CS 4501: Optimization - Assignment 2

Your name and email

The .tex file for generating this PDF can be found here. Please fill your answer into the .tex file and submit the compiled PDF. Only PDF generated by LaTex will be graded.

1 Equivalence between norms

Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms defined in $\mathbb{R}^N$. We say $\|\cdot\|_a$ is *equivalent* to $\|\cdot\|_b$ if there exist some constants $C_1, C_2 \in (0, \infty)$ such that for any $x \in \mathbb{R}^N$, the following holds:

$$C_1\|x\|_a \leq \|x\|_b \leq C_2\|x\|_a.$$

If $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$, we write $\|\cdot\|_a \sim \|\cdot\|_b$.

(a, 2pt) Prove that $\|\cdot\|_2$ is equivalent to $\|\cdot\|_\infty$.

Proof. Write your proof here $\square$

(b, 3pt) Prove that for any norm $\|\cdot\|$, there exists a constant $C_1 \in (0, \infty)$ such that

$$\forall x \in \mathbb{R}^N,\ \|x\| \leq C_1\|x\|_1.$$

Hint: Express x using a set of bases.

Proof. Write your proof here $\square$

(c, 3pt) Prove that for any norm $\|\cdot\|$, there exists a constant $C_2 \in (0, \infty)$ such that

$$\forall x \in \mathbb{R}^N,\ \|x\| \geq C_2\|x\|_1.$$

Hint: You can use the following two facts:

- $\|x\|$ is continous in x,
- the set $B \doteq \{x | \|x\|_1 = 1\}$ is complete.

Proof. Write your proof here $\square$

(d, 2pt) Prove that if $\|\cdot\|_a \sim \|\cdot\|_b$ and $\|\cdot\|_b \sim \|\cdot\|_c$, then $\|\cdot\|_a \sim \|\cdot\|_c$.

Proof. Write your proof here $\square$

Now it is easy to see that any two norms in $\mathbb{R}^N$ are equivalent.

2 Submultiplicity of Induced Norms

(a, 3pt) Let $A \in \mathbb{R}^{N\times N}$ and $\|\cdot\|$ denote both a vector norm and the corresponding induced matrix norm. Prove that $\|Ax\| \le \|A\|\|x\|$ holds for any $x \in \mathbb{R}^N$.

Proof. Write your proof here $\square$

(b, 2pt) Let $A, B \in \mathbb{R}^{N\times N}$ and $\|\cdot\|$ be an induced norm. Prove that $\|AB\| \le \|A\|\|B\|$

Proof. Write your proof here $\square$