Homework #3

Problem 1 Conductive Heat Transfer & Truncation Error

Consider a rod with circular cross-section of area A and length L. We will assume the rod is significantly longer than its diameter so that we can model the temperature in the rod as varying in x (the distance along the rod), but not radially. The rod is initially heated such that the temperature is highest at $L/2$ and decreases towards its ends. Specifically, the temperature distribution at time $t = 0$ is,

$$T(x, t = 0) = 20 + 100 \sin\left(\pi \frac{x}{L}\right)$$

where the rod is from $0 \leq x \leq L$ and the temperature is in degrees Celsius. In this problem, we will consider the analytic behavior of the temperature when the heat source is no longer applied and the temperature at the ends is maintained. As well, we will consider the error made in approximating the relevant PDE using a finite difference method.
The conservation of energy for this one-dimensional heat transfer problem reduces to the following PDE for diffusion,

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

where ρ, c, and k are the density, thermal heat capacity, and thermal conductivity of the material.

1. Prove that the solution to this one-dimensional heat transfer problem is,

$$T(x, t) = 20 + 100 \exp\left(-\pi^2 \frac{k}{\rho c L^2} t\right) \sin\left(\pi \frac{x}{L}\right).$$

   Specifically, do this by substituting this into the governing PDE and showing that it satisfies this PDE.

2. Determine an expression in terms of the rod properties for the time, t_{cool} at which the temperature is within 10 degrees of the final temperature ($T \leq 30°C$ for all x).

3. For the remainder of this problem, the bar is made of aluminum with the following properties,

   - $L = 20$ cm,
   - $\rho = 2.7$ g/cm^3, $c = 0.90$ J/g-$°C$, $k = 167$ W/m-$°C$

   Consider the following finite difference discretization

$$\rho c \frac{T_i^{n+1} - T_i^n}{\Delta t} = k \delta_x^2 T_i^n$$

   Develop a short Python/Julia script that implements this discretization to determine the evolution of the temperature in the aluminum rod starting from the given initial condition and running until a final time of $t = t_{cool}$. Calculate the mean of the absolute value of the error at the nodes at the final time, i.e.,

$$\text{Error} = \frac{1}{N_x + 1} \sum_{i=1}^{N_x+1} |T_i^{N_t} - T(x_i, t_{cool})|$$

where N_x is the number of spatial divisions (i.e. $N_x \Delta x = L$) and N_t is the final temporal iteration index (i.e. $N_t \Delta t = t_{cool}$). Fill in the error in following table:

	$N_x = 10$	$N_x = 20$	$N_x = 40$	$N_x = 80$
$N_t = 50$				
$N_t = 100$				
$N_t = 200$				
$N_t = 400$				
$N_t = 800$				
$N_t = 1600$				
$N_t = 3200$				

Note: you should see some very large error for some of these combinations.

4. The very large errors for some combinations are because the Δt chosen is not eigenvalue stable for the chosen Δx. Based on your results, how do you think the maximum eigenvalue stable timestep scales with the mesh size? Specifically, does the maximum timstep scale linearly with the mesh size, i.e. $\Delta t_{\max} = K \Delta x$ where K is some constant. If not, based on your results, can you determine a power of Δx that the maximum timestep scales with?

Problem 2 One-sided finite differences

Suppose we want to find a second-order accurate finite difference approximation of $\partial u/\partial x$ at $x = 0$ with a spatial discretization $x_i = (i-1)\Delta x$. The forward difference approximation is given by

$$ux\Big|_{x_1} = \frac{u_2 - u_1}{\Delta x} + O(\Delta x).$$

However, backward/central difference approximations need u_0, which is not available. Instead, we seek a one-sided approximation of the form

$$ux\Big|_{x_i} \approx \frac{\alpha u_i + \beta u_{i+1} + \gamma u_{i+2}}{\Delta x}.$$

(a) Second-Order Accuracy

Find values for α, β, and γ that will make this one-sided finite difference approximation second-order accurate.

(b) Von Neumann Stability Analysis

Perform a Von Neumann stability analysis for the finite difference scheme used to solve the linear advection equation:

$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0$$

using the forward Euler method in time and an appropriate spatial discretization (e.g., upwind or central difference). Derive the stability condition in terms of the time step Δt, spatial step Δx, and wave speed C. Based on your analysis, determine the combinations of Δx, Δt, and C for which the scheme remains stable. Clearly state the resulting stability criterion (such as a CFL condition) that must be satisfied.
