dylanadi.sty

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Contents

1 Theorems

Theorem 1.1 (Pythagorean Theorem)
For positive a, b, c,
$$a^2 + b^2 = c^2.$$

Example 1.2
What is $3^2 + 4^2$?

Walkthrough.

1. Where have we seen some of squares?
2. That's right – Pythagorean Theorem!
3. Thus, we can take advantage of a 3-4-5 triangle.
4. Alternatively, **just calculate it**.

Theorem
A *theorem* is a general proposition proved by a chain of reasoning.

Fact 1.4. This is important!

Remark 1.5. This is also important.

Paragraph Nice looking box.

This is a warning box, in case there's something important to talk about. For example, a bogus solution could go here. It helps denote what could possibly be incorrect: i.e.,

$$a^3 + b^3 = c^3$$

in a right triangle is wrong (if they represent the side lengths of the triangle).

Exercise 1.6 (AIME II 2020/3). The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Exercise 1.7 (AIME 1986/8). Let S be the sum of the base 10 logarithms of all the proper divisors (all divisors of a number excluding itself) of 1000000. What is the integer nearest to S?

Exercise 1.8 (AIME I 2020/2). There is a unique positive real number x such that the three numbers $\log_8 2x$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

1 This is a question. What is your question?

2 Problems

Minimum is [32 ♟]. Problems denoted with ♜ are required. (They still count towards the point total.)

> "I like cheeseburgers."
>
> *Anonymous*

[2 ♟] Problem 1 (AIME I 2007/7). Let $N = \sum_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor)$. Find the remainder when N is divided by 1000. ($\lfloor k \rfloor$ is the greatest integer less than or equal to k, and $\lceil k \rceil$ is the least integer greater than or equal to k.)

[3 ♟] Problem 2 (SMT 2020). If a is the only real number that satisfies $\log_{2020} a = 202020 - a$ and b is the only real number that satisfies $2020^b = 202020 - b$, what is the value of $a + b$?

[3 ♟] Problem 3 (AIME II 2013/2). Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

[3 ♜] Problem 4 (AIME II 2010/5). Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

[3 ♟] Problem 5 (AIME I 2006/9). The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

[4 ♟] Problem 6 (HMMT 2020). Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

[4 ♟] Problem 7 (AIME II 2007/12). The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that $\sum_{n=0}^{7} \log_3(x_n) = 308$ and $56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57$, find $\log_3(x_{14})$.

[4 ♟] Problem 8 (AIME I 2009/7). The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1}-a_n)} - 1 = \frac{1}{n+\frac{2}{3}}$ for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

[4 ♟] Problem 9 (AIME I 2010/14). For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \leq 300$. Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

[6 ♜] Problem 10 (AIME I 2005/8). The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

[6 ♟] Problem 11 (AIME I 2013/8). The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $\frac{1}{2013}$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m + n$ is divided by 1000.

[9 ♟] Problem 12 (AIME I 2012/9). Let x, y, and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where p and q are relatively prime positive integers. Find $p+q$.

Problem 13. Problem without points or source.