Optical Theory Derivations & Results.

Light rays orthogonal trajectories to wavefronts $\mathcal{S}$ (eqn 3) ~~Const~~ = Constant.
$\vec{r}$ position vector & s length of ray

$$n\frac{d\vec{r}}{ds} = \text{grad}\,\mathcal{S} = \nabla\mathcal{S}$$

Differentiate with s

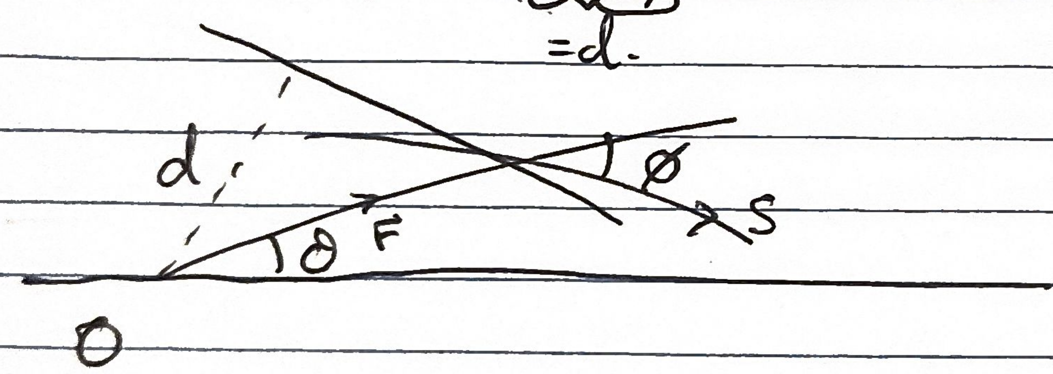
$$\frac{d}{ds}(\nabla\mathcal{S}) = \frac{1}{2n}\nabla n^2$$

$$\Rightarrow \frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \cancel{\text{grad}}\,\nabla n$$

Vector form of the differential equations for light rays.

--- See Born & Wolf for more ---

When $n = n(r)$, $n\underbrace{r\sin\phi}_{=d} = \text{Constant} = nd$.



$$\sin\phi = \frac{r(\theta)}{\sqrt{r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2}}.$$

$$\frac{d\vec{r}}{d\theta} = \frac{\vec{r}}{c}\sqrt{n^2r^2 - c^2} \qquad c \text{ constant.}$$

$$\Rightarrow \theta = c\int^{r}\frac{dr}{r\sqrt{n^2r^2 - c^2}} \qquad \text{Eq. (2) in report.}$$

Part 2

From Eq. (2)

$$\theta = u \int^{r} \frac{dr}{r\sqrt{n^2(\theta) r^2 - c^2}}$$

with $n(r) = \dfrac{n_0}{1 + \left(\frac{r}{R}\right)^2}$.

using $\rho = \frac{r}{a}$ and $\mathcal{K} = \frac{c}{a n_0}$

$$\theta = \int^{\rho} \frac{\mathcal{K}(1+\rho^2)}{\rho\sqrt{\rho^2 - \mathcal{K}^2(1+\rho^2)^2}} \, d\rho .$$

using $\dfrac{\mathcal{K}(1+\rho^2)}{\rho\sqrt{\rho^2 - \mathcal{K}^2(1+\rho^2)^2}} = \dfrac{d}{d\rho}\left[\arcsin\left(\dfrac{\mathcal{K}}{\sqrt{1-4\mathcal{K}^2}} \dfrac{\rho^2 - 1}{\rho}\right)\right]$

$$\Rightarrow \theta - C = \arcsin\left(\frac{\mathcal{K}}{\sqrt{1-4\mathcal{K}^2}} \frac{\rho^2-1}{\rho}\right)$$

C = integration constant.

$$\Rightarrow \sin(\theta - C) = \frac{\mathcal{K}}{\sqrt{1-4\mathcal{K}^2}} \frac{\rho^2-1}{\rho} .$$

Subbing $\rho = \frac{r}{a}$ and $\mathcal{K} = \frac{c}{a n_0}$ back in

$$\sin(\theta - C) = \frac{u}{\sqrt{R^2 n_0^2 - 4u^2}} \frac{r^2 - R^2}{Rr} \quad \text{Eq. (3)}$$

Part 3 from Eq. (3).

$$\sin(\theta - C) = \gamma \, \frac{r^2 - R^2}{r}$$

where $\gamma = \dfrac{u}{\sqrt{R^2 v_0^2 - 4c^2}} \dfrac{1}{R}$ is Constant.

Therefore we have that all values of C must satisfy

$$\frac{r^2 - R^2}{r\sin(\theta - C)} = \text{Constant}$$

So we choose r_0 θ_0 as our initial and we get

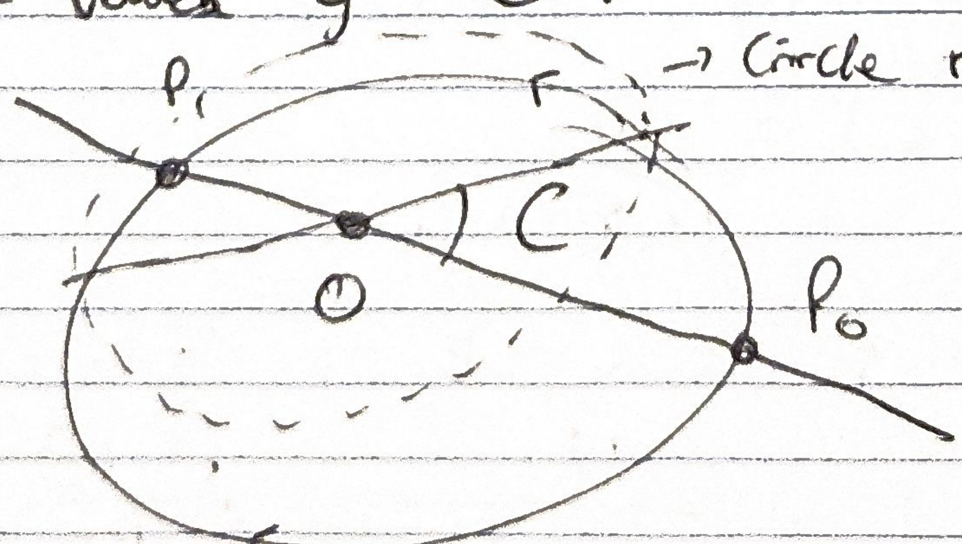
$$\frac{r_0^2 - R^2}{r_0 \sin(\theta_0 - C)} = \frac{r^2 - R^2}{r\sin(\theta - C)} \qquad \text{Eq. (7).}$$

Part 4

$\sin\theta_1 = -\sin\theta_0$

$r_1 = \dfrac{R^2}{r_0}$ and $\theta_1 = \pi + \theta_0$

for ~~all~~ any values of C.



Also, $r = u$ and $\theta = \pi + C$ are solutions.
Each ray intersects circle $r = uR$

~~Chart~~

$$\sin(\theta - C) = \sin\theta\cos C - \sin C\cos\theta .$$

So $\sin(\theta_0 - C) = \sin\theta_0\cos C - \sin C\cos\theta_0$.

$$\sin(\theta - C) = \sin\theta\cos C - \sin C\cos\theta$$

All rays from P_0 to go to P_1.

$OP_1 \cdot OP_0 = u^2$. and opposite sides.

$\Rightarrow$ absolute instrument of inversion focusing, imaging.

Part 5

Switching Eq. (3) to Cartesian.

$x = r\cos\theta \qquad y = r\sin\theta$.

Sub into Eq. (3) and use $\sin(\theta - C) = y\cos C - x\sin C$

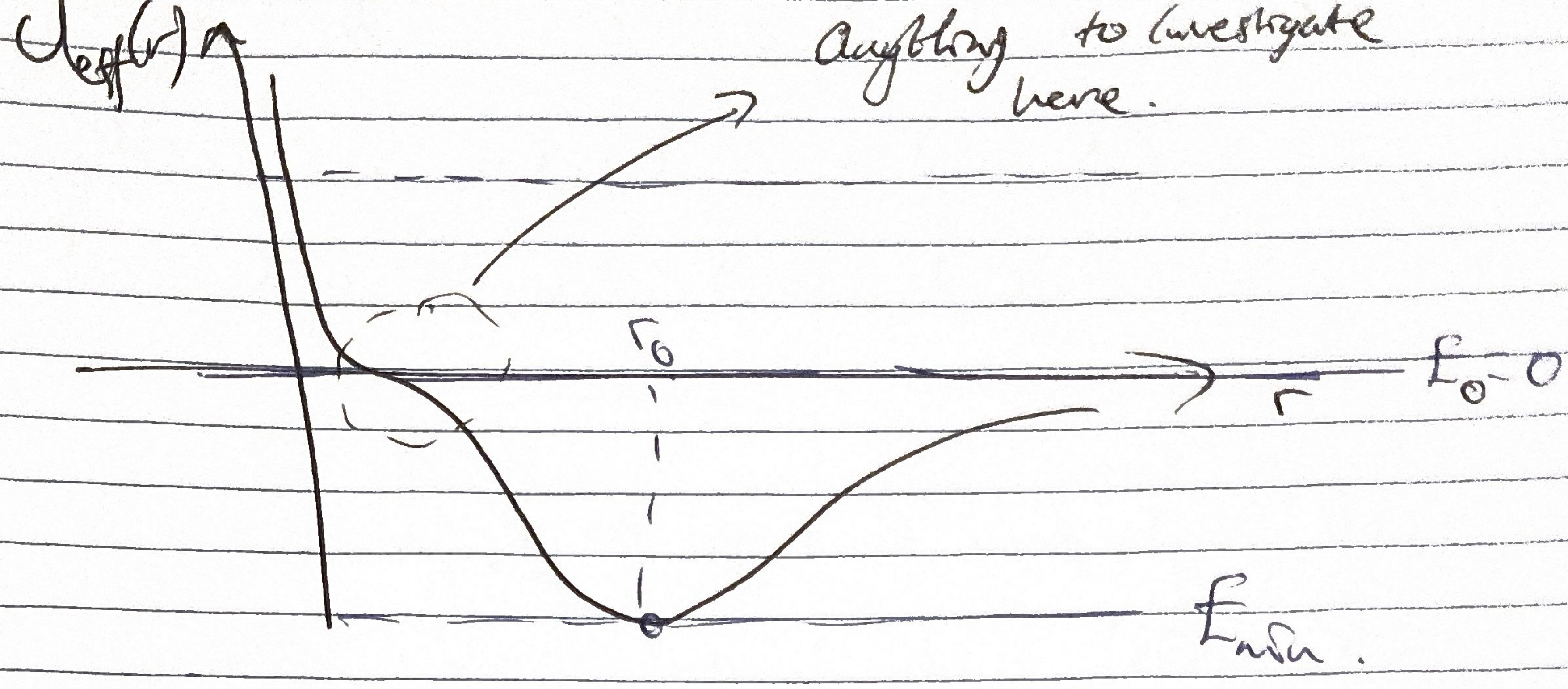
$$\Rightarrow y\cos C - x\sin C = \frac{u}{R\sqrt{R^2 n_0^2 - 4u^2}}(x^2+y^2-R^2).$$

$$\Rightarrow (x+\lambda\sin C)^2 + (y-\lambda\cos C)^2 = R^2+\lambda^2$$

where $\lambda = \frac{R}{2c}\sqrt{R^2 n_0^2 - 4u^2}$ Eq. (8)

This shows that each ray is a circle.

Central Potential Notes



r_0 can be found as $\frac{d\,U_{eff}(r)}{r} = 0$

using Sympy for $\frac{dU(r)}{dr} = 0$ we get,

$$r_0 = \left\{ \left(\delta R^{-\mu}\right)^{-1/\mu},\ \left(-\delta R^{-\mu}\right)^{-1/\mu},\ \sqrt[\mu]{\frac{R^{\mu}\sqrt{\mu^2-1}}{\mu+1}},\ \sqrt[\mu]{\frac{-R^{\mu}\sqrt{\mu^2-1}}{\mu+1}} \right\}.$$

This is ignoring angular term, but $U(r)$ gives

$$r_0 = \sqrt[\mu]{\frac{\pm R^{\mu}\sqrt{\mu^2-1}}{\mu+1}} \quad \text{for } \mu \geq 1$$

$\rightarrow$ whichever is Real

Classically under Kepler orbits.

$E = E_{min} \Rightarrow$ Circular

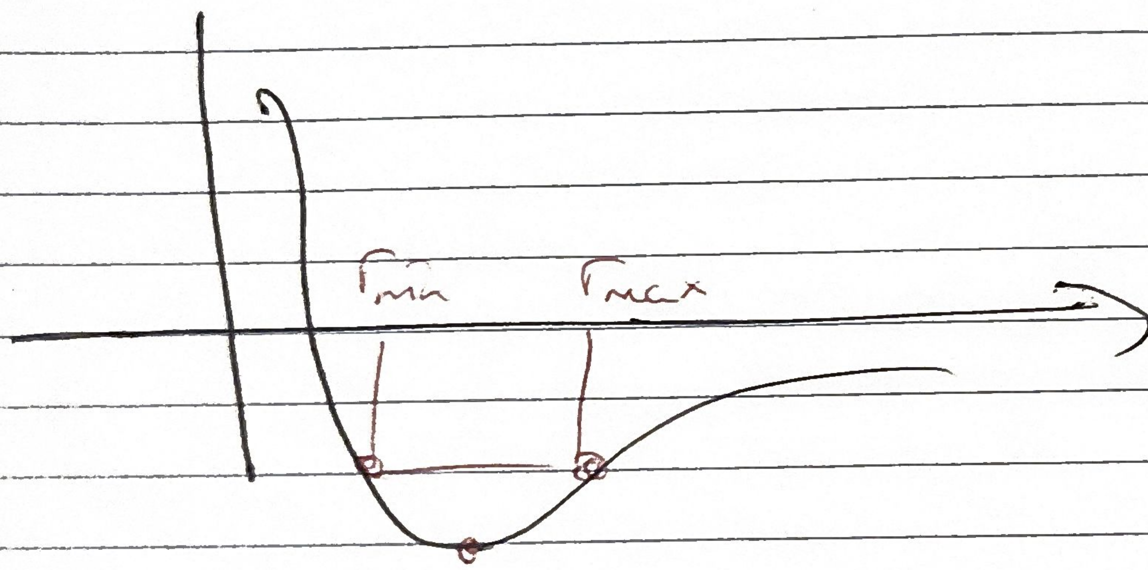
$E_{min} < E < E_0 \Rightarrow$ Elliptical

are Bounded
closed
TWO turning points

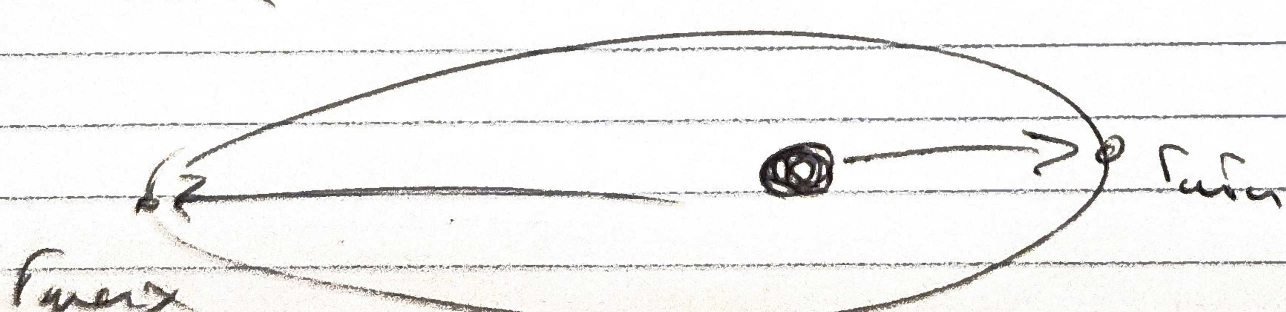
$E = E_0 = 0 \Rightarrow$ Parabolic

$E > E_0 \Rightarrow$ Hyperbola

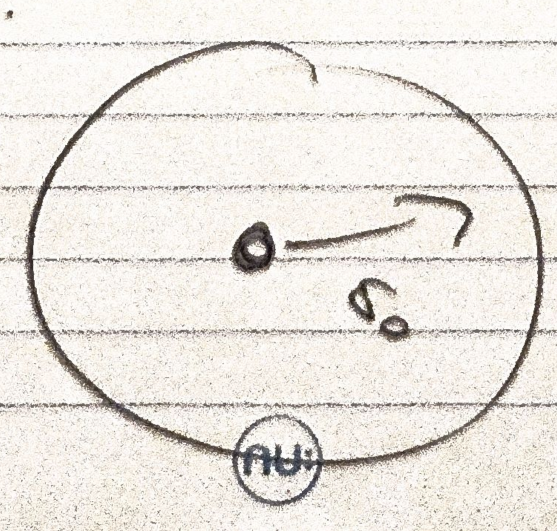
Unbounded,
only one
"turning point"



Particle can orbit between r_{min} and r_{max}



Hyperbola
"Deflected"



E_{min} orbit.