frac{{}^{5}C_1 * {}^{4}C_1 * {}^{3}C_1 * {}^{2}C_1 * {}^{1}C_1}{5!} = 1$$

$$1+1+1+2 = \frac{{}^{5}C_1 * {}^{4}C_1 * {}^{3}C_1}{3!} * {}^{2}C_2 = 10$$

$$1+1+3 = \frac{{}^{5}C_1 * {}^{4}C_1}{2!} * {}^{3}C_3 = 10$$

$$1+2+2 = {}^{5}C_1 * \frac{{}^{4}C_2 * {}^{2}C_2}{2!} = 15$$

$$1+4 = {}^{5}C_1 * {}^{4}C_4 = 5$$

$$2+3 = {}^{5}C_2 * {}^{3}C_3 = 10$$

$$5 = {}^{5}C_5 = 1$$

H.W:

7. Let $A = \{a, b, c, d, e, f\}$ i.e. $|A| = 6$ then Find the no. of Partitions of set A

i.e, $\boxed{\text{If } |A| = 6 \text{ then number of partitions of set } A = 203}$

$|A| = 6 \Rightarrow$

$$1+1+1+1+1+1 = 1$$

$$1+1+1+1+2 = \frac{6*5*4*3}{4!} * {}^2C_2 = 15$$

$$1+1+1+3 = \frac{6*5*4}{3!} * {}^3C_3 = 20$$

$$1+1+2+2 = \frac{6*5}{2!} * \frac{{}^4C_2 * {}^2C_2}{2!} = 45$$

$$1+1+4 = \frac{6*5}{2!} * {}^4C_4 = 15$$

$$1+2+3 = 6 * {}^5C_2 * {}^3C_3 = 60$$

$$1+5 = 6 * {}^5C_5 = 6$$

$$2+2+2 = \frac{{}^6C_2 * {}^4C_2 * {}^2C_2}{3!} = 15$$

$$2+4 = {}^6C_2 * {}^4C_4 = 15$$

$$3+3 \Rightarrow \frac{{}^6C_3 * {}^3C_3}{2!} = 10$$

$$6 = 1$$

Total no. of Partitions = 203

* Find the number of partitions of "set A" when

(1) $|A| = 0 \Rightarrow 1$

(2) $|A| = 1 \Rightarrow 1$

(3) $|A| = 2 \Rightarrow 2$

(4) $|A| = 3 \Rightarrow 5$

(5) $|A| = 4 \Rightarrow 15$

(6) $|A| = 5 \Rightarrow 52$

(7) $|A| = 6 \Rightarrow 203$

NOTE

* If we know the Equivalence relation on set A, then we can obtain the Partition of set A w.r.t. given Equivalence relation.

Partition of the set can be obtained by defining a set that contains all distinct Equivalence classes w.r.t. given Equivalence relation eg:- let $A = \{1, 2, 3, 4, 5, 6\}$

and $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (4,6), (6,4)\}$ is an Equivalence relation on set A.

Find the Partition of set A, w.r.t. Equivalence relation R

$[1] = \{1, 2, 3\}$

$[2] = \{2, 1, 3\}$

$[3] = \{3, 1, 2\}$

$[4] = \{4, 6\}$

$[5] = \{5\}$

$[6] = \{6, 4\}$

$\therefore$ Partition of set A w.r.t. Equivalence relation R is

$\{\{1, 2, 3\}, \{4, 6\}, \{5\}\}$

NOTE

* If we know the partition of set A, then we can identify the Equivalence relation on set A corresponding to which the given partition of set A is created.

* If we perform the self-cross-product of all the subsets of set A present in the partition, and if we union the result of all those self-cross products, then we get the desired Equivalence relation

eg: Let $A = \{1, 2, 3, 4, 5\}$
and R is an Equivalence relation on set A.

$$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (4,5), (5,4)\}$$

We know partition of set A w.r.t. Equivalence relation R is $= \{\{1, 2\}, \{3\}, \{4, 5\}\}$

Equivalence Rel^n $R = \{1,2\} \times \{1,2\} \cup \{3\} \times \{3\} \cup \{4,5\} \times \{4,5\}$

$= \{(1,1),(1,2),(2,1),(2,2)\} \cup \{(3,3)\} \cup \{(4,4),(4,5),(5,4),(5,5)\}$

$= \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,1),(4,5),(5,4)\}$

eg:- let $A = \{1, 2, 3, 4, 5, 6\}$

and $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (4,6), (6,4)\}$ is an Equivalence relation on set A.

We know,

Partition of set A w.r.t. relation R is $= \{ \{1,2,3\}, \{4,6\}, \{5\} \}$

$$\{1,2,3\} \quad \{4,6\} \quad \{5\}$$
$$\downarrow \qquad\quad \downarrow \qquad\quad \downarrow$$

Equivalence relation $R = \{1,2,3\} \times \{1,2,3\} \cup \{4,6\} \times \{4,6\} \cup \{5\} \times \{5\}$

NOTE

* For Every partition of set A, there is unique Equivalence relation, and with respect to Every Equivalence relation on set A there is unique partition of set A.

{ i.e. there is a one-to-one correspondance between the set of all Equivalence relations on set A and the set of all partitions of set A. }

i.e. $\boxed{\text{Number of Equivalence Relation on set A} = \text{Number of Partitions of set A.}}$

Topic : Number of equivalence relation on a set

Number of Equivalence relation on set A = Number of Partitions of set A.

Slide

* Find the number of Equivalence relations on "set A" when

(1) $|A| = 0$, then number of Equivalence relation on set $A = 1$

(2) $|A| = 1$, then number of Equivalence relation on set $A = 1$

(3) $|A| = 2$, then number of Equivalence relation on set $A = 2$

(4) $|A| = 3$, then number of Equivalence relation on set $A = 5$

(5) $|A| = 4$, then number of Equivalence relation on set $A = 15$

(6) $|A| = 5$, then number of Equivalence relation on set $A = 52$

6, 203

Topic : Bell Number

Bell number B_n will represent the number of partions of a set A of size = n.

↳ i.e., B_n will represent the number of Equivalence relation on a set A of cardinality = n

Slide

Bell Triangle :-

$B_0 =$	1					
$B_1 =$	1	2				
$B_2 =$	2	3	5			
$B_3 =$	5	7	10	15		
$B_4 =$	15	20	27	37	52	
$B_5 =$	52	67	87	114	151	203
$B_6 =$	203					

Q: Let $A = \{a, b, c, d, e\}$ i.e. $|A| = 5$

How many equivalence relations are possible on set A, such that cardinality of each equivalence relation is exactly '9'

$|A| = 5 \Rightarrow 1 + 1 + 1 + 1 + 1$

$(1\times1) + (1\times1) + (1\times1) + (1\times1) + (1\times1) = 5$

$1 + 1 + 1 + 2$

$(1\times1) + (1\times1) + (1\times1)\ (2\times2) = 7$

$1 + 1 + 3$

$(1\times1) + (1\times1) + (3\times3) = 11$

$$\boxed{\begin{array}{c} 1 + 2 + 2 \\ (1\times1) + (2\times2) + (2\times2) = 9 \end{array}}$$

Equivalence w.r.t. such Partition will Contain Exactly '9' order Pairs

No. of such Partition $= {}^5C_1 * \dfrac{{}^4C_2 * {}^2C_2}{2!}$

$= 15$

$\Downarrow$

$$\boxed{\text{No of Equivalence Relh} = \text{No. of Partition} = 15}$$

$1 + 4$

$(1\times1) + (4\times4) = 17$

$2 + 3$

$(2\times2) + (3\times3) = 13$

5

$(5\times5) = 25$

Q: Let $A = \{a, b, c, d, e, f\}$ i.e. $|A| = 6$

* How many equivalence relations are possible on set A, such that cardinality of each equivalence relation is exactly '12'.

$|A| = 6 \Rightarrow$

$1+1+1+1+1+1$

$1+1+1+1+2$

$1+1+1+3$

$1+1+2+2$

$1+1+4$

$1+2+3$

$1+5$

$2+2+2$

$2+4$

$3+3$

6

What is the possible number of reflexive relations on a set of 5 elements?

(A) 2^{10}

(B) 2^{15}

(C) 2^{20}

(D) 2^{25}

For a set A, the power set of A is denoted by 2^A. If A =$\{5,\{6\},\{7\}\}$, which of the following options are TRUE?

(1) $\emptyset \in 2^A$ (2) $\emptyset \subseteq 2^A$

(3) $\{5, \{6\}\} \in 2^A$ (4) $\{5, \{6\}\} \subseteq 2^A$

A. (1) and (3) only

B. (2) and (3) only

C. (1), (2) and (3) only

D. (1), (2) and (4) only

Let R be the relation on the set of positive integers such that aRb if and only if a and b are distinct and have a common divisor other than 1. Which one of the following statements about R is True?

A. R is symmetric and reflexive but not transitive

B. R is reflexive but not symmetric and not transitive

C. R is transitive but not reflexive and not symmetric

D. R is symmetric but not reflexive and not transitive

The cardinality of the power set of $\{0,1,2\ldots,10\}$ is_____.

Suppose U is the Power set of the set $S = \{1, 2, 3, 4, 5, 6\}$. For the any $T \in U$, let $|T|$ denote the number of elements in T and T' denote the complement of T. For any T,R $\in U$, let $T \setminus R$ be the set of all elements in T which are not in R. Which one of the following is true?

(A) $\forall X \in U, (|X| = |X'|)$

(B) $\exists X \in U, \exists Y \in U, (|X| = 5, |Y| = 5 \text{ and } X \cap Y = \emptyset)$

(C) $\forall X \in U, \forall Y \in U, (|X| = 2, |Y| = 3 \text{ and } X \setminus Y = \emptyset)$

(D) $\forall X \in U, \forall Y \in U, (X \setminus Y = Y' \setminus X')$

Let R be a relation on the set of ordered pairs of positive integers such that $((p, q), (r, s)) \in R$ if and only if $p - s = q - r$. Which one of the following is true about R?

(A) Both reflexive and symmetric

(B) Reflexive but not symmetric

(C) Not reflexive but symmetric

(D) Neither reflexive nor symmetric

A binary relation R on $N \times N$ is defined as follows: $(a, b)\ R\ (c, d)$ if $a \leq c$ or $b \leq d$. Consider the following propositions:

P: R is reflexive

Q: R is transitive

Which one of the following statements is TRUE

A. Both P and Q are true

B. P is true and Q is false

C. P is false and Q is true

D. Both P and Q are false

Let G be an arbitrary group.

Consider the following relations on G:

R_1: $\forall a, b \in G$, aR_1b if and only if $\exists g \in G$ such that $a = g^{-1}bg$

R_2: $\forall a, b \in G$, aR_2b if and only if $a = b^{-1}$

Which of the above is/are equivalence relation/relations?

(A) R_1 and R_2

(B) R_1 only

(C) R_2 only

(D) Neither R_1 nor R_2

Let R be the set of all binary relations on the set $\{1,2,3\}$. Suppose a relation is chosen from R at random. The probability that the chosen relation is reflexive (round off to 3 decimal places) is

______.

A relation R is said to be circular if aRb and bRc together imply cRa. Which of the following options is/are correct?

(A) If a relation S is reflexive and symmetric, then S is an equivalence relation.

(B) If a relation S is circular and symmetric, then S is an equivalence relation.

(C) If a relation S is reflexive and circular, then S is an equivalence relation.

(D) If a relation S is transitive and circular, then S is an equivalence relation.

Let S be a set consisting of 10 elements. The number of tuples of the form (A, B) such that A and B are subsets of S, and $A \subseteq B$ is______.

A pennant is a sequence of numbers, each number being 1 or 2. An n-pennant is a sequence of numbers with sum equal to n. For example, (1,1,2) is a 4-pennant. The set of all possible 1-pennants is {(1)}, the set of all possible 2-pennants is {(2), (1,1)}and the set of all 3-pennants is {(2,1), (1,1,1), (1,2)}.

Note that the pennant (1,2) is not the same as the pennant (2,1).

The number of 10-pennants is______.

2 mins Summary

Topic Number of partitions of a set

Topic Number of equivalence relation

Topic Bell number

Topic Questions

Slide

THANK - YOU

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