Validation of Gauss Quadrature against Monte Carlo Quadrature and Newton-Cotes Quadrature

1. Gauss Quadrature (Gauss-Legendre)

single integral:

$$I = \int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}t\right)dt \approx \frac{b-a}{2}\sum_{i=1}^{n} A_i f\left(\frac{a+b}{2} + \frac{b-a}{2}t_i\right)$$

double integral:

$$\begin{aligned} I = \iint_S f(x,y)dA &= \int_a^b \int_c^d f(x,y)dy\,dx \\ &= \int_{-1}^{1}\int_{-1}^{1} f\left[\frac{(b-a)v+a+b}{2}, \frac{(d-c)u+c+d}{2}\right]\frac{(b-a)(d-c)}{4}du\,dv \end{aligned}$$

$$\approx \frac{(b-a)(d-c)}{4}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} A_i B_j f\left[\frac{(b-a)u_i+a+b}{2}, \frac{(d-c)v_j+c+d}{2}\right]$$

weights and points at n = 5

n	weights	points
5	0.2369268851	± 0.9061798459
	0.4786286705	± 0.5384693101
	0.5688888889	0

2. Monte Carlo Quadrature

single integral:

$$I = \int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{i=1}^{m} f(x_i)$$

double integral:

$$I = \iint_S f(x,y)dA \approx \frac{(b-a)(d-c)}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)$$

where, uniform sampling is performed in the interval (a,b) or the area S.

3. Newton-Cotes Quadrature (Composite Simpson's rule)

single integral:

$$I = \int_a^b f(x)dx \approx \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(a+ih)\right]$$

double integral:

$$I = \iint_S f(x,y)dA = \frac{h\tau}{4}\sum_{i=0}^{m}\sum_{j=0}^{n}\lambda_{ij} f(x_i, y_j)$$

where,

$x_i = a + ih$ (i = 0,1,2,…,m) and $h = (b-a)/m$

$y_j = c + j\tau$ (j = 0,1,2,…,n) and $\tau = (d-c)/n$

$$\lambda_{00} = \lambda_{0n} = \lambda_{m0} = \lambda_{mn} = 1$$

$\lambda_{i0} = \lambda_{in} = 2$ (i = 1,2,…,m-1)

$\lambda_{0j} = \lambda_{mj} = 2$ (j = 1,2,…,n-1)

$\lambda_{ij} = 4$(i = 1,2,…,m-1; j = 1,2,…,n-1)

Comparison among three quadrature methods

$$\beta v = \frac{2\int_{\gamma_{min}}^{1}(1+\gamma)^2\gamma^{-11/3}\exp\left(-\frac{12c_f\sigma}{2.047\rho_b\varepsilon^{2/3}d^{5/3}\gamma^{11/3}}\right)d\gamma}{\int_0^1\int_{\gamma_{min}}^{1}(1+\gamma)^2\gamma^{-11/3}\exp\left(-\frac{12c_f\sigma}{2.047\rho_b\varepsilon^{2/3}d^{5/3}\gamma^{11/3}}\right)d\gamma df_{BV}}$$

Numerical Methods:

Gauss Legendre: 5 points for single integral and 5 times 5 for double integral

Newton Cotes: 50 points for single integral and 50 times 50 for double integral

Monte Carlo: 50 points for single integral and 50 times 50 for double integral

Monte Carlo: 500 points for single integral and 500 times 500 for double integral

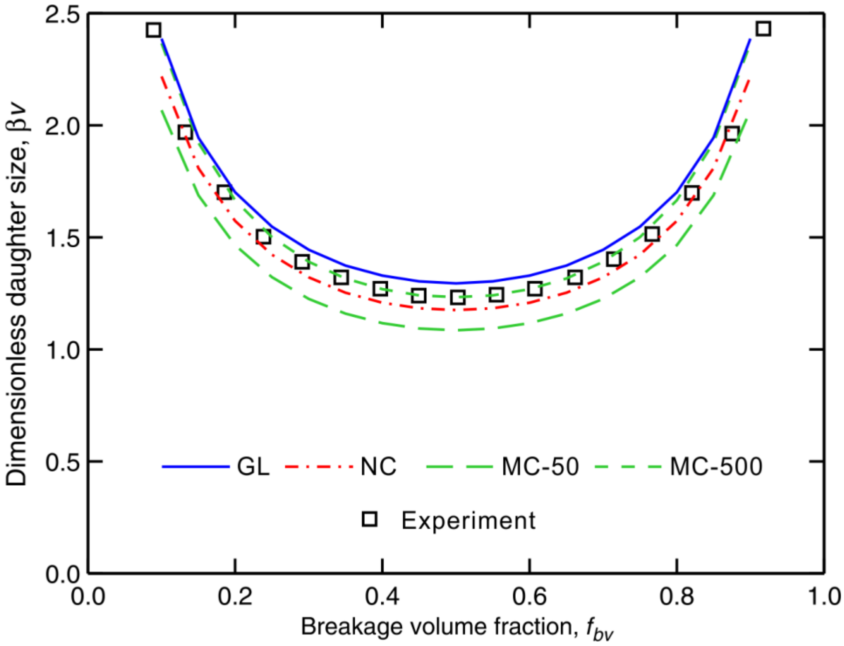

Fig. 1 Comparison between numerical results by different methods (Gauss Legendre: solid; Newton Cotes: dash-dotted; Monte Carlo with 50 points: long dashed; Monte Carlo with 500 points: short dashed) and experimental data (open markers).

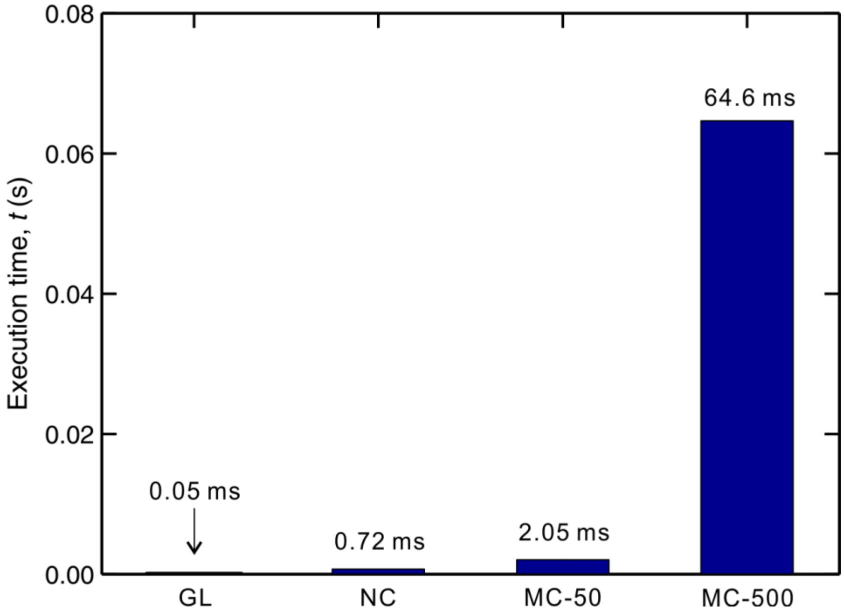

Fig. 2 Execution time for different numerical methods (Gauss Legendre: 0.05 ms; Newton Cotes: 0.72 ms; Monte Carlo with 50 points: 2.05 ms; Monte Carlo with 500 points: 64.6 ms).