# CH5170: Process Optimization HW2

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CH18B067

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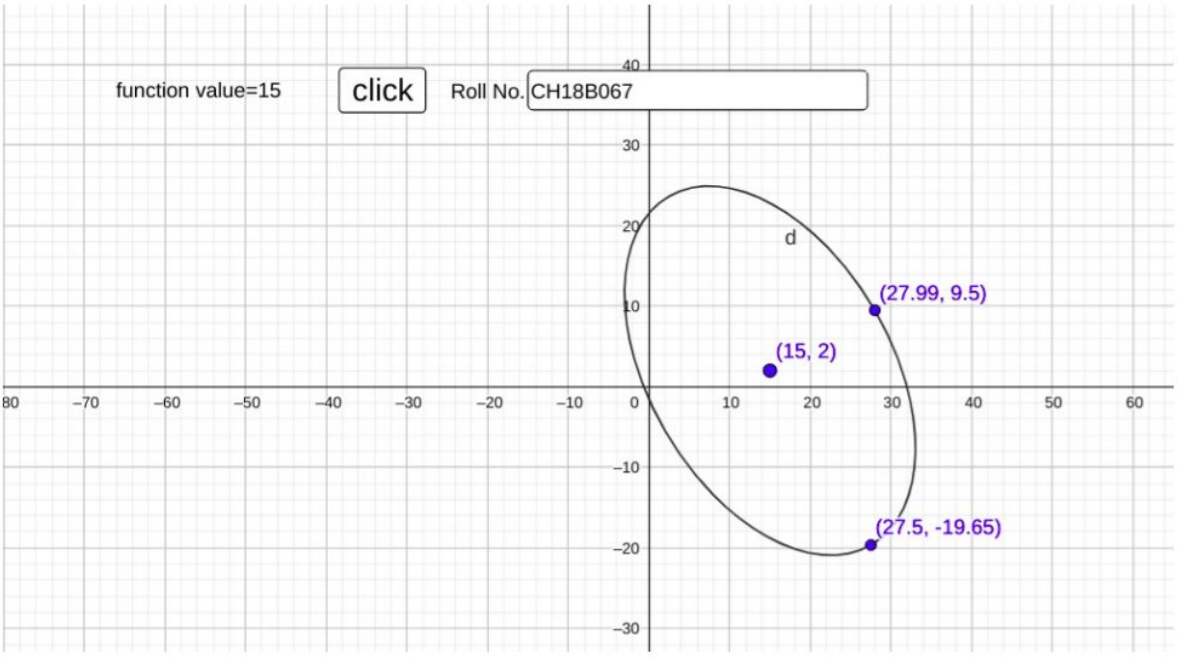

1) From the given graph,

$x^* =$ centre $= (15, 2)$

~~point~~

point on major axis $= (27.5, -19.65)$

point on minor axis $= (27.99, 9.5)$

Value of the function at the contour $= 15$

$\therefore a =$ semi-major axis $= 24.9993$

$b =$ semi-minor axis $= 14.9997$.

we know that,

$HV_1 = \lambda_1 V_1$ where $H =$ Hessian matrix of $f(x)$.

$V_1 =$ eigenvector corresponding to major axis

$\lambda_1 =$ eigenvalue corresponding to $V_1$

Similarly $HV_2 = \lambda V_2$.

$\therefore HV = V\Lambda$ where $V = [V_1 \; V_2]$

$H = V\Lambda V^{-1}$

$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

$V_1 = \begin{bmatrix} -12.5 \\ 21.65 \end{bmatrix}$

$V_2 = \cancel{\begin{bmatrix} -12.99 \\ -7.5 \end{bmatrix}} \quad \begin{bmatrix} 12.99 \\ 7.5 \end{bmatrix}$

$\therefore \cancel{H = \begin{bmatrix} -12.5 \\ 21.65 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}$

$$H = \begin{bmatrix} -12.5 & 12.99 \\ 21.65 & 7.5 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} -12.5 & 12.99 \\ 21.65 & 7.5 \end{bmatrix}^{-1}$$

we know that

$$\left(\frac{\lambda_1}{\lambda_2}\right) = \left(\frac{\text{semi-minor axis}}{\text{semi-major axis}}\right)^2$$

$$\frac{\lambda_1}{\lambda_2} = 0.36.$$

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$$\therefore H = \begin{bmatrix} 0.84 & 0.2771 \\ 0.2771 & 0.52 \end{bmatrix} \lambda_2$$

$$a = -Hx^*$$
$$= \begin{bmatrix} -13.1542 \\ -5.1970 \end{bmatrix} \lambda_2$$

From derivation in class we know that

$$f(x^*) = \text{contour value} - \tfrac{1}{2}$$
$$f(x^*) = 15 - 0.5$$
$$= 14.5$$

$$f(x) - f(x^*) = \frac{1}{2}(x - x^*)^T H (x - x^*) \quad (1)$$

Also,

$$f(x) = c + a^T x + \frac{1}{2} x^T H x \quad (2)$$

On comparing (1) and (2).

$$c = f(x^*) + \frac{1}{2} x^{*T} H x^*$$

$\therefore$ ~~c = 14.5 +~~ $\therefore\ c = 14.5 + 103.852\lambda_2$

Substituting ~~a~~ the major axis point in the ellipse equation

$$f(x) = c + a^T x + \frac{1}{2} x^T H x$$

$$15 = 14.5 + 103.852\lambda_2 + \begin{bmatrix} -13.1542 & -5.1970 \end{bmatrix} \begin{bmatrix} 27.5 \\ -19.65 \end{bmatrix} \lambda_2$$
$$+ \frac{1}{2} \begin{bmatrix} 27.5 & -19.65 \end{bmatrix} \begin{bmatrix} 0.84 & 0.2771 \\ 0.2771 & 0.52 \end{bmatrix} \begin{bmatrix} 27.5 \\ -19.65 \end{bmatrix} \lambda_2$$

On solving using MATLAB, we get

$$\lambda_2 = 0.0044$$

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$\therefore$ $a = \begin{bmatrix} -13.1542 \\ -5.1970 \end{bmatrix} \times 0.0044$

$a = \begin{bmatrix} -0.0585 \\ -0.0231 \end{bmatrix}$

||ly $H = \begin{bmatrix} 0.84 & 0.2771 \\ 0.2771 & 0.52 \end{bmatrix} \times 0.0044$

$H = \begin{bmatrix} 0.0037 & 0.0012 \\ 0.0012 & 0.0023 \end{bmatrix}$

$c = 14.5 + 103.852\, \lambda_2$

~~c = 14.0~~ (Using MATLAB)

$c = 14.9616$

$\therefore$ $f(x) = 14.9616 + \begin{bmatrix} -0.0585 & -0.0231 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0.0037 & 0.0012 \\ 0.0012 & 0.0023 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Q.2) $\nabla f|_{\text{major axis}} = H(x - x^*)$

Using MATLAB,

$\nabla f|_{\text{major axis}} = \begin{pmatrix} 0.02 \\ -0.0346 \end{pmatrix}$

$\therefore$ Direction of steepest ascent $= \nabla f = \begin{pmatrix} 0.02 \\ -0.0346 \end{pmatrix}$

direction of steepest descent $= -\nabla f = \begin{pmatrix} -0.02 \\ 0.0346 \end{pmatrix}$

direction of no change $=$ direction $\perp$ to $\nabla f = \begin{pmatrix} 0.0346 \\ 0.02 \end{pmatrix}$

Q.3) $\alpha^* = \left.\dfrac{(\nabla f)^T (\nabla f)}{(\nabla f^T) H (\nabla f)}\right|_{\text{major axis}}$

Using MATLAB,

$\alpha^* = 624.9725$

$\therefore$ maximum distance along which we can travel before function starts increasing $= \|\alpha^* p\|_2$

where $p = -\nabla f$

$= 24.976$ units

Q.4) Using (2,2)

$$\nabla f|_{(2,2)} = \begin{pmatrix} -0.0485 \\ -0.0160 \end{pmatrix}$$

Direction of steepest ascent $= \nabla f|_{2,2} = \begin{pmatrix} -0.0485 \\ -0.0160 \end{pmatrix}$

Direction of steepest descent $= -\nabla f|_{(2,2)} = \begin{pmatrix} 0.0485 \\ 0.0160 \end{pmatrix}$

Direction of no change $=$ direction $\perp$ to $\nabla f|_{2,2}$

$= \cancel{0.0160} \begin{pmatrix} 0.0160 \\ -0.0485 \end{pmatrix}$

$$\alpha^*|_{2,2} = \left.\frac{(\nabla f)^T (\nabla f)}{(\nabla f)^T H (\nabla f)}\right|_{2,2}$$

Using MATLAB,

$\alpha^*|_{2,2} = 231.1157$

Maximum distance $= \|\alpha^* p\|_2$

$= 11.80$ units

Q.5) For negative definite hessian, all eigen values can be replaced with negative of the eigen values.

$$H = V \Lambda V^{-1}$$

now $\Lambda = -\Lambda$

$\therefore H = -H.$

$$a = -Hx^*$$

But $H = -H$

$\therefore a = -a$

$$c = f(x^*) + \frac{1}{2} x^{*T} H_{new}\, x^*$$

~~Using~~

$f(x^*) = 14.5$ remains the same

& Using MATLAB,

$c = 14.0384.$

$$f(x) = 14.0384 + \begin{bmatrix} 0.0585 & 0.0231 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -0.037 & -0.0012 \\ -0.0012 & -0.0023 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

```
x_star = [15; 2];
major_axis = [27.5;-19.65];
minor_axis = [27.99; 9.5];
c0 = 15;
v1 = x_star - major_axis
v2 = x_star - minor_axis
a = sqrt(sum(v1.^2))
b = sqrt(sum(v2.^2))
lambda_ratio = b^2/a^2 %lambda1/lambda2 lambda1 -> v1
V = [v1 v2];
H = V*diag([lambda_ratio 1])*inv(V); %x lambda2
a = -H*x_star;
f_x_star = c0 - 0.5;
c = f_x_star + 0.5*x_star'*H*x_star; %x lambda2
x = major_axis;
lambda_sum = x_star'*H*x_star + 2*a'*x + x'*H*x;
lambda2 = 1/lambda_sum;
c = c*lambda2
a = a*lambda2
H = H*lambda2
% x = [16;22]
% x = [12;24]
diff = x - x_star;
f_x1 = f_x_star + 0.5*diff'*H*diff
%% Q2, 3
grad_f_major_axis= H*(major_axis - x_star)
dist = a
%% Q4
x2 = [2;2];
grad_f_2_2 = H*(x2-x_star);
alpha_2_2 = grad_f_2_2'*grad_f_2_2/(grad_f_2_2'*H*grad_f_2_2)
%% Q5
H_nd = -H
a_nd = -a
c_nd = f_x_star + 0.5*x_star'*H_nd*x_star;
```

```
v1 =

  -12.5000
   21.6500


v2 =

  -12.9900
   -7.5000


a =

   24.9994


b =

   14.9997


lambda_ratio =

    0.3600
```

```
lambda2 =

    0.0044


a =

   -0.0585
   -0.0231


H =

    0.0037    0.0012
    0.0012    0.0023


c =

   14.9616


f_x1 =

    15


grad_f_major_axis =

    0.0200
   -0.0346


alpha_major_axis =

  624.9725


dist_major_axis =

   24.9994
```

```
grad_f_major_axis =

    0.0200
   -0.0346


alpha_major_axis =

  624.9725


dist_major_axis =

   24.9994


grad_f_2_2 =

   -0.0485
   -0.0160


alpha_2_2 =

  231.1157


dist_2_2 =

   11.8119


H_nd =

   -0.0037   -0.0012
   -0.0012   -0.0023


a_nd =

    0.0585
    0.0231


c_nd =

   14.0384
```