Test 1:

Planck function: $B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}, \quad B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$

with units erg $s^{-1}Hz^{-1}cm^{-2}sr^{-1}$ and erg $s^{-1}cm^{-1}cm^{-2}sr^{-1}$, respectively. We also have $h = 6.63 \times 10^{-27}$erg s, $c = 3.00 \times 10^{10}$cm s^{-1}, $k = 1.38 \times 10^{-16}$ erg K^{-1}.

Note that $B_\nu(T) = I_\nu(T)$ at thermal equilibrium (when collisions dominate).

Rayleigh-Jeans law: $B_\lambda(T) \simeq \frac{2ckT}{\lambda^4}$, obtained when $hc \ll \lambda kT$

Wien's displacement law: $\lambda_{\max} = \frac{0.29}{T}$ cm

Stefan-Boltzmann law: $F = \sigma T^4, \quad \sigma = \frac{2\pi^5 k^4}{15h^3c^2} = 5.67$ erg $s^{-1}cm^{-2}K^{-4}$.

Also note that $F = \pi \int_o^\infty B_\lambda(T) d\lambda$

Luminosity: $L_* = 4\pi R_*^2 \sigma T_{\rm eff}^4$

Magnitude: $m_1 - m_2 = 2.5\log\left(\frac{F_1}{F_2}\right)$

Distance modulus: $m - M = 5\log\left(\frac{d}{10\text{ pc}}\right)$

Energy of hydrogen electron: $E_n = 13.6(1 - 1/n^2)$

Degeneracy for hydrogen: $g_n = 2n^2$

Boltzmann equation: $\frac{n_i}{n_j} = \frac{g_i}{g_j}e^{-(E_i-E_j)/kT}, \quad \frac{n_i}{n_{\rm ion}} = \frac{g_i}{Z}e^{-E_i/kT}$

Partition function: $Z = \sum_i g_i e^{-E_i/kT}$

Saha equation: $\frac{n_{i+1}n_{\rm e}}{n_i} = \left(\frac{2\pi m_{\rm e} kT}{h^2}\right)^{3/2}\frac{2Z_{i+1}}{Z_i}e^{-\chi_i/kT}$

where $m_{\rm e} = 9.109 \times 10^{-28}$, and χ_i is the ionization potential.

Fractional ionization: $f_i = \frac{n_i}{n_1+n_2+n_3+\dots} = \frac{\left(\frac{n_i}{n_{i-1}}\right)\left(\frac{n_{i-1}}{n_{i-2}}\right)\cdots\left(\frac{n_2}{n_1}\right)}{1+\left(\frac{n_2}{n_1}\right)+\left(\frac{n_3}{n_2}\right)\left(\frac{n_2}{n_1}\right)+\left(\frac{n_4}{n_3}\right)\left(\frac{n_3}{n_2}\right)\left(\frac{n_2}{n_1}\right)+\cdots}$

Mean molecular weight: $\mu = \frac{1}{n_{\rm tot}m_{\rm H}}\sum_i n_i m_i$

Maxwellian velocity distribution: $N(v) = N4\pi v^2\left(\frac{m}{2\pi kT}\right)^{3/2}e^{-mv^2/2kT}$

Hydrostatic equilibrium: $\frac{dP(r)}{dr} = -\rho(r)g(r)$

Universal law of graviation: $g(r) = \frac{GM(r)}{r^2}, \quad G = 6.67 \times 10^{-8}$ dyn cm^2 g^{-2}

Gravitational potetial energy of sphere: $\Omega = -\int_0^{M_*}\frac{GM(r)}{r}dM$

For a sphere: $V = \frac{4}{3}\pi r^3, \quad A_{\rm surface} = 4\pi r^2$

Equipartition theorem: $K = \frac{3}{2}kT$

Ideal gas law: $P(r) = n(r)kT = \frac{\rho(r)}{\mu m_{\rm H}}kT$

Column mass: $m(r) \equiv M(r)/dA$, where $M(r) = dA\int\rho(r)dr$

Pressure of an ideal gas: $P = \frac{2}{3}\varepsilon$ for energy per unit volume $\varepsilon = \frac{NK}{V} = \frac{3}{2}nkT$.

Virial theorem: $2U + \Omega = 0$ for a system in thermal equilibrium.

Jeans mass: $M_{\mathrm{J}} = \left(\frac{kT}{\mu m_{\mathrm{H}} G}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2}$

Jeans length: $\lambda_{\mathrm{J}} = \left(\frac{3M_{\mathrm{J}}}{4\pi\rho}\right)^{1/3}$

Free-fall time: $t_{\mathrm{ff}} = \sqrt{\frac{3\pi}{32G\rho}}$

Test 2:

Specific intensity change: $dI_\nu = -(\alpha_\nu + \varsigma_\nu) I_\nu ds$

where α_ν and ς_ν are the absorption and scattering coefficients, respectively.

Specific intensity profile: $I_\nu = I_\nu(0) e^{-(\alpha_\nu+\varsigma_\nu)s}$ for constant extinction and no emissivity.

Note I_ν has units of erg $\mathrm{s^{-1}Hz^{-1}cm^{-2}sr^{-1}}$.

Mean free path as the length-scale: $\ell_\nu = (\alpha_\nu + \varsigma_\nu)^{-1}$

Probability of absorption: $\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \varsigma_\nu}$

Mean displacement: $\ell_\nu^{*2} = \frac{\ell_\nu}{\sqrt{\epsilon_\nu}} = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \varsigma_\nu)}}$

Opacity: $k_\nu = \frac{1}{\rho}(\alpha_\nu + \varsigma_\nu) = \kappa_\nu + \sigma_\nu$ $[\mathrm{cm^2g^{-1}}]$

Mean intensity: $J_\nu(z) = \frac{1}{4\pi}\int I_\nu(z,\theta)\, d\Omega = \frac{1}{2}\int_{-1}^{1} I_\nu(z,u)\, du$ where $u = \cos(\theta)$.

Net flux: $F_\nu(z) = \int I_\nu(z,\theta)(\mathbf{n}\cdot\mathbf{e}_z)\, d\Omega = 2\pi\int_{-1}^{1} I_\nu(z,u) u du$ $[\mathrm{erg\ s^{-1}Hz^{-1}cm^{-2}}]$

Radiative moments: $M_\nu(z,n) = \frac{1}{2}\int_{-1}^{1} I_\nu(z,u) u^n du,$

where $M_\nu(z,0) = J_\nu(z),\ M_\nu(z,1) = H_\nu(z) = F\nu(z)/4\pi,$ and $M_\nu(z,2) = K_\nu(z)$.

Eddington's approximation: $J_\nu(z) = 3K_\nu(z)$

Eddington flux: $H_\nu(\tau_\nu) \simeq -\frac{1}{3k_\nu\rho}\frac{dB_\nu(T)}{dT}\frac{dT}{dz}$

Total Eddington flux: $H(z) = -\frac{1}{3k_{\mathrm{R}}\rho}\frac{dB(T)}{dT}\frac{dT}{dz}$

Rosseland mean opacity: $\frac{1}{k_{\mathrm{R}}} = \frac{\pi}{4\sigma T^3}\int_0^\infty \frac{1}{k_\nu}\frac{dB_\nu(T)}{dT} d\nu$

Radiative transfer equation: $\frac{1}{\rho}\frac{dI_\nu}{ds} = -k_\nu I_\nu + j_\nu,$

where ρ is mass density and j_ν is the emissivity in units of erg $\mathrm{s^{-1}Hz^{-1}sr^{-1}g^{-1}}$.

Radiative transfer in terms of optical depth: $u\frac{dI_\nu(\tau_\nu,u)}{d\tau_\nu} = I_\nu(\tau_\nu,u) - S_\nu(\tau_\nu),$

for optical depth $d\tau_\nu \equiv -k_\nu\rho dz$ and source function $S_\nu(\tau_\nu) \equiv \frac{j_\nu}{k_\nu}$.

General solution: $I_\nu(\tau_\nu,u) = I_\nu(0,u)e^{\tau_\nu/u} - \frac{e^{\tau_\nu/u}}{u}\int_0^{\tau_\nu} S(\lambda)e^{-\lambda/u} d\lambda$

For constant source: $I_\nu(\tau_\nu,u) = I_\nu(0,u)e^{\tau_\nu/u} + S_\nu\left(1 - e^{\tau_\nu/u}\right)$

Optically thick / LTE conditions: $I_\nu(|\tau_\nu| > 1, u) \approx S_\nu$

Optically thin / non-LTE conditions: $I_\nu(|\tau_\nu| \ll 1, u) \simeq I_\nu(0,u) + S_\nu\frac{|\tau_\nu|}{u}$

Grey atmosphere temperature profile: $T(\tau) = T_{\text{eff}}\left[\frac{3}{4}\left(\tau + \frac{2}{3}\right)\right]$

Heisenberg inequality: $\Delta E \Delta t \geq \frac{\hbar}{2}, \quad \Delta p_x \Delta x \geq \frac{\hbar}{2}$

Lorentz profile for natural broadening: $\varphi_\nu = \frac{\Gamma}{4\pi^2}\frac{1}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2},$

where $\Gamma/2\pi$ is the full-width-half-magnitude of the curve.

Opacity redefined: $k_\nu = \alpha(\nu)/\rho,$ where now $\alpha(\nu) = \frac{\pi e^2}{mc^2} f_{ij}\varphi_\nu$

Radial velocity: $n(v_z) = n\sqrt{\frac{m}{2\pi kT}}e^{-mv_z^2/2kT}$

Non-relativistic doppler effect: $\nu' = \nu\left(1 - \frac{U}{c}\right), \quad \frac{\Delta\nu}{\nu} = \frac{\Delta V}{c}$

Voigt profile: $\varphi_\nu = \frac{\mathsf{U}(a,v)}{\Delta v_D}$

Final:

Equation of hydrostatic equilibrium: $\rho\frac{d^2r}{dt^2} = -\frac{\rho(r)GM(r)}{r^2} - \frac{dP}{dr}$

Equation of mass conservation: $\frac{dM(r)}{dr} = 4\pi r^2\rho(r)$

Energy-transport equation: $\frac{dT(r)}{dr} = -\frac{3k_{\text{R}}\rho}{64\pi r^2\sigma T^3}L(r)$

Equation of energy conservation: $\frac{dL(r)}{dr} = 4\pi r^2\rho(r)\varepsilon(r),$

where $\varepsilon(r)$ is the rate of thermonuclear energy generated per unit mass in erg s^{-1} g^{-1}.

Temperature gradient: $\nabla \equiv \frac{d\ln(T)}{d\ln(P)}$

For pure radiative transport and hydrostatic equilibrium: $\nabla_{\text{rad}} = \frac{3k_{\text{R}}}{64\pi r^2 g}\frac{P}{\sigma T^4}L(r).$

Note that $\frac{dy}{dx} = y\frac{d\ln y}{dx}.$

Monochromatic Eddington flux: $H_\nu(r) = \frac{k^3}{8h^2c^2}\frac{k_{\text{R}}}{k_\nu}\frac{T_{\text{eff}}^4}{T}\left(\frac{R}{r}\right)^2\mathcal{P}(u),$ with $\mathcal{P}(u) = \frac{u^4e^u}{(e^u-1)^2}, \quad u = \frac{h\nu}{kT}$

Volume of a cell in phase space: $\Delta x\Delta y\Delta z\Delta p_x\Delta p_y\Delta p_z = h^3$

Non-degenerate: $n_e(p) \leq \frac{8\pi p^2 dp}{h^3}$ **Degenerate:** $n_e(p) \gtrsim \frac{8\pi p^2 dp}{h^3}$

Conductive Eddington flux: $H_{\text{cond}} = \frac{4\sigma T^3}{3\pi k_{\text{cond}}\rho}\frac{dT}{dr}$

Radiative Eddington flux: $H_{\text{rad}} = \frac{4\sigma T^3}{3\pi k_{\text{rad}}\rho}\frac{dT}{dr}$

Total Eddington flux: $H_{\text{tot}} = H_{\text{rad}} + H_{\text{cond}} = \frac{4\sigma T^3}{3\pi k_{\text{tot}}\rho}\frac{dT}{dr},$ where $\frac{1}{k_{\text{tot}}} = \frac{1}{k_{\text{rad}}} + \frac{1}{k_{\text{cond}}}$

Bouyant force: $ma = (\rho_{\text{f}} - \rho_{\text{c}})g\Delta V$

Condition for convection: $\left(\frac{d\rho}{dr}\right)_{\text{adi}} < \left(\frac{d\rho}{dr}\right)_{\text{rad}}$

Energy-transport in convective equilibrium: $\frac{dT}{dr} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho g T}{P}$

Schwarzchild criterion for convection: $\frac{3k_{\text{R}}}{64\pi r^2 g}\frac{P}{\sigma T^4}L(r) > \frac{\gamma-1}{\gamma},$ where $\gamma = c_{\text{P}}/c_{\text{V}}$

Poisson equation: $\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho(r)\frac{dP(r)}{dr}}\right] = -4\pi G\rho(r)$

Polytropic equation of state: $P = K\rho^{(n+1)/n}$

Lane-Emden equation: $\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\frac{d\theta(\xi)}{d\xi} + \theta^n(\xi) = 0$

Equation of state for an ideal gas: $P = P_{\rm ions} + P_{\rm e} = (n_{\rm ions} + n_{\rm e})kT = \left(\frac{1}{\mu_{\rm ions}} + \frac{1}{\mu_{\rm e}}\right)\frac{\rho kT}{m_{\rm H}}$

Mass fraction of a species: $X_i = \frac{\rho_i}{\rho} = \frac{n_i m_i}{\sum_j n_j m_j}$

Total mean molecular weight: $\frac{1}{\mu} = \frac{1}{\mu_{\rm ions}} + \frac{1}{\mu_{\rm e}}$,

where $\frac{1}{\mu_{\rm ions}} = \sum_i \frac{X_i}{A_i}$ and $\frac{1}{\mu_{\rm e}} = \sum_i \frac{z_i X_i}{A_i} = \frac{n_{\rm e}}{\rho/m_{\rm H}} = \frac{\text{number of free electrons}}{\text{number of nucleons}}$

For full ionization: $\mu \simeq \frac{2}{3X + Y/2 + 1}$,

For neutral: $\mu \simeq \frac{1}{X + Y/4}$

with mass fractions X and Y for hydrogen and helium, respectively.

Equation of state for degenerate gas: $P_{\rm e} \propto n_{\rm e}^{\gamma} \propto \left(\frac{\rho}{\mu_{\rm e}}\right)^{\gamma}$

Radiation pressure: $P_{\rm rad} = \frac{4\pi}{c}K_\nu$

If radiation pressure dominates over gas pressure: $P_{\rm rad} \approx \frac{4\sigma T^4}{3c}$

Atomic species: ${}^A_Z X$, $A =$ atomic mass, $Z =$ atomic number, $X =$ element symbol

Proton-proton chains:

$$\begin{array}{lrcl}
\text{PPI:} & {}^1\text{H} + {}^1\text{H} & \rightarrow & {}^2\text{H} + \beta^+ + \nu_{\rm e} \\
 & {}^2\text{H} + {}^1\text{H} & \rightarrow & {}^3\text{He} + \gamma \\
 & {}^3\text{He} + {}^2\text{He} & \rightarrow & {}^4\text{He} + {}^1\text{H} + {}^1\text{H} \\
 & & & \\
\text{Total:} & {}^4\text{H} & \rightarrow & {}^4\text{He} + {}^1\text{H} + {}^1\text{H}
\end{array}$$

Energy generated: $Q = [M_{\rm initial} - M_{\rm final}]c^2 = [4M({}^1\text{H}) - M({}^4\text{He})]c^2 = 26.73$ MeV

Energy conversion rate: $\varepsilon = \frac{Q}{M({}^A X)c^2}$

Energy produced through fusion: $E_{\rm ms} \sim L_* t_{\rm ms}$

Time on main sequence: $t_{\rm ms} \sim \varepsilon\frac{M_* c^2}{L_*}$