CS 4501: Optimization - Assignment 4

Your name and email

Task 1 (5 pts): Conjugate Function 1

Define

$$\Delta_n \doteq \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1, x_i > 0 \right\}.$$

Define $f : \Delta_n \to \mathbb{R}$ as

$$f(x) \doteq \sum_{i=1}^{n} x_i \ln x_i.$$

Then please compute the conjugate $f^*(y)$ of $f(x)$.

Proof.

$$f^*(y) = \max_{x \in \Delta_n} \left\{ y^\top x - f(x) \right\}.$$

Consider the Lagrangian

$$L(x, \lambda) = y^\top x - \sum_{i=1}^{n} x_i \ln x_i - \lambda(\sum x_i - 1).$$

Let (x_*, λ_*) satisfy

$$\begin{aligned} \nabla_\lambda L(x_*, \lambda_*) =& 0, \\ \nabla_x L(x_*, \lambda_*) =& 0. \end{aligned}$$

Then x_* is the maximizer of the constrained optimization problem. (SZ: Please complete the rest.) □

Task 2 (5 pts): Conjugate Function 2

Define a function $f : \mathbb{R}^n \to \mathbb{R}$ as

$$f(x) = \ln \sum_{i=1}^{n} \exp(x_i).$$

Please compute the conjugate $f^*(y)$ of $f(x)$.

Proof. (SZ: Please compete the computation.) □

Task 3 (5 pts): Smoothness

Let $\|\cdot\|$ be the ℓ_2 norm. Let C be a convex and closed set. Recall that

$$\begin{aligned} P_C(x) &\doteq \arg\min_{y \in C} \|y - x\|, \\ d_C(x) &\doteq \min_{y \in C} \|y - x\|, \\ \psi_C(x) &\doteq \frac{1}{2} d_C^2(x). \end{aligned}$$

Please prove that $\psi_C(x)$ is 1-smooth in $\|\cdot\|$. Hints:

1. You can use the analytical expression of $\nabla \psi_C(x)$ directly. We computed this in a lecture. This is meant to reward those actually attending lectures.
2. You can use the fact that $P_C(x)$ is firmly nonexpansive directly. We never discussed this property in lecutre and this is meant to practice your skill of "proof with Wikipedia" (cf. "coding with StackOverflow").

Proof. (SZ: Please complete the proof.) □