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Revisiting the Accelerated Expansion of the Universe

Shane Ackerley and Dakota Wolfe

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ABSTRACT

This paper uses data from observations of type Ia supernovae to obtain estimates of the cosmological parameters $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$. These represent the proportions of energy density in the universe contributed by matter and dark energy. Best fit model gives values of $\Omega_{M,0} = 0.27$ and $\Omega_{\Lambda,0} = 0.82$. Explicitly imposing the flat universe constraint (in which the two parameters sum to unity) gives a best fit value of $\Omega_{M,0} = 0.37$. The results indicate that dark energy is the dominant energy density in the universe and that the cosmological constant Λ is nonzero and positive. This signifies an accelerating expansion of the universe.

1. INTRODUCTION

For over a century, modern cosmology has produced exciting and surprising results that reshape the way scientists view the world at the largest scales. In a pivotal 1999 paper, "Measurements of Ω and Λ from 42 High-Redshift Supernovae," researchers presented results that implied the universe is expanding at an accelerating rate (Perlmutter et al. (1999)). This paper sets out to reproduce those results.

The starting point for the theory is the cosmological principle. It postulates that there is no privileged vantage point; the universe is homogeneous and isotropic on very large scales. This allows us to use Einstein's equations of general relativity to describe the relationship between the energy content of the universe and its geometry. The next assumption we make is that the universe is flat. This means that the whole of space-time is neither open nor closed, but is infinite and exhibits net zero curvature. This assumption is supported by the inflationary model in which small fluctuations of curvature were flattened due to a burst of expansion in the early universe.

Section 2 describes and derives some important parameters that are necessary for analysing apparent magnitude and redshift data from supernovae in the context of cosmic expansion. Section 3 displays the data in figures and outlines the methods used to obtain these results. In Section 4, the results are compared with those stated in Perlmutter et al. (1999) and are placed in context with the theoretical framework.

2. THEORY

2.1. *Cosmological parameters*

The rate of expansion of the universe depends on the energy content of the cosmos. Let us start with the Friedmann equation, which comes from the 00 component of the Einstein equation of general relativity:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a^2(t)} + \frac{\Lambda c^2}{3} \tag{1}$$

where $\kappa = -1, 0, 1$ gives the curvature of the universe, R_0 is the current radius of curvature, and Λ is the cosmological constant. Because we are working with a flat universe, we can set $\kappa = 0$ (zero curvature) so that the middle term vanishes. We can also make a substitution of the energy associated with the cosmological constant, $\varepsilon_\Lambda(t) = \Lambda c^4/8\pi G$, which, conveniently, can be absorbed into the total energy density $\varepsilon(t)$. Finally, we introduce the Hubble parameter, $H_0 \equiv \dot{a}/a$. With this, Equation (1) reduces to:

$$H^2(t) = \frac{8\pi G}{3c^2}\varepsilon(t)$$

This gives us what is called the critical energy density, that is, the energy density associated with a flat universe:

$$\varepsilon_c(t) = \frac{3c^2 H^2(t)}{8\pi G} \tag{2}$$

At this point, the total energy density can be split up into $\varepsilon(t) = \varepsilon_M(t) + \varepsilon_R(t) + \varepsilon_\Lambda(t)$, the sum of contributions from matter (M), radiation (R), and dark energy (Λ). We can now define the normalized energy density:

$$\Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)} \tag{3}$$
$$\Omega(t) = \frac{\varepsilon_M(t)}{\varepsilon_c(t)} + \frac{\varepsilon_R(t)}{\varepsilon_c(t)} + \frac{\varepsilon_\Lambda(t)}{\varepsilon_c(t)}$$
$$\Omega(t) = \Omega_M(t) + \Omega_R(t) + \Omega_\Lambda(t)$$

Note that in the flat universe, $\Omega(t) \equiv 1$. In our current epoch, the radiation energy parameter, $\Omega_{R,0}$, is on the order of 10^{-4} (Basu (2024)). What we wish to estimate in this paper are the current values of the two remaining parameters, $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$. Thus, we make the following approximation for the remainder of this paper: $\Omega_0 \simeq \Omega_{M,0} + \Omega_{\Lambda,0} = 1$.

2.2. *Luminosity distance*

It is useful to define something called luminosity distance in terms of the redshift and the normalized energy density parameters. We begin with the following definition:

$$r(t) = a(t)r_0$$

where $r(t)$ is the proper distance between two objects, and $a(t)$ is a dimensionless scale factor which accounts for the expansion of space at a given time t. Note that $r(t = t_0) = t_0$ is the comoving radius in our current epoch. Again taking the Friedmann equation, Equation (1), we can substitute our critical energy density (Equation (2)), and our normalized energy density (Equation (3)):

$$H^2(t) = \Omega(t)H^2(t) - \frac{\kappa c^2}{R_0^2 a^2(t)} \tag{4}$$

For our present epoch, $\Omega(t_0) = \Omega_0$, $\varepsilon_c(t) = \varepsilon_{c,0}$, $H(t) = H_0$, and $a(t_0) = 1$. Equation (4) becomes:

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1)$$

Substituting this back into Equation (4) and simplifying, we get the following expression:

$$\frac{H^2(t)}{H_0^2} = \frac{\varepsilon(t)}{\varepsilon_{c,0}} + \frac{1 - \Omega_0}{a^2(t)} \tag{5}$$

We now note that the individual energy densities scale with the expansion of the universe like so (Basu (2024)):

$$\varepsilon(t) = \frac{\varepsilon_{M,0}}{a^3(t)} + \frac{\varepsilon_{R,0}}{a^4(t)} + \varepsilon_{\Lambda,0}$$

Substituting this into Equation (5) gives:

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_{R,0}}{a^4(t)} + \frac{\Omega_{M,0}}{a^3(t)} + \Omega_{\Lambda,0} + \frac{1 - \Omega_0}{a^2(t)} \tag{6}$$

At this point we want to incorporate the redshift (something that is directly measurable) into our definitions. This can be done by noting the relationship $a = 1/(1 + z)$. Substituting this into Equation (6), and rearranging for $H(t)$, we get:

$$H(t) = H_0\sqrt{\Omega_{R,0}(1 + z)^4 + \Omega_{M,0}(1 + z)^3 + \Omega_{\Lambda,0} + (1 - \Omega_0)(1 + z)^2} \tag{7}$$

Let us now write $H(t) = H_0F(z)$. For the flat universe, $\Omega_0 = 1$, and with negligible $\Omega_{R,0}$, this greatly simplifies:

$$F(z) = \sqrt{\Omega_{M,0}(1 + z)^3 + \Omega_{\Lambda,0}} \tag{8}$$

Meanwhile Basu (2024) states that, for a flat universe, the Friedmann-Lemaître-Robertson-Walker metric gives the following formula for the comoving radius:

$$
\begin{aligned}
r_0 &= \int_{t_e}^{t_0} \frac{cdt'}{a(t')} \\
r_0 &= c\int_{t_e}^{t_0} \frac{dt'}{a(t')}\frac{da}{da} \\
r_0 &= c\int_{t_e}^{t_0} \frac{da}{a(t')}\frac{1}{da/dt'}
\end{aligned}
$$

But we know that $a = 1/(1+z)$ and so $da = -a^2dz$. From our previous definition of the Hubble parameter, $da/dt = aH(t) = aH_0F(z)$. When changing integration variables from t' to z', we must realize that for the time of emission $t' = t_e$, the redshift is what we observe i.e. $z' = z$. Meanwhile the time of observation $t' = t_0$ corresponds to no redshift, so $z' = 0$. Substituting all this into our integral:

$$
\begin{aligned}
r_0 &= c\int_{z}^{0} \frac{-a^2dz'}{a^2}\frac{1}{H_0F(z')} \\
r_0 &= \frac{c}{H_0}\int_{0}^{z} \frac{dz'}{F(z')} \qquad (9)
\end{aligned}
$$

This is the comoving distance in integral form, and it can be related to the luminosity distance by considering how the redshift affects the luminosity we observe. In particular, the energy that reaches us is related to the the energy emitted by a distant object by $dE_0 = dE_e/(1+z)$. The time for the photons to reach us is also affected by the expansion, so that $dt_0 = dt_e(1+z)$. Since luminosity is the rate of change of the energy with respect to time, the luminosity we observe must be related to the luminosity that was emitted as follows:

$$
L_0 = \frac{dE_0}{dt_0} = \frac{dE_e/(1+z)}{dt_e(1+z)} = \frac{L_e}{(1+z)^2}
$$

We are now prepared to define the luminosity distance as the distance we expect if the flux we observe corresponds to the luminosity that was emitted by that object. Thus,

$$
\begin{aligned}
\frac{L_e}{4\pi d_L^2} =& f = \frac{L_0}{4\pi r^2} \\
\frac{L_e}{d_L^2} &= \frac{L_0}{r^2} \\
d_L^2 &= r^2\frac{L_0}{L_e} \\
d_L &= r(1+z)
\end{aligned}
$$

Given our expression in Equation (9), this means our luminosity distance can be written as:

$$
d_L = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_{M,0}(1+z')^3 + \Omega_{\Lambda,0}}} \qquad (10)
$$

2.3. *Apparent magnitude*

The data we work with in Section 3 involves measurements of the apparent magnitude of supernovae in the B-band. This is denoted by m_B and can be written as a function of the redshift, z, which is also measured. Starting with the distance modulus in terms of luminosity distance:

$$
m_B(z) = M + 5\log\left(\frac{d_L}{10 \text{ pc}}\right)
$$

Let us introduce a new parameter $\mathscr{D}_L \equiv H_0 d_L$ such that:

$$m_B(z) = M + 5\log\left(\frac{\mathscr{D}_L}{10\ \text{pc}\cdot H_0}\right)$$
$$m_B(z) = M + 5\log\mathscr{D}_L - 5\log(10^{-5}\ \text{Mpc}) - 5\log H_0$$
$$m_B(z) = M + 5\log\mathscr{D}_L + 25 - 5\log H_0$$

We now use the modified absolute magnitude, denoted by $\mathscr{M}_B$. In Perlmutter et al. (1997), this is called the magnitude zero point, and it is defined as $\mathscr{M}_B \equiv M - 5\log H_0 + 25$ where H_0 is in units km s^{-1} Mpc^{-1}. Using this definition, we find that:

$$m_B(z) = \mathscr{M}_B + 5\log\mathscr{D}_L(z;\Omega_{M,0},\Omega_{\Lambda,0}) \tag{11}$$

3. RESULTS

To estimate the cosmological parameters for our current epoch, $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$, we fit the data from observations of supernovae listed in Tables 1 and 2 from Perlmutter et al. (1999) to the expression for $m_B(z)$ given in Equation (11). To begin, we require an estimate of the modified absolute magnitude, $\mathscr{M}_B$.

3.1. *Modified absolute magnitude*

The modified absolute magnitude (i.e. the magnitude zero point, since it is the intercept of the magnitude) can be estimated by utilizing Equation (11) in the low-redshift limit:

$$m_B(z) = \mathscr{M}_B + 5\log cz \tag{12}$$

This simplification is made by noting that the integrand of Equation (10) goes to unity in the limit of $z \to 0$. Meanwhile the $(1+z)$ factor goes to 1, so that all that is left in $\mathscr{D}_L$ is the speed of light times the redshift.

The low-redshift data comes from Table 2 of Perlmutter et al. (1999). We obtain a set of points, $(z, m_{B,corr})$, for each supernova observed. To fit this data to Equation (12), the least squares method was used via the `curve_fit` function from `scipy`. The data points, as well as the fitted curve, are plotted in Figure 1. The estimated modified absolute magnitude is $\mathscr{M}_B = -3.31$. This is what is used to estimate the cosmological parameters in the following section.

3.2. *Estimate of cosmological parameters*

The cosmological parameters are the components of the total normalized energy density, Ω as given in Section 3.2. Using supernova observations and Equation (11), we can make an estimate of the current energy densities associated with matter and dark energy, $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$. Now using a combination of low-redshift and high-redshift data, given by Tables 1 and 2 in Perlmutter et al. (1999). Again using the `scipy` function `curve_fit` with the least squares method, we get estimates of the cosmological parameters $\Omega_{M,0} = 0.27$ and $\Omega_{\Lambda,0} = 0.82$. Note that, although Equation (11) was derived using the flat universe constraint $\Omega_{M,0} + \Omega_{\Lambda,0} = 1$, this clearly has not been strongly imposed given the values we get.

3.3. *Flat universe constraint*

To more explicitly enforce the flat universe constraint, the replacement $\Omega_{\Lambda,0} = 1 - \Omega_{M,0}$ is made in Equation (11). The resulting expression for $m_B(z)$ can be fit to the data in Tables 1 and 2 of Perlmutter et al. (1999) using `curve_fit` to estimate only the parameter $\Omega_{M,0}$. This is presumably more accurate, since Equation (11) was derived using the flat universe assumption, and now the `curve_fit` function only needs to calculate one best fit parameter instead of two. The result is an estimate of the mass energy density of $\Omega_{M,0} = 0.37$. The dark energy density, meanwhile, is calculated simply as $\Omega_{\Lambda,0} = 1 - \Omega_{M,0} = 0.63$. To further examine the flat universe model, we compare the above results with other possible values of the cosmological parameters in Figure 2. The plot contains the data from Perlmutter et al. (1999), with corresponding uncertainties. It also contains the $m_B(z)$ curve for our estimated cosmological parameters in a flat universe $(\Omega_{M,0}, \Omega_{\Lambda,0}) = (0.37, 0.63)$, along with three other cases: a matter dominated universe, $(\Omega_{M,0}, \Omega_{\Lambda,0}) = (1, 0)$; equal parts matter and dark energy, $(\Omega_{M,0}, \Omega_{\Lambda,0}) = (0.5, 0.5)$; and a dark energy dominated universe, $(\Omega_{M,0}, \Omega_{\Lambda,0}) = (0, 1)$.

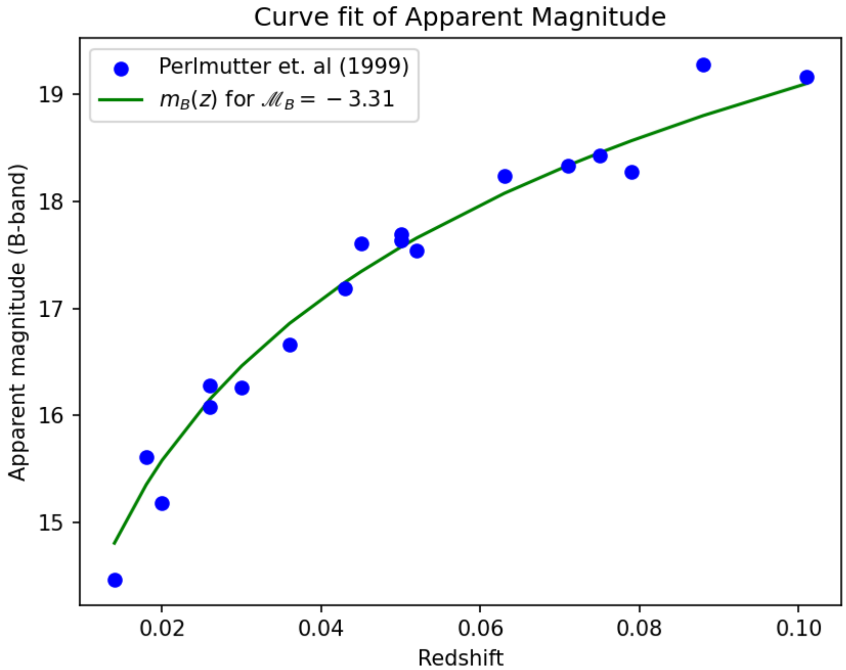

Figure 1. Curve fit of low-redshift supernova data from Table 2 of Perlmutter et al. (1999).

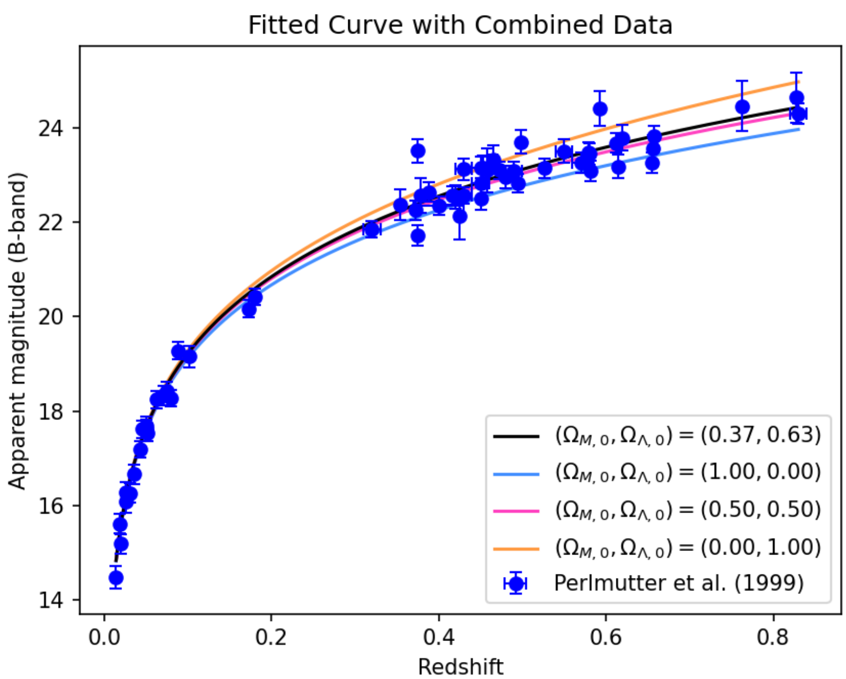

Figure 2. Reproduction of Fig. 2 from Perlmutter et al. (1999). Data points and uncertainties taken from Tables 1 and 2.

4. DISCUSSION

4.1. *Comparison with literature*

Perlmutter et al. (1997) refers to the intercept of Equation (11) as the magnitude zero point, denoted by $\mathscr{M}_B$. We call this the modified absolute magnitude, and in Section 3.1 give an estimate of $\mathscr{M}_B = -3.31$. This is consistent with the value given in Perlmutter et al. (1997) of $\mathscr{M}_{B,corr} = -3.32 \pm 0.05$.

Before explicitly enforcing the flat universe constraint, the least squares curve fit method gave cosmological parameters of $\Omega_{M,0} = 0.27$ and $\Omega_{\Lambda,0} = 0.82$. The primary fit given in Perlmutter et al. (1999) has best values of $\Omega_{M,0} = 0.73$ and $\Omega_{\Lambda,0} = 1.32$, which differ greatly from ours. Apart from the simple fact that we have used different methods for fitting the data, the cited paper also excludes two supernovae that were deemed anomalous (likely due to reddening). Our fit, on the other hand, used all of the supernova data.

After explicitly enforcing the flat universe constraint, the least squares curve fit method gave cosmological parameters of $\Omega_{M,0} = 0.37$ and $\Omega_{\Lambda,0} = 0.63$. This is just barely within the uncertainty of the value $\Omega^{\text{flat}}_{M,0} = 0.28^{+0.09}_{-0.08}$ given in Perlmutter et al. (1999). Notably, however, our result for $\Omega_{M,0}$ before explicitly imposing the flat universe constraint is closer to that cited value, though the sum did not satisfy the flat universe model—we interpret this to be a coincidence. Perlmutter et al. (1999) does, however, give several fit results in Table 3, ranging from $\Omega_{M,0} = 0.24^{+0.09}_{-0.08}$ to $\Omega_{M,0} = 0.35^{+0.12}_{-0.10}$.

4.2. *Conceptual implications*

The curves in Figure 2 imply that if more of the energy content of the universe is contributed by dark energy—and thus less is contributed by matter—we expect to observe greater magnitude (dimmer) supernovae at the same redshift (distance). At first thought, one might expect this to be due to the fact that dark energy is not associated with the emission of radiation, while matter is. However, the contribution of matter to the energy content of the universe is dominated by dark matter, which does not interact with light. So the relationship between supernova apparent magnitude and distribution of energy content may be more accurately explained by the fact that a scarcity of dark matter (and an abundance of dark energy!) would not allow galaxies to endure gravitationally, preventing stellar evolution and, ultimately, the supernova events we observe.

The more significant interpretation of our results is that our flat universe value for the current energy density due to matter, $\Omega_{M,0} = 0.37$, is less than half of the total energy energy density. The rest is essentially made up by contributions from dark energy, $\Omega_{\Lambda,0} = 0.63$. In Section 3.2, we noted the energy density associated with the cosmological constant $\varepsilon_\Lambda(t) = \Lambda c^4/8\pi G$, as well as the critical energy density for the flat universe $\varepsilon_c(t) = 3c^2 H^2(t)/8\pi G$. So we have direct proportionality between our dark energy density parameter and the cosmological constant as $\Omega_{\Lambda,0} = \varepsilon_{\Lambda,0}/\varepsilon_{c,0} = \Lambda c^2/3H_0$. The current accepted value for the Hubble constant, H_0, is positive and nonzero. Thus we conclude that our result—a positive, nonzero dark energy parameter—directly implies a positive, nonzero cosmological constant, which corresponds to an expanding universe. Furthermore, since dark energy increases with expansion while the amount of matter remains constant, the universe is expected to be expanding at an accelerating rate. These are the same conclusions that were drawn in the 1999 paper, albeit from slightly different values for the energy density.

All of our discussions assume the cosmological principle and a flat universe. The derivations of the equations also depend crucially on these assumptions. The consequence of this dependence is that we can only discussed the implications of our results in the context of the best theory we have. Some mysteries remain, such as the source of dark energy.

5. CONCLUSION

The exercises carried out in this paper are considered to have been successful. Taking the cosmological principle and the assumption of a flat universe, an expression for the apparent magnitude in the B-band as a function of the redshift was derived. With $m_B(z)$ written in terms of the cosmological parameters $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$, the supernova data listed in Perlmutter et al. (1999) could be used to obtain best fit values. The resulting energy densities are $(\Omega_{M,0}, \Omega_{\Lambda,0}) = (0.27, 0.82)$. More strictly imposing the flat universe constraint gives $\Omega_{M,0} = 0.37$ and consequently $\Omega_{\Lambda,0} = 0.63$. In either case, the energy density attributable to dark energy is dominant. A positive, nonzero value for this parameter means the cosmological constant, Λ, is also positive and nonzero. Thus the universe is expanding at an accelerating rate.

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