# Eigenpairs

DSTA

## Eigenpairs

### Study materials

I. Goodfellow, Y. Bengio and A. Courville:

Deep Learning, MIT Press, 2016.

J. Lescovec, A. Rajaraman, J. Ullmann:

Mining of Massive datasets, MIT Press, 2016.

The material covered here is presented in the excerpts available for download.

## Spectral Analysis

### Eigenpairs

If, given a matrix $A$ we find a real $\lambda$ and a vector $\mathbf{e}$ s.t.

$$A\mathbf{e} = \lambda\mathbf{e}$$

then $\lambda$ and $\mathbf{e}$ will be an eigenpair of A.

. . .

In principle, if A has rank n there should be n such pairs.

. . .

In practice, eigenpairs

- are always *costly* to find.

- they might have $\lambda = 0$: no information, or
- $\lambda$ might not be a real number: no interpretation.

## Conditions for ***good*** eigen-

A square matrix $A$ is called *positive semidefinite* when for any $\mathbf{x}$ we have

$$\mathbf{x}^T A\mathbf{x} \geq 0$$

In such case its eigenvalues are non-negative: $\lambda_i \geq 0$.

## Underlying idea, I

In Geometry, applying a matrix to a vector, $A\mathbf{x}$, creates all sorts of alteration to the space, e.g,

- rotation
- deformation

Eigenvectors, i.e., solutions to $A\mathbf{e} = \lambda\mathbf{e}$

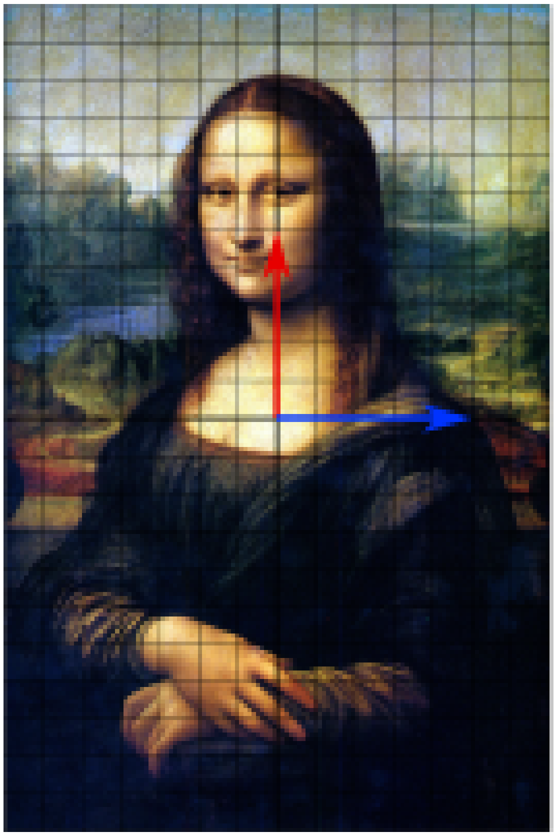
describe the direction along which matrix A operates an **expansion**

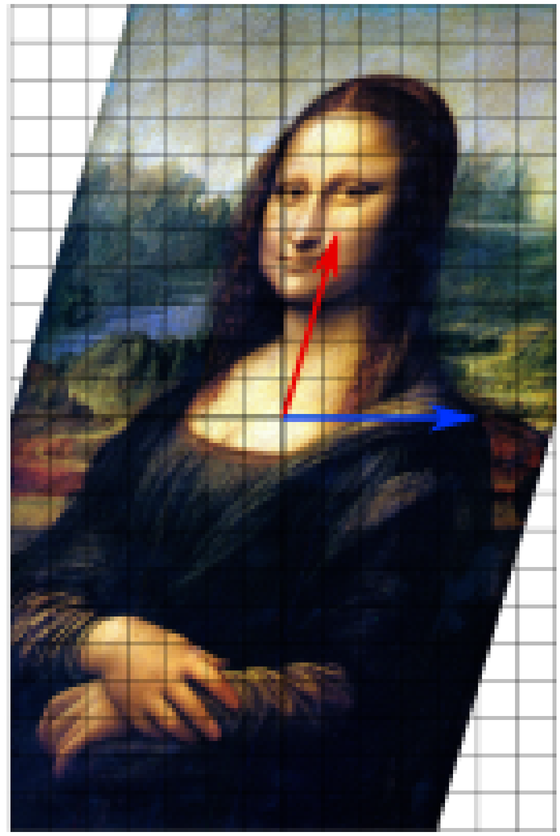
## Example: shear mapping

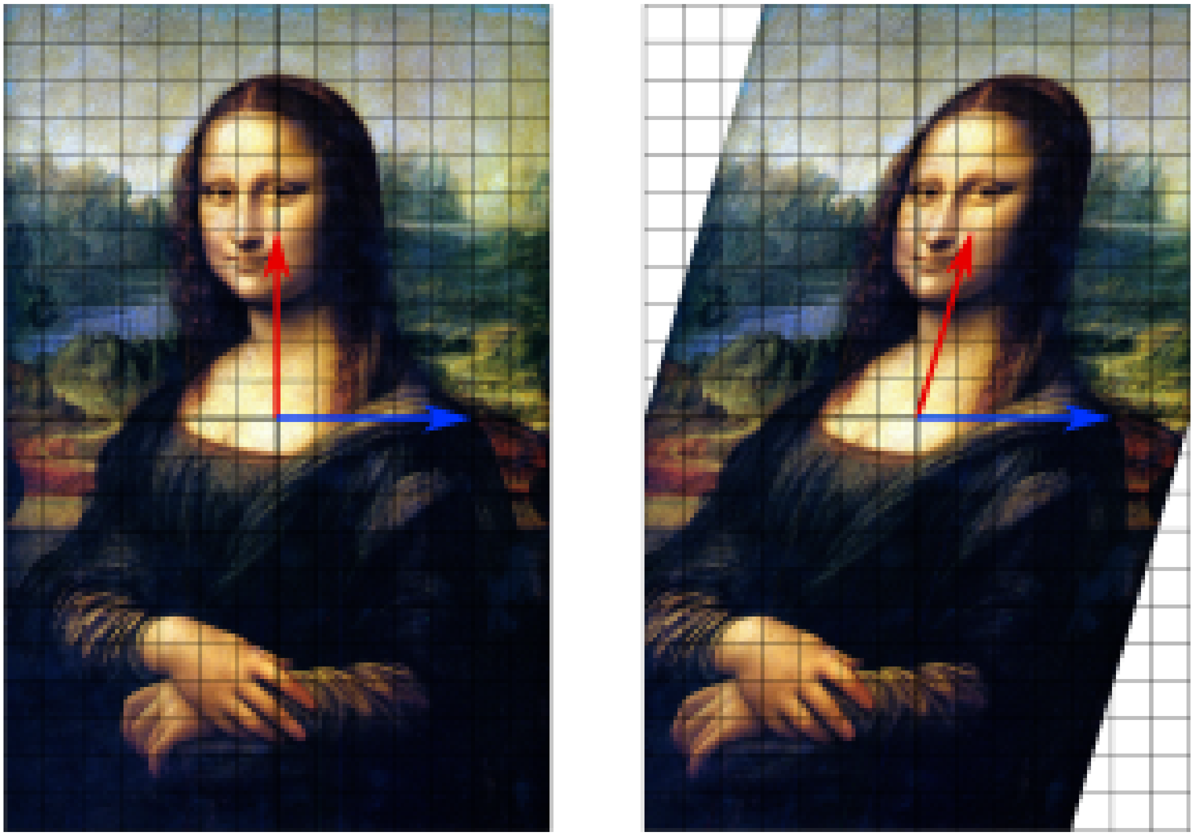
```
A = [[1, .27],
     [0,   1]
    ]
```

deforms a vector by increading the first dimension by a quantity proportional to the value of the second dimension:

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x + \frac{3}{11}y \\ y \end{bmatrix}$$







The blue line is unchanged:

- an $[x, 0]^T$ eigenvector
- corresponding to $\lambda = 1$

## Activity matrices, I

Under certains conditions:

-the eigenpairs exists,

-e-values are real, non-negative numbers (0 is ok), and

-e-vectors are orthogonal with each other:

. . .

User-activity matrices normally meet those conditions!

### Activity matrices, II

If an activity matrix has *good* eigenpairs,

. . .

each e-vector represents a *direction*

we interpret those directions as *topics* that hidden (latent) within the data.

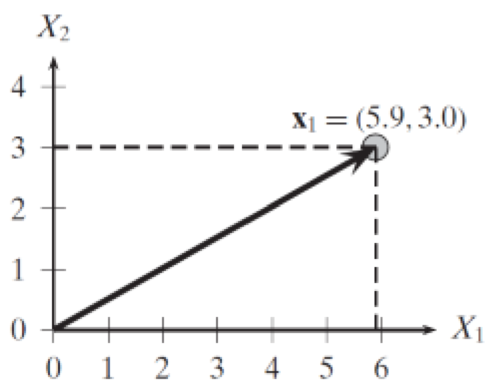
e-values *expand* one's affiliation to a specific *topic*.

## Norms and distances

### Euclidean norm

Pythagora's theorem, essentially.

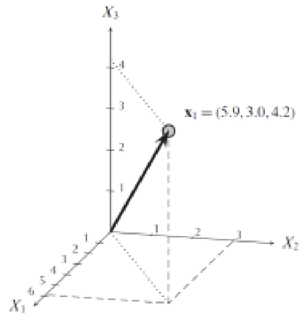
$||\mathbf{x}|| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{\sum_{i=1}^{m} x_i^2}$




. . .

Generalisation:

$||\mathbf{x}||_p = (|x_1|^p + |x_1|^p + ... |x_m|^p)^{\frac{1}{p}} = (\sum_{i=1}^{m} |x_i|^p)^{\frac{1}{p}}$




. . .

The Frobenius norm $||\cdot||_F$ extends $||\cdot||_2$ to matrices:

$||\mathbf{A}||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}$

### Normalization

The *unit* or *normalized* vector of $\mathbf{x}$

$$\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||} = (\frac{1}{||\mathbf{x}||})\mathbf{x}$$

- has the same direction of the original
- its norm is constructed to be 1.

## Computing Eigenpairs

### With Maths

$$M\mathbf{e} = \lambda\mathbf{e}$$

. . .

Handbook solution: solve the equivalent system

$$(M - \lambda\mathbf{I})\mathbf{e} = \mathbf{0}$$

. . .

Either of the two factors should be 0. Hence, a non-zero vector $\mathbf{e}$ is associated to a solution of

$$|M - \lambda \mathbf{I}| = 0$$

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$$|M - \lambda \mathbf{I}| = 0$$

In Numerical Analysis many methods are available.

Their general algorithmic structure:

-find the $\lambda$s that make $| \dots | = 0$, then

-for each $\lambda$ find its associated vector $\mathbf{e}$.

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## With Computer Science

At the scale of the Web, few methods will still work!

Ideas:

1. find the e-vectors first, with an iterated method.
2. interleave iteration with control on the *expansion in value*

. . .

$\mathbf{x_0} = [1, 1, \dots 1]^T$

. . .

$\mathbf{x_{k+1}} = \frac{M\mathbf{x}_k}{||M\mathbf{x}_k||}$

. . .

until an approximate fix point: $x_{l+1} \approx x_l$.

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Now, eliminate the contribution of the first eigenpair:

$$M^* = M - \lambda_1' \mathbf{x}_1 \mathbf{x}_1^T$$

(since $\mathbf{x}_1$ is a column vector, $\mathbf{x}_1^T \mathbf{x}_1$ will be a scalar: its norm. Vice versa, $\mathbf{x}_1 \mathbf{x}_1^T$ will be a matrix)

. . .

Now, we repeat the iteration on $M^*$ to find the second eigenpair.

Times are in $\Theta(dn^2)$.

For better scalability, we will cover Pagerank later.

## Eigenpairs in Python

### E-pairs with Numpy

```
import numpy as np

# this is the specific submodule
from numpy import linalg as la
```

```
# create a 'blank' matrix
m = np.zeros([7, 5])

m = [[1, 1, 1, 0, 0],
     [3, 3, 3, 0, 0],
     [4, 4, 4, 0, 0],
     [5, 5, 5, 0, 0],
     [0, 0, 0, 4, 4],
     [0, 0, 0, 5, 5],
     [0, 0, 0, 2, 2]
    ]
```

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```python
def find_eigenpairs(mat):
    """Test the quality of Numpy eigenpairs"""
    n = len(mat)

    # is it squared?
    m = len(mat[0])

    eig_vals, eig_vects = la.eig(mat)

    # they come in ascending order, take the last one on the right
    dominant_eig = abs(eig_vals[-1])
    return dominant_eig
```

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E-values come normalized: $\sqrt{\lambda_1^2 + ... \lambda_n^2} = 1$; hence we later multiply them by $\frac{1}{\sqrt{n}}$

```python
# lambda_1 = find_eigenpairs(m)

# lambda_1
```

### Coda: non-norms

$||\mathbf{x}||_0 = \#$ of non-zero scalar values in $\mathbf{x}$

$||\mathbf{x}||_\infty = max\{|x_i|\}$