16.S091: Homework #2

Due Date: 11:59 PM on April 18

Problem 1: Upwind Differencing and Numerical Dissipation in Hyperbolic PDEs

In this question we will investigate the effects of upwind differencing and numerical dissipation.

(a) Consider the 1-D partial differential equation,

$$Ut + uUx = 0.$$

To apply upwind-differencing (i.e. one-sided) to this equation in the general case where the velocity u can be positive or negative, the following approximations are used,

$$u_i \, Ux|_i = \begin{cases} u_i \delta_x^+ U_i & \text{if } u_i < 0 \\ \\ u_i \delta_x^- U_i & \text{if } u_i > 0 \end{cases}$$

Prove that this direction-dependent differencing scheme can be written as

$$u_i \, Ux|_i = u_i \, \delta_{2x} U_i - \frac{1}{2}|u_i| \Delta x \, \delta_x^2 U_i, \tag{1}$$

where δ_{2x} and δ_x^2 are standard central-difference operators for 1st and 2nd derivatives.

(b) In Computational Fluid Dynamics (CFD) algorithms to solve inviscid flows, an early approach was to add numerical (non-physical) dissipation to improve the stability of the discretizations. Specifically, consider the modified equation,

$$Ut + uUx = {k_{num}}^2 Ux^2,$$

where k_{num} is a numerical viscosity coefficient. If the x-derivatives are all approximated with standard central-difference approximations, what is the value of k_{num} such that the resulting finite difference method will be identical to the previous upwind discretization given by Equation (1).

Problem 2: Diffusion-Convection in an L-Shaped Domain

Consider the steady-state convection-diffusion equation:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.1 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + 1$$

posed on the **L-shaped domain**

$$\Omega = [-1, 1] \times [-1, 1] \setminus [0, 1] \times [0, 1].$$

The boundary condition is:

$$u = 0 \quad \text{on } \partial\Omega.$$

Discretize and solve the above PDE numerically on a uniform grid. Clearly describe your discretization scheme, implementation, and solver strategy. Visualize the resulting solution and comment on any features observed due to the geometry or boundary conditions.