CS 4501: Optimization - Assignment 2

Your name and email

1 Equivalence between norms

Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms defined in $\mathbb{R}^N$. We say $\|\cdot\|_a$ is *equivalent* to $\|\cdot\|_b$ if there exist some constants $C_1, C_2 \in (0, \infty)$ such that for any $x \in \mathbb{R}^N$, the following holds:

$$C_1\|x\|_a \leq \|x\|_b \leq C_2\|x\|_a.$$

If $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$, we write $\|\cdot\|_a \sim \|\cdot\|_b$.

(a, 2pt) Prove that $\|\cdot\|_2$ is equivalent to $\|\cdot\|_\infty$.

Proof. Write your proof here $\square$

(b, 3pt) Prove that for any norm $\|\cdot\|$, there exists a constant $C_1 \in (0, \infty)$ such that

$$\forall x \in \mathbb{R}^N,\ \|x\| \leq C_1\|x\|_1.$$

Hint: Express x using a set of bases.

Proof. Write your proof here $\square$

(c, 3pt) Prove that for any norm $\|\cdot\|$, there exists a constant $C_2 \in (0, \infty)$ such that

$$\forall x \in \mathbb{R}^N,\ \|x\| \geq C_2\|x\|_1.$$

Hint: You can use the following two facts:

- $\|x\|$ is continous in x,
- the set $B \doteq \{x \mid \|x\|_1 = 1\}$ is complete.

Proof. Write your proof here $\square$

(d, 2pt) Prove that if $\|\cdot\|_a \sim \|\cdot\|_b$ and $\|\cdot\|_b \sim \|\cdot\|_c$, then $\|\cdot\|_a \sim \|\cdot\|_c$.

Proof. Write your proof here $\square$

Now it is easy to see that any two norms in $\mathbb{R}^N$ are equivalent.

2 Submultiplicity of Induced Norms

(a, 3pt) Let $A \in \mathbb{R}^{N\times N}$ and $\|\cdot\|$ denote both a vector norm and the corresponding induced matrix norm. Prove that $\|Ax\| \leq \|A\|\|x\|$ holds for any $x \in \mathbb{R}^N$.

Proof. Write your proof here $\square$

(b, 2pt) Let $A, B \in \mathbb{R}^{N\times N}$ and $\|\cdot\|$ be an induced norm. Prove that $\|AB\| \leq \|A\|\|B\|$

Proof. Write your proof here $\square$