Vorticity streamfunction Method for a Lid-driven Flow in a Cavity

1. Numerical Methodology

Assuming that a flow is incompressible and Newtonian, the governing equations read as

$$\nabla \cdot \vec{\boldsymbol{u}} = 0$$

$$\frac{\partial \vec{\boldsymbol{u}}}{\partial t} + \vec{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \vec{\boldsymbol{u}} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{\boldsymbol{u}}$$

For a two dimensional flow, the governing equations in the Cartesian coordinate system are described by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Introduce the concepts of stream function $\psi(t, x, y)$ and vorticity $\omega(t, x, y)$, which are defined by

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Thus, the governing equations in the form of stream function and vorticity are given by,

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

$$\omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

where Re is the Reynolds number, $\mathrm{Re} = UL/\nu$.

Finite difference method: forward time centered space (FTCS) scheme, which is first-order and explicit in time, and second-order in space.

$$u_{i,j}^n = \frac{\psi_{i,j+1}^n - \psi_{i,j-1}^n}{2\Delta y}$$

$$v_{i,j}^n = -\frac{\psi_{i+1,j}^n - \psi_{i-1,j}^n}{2\Delta x}$$

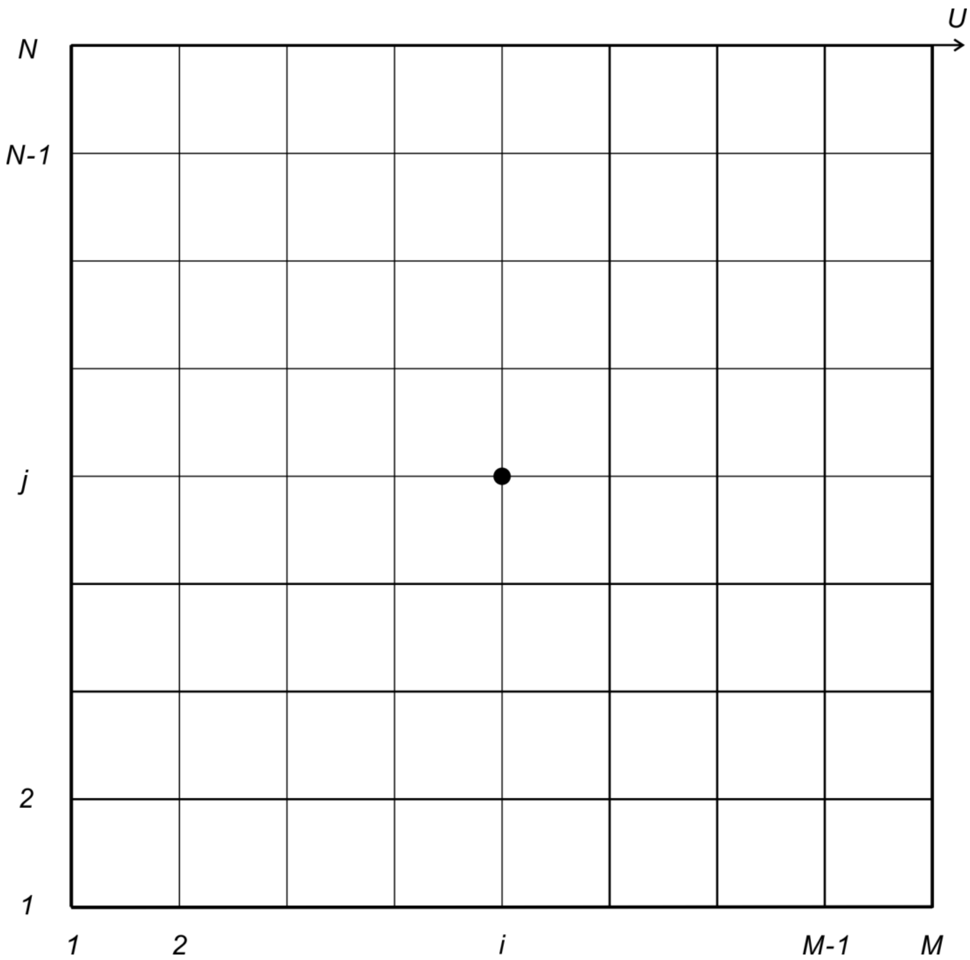
$$\frac{\omega_{i,j}^{n+1}-\omega_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\frac{\omega_{i+1,j}^{n}-\omega_{i-1,j}^{n}}{2\Delta x}+v_{i,j}^{n}\frac{\omega_{i,j+1}^{n}-\omega_{i,j-1}^{n}}{2\Delta y}$$
$$=\frac{1}{\mathrm{Re}}\left[\frac{\omega_{i+1,j}^{n}-2\omega_{i,j}^{n}+\omega_{i-1,j}^{n}}{(\Delta x)^2}+\frac{\omega_{i,j+1}^{n}-2\omega_{i,j}^{n}+\omega_{i,j-1}^{n}}{(\Delta y)^2}\right]$$

$$\psi_{i,j}^{n+1}=\frac{(\Delta x\Delta y)^2}{2(\Delta x)^2+2(\Delta y)^2}\left[\omega_{i,j}+\frac{\psi_{i+1,j}+\psi_{i-1,j}}{(\Delta x)^2}+\frac{\psi_{i,j+1}+\psi_{i,j-1}}{(\Delta y)^2}\right]$$

2. Case Study

2.1 Problem Statement

A lid-driven flow in a cavity with length *L* and discretized at Δh



2.2 Boundary Conditions

The stream function ψ on all boundary walls

$$\psi_b = 0$$

The vorticity ω on the boundary walls (Thom model – derived from second order Taylor series).

Left, right and bottom boundary:

$$\omega_b = -\frac{2(\psi_b' - \psi_b)}{(\Delta h)^2}$$

Top boundary:

$$\omega_b = -\frac{2(\psi_b' - \psi_b + \Delta h)}{(\Delta h)^2}$$

where ψ_b and ψ_b' represent the stream function on the boundary wall and the nodes adjacent to the boundary wall, respectively.

The stream function at a node adjacent to the left/right/bottom boundary can be approximated according to the Taylor series,

$$\phi_{2,j} = \phi_{1,j} + \left.\frac{\partial \phi}{\partial y}\right|_{1,j} \Delta y + \left.\frac{\partial^2 \phi}{\partial y^2}\right|_{1,j} \frac{(\Delta y)^2}{2} + O[(\Delta y)^3]$$

Thus, the vorticity on the left/right/bottom boundary can be described by

$$\omega_{1,j} = 2\frac{\phi_{1,j} - \phi_{2,j}}{(\Delta y)^2} + \frac{2u_{1,j}}{\Delta y} + O(\Delta y)$$

Similarly, the stream function at a node adjacent to the top boundary (lid) is given by

$$\phi_{N-1,j} = \phi_{N,j} - \left.\frac{\partial \phi}{\partial y}\right|_{N,j} \Delta y + \left.\frac{\partial^2 \phi}{\partial y^2}\right|_{N,j} \frac{(\Delta y)^2}{2} + O[(\Delta y)^3]$$

Thus, the vorticity on the top boundary (lid) is given by

$$\omega_{N,j} = 2\frac{\phi_{N,j} - \phi_{N-1,j}}{(\Delta y)^2} - \frac{2u_{N,j}}{\Delta y} + O(\Delta y)$$

where, $u_{N,j}$ is the moving velocity of the lid.

2.3 Results

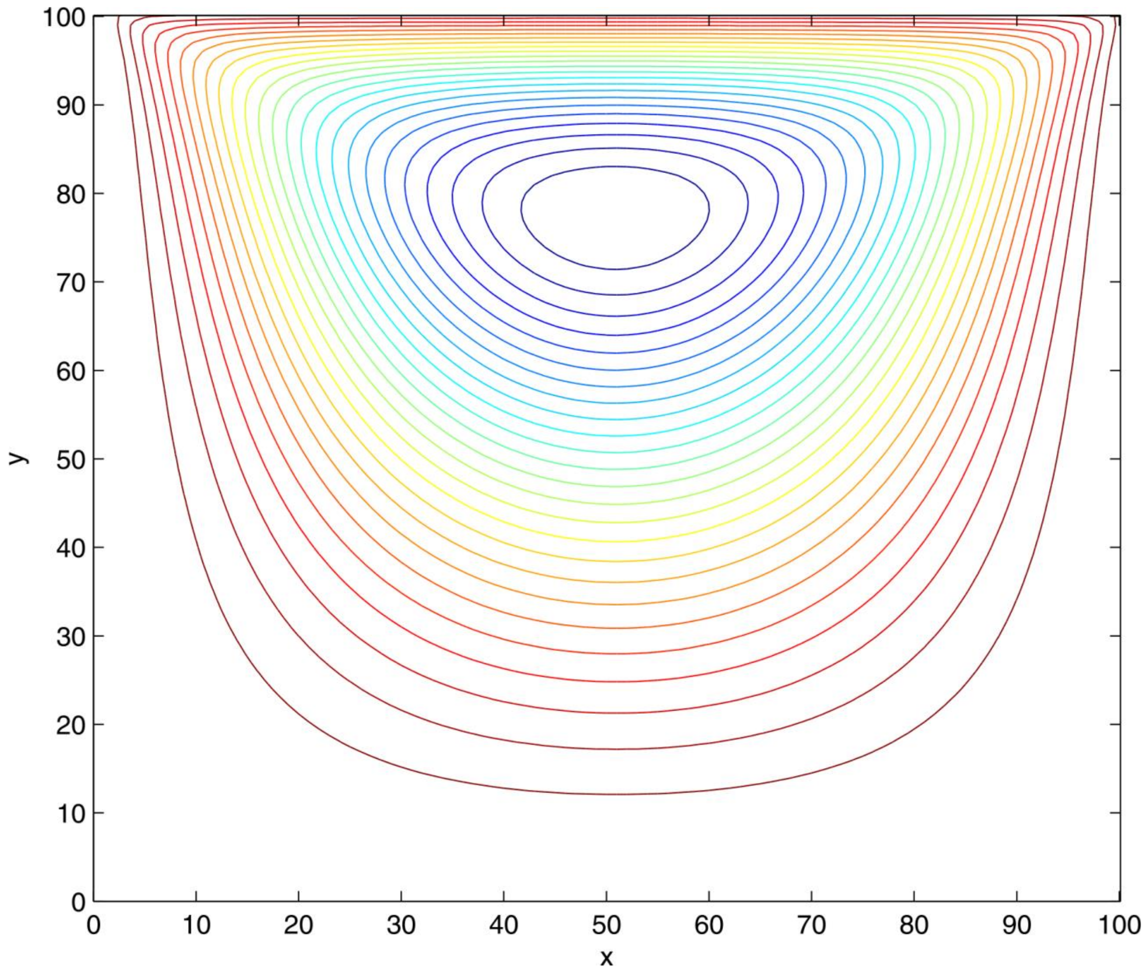


Fig. 1 Streamlines in a cavity driven by a lid at Re = 1

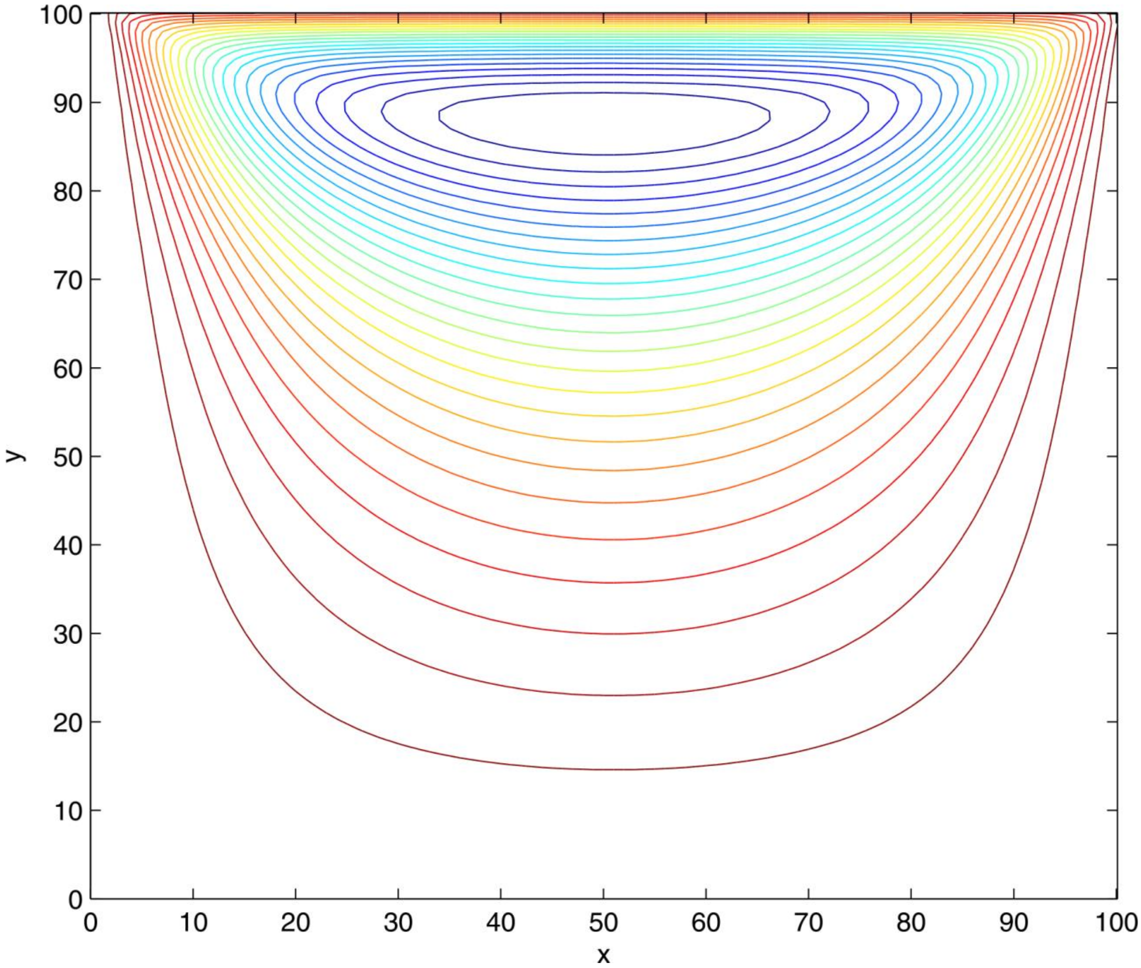


Fig. 2 Streamlines in the cavity driven by a lid at Re = 10