Midterm:

$\boldsymbol{\mu} = \frac{q}{2mc}\mathbf{L} = \frac{gq}{2mc}\mathbf{S}$

$F_z = \boldsymbol{\mu}\cdot\frac{\partial \mathbf{B}}{\partial z} \simeq \mu_z \frac{\partial B_z}{\partial Z}$

$|\psi\rangle = |+\mathbf{z}\rangle\langle +\mathbf{z}|\psi\rangle + |-\mathbf{z}\rangle\langle -\mathbf{z}|\psi\rangle$

$|\psi\rangle = c_+|+\mathbf{z}\rangle + c_-|-\mathbf{z}\rangle, \quad \langle\psi| = c_+^*\langle +\mathbf{z}| + c_-^*\langle -\mathbf{z}|$

$|\langle \pm\mathbf{z}|\psi\rangle|^2 \equiv \langle \pm\mathbf{z}|\psi\rangle^*\langle \pm\mathbf{z}|\psi\rangle$

$|\pm\mathbf{x}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle \pm \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle$

$|\pm\mathbf{y}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle \pm \frac{i}{\sqrt{2}}|-\mathbf{z}\rangle$

$\langle S_z\rangle = c_+^*c_+\left(\frac{\hbar}{2}\right) + c_-^*c_-\left(-\frac{\hbar}{2}\right)$

$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2}$

$|\psi\rangle \xrightarrow[S_z\,\text{basis}]{} \begin{pmatrix}\langle +\mathbf{z}|\psi\rangle\\ \langle -\mathbf{z}|\psi\rangle\end{pmatrix} = \begin{pmatrix}c_+\\ c_-\end{pmatrix}$

$\langle\psi| \xrightarrow[S_z\,\text{basis}]{} (\langle\psi|+\mathbf{z}\rangle,\ \langle\psi|-\mathbf{z}\rangle)$

$\quad = (\langle +\mathbf{z}|\psi\rangle^*,\ \langle -\mathbf{z}|\psi\rangle^*) = (c_+^*,\ c_-^*)$

$|+\mathbf{x}\rangle = \hat{R}(\frac{\pi}{2}\mathbf{j})|+\mathbf{z}\rangle, \quad \langle +\mathbf{x}| = \langle +\mathbf{z}|\hat{R}^\dagger(\frac{\pi}{2}\mathbf{j})$

$\langle i|\hat{A}|j\rangle^* = \langle j|\hat{A}^\dagger|i\rangle$

$\hat{A}^\dagger_{ij} = A^*_{ji}$

$(\hat{A}\hat{B})^\dagger = \hat{A}^\dagger\hat{B}^\dagger$

Unitary: $\hat{A}^\dagger\hat{A} = \mathbb{I}$

Hermitian/Self-adjoint: $\hat{A}^\dagger = \hat{A}, \quad \langle i|\hat{A}|j\rangle^* = \langle j|\hat{A}|i\rangle$

$\hat{R}(d\phi\mathbf{k}) = 1 - \frac{i}{\hbar}\hat{J}_z d\phi, \quad \hat{R}^\dagger(d\phi\mathbf{k}) = 1 + \frac{i}{\hbar}\hat{J}_z^\dagger d\phi$

$\hat{R}(\phi\mathbf{k}) = e^{-i\hat{J}_z\phi/\hbar}, \quad \hat{R}^\dagger(\phi\mathbf{k}) = e^{i\hat{J}_z\phi/\hbar}$

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}, \quad e^{ix} = \cos x + i\sin x$

$\hat{P}_+ = |+\mathbf{z}\rangle\langle +\mathbf{z}|, \quad \hat{P}_- = |-\mathbf{z}\rangle\langle -\mathbf{z}|$

$\mathbb{I} \equiv |+\mathbf{z}\rangle\langle +\mathbf{z}| + |-\mathbf{z}\rangle\langle -\mathbf{z}|$

$\hat{A} \xrightarrow[S_z\,\text{basis}]{} \begin{pmatrix}\langle +\mathbf{z}|\hat{A}|+\mathbf{z}\rangle & \langle +\mathbf{z}|\hat{A}|-\mathbf{z}\rangle\\ \langle -\mathbf{z}|\hat{A}|+\mathbf{z}\rangle & \langle -\mathbf{z}|\hat{A}|-\mathbf{z}\rangle\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$

$|\psi'\rangle \xrightarrow[S_z\,\text{basis}]{} \begin{pmatrix}\langle +\mathbf{z}|\psi'\rangle\\ \langle -\mathbf{z}|\psi'\rangle\end{pmatrix}$

$\quad = \begin{pmatrix}\langle +\mathbf{z}|\hat{R}^\dagger(\frac{\pi}{2}\mathbf{j})|\psi\rangle\\ \langle -\mathbf{z}|\hat{R}^\dagger(\frac{\pi}{2}\mathbf{j})|\psi\rangle\end{pmatrix} = \begin{pmatrix}\langle +\mathbf{x}|\psi\rangle\\ \langle -\mathbf{x}|\psi\rangle\end{pmatrix} \xleftarrow[S_x\,\text{basis}]{} |\psi\rangle$

$\mathbb{S} = \begin{pmatrix}\langle +\mathbf{z}|+\mathbf{x}\rangle & \langle +\mathbf{z}|-\mathbf{x}\rangle\\ \langle -\mathbf{z}|+\mathbf{x}\rangle & \langle -\mathbf{z}|-\mathbf{x}\rangle\end{pmatrix} \quad [S_x \to S_z \text{ basis}]$

$\hat{A} \xrightarrow[S_x\,\text{basis}]{} \mathbb{S}^\dagger\mathbb{A}\mathbb{S}, \quad \text{where } \hat{A} \xrightarrow[S_z\,\text{basis}]{} \mathbb{A}$

$[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

$\langle S_z\rangle = \langle\psi|\hat{J}_z|\psi\rangle$

$\hat{S}_+|+\mathbf{z}\rangle = 0, \quad \hat{S}_+|-\mathbf{z}\rangle = \hbar|+\mathbf{z}\rangle$

$\hat{S}_-|+\mathbf{z}\rangle = \hbar|-\mathbf{z}\rangle, \quad \hat{S}_-|-\mathbf{z}\rangle = 0$

$\hat{J}_z|\pm\mathbf{z}\rangle = \pm\frac{\hbar}{2}|\pm\mathbf{z}\rangle$

$[\hat{J}_x,\hat{J}_y] = i\hbar\hat{J}_z, \quad [\hat{J}_y,\hat{J}_z] = i\hbar\hat{J}_x, \quad [\hat{J}_z,\hat{J}_x] = i\hbar\hat{J}_y$

$[\hat{J}_i,\hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$

$$\epsilon_{ijk} = \begin{cases}1 & \text{if } ijk \text{ is any even permutation of 123}\\ 0 & \text{if } i=j,\ i=k, \text{ or } j=k\\ -1 & \text{if } ijk \text{ is any odd permutation of 123}\end{cases}$$

Final:

$[\hat{A},\hat{B}\hat{C}] = \hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]\hat{C}$

$\hat{\mathbf{J}} = \hat{J}_x\mathbf{i} + \hat{J}_y\mathbf{j} + \hat{J}_z\mathbf{k}, \quad \hat{\mathbf{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$

$\hat{\mathbf{J}}^2|\lambda,m\rangle = \lambda\hbar^2|\lambda,m\rangle, \quad \hat{J}_z|\lambda,m\rangle = m\hbar|\lambda,m\rangle$

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$

$\hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2}, \quad \hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}$

$\hat{J}_\pm^\dagger = \hat{J}_\mp, \quad [\hat{J}_z,\hat{J}_\pm] = \pm\hbar\hat{J}_\pm,$

$[\hat{J}_+,\hat{J}_-] = i\hbar\hat{J}_z$

$\hat{J}_z\hat{J}_\pm|\lambda,m\rangle = (m\pm 1)\hbar\hat{J}_\pm|\lambda,m\rangle$

$\hat{\mathbf{J}}^2\hat{J}_\pm|\lambda,m\rangle = \lambda\hbar^2\hat{J}_\pm|\lambda,m\rangle, \quad m^2 \le \lambda$

$\hat{\mathbf{J}}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle, \quad \hat{J}_z|j,m\rangle = m\hbar|j,m\rangle$

$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

$m = j, j-1, j-2, \ldots, -j+1, -j$

Magnitude of angular momentum: $\sqrt{j(j+1)}\hbar$

Projection onto z-axis: $j\hbar$

$\langle j,m'|\hat{J}_\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\hbar\delta_{m',m\pm 1}$

where $\delta_{m',m\pm 1} = \langle j,m'|j,m\pm 1\rangle$

$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \ge |\langle\alpha|\beta\rangle|^2$

$[\hat{A},\hat{B}] = i\hat{C} \ \rightarrow\ \Delta A\Delta B \ge \frac{|\langle C\rangle|}{2}$

$\Delta J_x\Delta J_y \ge \frac{\hbar}{2}|\langle J_z\rangle|$

$\hat{\mathbf{S}}^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle, \quad \hat{S}_z|s,m\rangle = m\hbar|s,m\rangle$

$s = \frac{1}{2}, \quad m = \pm\frac{1}{2}, \quad |\frac{1}{2},\frac{1}{2}\rangle = |+\mathbf{z}\rangle, \quad |\frac{1}{2},-\frac{1}{2}\rangle = |-\mathbf{z}\rangle$

$\hat{\mathbf{S}} \rightarrow \frac{\hbar}{2}\boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = \sigma_x\mathbf{i} + \sigma_y\mathbf{j} + \sigma_z\mathbf{k},$

$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$\hat{S}_n = \hat{\mathbf{S}}\cdot\mathbf{n} \ \rightarrow\ \hat{S}_n|\mu\rangle = \mu\frac{\hbar}{2}|\mu\rangle$

$$\hat{U}(t)\left|\psi(0)\right\rangle = \left|\psi(t)\right\rangle, \quad \hat{U}^\dagger(t)\hat{U}(t) = 1$$

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}dt$$

$$\left|\psi(t)\right\rangle = e^{-i\hat{H}t/\hbar}\left|\psi(0)\right\rangle$$

$$\hat{H} = -\hat{\boldsymbol{\mu}}\cdot\mathbf{B} = \omega_0\hat{S}_n \quad \text{where} \quad \mathbf{B} = B_0\mathbf{k}, \quad \omega_0 = geB_0/2mc$$

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\hat{S}_z\phi/\hbar} = \hat{R}(\phi\mathbf{k}) \quad \text{where} \quad \phi = \omega_0 t$$

$$\frac{\partial}{\partial t}\left\langle A\right\rangle = \frac{i}{\hbar}\left\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\right\rangle + \left\langle\psi(t)\right|\frac{\partial\hat{A}}{\partial t}\left|\psi(t)\right\rangle$$

MRI: $\hat{H} = \omega_0\hat{S}_z + \omega_1(\cos\omega t)\hat{S}_x$

$\mathbf{B} = B_1\cos\omega t\mathbf{i} + B_0\mathbf{k}, \quad \omega_0 = egB_0/2mc, \quad \omega_1 = egB_1/2mc$

Energy eigenvalue problem: $\hat{H}\left|\psi\right\rangle = E\left|\psi\right\rangle$

Stationary state: $e^{-i\hat{H}t/\hbar}\left|E\right\rangle = e^{-iEt/\hbar}\left|E\right\rangle$

$$\hat{H} \to \begin{pmatrix} \langle 1|\hat{H}|2\rangle & \langle 1|\hat{H}|2\rangle \\ \langle 2|\hat{H}|1\rangle & \langle 2|\hat{H}|2\rangle \end{pmatrix} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}$$

$$\left|I\right\rangle = \frac{1}{\sqrt{2}}\left|1\right\rangle + \frac{1}{\sqrt{2}}, \quad \left|II\right\rangle = \frac{1}{\sqrt{2}}\left|1\right\rangle - \frac{1}{\sqrt{2}}\left|2\right\rangle$$

$$\hat{x}\left|x\right\rangle = x\left|x\right\rangle$$

$$\langle x|x'\rangle = \delta(x-x')$$

$$\int_{-\infty}^{\infty} dx\delta(x-x') = 1, \quad \int_{-\infty}^{\infty} dx\delta(x-x')f(x) = f(x')$$

$$\left|\psi\right\rangle = \int_{-\infty}^{\infty}\left|x\right\rangle\left\langle x|\psi\right\rangle dx, \quad \left\langle\psi\right| = \int_{-\infty}^{\infty}\left\langle x'\right|\left\langle\psi|x'\right\rangle dx'$$

$$\mathbb{I} = \int_{-\infty}^{\infty}\left|x\right\rangle\left\langle x\right| dx, \quad \mathbb{I} = \int_{-\infty}^{\infty}\left|p\right\rangle\left\langle p\right| dp$$

$$\psi(x) = \left\langle x|\psi\right\rangle, \quad \left\langle\psi|\psi\right\rangle = \int_{-\infty}^{\infty}\left|\left\langle x|\psi\right\rangle\right|^2 dx = 1$$

$$\hat{T}(dx) = 1 - \frac{i}{\hbar}\hat{p}_x dx, \quad \hat{T}(dx)\left|x\right\rangle = \left|x+dx\right\rangle$$

$$\hat{T}(a) = e^{-i\hat{p}_x a/\hbar}, \quad \hat{T}(a)\left|x\right\rangle = \left|x+a\right\rangle, \quad \hat{T}^\dagger\hat{T} = 1$$

$$[\hat{x},\hat{p}_x] = i\hbar \ \to\ \Delta x\Delta p_x \geq \frac{\hbar}{2}$$

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x})$$

$$\frac{d\left\langle p_x\right\rangle}{dt} = \frac{i}{\hbar}\left\langle\psi|[\hat{H},\hat{p}_x]|\psi\right\rangle = \left\langle -\frac{dV}{dx}\right\rangle$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$

$$\hat{p}_x\left|\psi\right\rangle = -i\hbar\int dx'\left|x'\right\rangle\frac{\partial}{\partial x'}\left\langle x'|\psi\right\rangle$$

$$\langle x|\hat{p}_x|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\left\langle x|\psi\right\rangle$$

$$\langle p|\psi\rangle = \int dx\frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar}\left\langle x|\psi\right\rangle$$

$$\langle x|\psi\rangle = \int dp\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}\left\langle p|\psi\right\rangle$$

$$p = \frac{h}{\lambda}$$

Harmonic oscillator: $\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right)$$

$$[\hat{a},\hat{a}^\dagger] = 1$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{p}_x = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^\dagger)$$

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})$$

$$\hat{N} = \hat{a}^\dagger\hat{a}$$

$$[\hat{N},\hat{a}] = -\hat{a}, \quad [\hat{N},\hat{a}^\dagger] = \hat{a}^\dagger$$

$$\hat{N}\left|\eta\right\rangle = \eta\left|\eta\right\rangle, \quad \eta \geq 0$$

$$\hat{N}\hat{a}\left|\eta\right\rangle = (\eta-1)\hat{a}\left|\eta\right\rangle \ \to\ \hat{a}\left|\eta\right\rangle = c_+\left|\eta-1\right\rangle$$

$$\hat{N}\hat{a}^\dagger\left|\eta\right\rangle = (\eta+1)\hat{a}^\dagger\left|\eta\right\rangle \ \to\ \hat{a}^\dagger\left|\eta\right\rangle = c_-\left|\eta+1\right\rangle$$

$$\hat{a}\left|\eta_{\min}\right\rangle = 0$$

$$\hat{N}\left|n\right\rangle = n\left|n\right\rangle, \quad n = 0,1,2,\ldots$$

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \quad \hat{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

$$\begin{aligned}\hat{H}\left|n\right\rangle &= \hbar\omega(\hat{N}+\tfrac{1}{2})\left|n\right\rangle \\ &= \hbar\omega(n+\tfrac{1}{2})\left|n\right\rangle = E_n\left|n\right\rangle, \quad n = 0,1,2,\ldots\end{aligned}$$

$$\langle n'|\hat{a}|n\rangle = \sqrt{n}\delta_{n',n-1}, \quad \langle n'|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\delta_{n',n+1}$$

$$\left|n\right\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\left|0\right\rangle$$

$$\langle x|\hat{p}_x|0\rangle = -i\hbar\frac{\partial\left\langle x|0\right\rangle}{\partial x}$$

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}$$

$$\langle x|n\rangle = \frac{1}{\sqrt{n!}}\left\langle x|(\hat{a}^\dagger)^n|0\right\rangle$$

Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin(\theta\pm\phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta\pm\phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$