16.S091: Homework #1

Due Date: 11:59 PM on April 11

Problem #1: Numerical Domain of Dependence and CFL Condition

Consider the 1-D convection equation:

$$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} = 0$$

Assume $u > 0$.

1. Plot the numerical domain of dependence for the first-order upwind discretization using forward Euler to integrate in time:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + u\frac{U_i^n - U_{i-1}^n}{\Delta x} = 0$$

2. For the discretization above, what is the maximum allowable timestep that satisfies the Courant-Friedrichs-Lewy (CFL) condition?

3. Plot the numerical domain of dependence for the first-order upwind discretization using backward Euler to integrate in time:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + u\frac{U_i^{n+1} - U_{i-1}^{n+1}}{\Delta x} = 0$$

4. For the discretization above, what is the maximum allowable timestep that satisfies the CFL condition?

Problem #2: Backward Euler

Solve the 1-D convection equation:

$$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} = 0$$

using a central difference spatial approximation and backward Euler time integration. Assume $u > 0$. The scheme is:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + u\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2\Delta x} = 0$$

Consider a domain from $x_L = -4$ to $x_R = 4$. Use:

$$U^0(x) = e^{-x^2}, \quad U(t, x_L) = e^{-x_L^2}$$

1. Analysis shows that the scheme is stable for any Δt. Run the code for CFL $= 10$ with $N_x = 401$ divisions. Plot the solution at $t = 3$. Is it stable? Accurate?

2. Run for CFL values: 0.25, 0.5, 1.0, and 2.0. For $t = 3$, compute:

   - Norm of error
   - Maximum error over x
   - $U(3, x = 3)$

   Present results in a table. How does error change with timestep?

3. Submit a copy of your commented code.