CS 4501: Optimization - Assignment 3

Your name and email

Use Projected Gradient Descent (PGD) to minimize $f(x)$ starting from a given x_0 for n iterations.

$$x_{n+1} = P_C\left(x_n - \alpha_n \nabla f(x_n)\right).$$

Task 1

Projection to a centered ball $(x \in \mathbb{R}^N, r > 0)$

$$C = \{x | \|x\|_2 \le r\}$$

Proof. If $\|x\|_2 \le r$, then objectively we have

$$P_C(x) = x.$$

Now suppose $\|x\|_2 > r$. Then we have $P_C(x) = y$, where y is the solution for the following optimization problem.

$$\begin{aligned} \min_y \quad & \|x - y\|_2^2 \\ \text{subject to} \quad & \|y\|_2^2 \le r^2 \end{aligned}$$

The Lagrangian of this optimization problem is

$$L(y, \lambda) \doteq \|x - y\|_2^2 + \lambda(\|y\|_2^2 - r^2)$$

where $x \in \mathbb{R}^N, y \in \mathbb{R}^N, \lambda \in \mathbb{R}$. Then we need to find (y_*, λ_*) such that

$$\begin{aligned} \nabla_y L(y_*, \lambda_*) =& 0, \\ \nabla_\lambda L(y_*, \lambda_*) =& 0. \end{aligned}$$

Then this y_* is the solution to the optimization problem, i.e.,

$$P_C(x) = \begin{cases} x, & \|x\|_2 \le r, \\ y_*, & \|x\|_2 > r \end{cases}.$$

(SZ: Please compute y_* here) $\square$

Task 2

Projection to a noncentered ball ($x \in \mathbb{R}^N, c \in \mathbb{R}^N, r > 0$)

$$C = \{x | \|x - c\|_2 \leq r\}.$$

We have

$$P_C(x) = \begin{cases} x, & \|x - c\|_2 \leq r \\ c + \frac{r}{\|x-c\|_2}(x - c), & \text{otherwise} \end{cases}.$$

(SZ: No proof is required.)

Task 3

Projection to column space ($A \in \mathbb{R}^{N \times M}$ **has full column rank but not necessarily square)**

$$C = \left\{x | \exists y \in \mathbb{R}^M \quad \text{such that} \quad Ay = x\right\}.$$

Proof. (SZ: Please compute the analytical expression of $P_C(x)$ here) □

Notes

1. Third-party packages, excluding numpy, are not allowed.
2. The proof of Task 1 is 4 points. The proof of Task 3 is 2 points. The points for implementation are documented in the python script.