Numerical Solution for the Fokker Planck Equation

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Introduction

The Brownian motion of particles can be treated as the rate of Brownian particles approaching a state of equilibrium which is provided by the Master equation. A simplified version of this is the Fokker Planck Equation.

Fokker Planck Equation

From the master equation, we arise with the FPE where we have some function f, λ being the restoring force, B being the modulus, T being the temperature and k being the Boltzmann constant

$$\frac{\partial f}{\partial t} = \lambda B \nabla (rf) + kBT\nabla^2 f$$

Methods

The method used was the Forward Time Centered Space (FTCS) method. Where we can write the FPE numerically as such.

Boundary and initial conditions

Considering a function where the parameters of its position x and y to be given by

$$f = f(r, t)$$
$$r = r(x, y) = \sqrt{x^2 + y^2}$$

With a grid space s and time step h. Numerically our position functions are defined as.

$$x_j = x_0 + j\Delta x$$
$$y_k = y_0 + k\Delta y$$
$$r_j^k = \sqrt{x_j^2 + y_k^2}$$

Where the change in x and y are defined by the following. Where we have n being the number of grid spaces in i/j. Along with a length of L = 2.

$$\Delta x = \frac{h}{n_j}$$
$$\Delta y = \frac{s}{n_k}$$
$$f(0, t) = f(L, t) = 0$$

Results

From Figure 1 and 2, our computations shows a generated solution using a exponential test function decreases the density function along the y-axis.

Conclusions

We can see that the FPE can be described for a Brownian particle approaching equilibrium over time. Further work would be to look at an animation of the function over time with a gaussian approaching to an equilibrium state.

Link to Github for code:

https://github.com/ShaneGervais/Fokker-Planck-Equation-Using-FTCS-Method

References

1. Statistical Mechanics by R.K. Pathria, Butterworth Heinemann, Elsevier Science & Technology, 1996
2. Computational Physics by Mark Newmann, University of Michigan, 2012

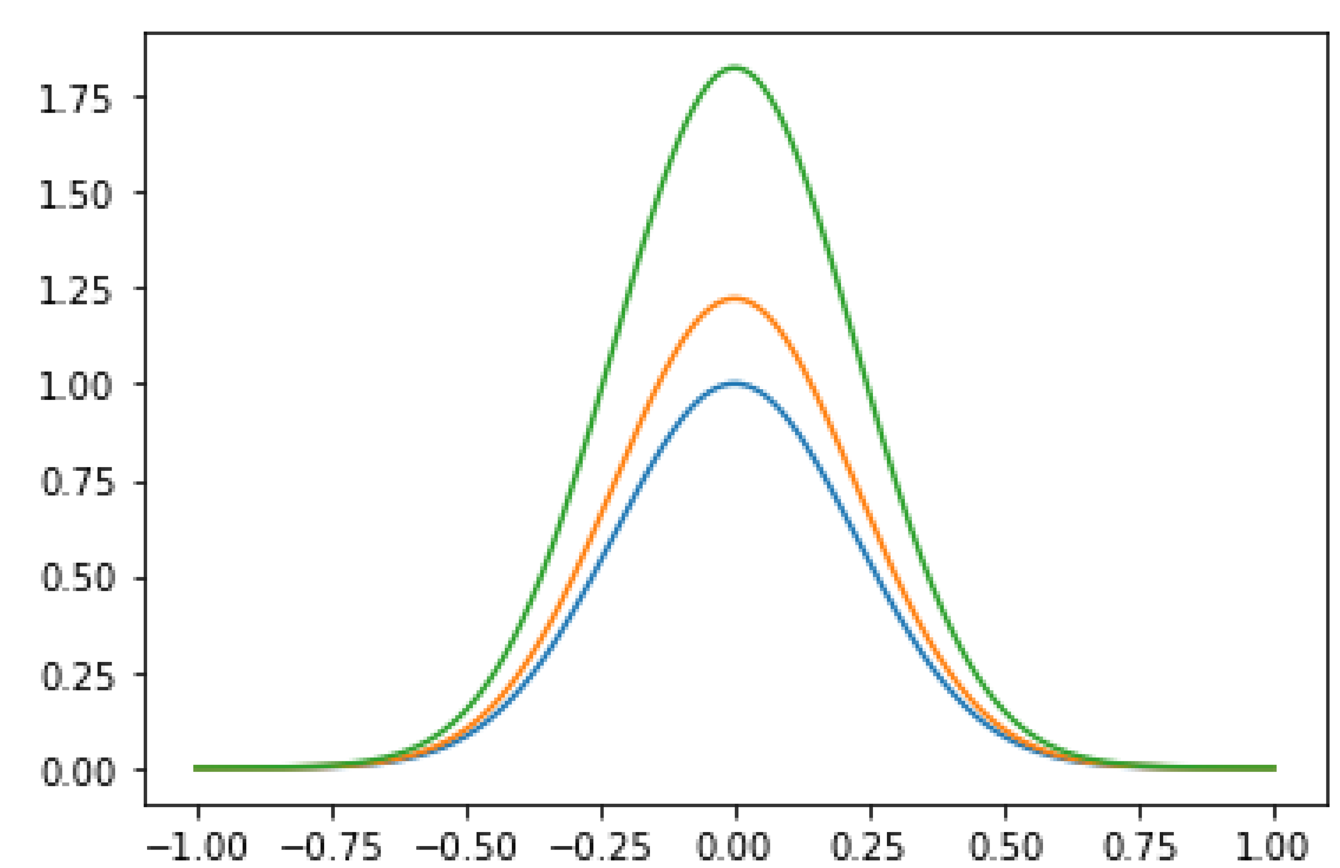

Figure 1. Numerical Sol. For FPE in 1D

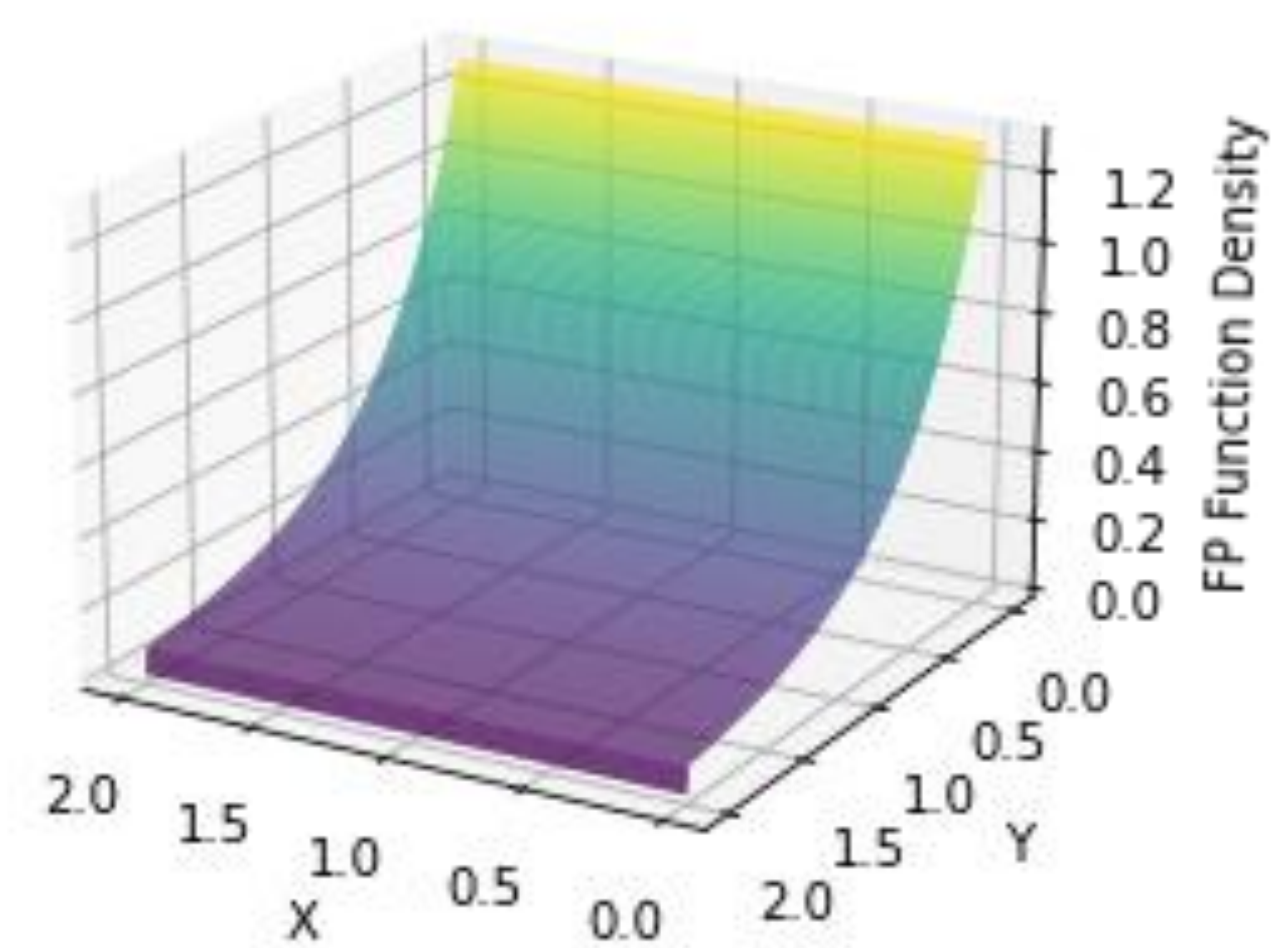

Figure 2. 3D representation of FPE

$$\frac{f^k_{i+1,j} - f^k_{i-1,j}}{s} = \lambda B\left[\frac{r^k_j(f^k_{i,j+1} - f^k_{i,j-1})}{2\Delta x} + \frac{f^k_{i,j}(r^k_{j+1} - r^k_{j-1})}{2\Delta x} + \frac{r^k_j(f^{k+1}_{i,j} - f^{k-1}_{i,j})}{2\Delta y} + \frac{f^k_{i,j}(r^{k+1}_j - r^{k-1}_j)}{2\Delta y}\right] + kBT\left(\frac{f^k_{i,j+1} - 2f^k_{i,j} + f^k_{i,j-1}}{\Delta x^2} + \frac{f^{k+1}_{i,j} - 2f^k_{i,j} + f^{k-1}_{i,j}}{\Delta y^2}\right)$$