Digital Image Processing

Chap 9: Morphological Image Processing

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Morphological Image Processing

- **Mathematic morphology**: a tool for extracting image components, such as boundaries, skeletons, and the convex hull.
- Morphological filtering
- Morphological Thinning
- Morphological Pruning

Preliminaries

- The language in mathematical morphology is **set theory**
- **Sets** in mathematical morphology represents **objects** in image.
- In **binary images**, the sets are members of the 2-D integer space Z^2, where each element of a set is a tuple (2-D vector) whose coordinates are the (x, y) coordinates of a black (or white) pixel in the image.

Preliminaries

- Let *A* be a set in Z^2, if $a = (a_1, a_2)$ is an element of *A* then $a \in A$.
- The set with no element is called the null or empty set and is denoted as $\emptyset$.
- If every element of a set A is also an element of B then *A* is a subset of *B*, denoted as $A \subseteq B$
- The **union**: $C = A \cup B$
- The **intersection**: $D = A \cap B$
- Two set are **mutually exclusive** or **disjoint** (they have no common element) then $A \cap B = \emptyset$

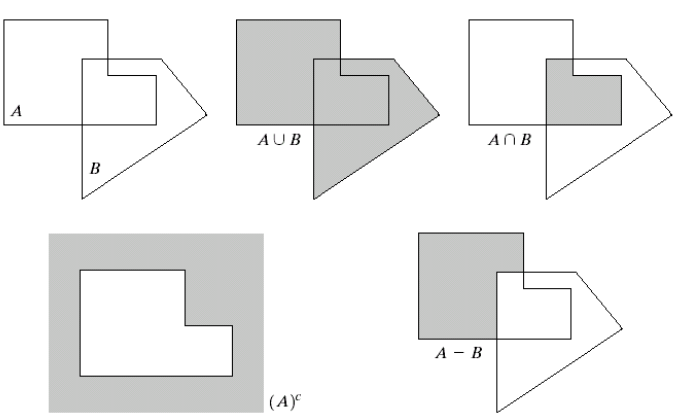
Preliminaries

- The **complement** of a set A is the set of elements not contained in A as $A^c = \{w | w \notin A\}$
- The **difference** of two sets is the set of elements that belong to A but not to B, denoted as

$$A - B = \{w | w \in A, w \notin B\} = A \cap B^c$$

- The **reflection** of set A is $\hat{A} = \{w | w = -a, \text{for } a \in A\}$
- The **translation** of set A by a point $z = (z_1, z_2)$ as $(A)_z = \{c | c = a + z, \text{for } a \in A\}$

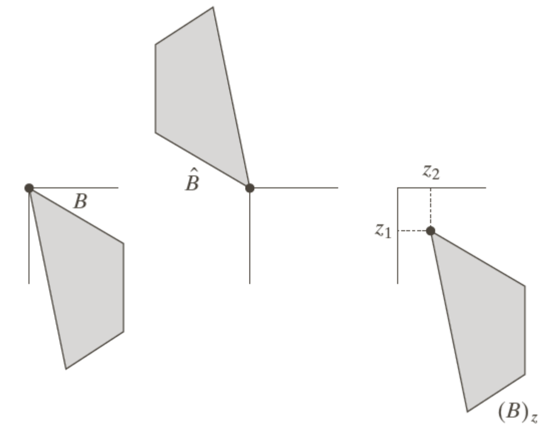
Preliminaries



a b c
d e

(a) Two sets A and B. (b) The union of A and B. (c) The intersection of A and B. (d) The complement of A. (e) The difference between A and B.

Preliminaries



a b c

FIGURE 9.1 (a) A set, (b) its reflection, and (c) its translation by z.

Dilation and Erosion

- The **dilation** and **erosion** are two fundamental operations in morphological processing
- The **dilation** of A by B is $A \oplus B = \left\{ z \middle| \left(\hat{B}\right)_z \cap A \neq \emptyset \right\}$
- The set of all displacements z, such that $\hat{B}$ and A overlap by at least one element.
- It can be rewritten as $A \oplus B = \left\{ z \middle| \left[\left(\hat{B}\right)_z \cap A \right] \subseteq A \right\}$
- Set B is commonly referred to as the **structuring element** in dilation.

Dilation and Erosion

- The **erosion** of A by B is denoted as
$$A \ominus B = \{z | (B)_z \subseteq A\}$$
- The set of all points z such that B, translated by z, is contained in A.
- **Dilation** and **erosion** are **duals** of each other
$$(A \ominus B)^c = A^c \oplus \hat{B}$$
- Proof:
  - Starting with $(A \ominus B)^c = \{z | (B)_z \subseteq A\}^c$
  - $(B)_z \subseteq A \longrightarrow (B)_z \cap A^c = \emptyset$
  - Then $(A \ominus B)^c = \{z | (B)_z \cap A^c = \emptyset\}^c = \{z | (B)_z \cap A^c \neq \emptyset\}$
  - Therefore $(A \ominus B)^c = A^c \oplus \hat{B}$

Structuring Elements

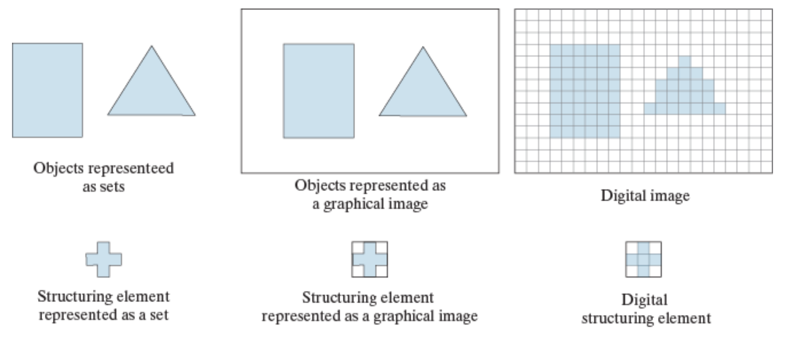
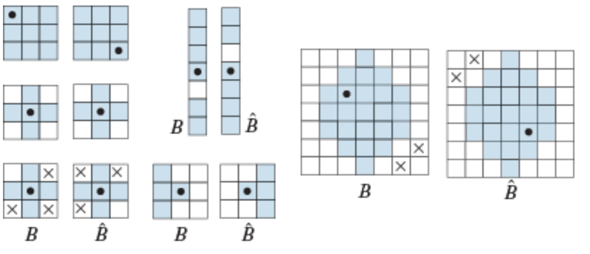


FIGURE 9.1 Top row. *Left:* Objects represented as graphical sets. *Center:* Objects embedded in a background to form a graphical image. *Right:* Object and background are digitized to form a digital image (note the grid). Second row: Example of a structuring element represented as a set, a graphical image, and finally as a digital SE.

Structuring Elements

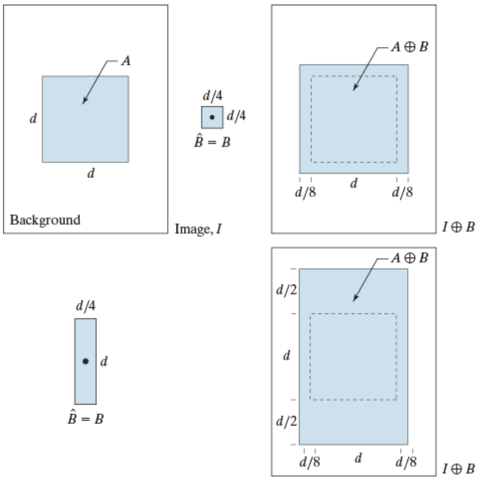
FIGURE 9.2 Structuring elements and their reflections about the origin (the ×'s are don't care elements, and the dots denote the origin). Reflection is rotation by 180° of an SE about its origin.



Dilation

a b c
d e

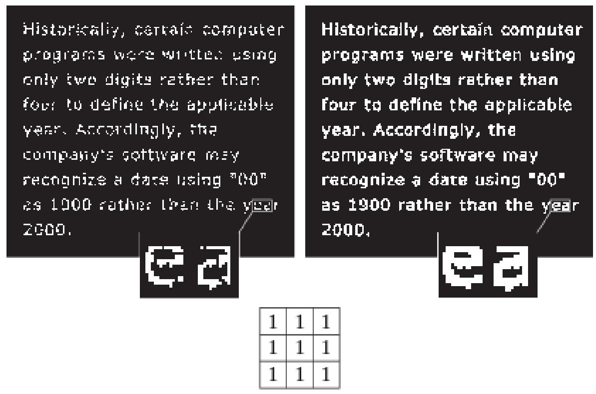
FIGURE 9.6 (a) Image I, composed of set (object) A and background. (b) Square SE (the dot is the origin). (c) Dilation of A by B (shown shaded). (d) Elongated SE. (e) Dilation of A by this element. The dotted line in (c) and (e) is the boundary of A, shown for reference.



Dilation

a b c

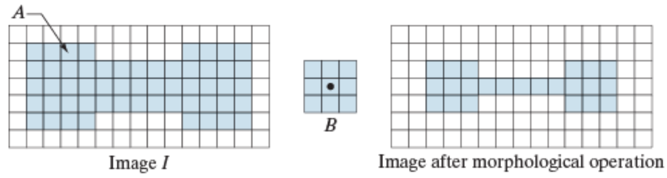
FIGURE 9.7
(a) Low-resolution text showing broken characters (see magnified view).
(b) Structuring element.
(c) Dilation of (a) by (b). Broken segments were joined.



Erosion

a b c

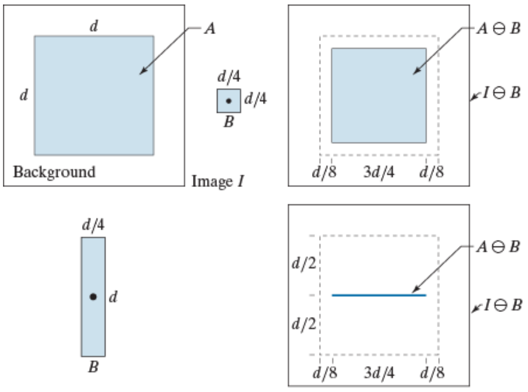
FIGURE 9.3
(a) A binary image containing one object (set), A. (b) A structuring element, B.
(c) Image resulting from a morphological operation (see text).



Erosion

a b c
d e

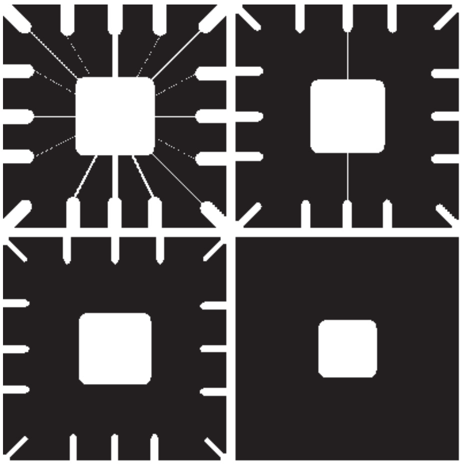
FIGURE 9.4
(a) Image I, consisting of a set (object) A, and background.
(b) Square SE, B (the dot is the origin).
(c) Erosion of A by B (shown shaded in the resulting image).
(d) Elongated SE.
(e) Erosion of A by B. (The erosion is a line.) The dotted border in (c) and (e) is the boundary of A, shown for reference.



Erosion

a b
c d

FIGURE 9.5
Using erosion to remove image components.
(a) A 486×486 binary image of a wire-bond mask in which foreground pixels are shown in white.
(b)–(d) Image eroded using square structuring elements of sizes 11×11, 15×15, and 45×45 elements, respectively, all valued 1.



Opening and Closing

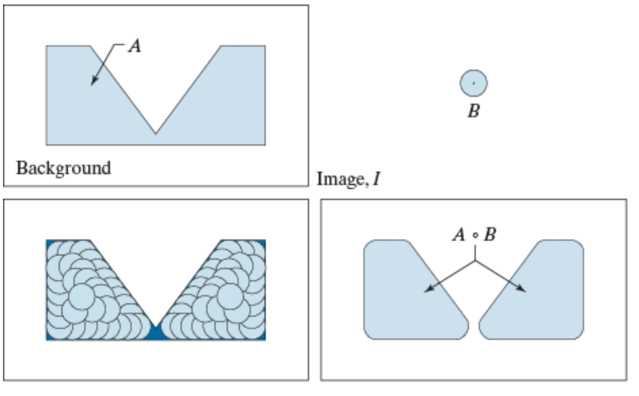
- **Opening** smooths the contour of an object, breaks narrow isthmuses (地狹) and eliminates thin protrusion (突出).
- Opening: $A \circ B = (A \ominus B) \oplus B$
- Geometric interpretation for **opening**: the boundary of $A \circ B$ is established by the point in B that reaches the farthest into boundary of A as B is rolled around the inside of this boundary.
- Opening A by B is obtained by taking the union of all translates of B that fit into A.

$$A \circ B = \cup \{(B)_z | (B)_z \subseteq A\}$$

Opening and Closing

a b
c d

FIGURE 9.8
(a) Image I, composed of set (object) A and background.
(b) Structuring element, B.
(c) Translations of B while being contained in A. (A is shown dark for clarity.)
(d) Opening of A by B.

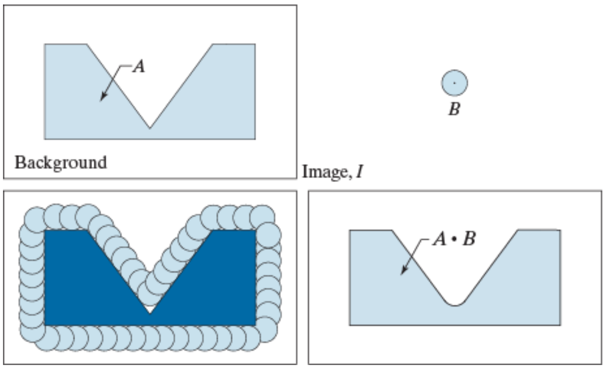


Opening and Closing

- **Closing** also tends to smooth contour, but it fuses narrow breaks and long thin gulfs, eliminate small holes, and fill gaps in the contour.
- Closing: $A \cdot B = (A \oplus B) \ominus B$
- Opening and closing are duals of each other.
- Geometrical interpretation of **closing**: a point w is an element of $A \cdot B$ if and only if $(B)_z \cap A \neq \emptyset$ for any translation of $(B)_z$ that contains w.

$$(A \cdot B)^c = A^c \circ \hat{B}$$

Opening and Closing



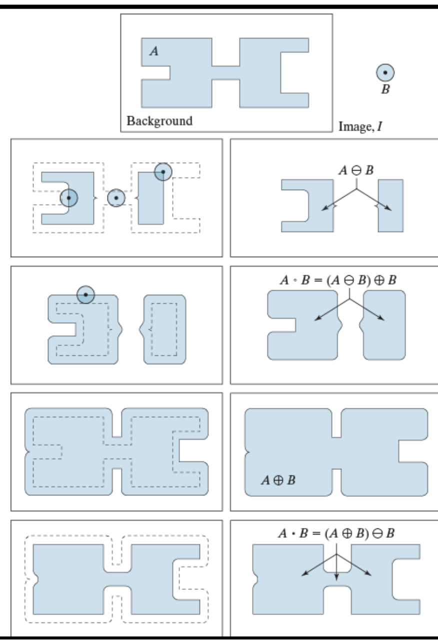
a b
c d

FIGURE 9.9 (a) Image I, composed of set (object) A, and background. (b) Structuring element B. (c) Translations of B such that B does not overlap any part of A. (A is shown dark for clarity.) (d) Closing of A by B.

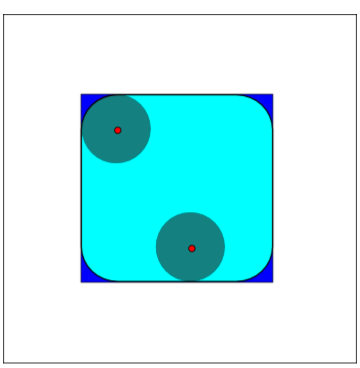
Opening and Closing

a
b c
d e
f g
h i

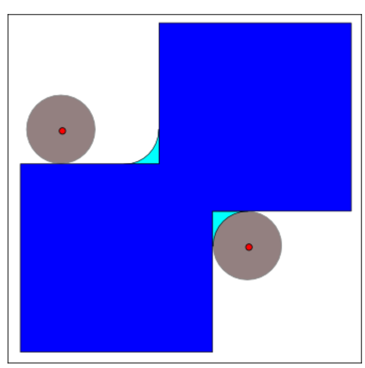
FIGURE 9.10 Morphological opening and closing. (a) Image I, composed of a set (object) A and background; a solid, circular structuring element is shown also. (The dot is the origin.) (b) Structuring element in various positions. (c)-(i) The morphological operations used to obtain the opening and closing.



Opening and Closing



Opening



Closing

Opening and Closing

Opening

Closing

Opening and Closing

Properties of opening and closing

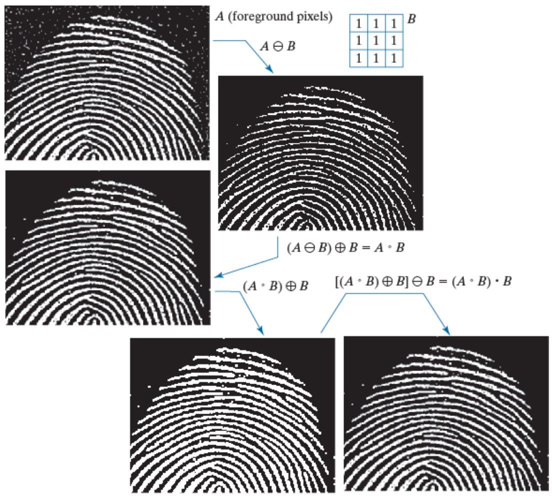
Opening

- $A \circ B$ is a subset of A
- If C is a subset of D, then $C \circ B$ is a subset of $D \circ B$
- $(A \circ B) \circ B = A \circ B$

Closing

- A is a subset of $A \cdot B$
- If C is a subset of D, then $C \cdot B$ is a subset of $D \cdot B$.
- $(A \cdot B) \cdot B = A \cdot B$

Opening and Closing



a b
d c
e f

FIGURE 9.11
(a) Noisy image. (b) Structuring element. (c) Eroded image. (d) Dilation of the erosion (opening of A). (e) Dilation of the opening. (f) Closing of the opening. (Original image courtesy of the National Institute of Standards and Technology.)

Hit or Miss Transformation

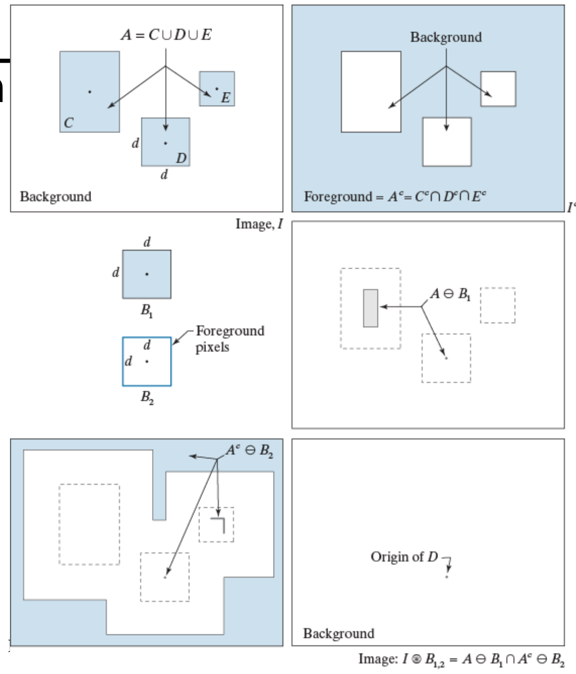
- Goal: Find the **location** of the shape X in $A = X \cup Y \cup Z$
- Let X be enclosed by a small window W.
- $W - X$: the **local background** of X with respect to W
- $A \ominus X$ may be viewed geometrically as the set of all locations of the origin X at which X found a **match** (or **hit**) in A.
- Let $B = (B_1, B_2)$, $B_1 = X$, $B_2 = W - X$
- The match of B in A is denoted as

  $A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2) = (A \ominus X) \cap [A^c \ominus (W - X)]$
- By using the definition of set difference, i.e.,

  $A - B = A \cap B^c$ and the duality between the erosion and dilation, i.e., $(A \ominus B)^c = A^c \oplus \hat{B}$

  we have $A \circledast B = (A \ominus B_1) - (A \oplus \hat{B}_2)$

Hit or Miss Transformation



a b
c d
e f

FIGURE 9.12 (a) Image consisting of a foreground (1's) equal to the union, A, of set of objects, and a background of 0's. (b) Image with its foreground defined as A^c. (c) Structuring elements designed to detect object D. (d) Erosion of A by B_1. (e) Erosion of A^c by B_2. (f) Intersection of (d) and (e), showing the location of the origin of D, as desired. The dots indicate the origin of their respective components. Each dot is a single pixel.

Basic Morphological Algorithms

- **Operations**: extracting boundaries, connected components, the convex hull, and the skeleton of a region,……
- **Examples**:

  Boundary extraction

  Hole filling

  Thinning

  Thickening

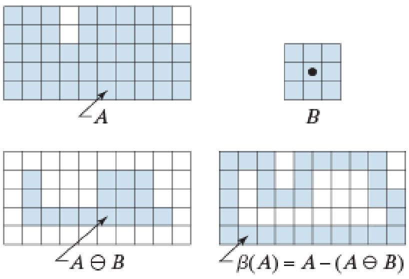
  Pruning

  etc.

Boundary Extraction

- The boundary of set A, denoted as $\beta(A)$, can be obtained by first eroding A by B and then performing the set difference between A and its erosion as

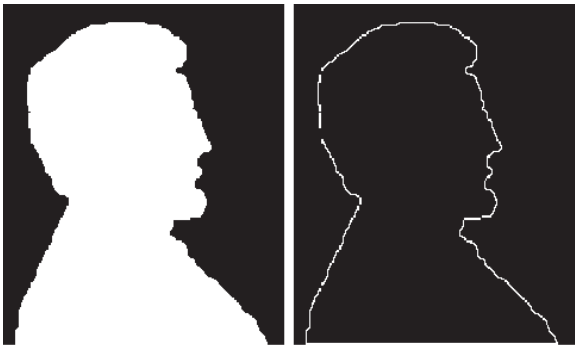
$$\beta(A) = A - (A \ominus B)$$



a b
c d

FIGURE 9.15 (a) Set, A, of foreground pixels. (b) Structuring element. (c) A eroded by B. (d) Boundary of A.

Boundary Extraction



a b

FIGURE 9.16 (a) A binary image. (b) Result of using Eq. (9-18) with the structuring element in Fig. 9.15(b).

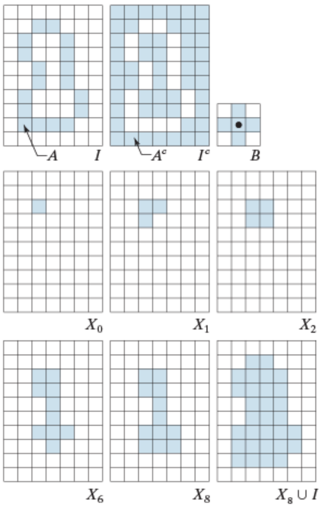
Hole Filling

- A denotes a set containing a subset whose elements are 8-connected **boundary points** of a region.
- Beginning with a point p inside the boundary, the objective is to fill the entire region with 1's.
- Assume that all non-boundary points are labeled 0.
- The filling iteration as

$$X_k = (X_{k-1} \oplus B) \cap A^c \quad k = 1, 2, 3, \ldots.$$

  where $X_0 = p$ and B is a **symmetric structure element**.
- The iteration stops at step k when $X_k = X_{k-1}$.
- The set union of X_k and A contains the filled holes and their boundaries.
- The dilation process $(X_{k-1} \oplus B)$ is constrained by A^c, which limits the result to inside the region of interest.

Hole Filling



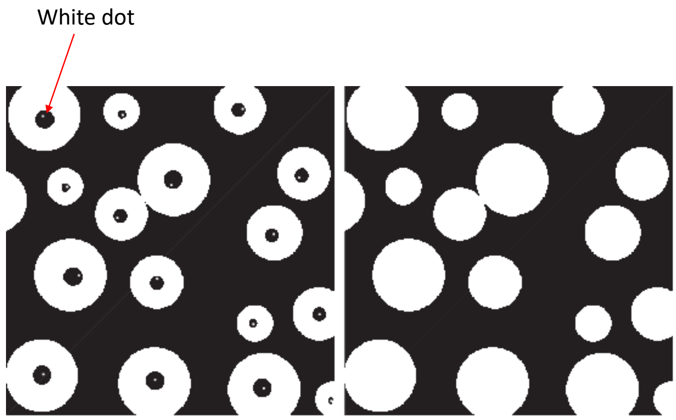
a b c
d e f
g h i

FIGURE 9.17 Hole filling. (a) Set A (shown shaded) contained in image I. (b) Complement of I. (c) Structuring element B. *Only the foreground elements are used in computations* (d) Initial point inside hole, set to 1. (e)–(h) Various steps of Eq. (9-19). (i) Final result [union of (a) and (h)].

Hole Filling

a b

FIGURE 9.18
(a) Binary image. The white dots inside the regions (shown enlarged for clarity) are the starting points for the hole-filling algorithm.
(b) Result of filling all holes.



Extraction of Connected Components

- Let Y be a connected component contained in set *A* and point p of Y is known.
- The following iteration yields all the elements of Y:

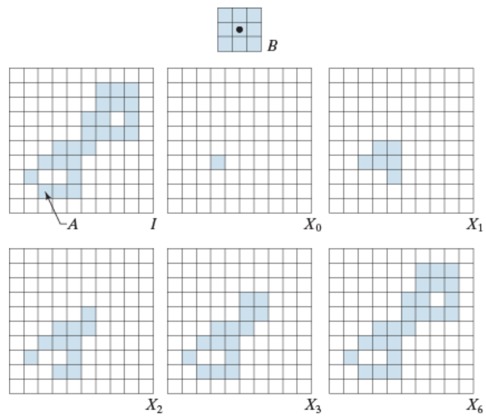
$$X_k = (X_{k-1} \oplus B) \cap A \quad k = 1,2,3,\ldots$$

  where $X_0 = p$ and B is a suitable structuring element
- If $X_k = X_{k-1}$, then the iteration converges and we let $Y = X_k$.

Extraction of Connected Components

a
b c d
e f g

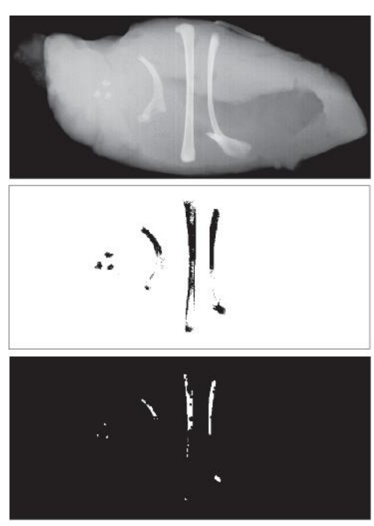
FIGURE 9.19
(a) Structuring element.
(b) Image containing a set with one connected component.
(c) Initial array containing a 1 in the region of the connected component.
(d)–(g) Various steps in the iteration of Eq. (9-20)



Extraction of Connected Components

a
b
c d

FIGURE 9.20
(a) X-ray image of a chicken filet with bone fragments.
(b) Thresholded image (shown as the negative for clarity).
(c) Image eroded with a 5×5 SE of 1's.
(d) Number of pixels in the connected components of (c).
(Image (a) courtesy of NTB Elektronische Geraete GmbH, Diepholz, Germany, www.ntbxray.com.)



Connected component	No. of pixels in connected comp
01	11
02	9
03	9
04	39
05	133
06	1
07	1
08	743
09	7
10	11
11	11
12	9
13	9
14	674
15	85

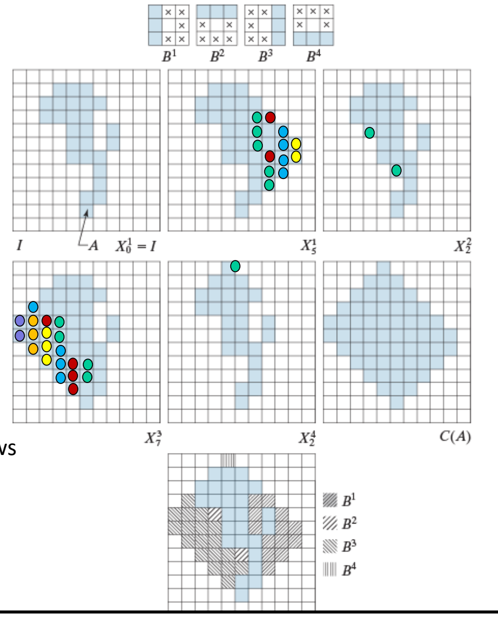
Convex Hull

- A set A is said to be **convex** if the straight line segment joining any two points in A lines entirely within A
- The **convex hull** H of an arbitrary set S is the smallest convex set containing S
- The set difference $H - S$ is called convex deficiency of S
- Let B^i, $i = 1{\sim}4$ represent four structure elements, we have $X_k^i = \left(X_{k-1}^i \circledast B^i\right) \cup A$ where k =1, 2, 3,… "$\circledast$" is the **hit-or-miss** operation, and $X_0^i = A$.
- Now let $D^i = X_{\text{conv}}^i$
- Subscript "conv" indicates the convergence ($X_k^i = X_{k-1}^i$)
- The convex hull of A is $C(A) = \bigcup_{i=1}^{4} D^i$

Convex Hull

a
b c d
e f g
h

FIGURE 9.21 (a) Structuring elements. (b) Set A. (c)–(f) Results of convergence with the structuring elements shown in (a). (g) Convex hull. (h) Convex hull showing the contribution of each structuring element.

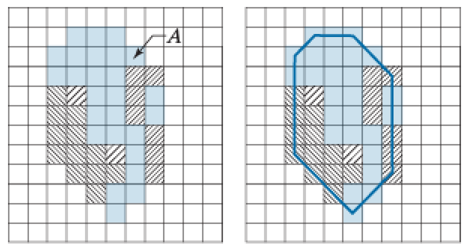


Shortcoming: the convex hull grows beyond the minimum dimensions required to guarantee convexity.

Convex Hull

a b

FIGURE 9.22
(a) Result of limiting growth of the convex hull algorithm.
(b) Straight lines connecting the boundary points show that the new set is convex also.



Thinning

- The thinning of a set A by a structuring element, denoted $A \otimes B$, can be defined in terms of **hit-or-miss** operation:

  $$A \otimes B = A - (A \circledast B) = A \cap (A \circledast B)^c$$

- Thinning A (**symmetrically**) is based on a sequence of structuring elements: $\{B\} = \{B^1, B^2, B^3, \cdots, B^n\}$ where B^i is a rotated version of B^{i-1}.
- **Thinning** by a sequence of structure elements as $A \otimes \{B\} = ((\cdots((A \otimes B^1) \otimes B^2)\cdots) \otimes B^n)$

Thinning

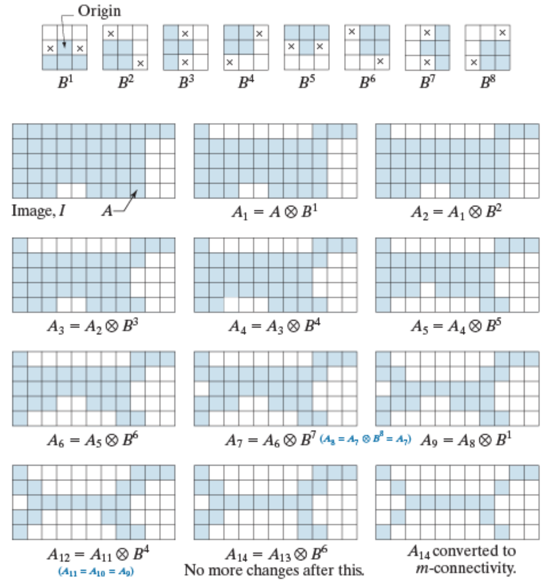


FIGURE 9.23 (a) Structuring elements. (b) Set A. (c) Result of thinning A with B^1 (shaded). (d) Result of thinning A_1 with B_2. (e)–(i) Results of thinning with the next six SEs. (There was no change between A_7 and A_8.) (j)–(k) Result of using the first four elements again. (l) Result after convergence. (m) Result converted to m-connectivity.

Thickening

- **Thickening** is dual of thinning, and is defined as $A \odot B = A \cup (A \circledast B)$ where B is a structure element suitable for thickening.
- The structure elements for thickening are the same as the ones for thinning with all 1's *and* 0's interchanged.
- Thickening of A = Thinning the background of A and then complement the results (see Fig. 9.24)

Thickening

a b
c d
e

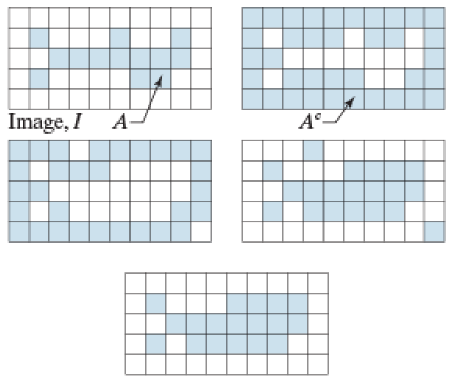


FIGURE 9.24
(a) Set A.
(b) Complement of A.
(c) Result of thinning the complement.
(d) Thickened set obtained by complementing (c).
(e) Final result, with no disconnected points.

Skeleton

- The **skeleton** of set A is denoted as $S(A)$ as shown in Fig. 9.25. It has the properties as
  - (a) If z is a point of $S(A)$, and $(D)_z$ is the largest disk centered at z and contained in A. If one cannot find a larger disk, then $(D)_z$ is called a **maximum disk**.
  - (b) The disk $(D)_z$ touches the boundary of A at two or more different places.
- The skeleton of A can be expressed in terms of **erosions** and **openings**, as

$S(A) = \bigcup_{k=0}^{K} S_k(A)$ with $S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B$

where B is a structure element, and $(A \ominus kB)$ indicates k successive erosions of A as

$$(A \ominus kB) = (\cdots((A \ominus B) \ominus B) \ominus \cdots) \ominus B$$

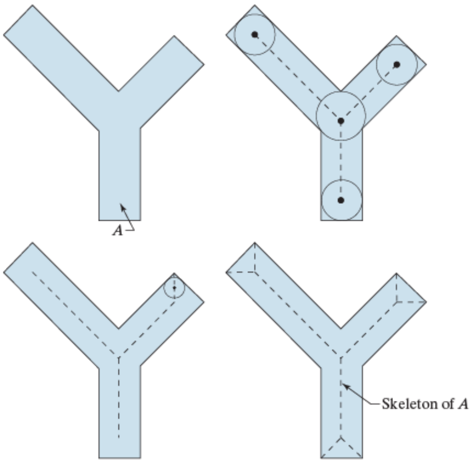
Skeleton

- K is the last iteration step before A erodes to an empty set, i.e., $K = \max\{k \mid (A \ominus kB) \neq \emptyset\}$.
- $S(A)$ is a union of the skeleton subset $S_k(A)$
- A can be reconstructed from these subsets as

$$A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$$

where $S_k(A) \oplus kB$ denotes the k successive dilations of $S_k(A)$ as
$S_k(A) \oplus kB = ((\cdots(S_k(A) \oplus B) \oplus B) \oplus \cdots) \oplus B$

Skeleton



a b
c d

FIGURE 9.25 (a) Set A. (b) Various positions of maximum disks whose centers partially define the skeleton of A. (c) Another maximum disk, whose center defines a different segment of the skeleton of A. (d) Complete skeleton (dashed).

Skeleton

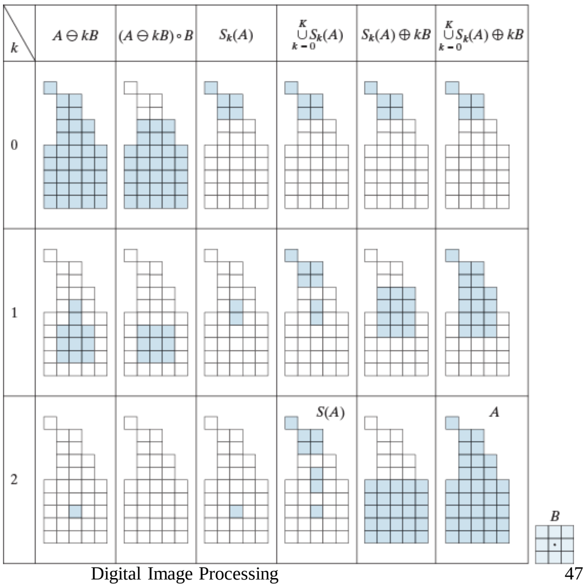


FIGURE 9.26 Implementation of Eqs. (9-28) through (9-33). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.

Pruning

- Pruning method is an essential process to "**clean-up**" the **parasitic components** after thinning and skeletonizing algorithms.
- Use **thinning** to detect the end points

  $$X_1 = A \otimes \{B\}$$

  where $\{B\}$ is a set of **structure elements**
- *Restore* the character to its original form

  → it requires forming a set X_2 containing **end points** in X_1 (Fig. 9.27(e))

  $$X_2 = \bigcup_{i=1}^{8} \left(X_1 * B^k \right)$$

Pruning

- Apply dilation of the end point three times and use A as a delimiter as (Fig. 9.27(f))

$$X_3 = (X_2 \oplus H) \cap A$$

where H is a 3×3 structure element of 1's

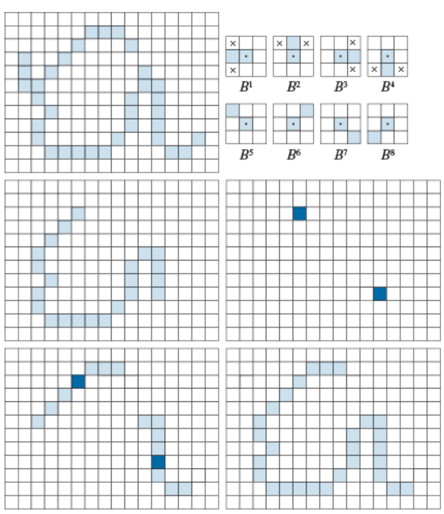
- Finally the union of X_3 and X_1 yields the desired result.

$$X_4 = X_3 \cup X_1$$

Pruning

a b
c d
e f

FIGURE 9.27 (a) Set A of foreground pixels (shaded). (b) SEs used for deleting end points. (c) Result of three cycles of thinning. (d) End points of (c). (e) Dilation of end points conditioned on (a). (f) Pruned image.



Morphological Reconstruction

- Morphological Reconstruction: two images and one structure element.
- **Marker**: Contains the starting point for the transformation
- **Mask**: Constraints the transformation
- **Structure element**: Defines the connectivity

Morphological Reconstruction

Geodesic dilation

- Let F denote the Marker image, G is the Mask image. F and G are binary image and $F \subseteq G$
- The **geodesic dilation** of size 1 of the marker image (F) with respect to mask (G) is defined as

$$D_G^1(F) = (F \oplus B) \cap G$$

where $\cap$ denotes the set intersection (or logical AND)

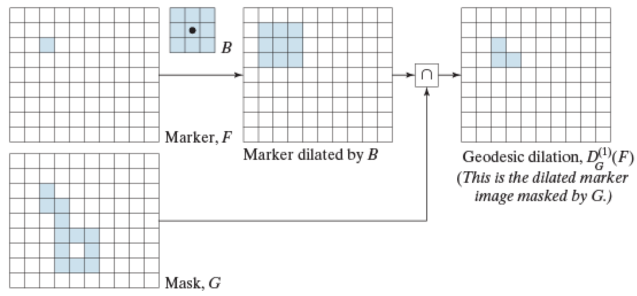
- The **geodesic dilation** of size n of F w.r.t. to G is

$$D_G^n(F) = D_G^1\left[D_G^{n-1}(F)\right]$$

with $D_G^0(F) = F$

Morphological Reconstruction

FIGURE 9.28 Illustration of a geodesic dilation of size 1. Note that the marker image contains a point from the object in G. If continued, subsequent dilations and maskings would eventually result in the object contained in G.



Morphological Reconstruction

Geodesic Erosion

- The Geodesic erosion of size 1 of the marker image (F) with respect to mask (G) is defined as

$$E_G^1(F) = (F \ominus B) \cup G$$

where $\cup$ denotes the set intersection (or logical OR)

- The geodesic dilation of size n of F w.r.t. to G is

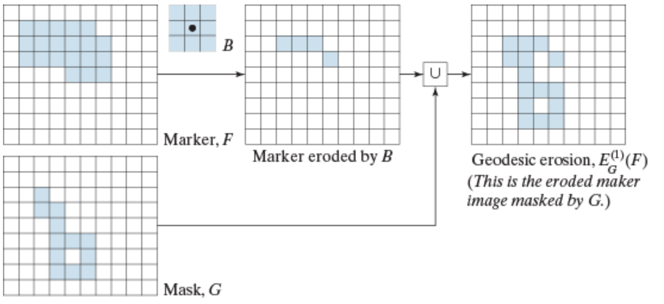
$$E_G^n(F) = E_G^1\left[E_G^{n-1}(F)\right]$$

with $E_G^0(F) = F$

- Geodesic dilation and erosion are duals w.r.t. **set complement**. (Prob. 9.29)

Morphological Reconstruction

FIGURE 9.29 Illustration of a geodesic erosion of size 1.



Morphological Reconstruction

Morphological Reconstruction by Dilation

- The morphological reconstruction by dilation of a mask image G from a marker image F is defined as

$$R_G^D(F) = D_G^{(k)}(F)$$

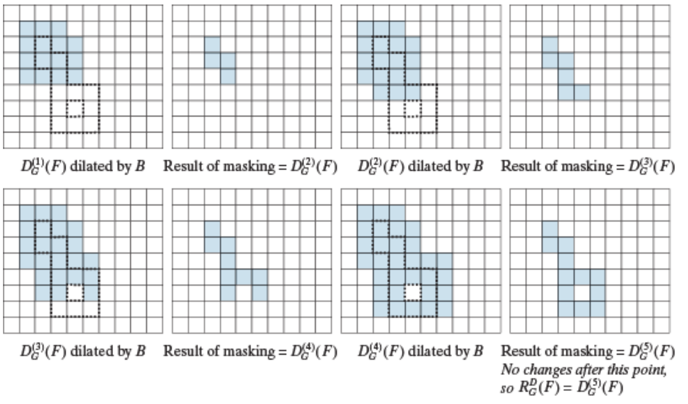
with k such that

$$D_G^{(k)}(F) = D_G^{(k+1)}(F)$$

Morphological Reconstruction

a b c d
e f g h

FIGURE 9.30 Illustration of morphological reconstruction by dilation. Sets $D_G^{(1)}(F)$, G, B and F are from Fig. 9.28. The mask (G) is shown dotted for reference.



Morphological Reconstruction

Morphological Reconstruction by Erosion

- The morphological reconstruction by erosion of a mask image G from a marker image F is defined as

$$R_G^E(F) = E_G^{(k)}(F)$$

with k such that

$$E_G^{(k)}(F) = E_G^{(k+1)}(F)$$

Morphological Reconstruction

Opening by Reconstruction

- The opening by reconstruction restores *exactly* the shape of objects that remains after erosion.
- The opening by reconstruction of size n of an image F is defined as

$$O_G^{(n)}(F) = R_F^D[(F \ominus nB)]$$

where $(F \ominus nB)$ indicate n erosions of F by B.

Morphological Reconstruction

Extracting the characters that contain long, vertical strokes

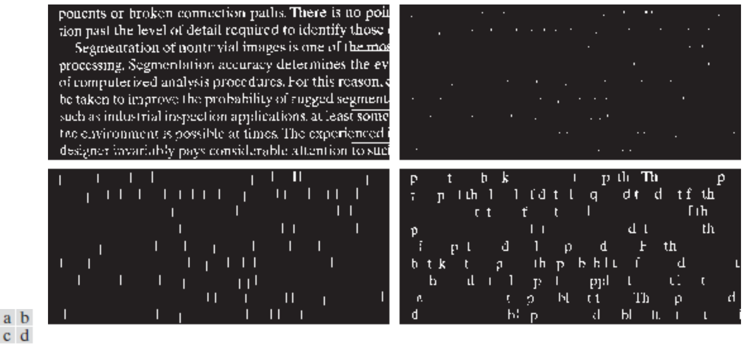


FIGURE 9.31 (a) Text image of size 918×2018 pixels. The approximate average height of the tall characters is 51 pixels. (b) Erosion of (a) with a structuring element of size 51×1 elements (all 1's). (c) Opening of (a) with the same structuring element, shown for comparison. (d) Result of opening by reconstruction.

Morphological Reconstruction

Filling Holes

- Let $I(x, y)$ denote a binary image and form a marker image F that is 0 everywhere, except at the image border, where it is set to $1 - I$, that is

$$F(x, y) = \begin{cases} 1 - I(x, y) & \text{if } (x, y) \text{ is on the border of } I \\ 0 & \text{otherwise} \end{cases}$$

- Then

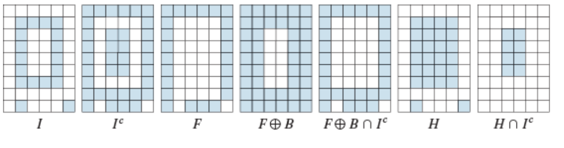
$$H = \left[R_{I^c}^{D}(F)\right]^c$$

is a binary image equal to I with all holes filled.

Morphological Reconstruction

a b c d e f g

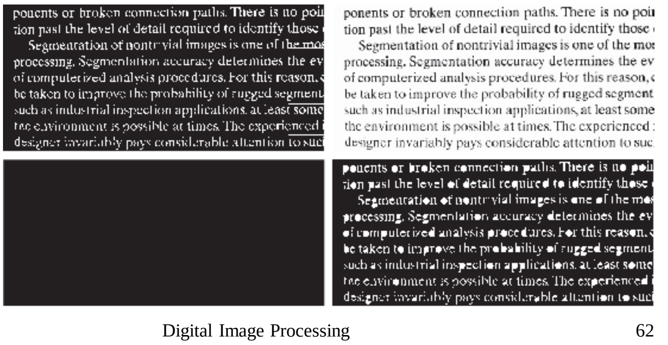
FIGURE 9.32 Hole filling using morphological reconstruction.



with 3×3 SE all 1s

a b
c d

FIGURE 9.33 (a) Text image of size 918 × 2018 pixels. (b) Complement of (a) for use as a mask image. (c) Marker image. (d) Result of hole-filling using Eqs. (9-45) and (9-46).



Morphological Reconstruction

Border Cleaning

- Let $I(x, y)$ be a mask image and a marker image F defined as

$$F(x,y) = \begin{cases} I(x,y) & \text{if } (x,y) \text{ is on the border of } I \\ 0 & \text{otherwise} \end{cases}$$

- Then, we may have

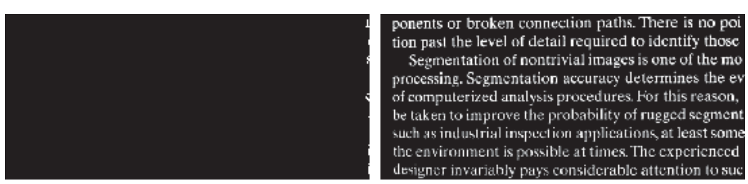
$$X = I - R_I^D(F)$$

where X is an image with no objects touching the border.

Morphological Reconstruction

a b

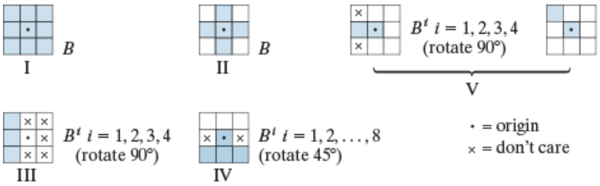
FIGURE 9.34 (a) Reconstruction by dilation of marker image. (b) Image with no objects touching the border. The original image is Fig. 9.31(a).



with 3×3 SE all 1s

Summary of Morphological Operations

FIGURE 9.35 Five basic types of structuring elements used for binary morphology.



Summary of Morphological Operations

TABLE 9.1 Summary of binary morphological operations and their properties. A is a set of foreground pixels contained in binary image I, and B is a structuring element. I is a binary image (containing A), with 1's corresponding to the elements of A and 0's elsewhere. The Roman numerals refer to the structuring elements in Fig. 9.35.

Operation	Equation	Comments
Translation	$(B)_z = \{c \mid c = b + z, \text{ for } b \in B\}$	Translates the origin of B to point z.
Reflection	$\hat{B} = \{w \mid w = -b, \text{ for } b \in B\}$	Reflects B about its origin.
Complement	$A^c = \{w \mid w \notin A\}$	Set of points not in A.
Difference	$A - B = \{w \mid w \in A, w \notin B\} = A \cap B^c$	Set of points in A, but not in B.
Erosion	$A \ominus B = \{z \mid (B)_z \subseteq A\}$	Erodes the boundary of A. (I)
Dilation	$A \oplus B = \{z \mid (\hat{B})_z \cap A \neq \varnothing\}$	Dilates the boundary of A. (I)
Opening	$A \circ B = (A \ominus B) \oplus B$	Smoothes contours, breaks narrow isthmuses, and eliminates small islands and sharp peaks. (I)
Closing	$A \bullet B = (A \oplus B) \ominus B$	Smoothes contours, fuses narrow breaks and long thin gulfs, and eliminates small holes. (I)
Hit-or-miss transform	$I \circledast B = \{z \mid (B)_z \subseteq I\}$	Finds instances of B in image I. B contains *both* foreground and background elements.
Boundary extraction	$\beta(A) = A - (A \ominus B)$	Set of points on the boundary of set A. (I)
Hole filling	$X_k = (X_{k-1} \oplus B) \cap I^c$ $k = 1, 2, 3, \ldots$	Fills holes in A. X_0 is of same size as I, with a 1 in each hole and 0's elsewhere. (II)
Connected components	$X_k = (X_{k-1} \oplus B) \cap I$ $k = 1, 2, 3, \ldots$	Finds connected components in I. X_0 is a set, the same size as I, with a 1 in each connected component and 0's elsewhere. (I)
Convex hull	$X_k^i = (X_{k-1}^i \circledast B^i) \cup X_{k-1}^i$; $i = 1, 2, 3, 4$ $k = 1, 2, 3, \ldots$ $X_0^i = I$; $D^i = X_{conv}^i$; $C(A) = \bigcup_{i=1}^{4} D^i$	Finds the convex hull, $C(A)$, of a set, A, of foreground pixels contained in image I. X_{conv}^i means that $X_k^i = X_{k-1}^i$. (III)

Summary of Morphological Operations

TABLE 9.1
(Continued)

Operation	Equation	Comments
Thinning	$A \otimes B = A - (A \circledast B)$ $= A \cap (A \circledast B)^c$ $A \otimes \{B\} =$ $((\ldots((A \otimes B^1) \otimes B^2)\ldots) \otimes B^n)$ $\{B\} = \{B^1, B^2, B^3, \ldots, B^n\}$	Thins set A. The first two equations give the basic definition of thinning. The last two equations denote thinning by a sequence of structuring elements. This method is normally used in practice. (IV)
Thickening	$A \odot B = A \cup (A \circledast B)$ $A \odot \{B\} =$ $((\ldots((A \odot B^1) \odot B^2)\ldots) \odot B^n)$	Thickens set A using a sequence of structuring elements, as above. Uses (IV) with 0's and 1's reversed.
Skeletons	$S(A) = \bigcup_{k=0}^{K} S_k(A)$ $S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B$ Reconstruction of A: $A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$	Finds the skeleton $S(A)$ of set A. The last equation indicates that A can be reconstructed from its skeleton subsets $S_k(A)$. K is the value of the iterative step after which the set A erodes to the empty set. The notation $(A \ominus kB)$ denotes the kth iteration of successive erosions of A by B. (I)
Pruning	$X_1 = A \otimes \{B\}$ $X_2 = \bigcup_{k=1}^{8} (X_1 \circledast B^k)$ $X_3 = (X_2 \oplus H) \cap A$ $X_4 = X_1 \cup X_3$	X_4 is the result of pruning set A. The number of times that the first equation is applied to obtain X_1 must be specified. Structuring elements (V) are used for the first two equations. In the third equation H denotes structuring element. (I)
Geodesic dilation–size 1	$D_G^{(1)}(F) = (F \oplus B) \cap G$	F and G are called the *marker* and the *mask* images, respectively. (I)
Geodesic dilation–size n	$D_G^{(n)}(F) = D_G^{(1)}(D_G^{(n-1)}(F))$	Same comment as above.
Geodesic erosion–size 1	$E_G^{(1)}(F) = (F \ominus B) \cup G$	Same comment as above.
Geodesic erosion–size n	$E_G^{(n)}(F) = E_G^{(1)}(E_G^{(n-1)}(F))$	Same comment as above.
Morphological reconstruction by dilation	$R_G^D(F) = D_G^{(k)}(F)$	With k is such that $D_G^{(k)}(F) = D_G^{(k+1)}(F)$.

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Summary of Morphological Operations

TABLE 9.1
(Continued)

Operation	Equation	Comments
Morphological reconstruction by erosion	$R_G^E(F) = E_G^{(k)}(F)$	With k such that $E_G^{(k)}(F) = E_G^{(k+1)}(F)$.
Opening by reconstruction	$O_R^{(n)}(F) = R_F^D(F \ominus nB)$	$F \ominus nB$ indicates n successive erosions by B, starting with F. The form of B is application-dependent.
Closing by reconstruction	$C_R^{(n)}(F) = R_F^E(F \oplus nB)$	$F \oplus nB$ indicates n successive dilations by B, starting with F. The form of B is application-dependent.
Hole filling	$H = [R_F^D(F)]^c$	H is equal to the input image I, but with all holes filled. See Eq. (9-45) for the definition of marker image F.
Border clearing	$X = I - R_I^D(F)$	X is equal to the input image I, but with all objects that touch (are connected to) the boundary removed. See Eq. (9-47) for the definition of marker image F.

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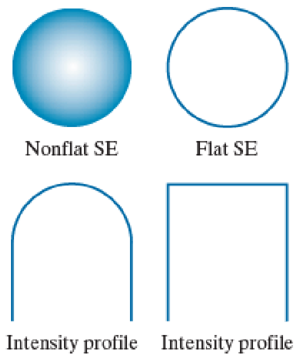
Gray-Level Morphology – SEs

a b
c d

FIGURE 9.36 Nonflat and flat structuring elements, and corresponding horizontal intensity profiles through their centers. All examples in this section are based on flat SEs.

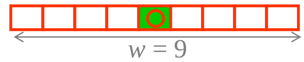


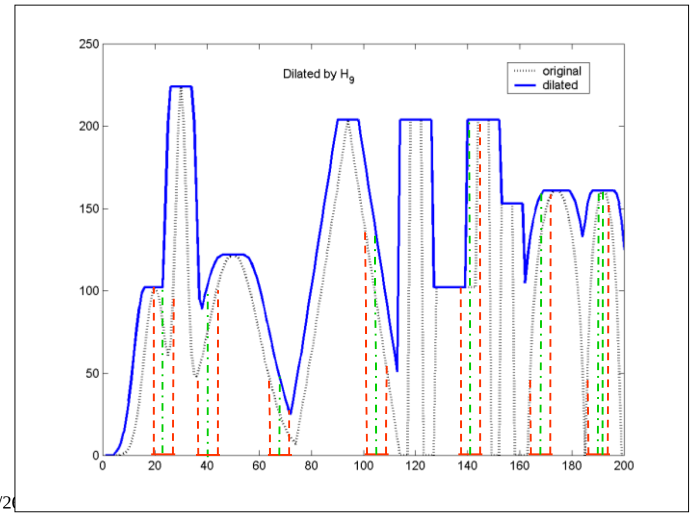
Extension to Gray-Level Image – Dilation

- **Gray-scale dilation** of f by b_N is defined as
  $$[f \oplus b_N](x, y) = \max_{(s,t)\in b_N} \{f(x-s, y-t) + b_N(s,t)\}$$
- Simplified 1-D function as
  $$[f \oplus b_N](x) = \max_{s\in b_N}\{f(x-s) + b_N(s)\}$$
- If the structuring element b is flat
  $$[f \oplus b](x, y) = \max_{(s,t)\in b} \{f(x-s, y-t)\}$$

Extension to Gray-Level Image – Dilation

$I \oplus H_9$



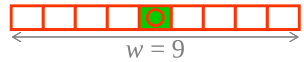


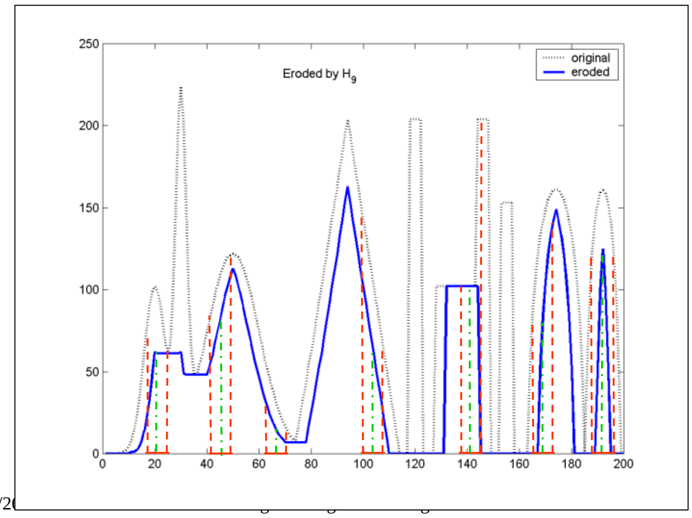
Extension to Gray-Level Image – Erosion

- **Gray-scale erosion** of f by b_N is defined as
  $$[f \ominus b_N](x,y) = \min_{(s,t)\in b_N}\{f(x+s,y+t) - b_N(s,t)\}$$
- Simplified 1-D function becomes
  $$[f \ominus b_N](x) = \min_{s\in b_N}\{f(x+s) - b_N(s)\}$$
- If the structuring element b is flat
  $$[f \ominus b](x,y) = \min_{(s,t)\in b}\{f(x+s,y+t)\}$$
- Erosion and dilation are dual operations
  $$(f \ominus b)^c(x,y) = \left(f^c \oplus \hat{b}\right)(x,y)$$
  $$f^c = -f(x,y);\ \hat{b} = b(-x,-y)$$

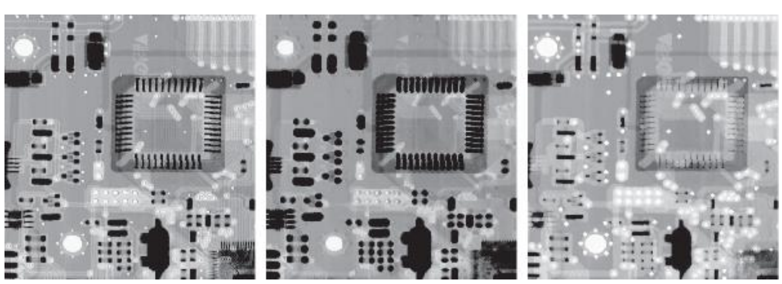
Extension to Gray-Level Image – Erosion

$I \ominus H_9$





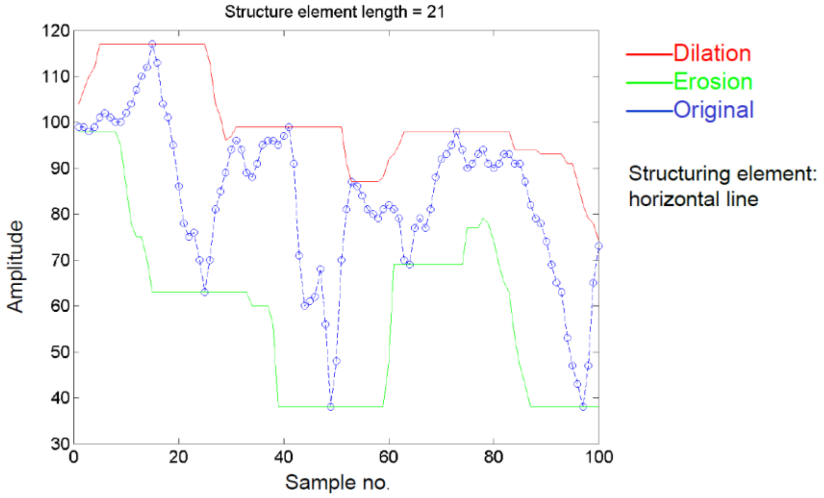
Extension to Gray-Level Image – Erosion



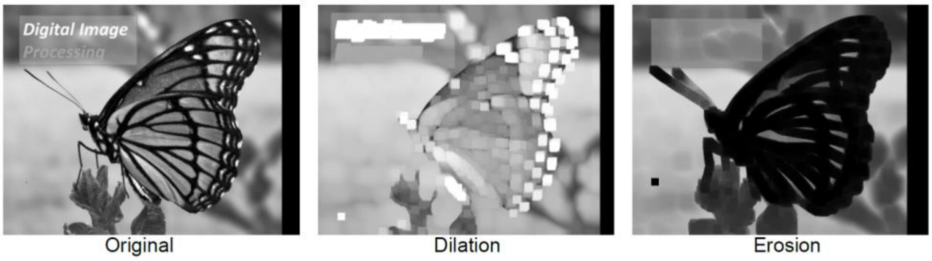
a b c

FIGURE 9.37 (a) Gray-scale X-ray image of size 448 × 425 pixels. (b) Erosion using a flat disk SE with a radius of 2 pixels. (c) Dilation using the same SE. (Original image courtesy of Lixi, Inc.)

Dilation vs. Erosion with Flat SE



Dilation vs. Erosion with Flat SE



Opening and Closing

- Opening: $f \circ b = (f \ominus b) \oplus b$
- Closing: $f \cdot b = (f \oplus b) \ominus b$

$$(f \cdot b)^c = f^c \circ \hat{b}$$

- $f^c = -f(x, y)$ and $\hat{b} = b(-x, -y)$

$$-(f \cdot b) = -f \circ \hat{b}$$

- Viewing $f(x, y)$ in 3-D perspective as a 2-D surface.
- Opening f by a spherical structure element, b, may be interpreted geometrically as the process of pushing the ball against the underside of the surface

Opening and Closing

- The opening of f by b is the surface of the highest points reached by any part of the sphere as it slides over the entire **under-surface** of f.
- Opening is to **remove the light details** of the image.
- The closing operation can be viewed as slide the ball on the top of the surface.
- Closing is to **remove the dark details** of the image.

Opening and Closing

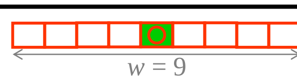
- The opening operation properties

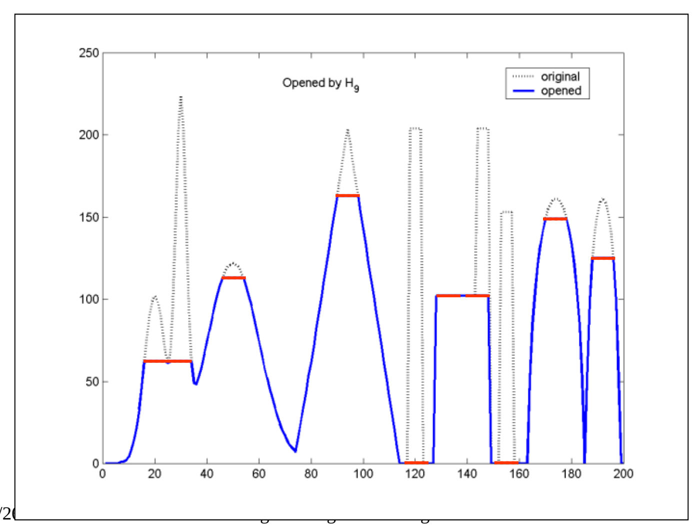
  1) $(f \circ b)$↲ f
  2) If f_1 ↲ f_2 then $(f_1 \circ b)$ ↲ $(f_2 \circ b)$
  3) $(f \circ b) \circ b = f \circ b$

  where e↲r indicates that the domain of e is a subset of the domain of r, and also that $e(x, y) \leq r(x, y)$ in the domain of e

Opening

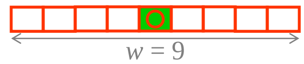
$I \circ H_9$

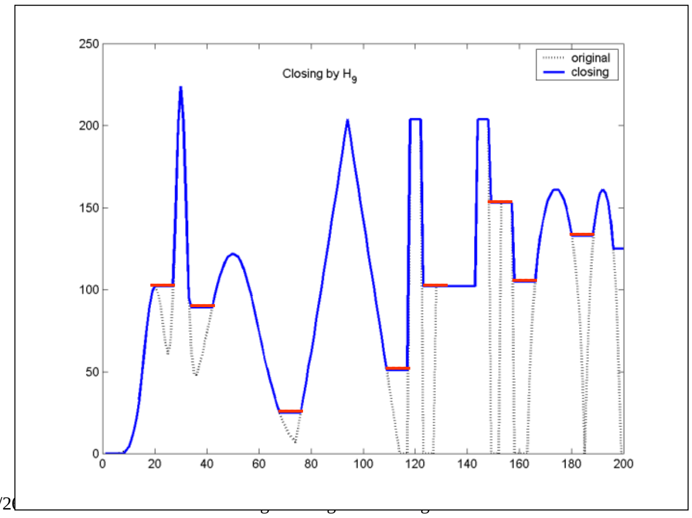




Closing

$$I \bullet H_9$$

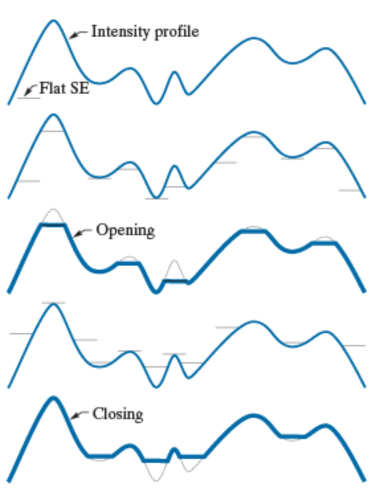




Opening and Closing

a
b
c
d
e

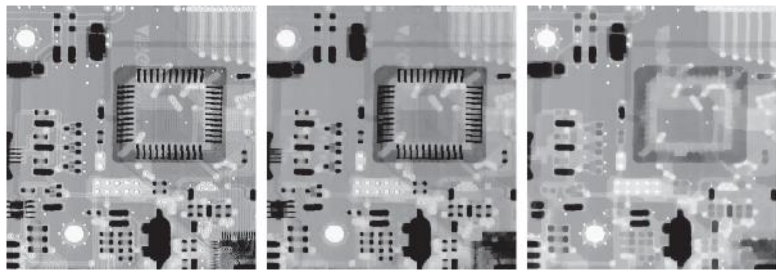
FIGURE 9.38 Grayscale opening and closing in one dimension. (a) Original 1-D signal. (b) Flat structuring element pushed up underneath the signal. (c) Opening. (d) Flat structuring element pushed down along the top of the signal. (e) Closing.



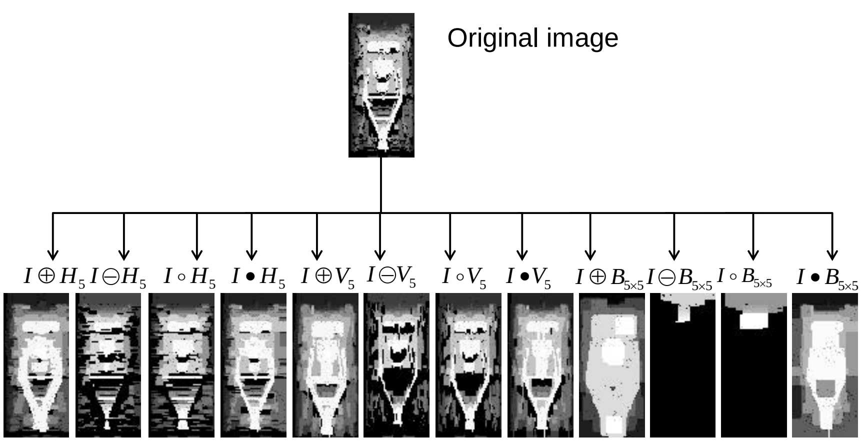
Opening and Closing

a b c

FIGURE 9.39
(a) A grayscale X-ray image of size 448×425 pixels.
(b) Opening using a disk SE with a radius of 3 pixels.
(c) Closing using an SE of radius 5.



Opening & Closing



Morphological Smoothing

- Morphological smoothing: apply opening and then closing to remove the bright and dark noise.
- Morphological Gradient

$$g = (f \oplus b) - (f \ominus b)$$

- Top-hat (white top-hat) transformation

$$T_{\text{hat}}(f) = f - (f \circ b)$$

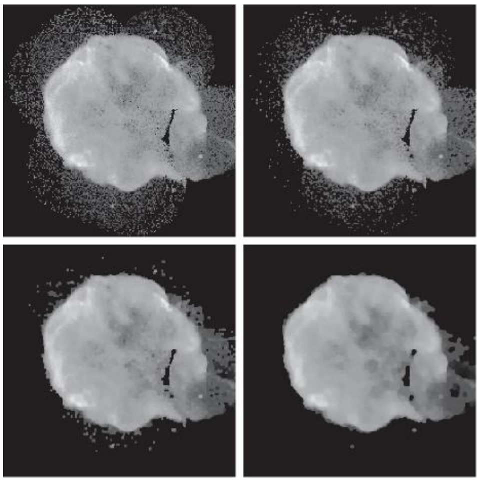
- Bottom-hat (black top-hat) transformation

$$B_{\text{hat}}(f) = (f \cdot b) - f$$

Morphological Smoothing

a b
c d

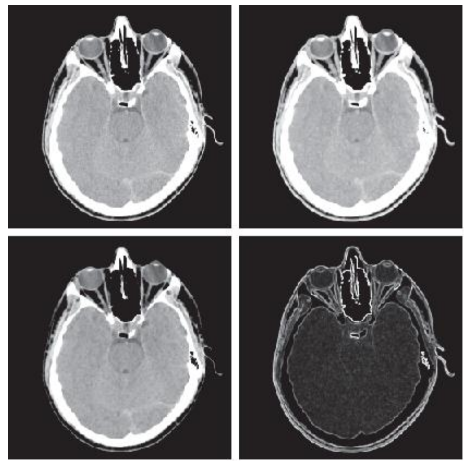
FIGURE 9.40 (a) 566 × 566 image of the Cygnus Loop supernova, taken in the X-ray band by NASA's Hubble Telescope. (b)–(d) Results of performing opening and closing sequences on the original image with disk structuring elements of radii, 1, 3, and 5, respectively. (Original image courtesy of NASA.)



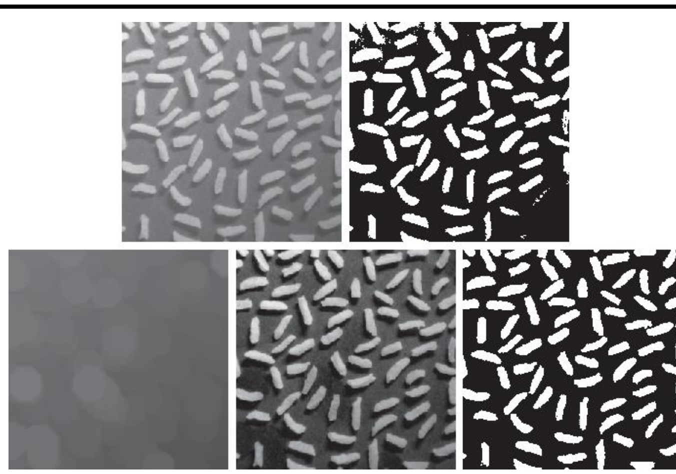
Morphological Gradient

a b
c d

FIGURE 9.41 (a) 512 × 512 image of a head CT scan. (b) Dilation. (c) Erosion. (d) Morphological gradient, computed as the difference between (b) and (c). (Original image courtesy of Dr. David R. Pickens, Vanderbilt University.)



Top-hat Transformation



a b
c d e

FIGURE 9.42 Using the top-hat transformation for *shading correction*. (a) Original image of size 600 × 600 pixels. (b) Thresholded image. (c) Image opened using a disk SE of radius 40. (d) Top-hat transformation (the image minus its opening). (e) Thresholded top-hat image.

Granulometry

- Granulometry: determining the size distribution of particles in an images:
  1) Opening with increasing size structure elements.
  2) Each difference between the original and the opened image is computed after each pass.
  3) These differences are normalized and used to construct a histogram of particle-size distribution.

Granulometry

a b c
d e f

FIGURE 9.43
(a) 531 × 675 image of wood dowels. (b) Smoothed image. (c)–(f) Openings of (b) with disks of radii equal to 10, 20, 25, and 30 pixels, respectively. (Original image courtesy of Dr. Steve Eddins, MathWorks, Inc.)

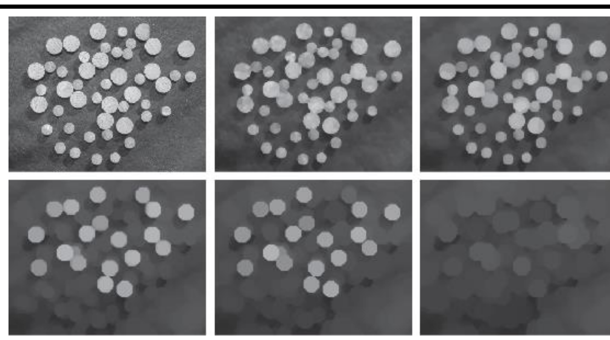
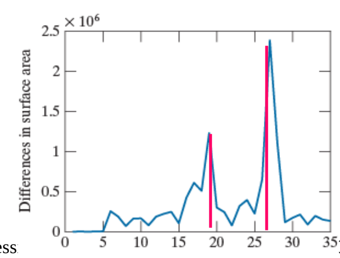


FIGURE 9.44
Differences in surface area as a function of SE disk radius, r. The two peaks indicate that there are two dominant particle sizes in the image.



Texture Segmentation

- Textural Segmentation:
  1) Close the input image by using successively larger structure elements.
  2) When the structure element=small blobs, they are removed and leaving only light background.
  3) A single *opening* is performed with a structure element that is large in relation to the separation between large blobs.
  4) Remove light patches between the blobs, and leave a dark region on the right.
  5) A light region on the left and dark region on the right.
  6) A simple threshold then yields the boundary between two texture regions.

Texture Segmentation

a b
c d

FIGURE 9.45 Textural segmentation. (a) A 600 × 600 image consisting of two types of blobs. (b) Image with small blobs removed by closing (a). (c) Image with light patches between large blobs removed by opening (b). (d) Original image with boundary between the two regions in (c) superimposed. The boundary was obtained using a morphological gradient.