# Steady and Transient Temperature Fields by Conduction

## 1. 2D steady-state temperature distribution in an area Ω

### 1.1 Numerical methodology

Assuming that the thermal diffusivity is equal to unity, the governing equation is a Poisson equation (boundary value problem),

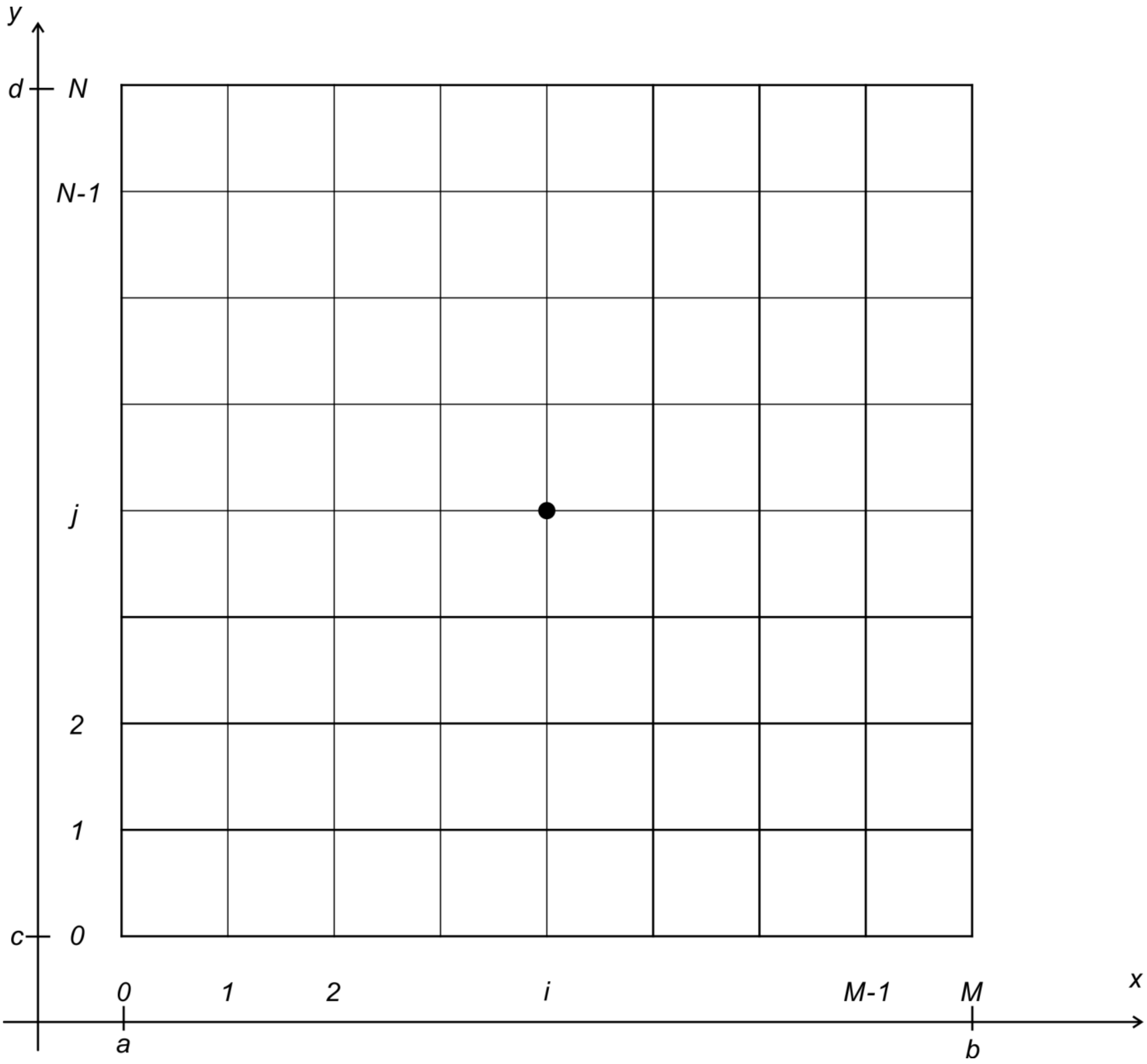
$$\begin{cases} -\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)T = f(x, y), & (x, y) \in \Omega \\ T|_{\partial\Omega} = \phi(x, y), & \end{cases}$$

$$\partial\Omega = [a < x < b, c < y < d]$$

The domain is discretized as,

$$\Delta x = (b - a)/M, \Delta y = (d - c)/N$$

$$x_i = a + i\Delta x,\ y_j = a + j\Delta y \text{ (where } 0 \le i \le M,\ 0 \le j \le N)$$




Internal nodes:

$$-\left[\frac{\partial^2 T(x_i, y_j)}{\partial x^2} + \frac{\partial^2 T(x_i, y_j)}{\partial y^2}\right] = f(x_i, y_j)$$

$$(1 \le i \le M-1,\ 1 \le j \le N-1)$$

Thus, the finite difference equation reads as

$$-\frac{1}{(\Delta y)^2} T_{i,j-1} - \frac{1}{(\Delta x)^2} T_{i-1,j} + 2\left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right] T_{i,j} - \frac{1}{(\Delta x)^2} T_{i+1,j} - \frac{1}{(\Delta y)^2} T_{i,j+1} = f_{i,j}$$

$$qT_{i,j-1} + pT_{i-1,j} + rT_{i,j} + pT_{i+1,j} + qT_{i,j+1} = f_{i,j}$$

$$p = -\frac{1}{(\Delta x)^2}$$

$$q = -\frac{1}{(\Delta y)^2}$$

$$r = 2\left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right]$$

$$(1 \leq i \leq M-1,\ 1 \leq j \leq N-1),$$

A vector is defined by,

$$\boldsymbol{T}_j = \left(T_{1,j}, T_{2,j}, \cdots, T_{M-1,j}\right)^{\mathrm{T}}$$

$$\text{where } 1 \leq j \leq N-1$$

Thus, the finite difference equation can be rewritten in the matrix form,

$$\boldsymbol{D}\boldsymbol{T}_{j-1} + \boldsymbol{C}\boldsymbol{T}_j + \boldsymbol{D}\boldsymbol{T}_{j+1} = \boldsymbol{f}_j$$

$$\text{where } 1 \leq j \leq N-1$$

$$\boldsymbol{C} = \begin{bmatrix} r & p & 0 & 0 & 0 \\ p & r & p & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & p & r & p \\ 0 & 0 & 0 & p & r \end{bmatrix}_{M-1\times M-1}$$

$$\boldsymbol{D} = \begin{bmatrix} q & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & q \end{bmatrix}_{M-1\times M-1}$$

$$\boldsymbol{f}_j = \begin{bmatrix} f_{1,j} - pT_{0,j} \\ f_{2,j} \\ \vdots \\ f_{M-2,j} \\ f_{M-1,j} - pT_{M,j} \end{bmatrix}_{M-1}$$

Further, the finite difference equation can be summarized,

$$\begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{D} & \boldsymbol{C} & \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & \ddots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D} & \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D} & \boldsymbol{C} \end{bmatrix} \begin{pmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \\ \vdots \\ \boldsymbol{T}_{N-2} \\ \boldsymbol{T}_{N-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 - \boldsymbol{D}\boldsymbol{T}_0 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_{N-2} \\ \boldsymbol{f}_{N-1} - \boldsymbol{D}\boldsymbol{T}_N \end{pmatrix}$$

As a result, the PDE is converted into a matrix problem as $\boldsymbol{A}\boldsymbol{T} = \boldsymbol{f}$.

1.2 Solution of matrix

1.2.1 Jacobi method

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{A} = \boldsymbol{D} + \boldsymbol{L} + \boldsymbol{U}$$

$$\boldsymbol{D}\boldsymbol{x} = -(\boldsymbol{L} + \boldsymbol{U})\boldsymbol{x} + \boldsymbol{b}$$

$$\boldsymbol{x}^{k+1} = -\boldsymbol{D}^{-1}(\boldsymbol{L} + \boldsymbol{U})\boldsymbol{x}^{k} + \boldsymbol{D}^{-1}\boldsymbol{b}$$

1.2.2 Gauss-Seidel method

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{A} = \boldsymbol{D} + \boldsymbol{L} + \boldsymbol{U}$$

$$(\boldsymbol{D} + \boldsymbol{L})\boldsymbol{x} = -\boldsymbol{U}\boldsymbol{x} + \boldsymbol{b}$$

$$\boldsymbol{x}^{k+1} = -\boldsymbol{D}^{-1}\boldsymbol{L}\boldsymbol{x}^{k+1} - \boldsymbol{D}^{-1}\boldsymbol{U}\boldsymbol{x}^{k} + \boldsymbol{D}^{-1}\boldsymbol{b}$$

1.2.3 Successive Over Relaxation (SOR) method

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{A} = \frac{1}{\omega}\boldsymbol{D} + \boldsymbol{L} + \left(1 - \frac{1}{\omega}\right)\boldsymbol{D} + \boldsymbol{U}$$

$$\left(\frac{1}{\omega}\boldsymbol{D} + \boldsymbol{L}\right)\boldsymbol{x} = -\left[\left(1 - \frac{1}{\omega}\right)\boldsymbol{D} + \boldsymbol{U}\right]\boldsymbol{x} + \boldsymbol{b}$$

$$\boldsymbol{x}^{k+1} = \omega\left\{-\boldsymbol{D}^{-1}\boldsymbol{L}\boldsymbol{x}^{k+1} - \left[\left(1 - \frac{1}{\omega}\right)\boldsymbol{I} + \boldsymbol{D}^{-1}\boldsymbol{U}\right]\boldsymbol{x}^{k} + \boldsymbol{D}^{-1}\boldsymbol{b}\right\}$$

where, $\omega$ is the relaxation factor.

1.3 Case study

$$\begin{cases} -\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)T = (\pi^2 - 1)e^{x}\sin(\pi y), \\ T(0, y) = T_0 + \sin(\pi y) \\ T(2, y) = T_0 + e^{2}\sin(\pi y) \\ T(x, 0) = T_0 \\ T(x, 1) = T_0 \end{cases}$$

$$\partial\Omega = [0 < x < 2, 0 < y < 1]$$

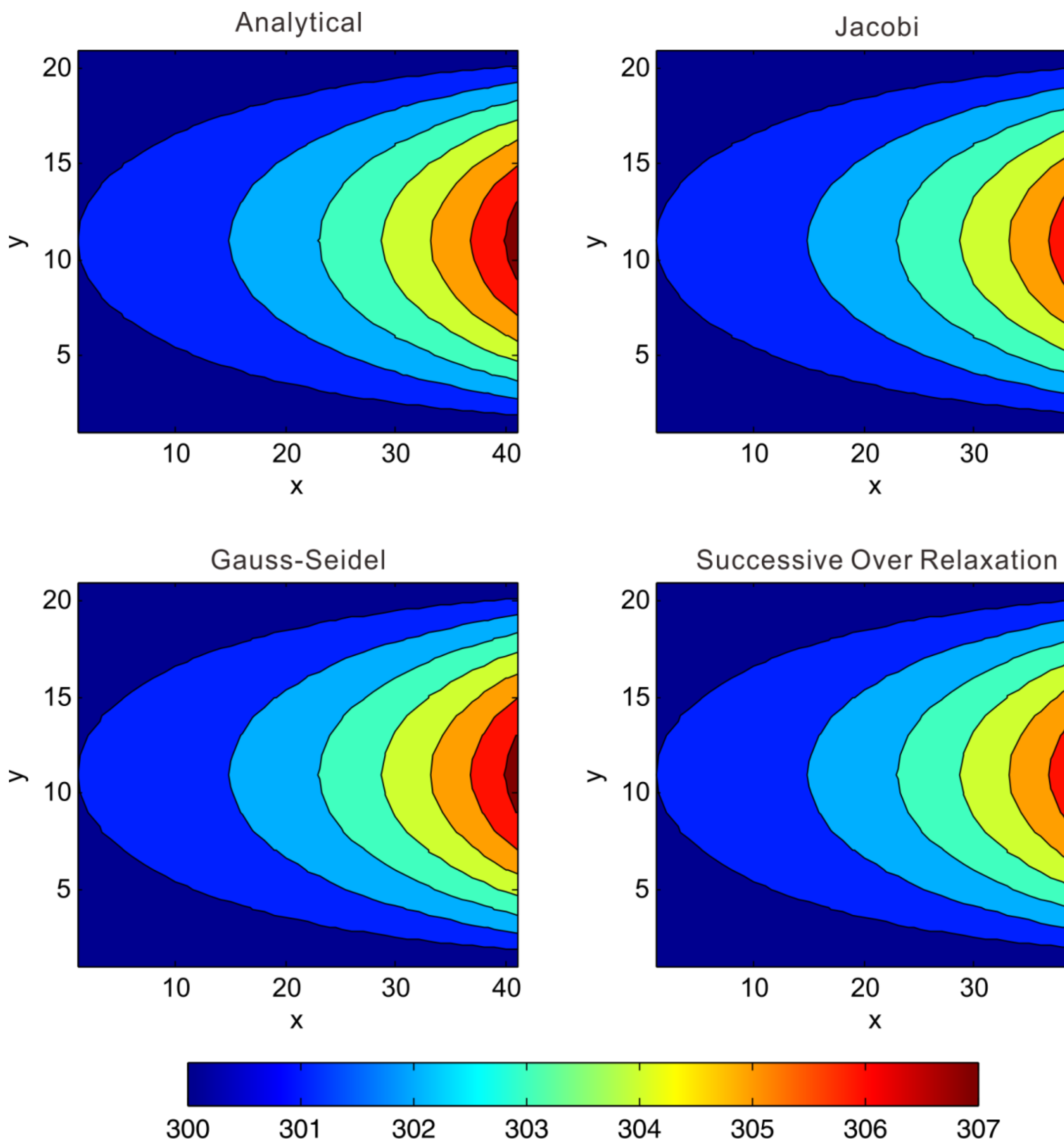



Fig. 1 Steady-state temperature fields simulated by different iteration methods.

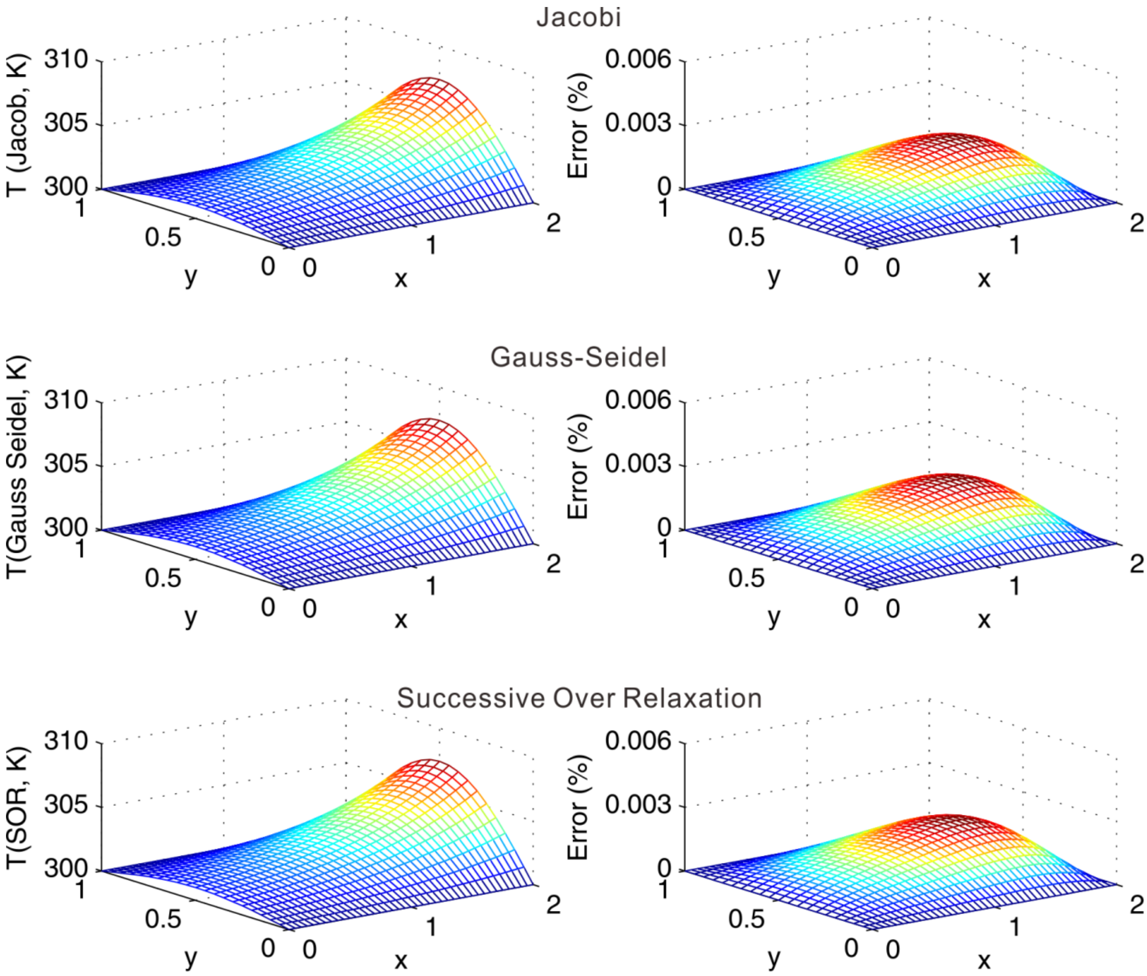



Fig. 2 3D visualization of steady-state temperature fields simulated by different iteration methods and corresponding errors.

2. 2D transient temperature distribution in an area Ω

2.2.1 Numerical methodology

Assuming that the thermal diffusivity is equal to unity, the governing equation reads as,

$$\begin{cases} \dfrac{\partial T}{\partial t} - \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) T = f(x,y), \quad (x,y) \in \Omega \\ T|_{\partial\Omega} = \phi(t,x,y), \\ T(0,x,y) = \psi(x,y), \end{cases}$$

$$\partial\Omega = [a < x < b, c < y < d]$$

Forward time centered space (FTCS) scheme:

Time → Euler forward time; Space → central differencing

The finite difference equation is described by

$$\frac{T_{i,j}^{k+1}-T_{i,j}^{k}}{\Delta t}-\frac{1}{(\Delta y)^2}T_{i,j-1}^{k+1}-\frac{1}{(\Delta x)^2}T_{i-1,j}^{k+1}+2\left[\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}\right]T_{i,j}^{k+1}-\frac{1}{(\Delta x)^2}T_{i+1,j}^{k+1}-\frac{1}{(\Delta y)^2}T_{i,j+1}^{k+1}=f_{i,j}$$

$$qT_{i,j-1}^{k+1}+pT_{i-1,j}^{k+1}+r'T_{i,j}^{k+1}+pT_{i+1,j}^{k+1}+qT_{i,j+1}^{k+1}=f'_{i,j}$$

$$p=-\frac{1}{(\Delta x)^2}$$

$$q=-\frac{1}{(\Delta y)^2}$$

$$r=2\left[\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}\right]+\frac{1}{\Delta t}$$

$$f'_{i,j}=f_{i,j}+\frac{T_{i,j}^{k}}{\Delta t}$$

$$(1\leq i\leq M-1,\ 1\leq j\leq N-1),$$

2.2.2 Case study

$$\begin{cases}\frac{\partial T}{\partial t}-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)T=(\pi^2-1)e^{x}\sin(\pi y),\\ T(t,0,y)=T_0+\sin(\pi y)\\ T(t,2,y)=T_0+e^2\sin(\pi y)\\ T(t,x,0)=T_0\\ T(t,x,1)=T_0\\ T(0,x,y)=T_0\end{cases}$$

$$\partial\Omega=[0<x<2,0<y<1]$$

Evolution of temperature fields simulated by Jacobi, Gauss-Seidel and SOR method.

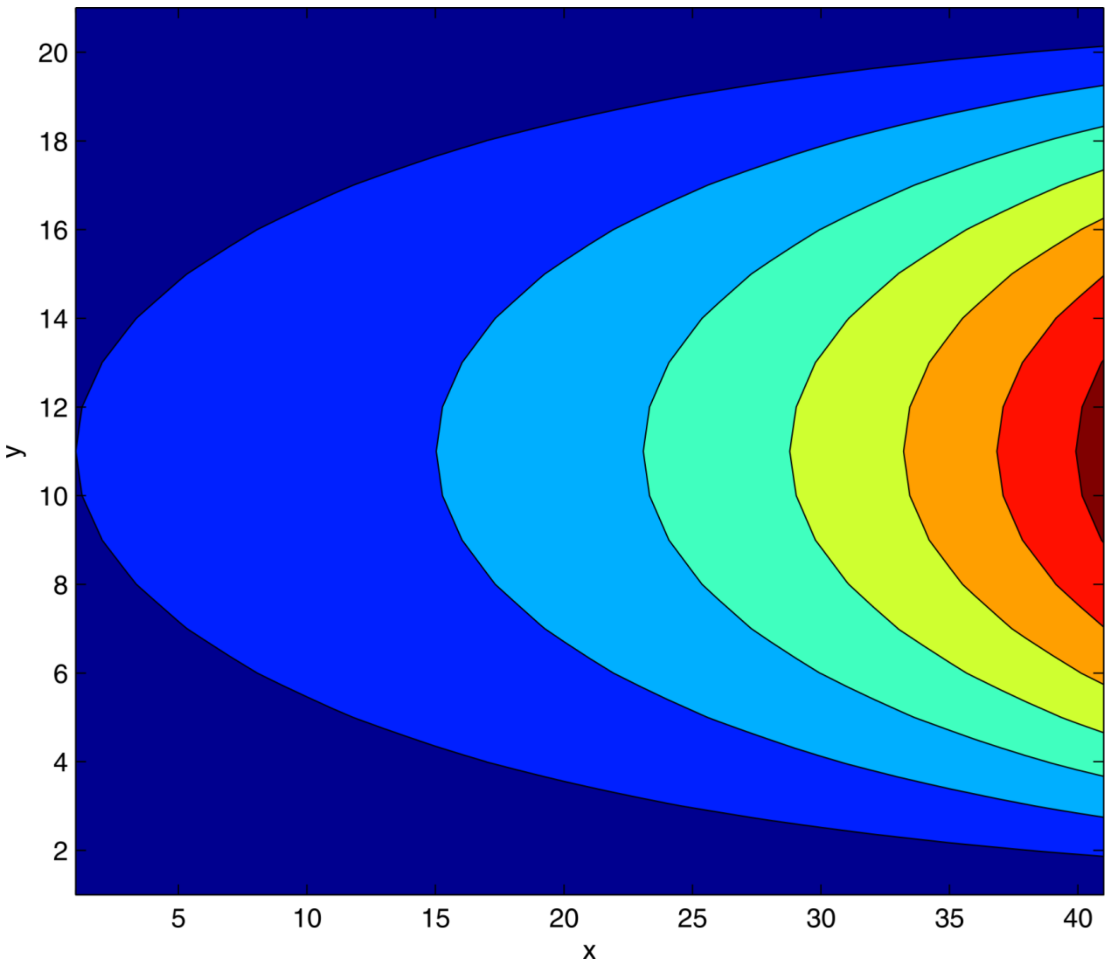



Fig. 3 Temperature field simulated by the Jacobi method at t = 0.44 second (see animations).

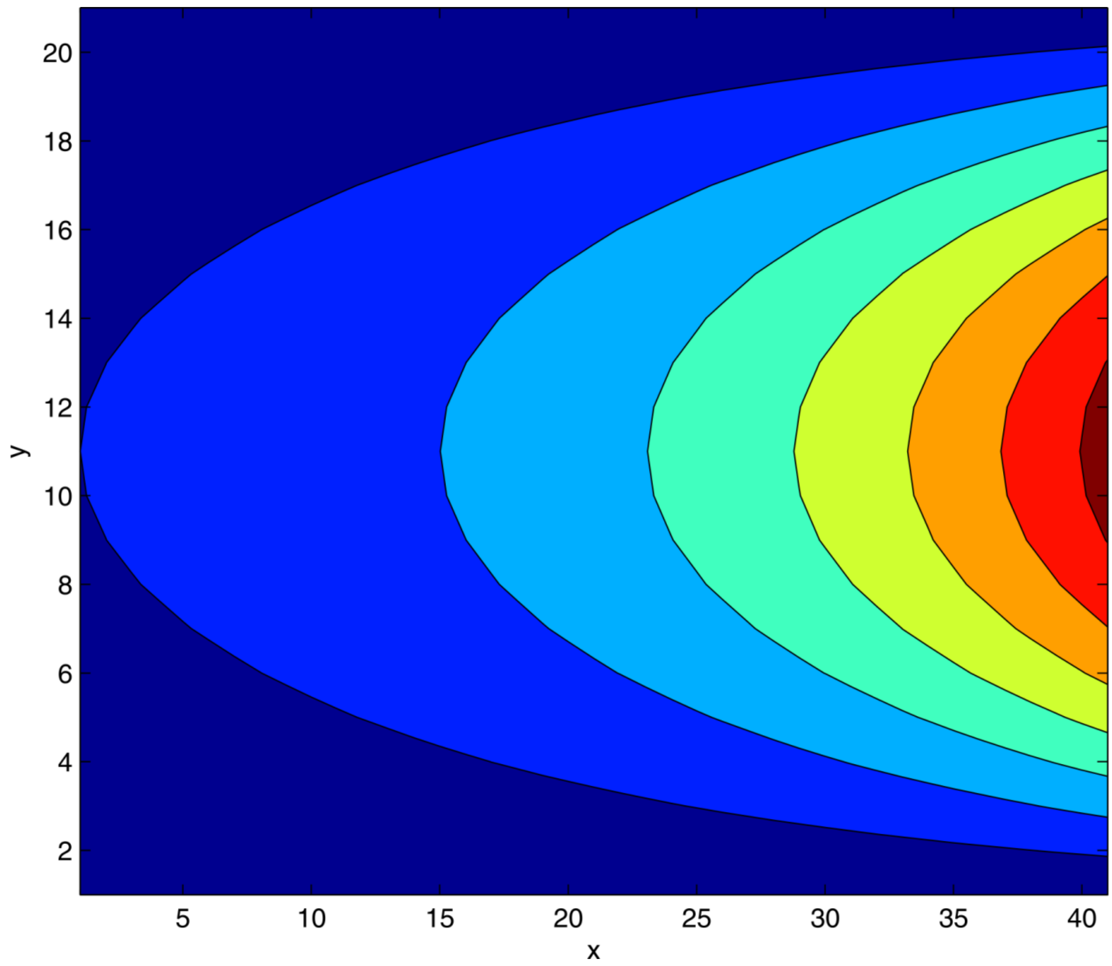



Fig. 4 Temperature field simulated by the Gauss-Seidel method at t = 0.44 second (see animations).

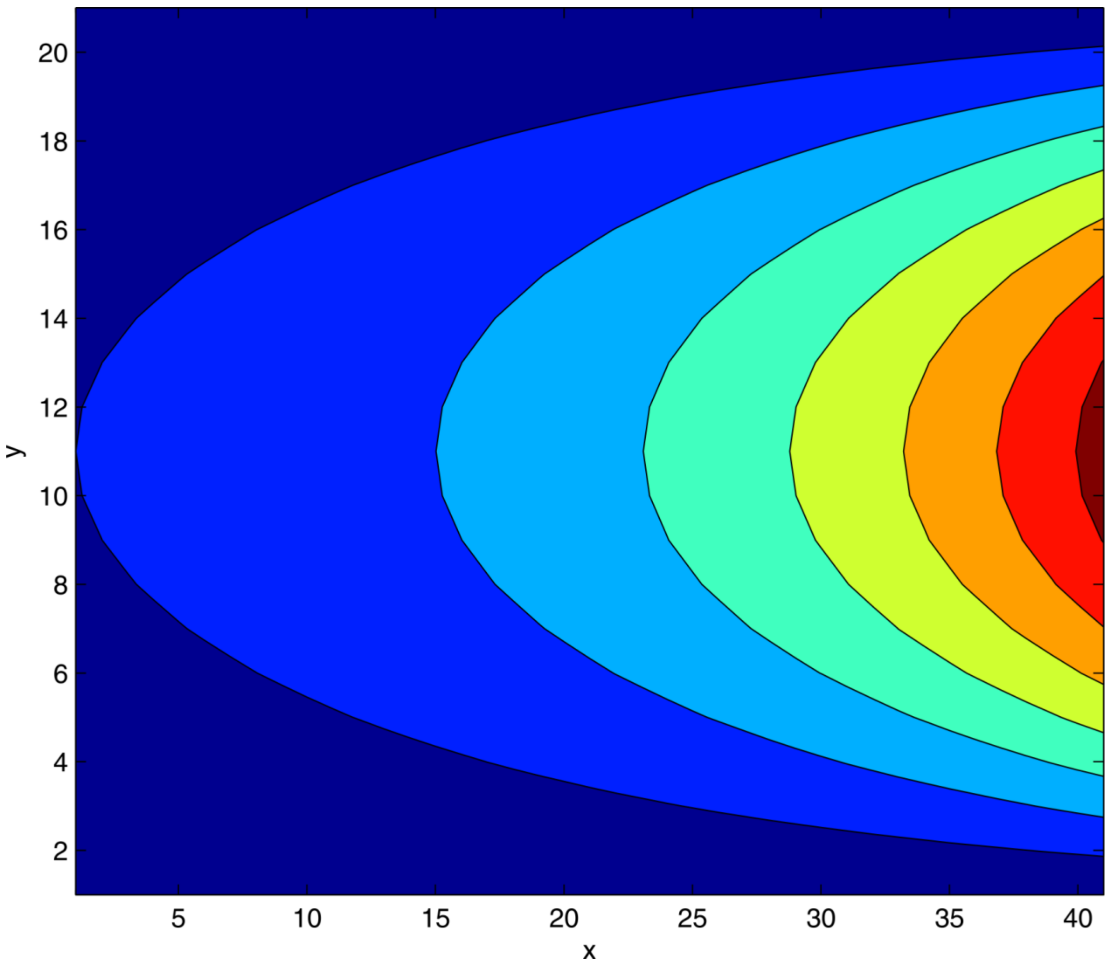



Fig. 5 Temperature field simulated by the successive over relaxation (SOR) method at t = 0.44 second (see animations).