Performance analysis and optimal selection of large mean-variance portfolios under estimation risk

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Abstract

We study the consistency of sample mean-variance portfolios of arbitrarily high dimension that are based on Bayesian or shrinkage estimation of the input parameters as well as weighted sampling. In an asymptotic setting where the number of assets remains comparable in magnitude to the sample size, we provide a characterization of the estimation risk by providing deterministic equivalents of the portfolio out-of-sample performance in terms of the underlying investment scenario. The previous estimates represent a means of quantifying the amount of risk underestimation and return overestimation of improved portfolio constructions beyond standard ones. Well-known for the latter, if not corrected, these deviations lead to inaccurate and overly optimistic Sharpe-based investment decisions. Our results are based on recent contributions in the field of random matrix theory. Along with the asymptotic analysis, the analytical framework allows us to find bias corrections improving on the achieved out-of-sample performance of typical portfolio constructions. Some numerical simulations validate our theoretical findings.

Index Terms

mean-variance portfolio optimization, asymptotic performance analysis, consistent estimation, stochastic convergence, random matrix theory

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I. INTRODUCTION

A. Background and research motivations

The foundations of modern portfolio theory were laid by Markowitz's ground-breaking article [1], where the idea of diversifying a portfolio by spreading bets across a universe of risky financial assets was refined and generalized by the more sophisticated one of combining the assets so as to optimize the risk-return tradeoff. In practice, Markowitz's mean-variance optimization framework for solving the canonical wealth allocation problem relies on the statistical estimation of the unknown expected values and covariance matrix of the asset returns from sample market observations.

In general, the uncertainty inherently associated with imperfect moments estimates represents a major drawback in the application of the classical Markowitz framework. Indeed, the optimal mean-variance solution has been empirically observed to be significantly sensitive to deviations from the true input parameters. In addition, and aside from computational complexity issues, the estimation of the parameters is involved, mainly due to the instability of the parameter estimates through time. Generally, estimates of the covariance matrix are more stable than those of the mean returns, and so many studies disregard the estimation of the latter and concentrate on improving the sample performance of the so-called global minimum variance portfolio (GMVP); see arguments in [2].

In the financial literature, the previous source of portfolio performance degradation is referred to as estimation risk. Especially when the number of securities is comparable to the number of observations, estimation errors may in fact prevent the mean-variance optimization framework from being of any practical use. In fact, for severe levels of estimation risk, the naive portfolio allocation rule namely obtained by equally weighting the assets without incorporating any knowledge about their mean and covariance turns out to represent a firm candidate choice [3]. The consistency and distributional properties of sample optimal mean-variance portfolios and their Sharpe ratio performance has been analyzed and characterized for finite samples and asymptotically (see, most recently, [4], [5], [6], and also the list of references therein for earlier contributions).

Commencing with particularly high activity and contribution levels in the 80's, there exists a vast literature on portfolio selection methods accounting for estimation risk by explicitly dealing with the lack of robustness and stability of the sample optimal mean-variance solution, which we do not intend to exhaustively review here; we refer the reader to [7], [8] for a thorough treatment of the subject. Some remarks on the two main lines of approach are in order. One class of methods based on convex analysis and nonlinear optimization techniques focuses on formulations of the allocation problem where

robustness to estimation errors is achieved by means of the explicit modeling of parameter uncertainty regions. Conceptually, assuming worst-case bounds on the input parameters may not be effective in practice since no information is available about the distribution of the estimated parameters within the uncertainty boundaries.

On the other hand, instances of a second family of methods of statistical or probabilistic nature are approaches based on Bayesian and Steinian shrinkage estimation seeking efficiency by weighting a sensible prior belief and the classical sample estimator in inverse proportion to their dispersion (see, e.g., [9], [10]). As a matter of fact, this class of techniques provides a rather general framework for understanding different forms of portfolio corrections and performance improvements tackling estimation risk. Indeed, explicit links have been found between the latter and the robust optimization solutions introduced above, which turn out to be possibly interpreted from a shrinkage estimation perspective [11]. Furthermore, constraining[1] the portfolio weights has been additionally noted to be equivalent to adding some structure to the covariance estimation problem as obtained through Bayesian or shrinkage-based procedures [2]. Arguably, the latter constitutes an effective way helping to avoid overfitting the sample data and to improve the stability of the realized portfolio solution out of sample and over time (see also [12], where the authors investigate the effects of norm-constrains in the solution of the weight vector). The application of Bayesian and shrinkage approaches is not limited to the moment estimation problem alone, but can indeed be extended to incorporate any prior belief directly on the portfolio weights. Linear shrinkage solutions optimally combining different portfolio allocation rules, such as the GMVP, the portfolio with equal weights and the tangency portfolio (cf. Section II) have been reported in [13], [14], [15], [16].

Alternative approaches have been based on resampling techniques [17], [18], as well as stochastic programming and also robust estimation, where the emphasis is on robustifying estimators that are efficient under the assumption of Gaussian asset-returns, and which are usually highly sensitive to deviations from the distributional assumption (see, e.g., [19] and references therein). Finally, a line of contributions from statistical physics initiated by [20], [21] have been reporting on a methodology based on random matrix theory that consists of preserving the stability over time of the covariance matrix estimator by filtering noisy eigenvalues conveying no valuable information. The cleaning mechanism relies on the empirical fact that relevant information is structurally captured by some few eigenvalues, while the rest can be

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[1]Usual constraints on the allocation weights that are typically considered in the portfolio construction process are those modelling the self-financing characteristic of the investment rule, as well as budget and short-selling restrictions.

ascribed to noise and measurement errors and resemble the spectrum of a white covariance matrix (see also [22]).

In this paper, we are interested in the class of structured portfolio estimators based on the combination of Bayesian or James-Stein shrinkage and sample weighting. Motivated by the widespread application of this class of statistical methods in the practice of portfolio and risk management, our focus is on the performance of portfolio constructions as a function of the set of weights as well as the shrinkage targets and intensity coefficients parameterizing the improved moment forecasts. The extension of the statistical performance analysis of sample optimal portfolios with standard moment estimates to the case of improved shrinkage estimators is not straightforward. We concentrate on the consistency analysis by considering a limiting regime that is defined by both the number of samples and the portfolio dimension going to infinity at the same rate. Such an asymptotic setting will prove to be more convenient to characterize realistic, finite-dimensional practical conditions, where sample-size and number of assets are comparable in magnitude. In particular, we resort to some recent results from the theory of the spectral analysis of large random matrices, which as in [18] and contrary to the random matrix theoretical contributions from statistical physics cited above, are based on Stieltjes transform methods and stochastic convergence theory.

Before outlining the contributions and structure of the work, we draw some connections between the subject of the paper and classical methods in the statistical signal processing literature. As a matter of fact, (1) encompasses a broad range of system configurations described by the general vector channel model. In fact, as for the mean-variance portfolio optimization problem, usual linear filtering schemes solving typical signal waveform estimation and detection problems in sensor array processing and wireless communications are based on the estimation of the unknown observation covariance matrix as well as possibly a vector of cross-correlations with a pilot training sequence. Prominent examples are the Capon or minimum variance spatial filter as well as the minimum mean-square error beamformer and detector [23], [24], and also adaptive filtering[2] applications [26], in all of which both Bayesian and regularization (shrinkage) methods are widely applied. Indeed, robust methods are similarly well-known and extensively used in signal processing applications (see examples in, e.g., [27], [28]). In particular, norm-constrains have been extensively investigated in the sensor array signal processing literature (see, e.g., [29]). Finally, analyses of weighted sample estimators of covariance matrices can be found in [30], [31] and applications

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[2]In particular, typical formulations of this problem based on (weighted) least-squares regression are intimately related to the passive investment strategy of index tracking (see, e.g., [25, Chapter 4]).

of the bootstrap in [32].

B. Contributions and structure of the work

The main contributions of the paper are as follows. We first characterize the consistency of sample mean-variance portfolios based on the aforementioned improved moment estimators by providing asymptotic deterministic equivalents of the achieved out-of-sample performance in the more meaningful double-limit regime introduced above. Our analytical framework allows us to quantify and better understand the impact of estimation errors on the out-of-the-sample performance of optimal portfolios. Specifically, we provide a precise quantitative description of the amount of risk underestimation and return overestimation of portfolio constructions based on improved estimators, in a way depending on the ratio of the portfolio dimension to sample-size as well as the underlying investment scenario. This phenomena, which render overly optimistic any investment assessment and decision based on estimated Sharpe ratios, has already been observed in the financial literature for standard portfolio implementations.

Furthermore, we propose a class of mean-variance portfolio estimators defined in terms of a set of weights and shrinkage parameters calibrated so as to optimize the achieved out-of-sample performance. In essence, an optimal parameterization is obtained by effectively correcting the analytically derived asymptotic deviations of the performance of sample portfolios.

The structure of the work is as follows. After the brief literature account and introductory research motivations in this section, Section II introduces the modeling details and the moment forecasting schemes considered in this paper. The problem of evaluating the out-of-sample performance of large portfolios is also explained. In Section III-A, we provide a characterization of the performance of improved estimators based on sample weighting and James-Stein shrinkage. Observed deviations from optimal performance are corrected in Section III-B, where we propose a class of improved portfolios for high-dimensional settings. Section IV presents some simulation work validating our theoretical findings and Section V concludes the contribution by summarizing the paper. Technical results and proofs are relegated to the appendices.

II. DATA MODEL AND PROBLEM FORMULATION

Consider the time series with the logarithmic differences of the prices of M financial assets at the edges of an investment period with time-horizon t. Generally enough, we can define the data generating

process of the previous compound or log returns by the following vector stochastic process[3]:

$$\mathbf{y}_t = \boldsymbol{\mu}_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t = \boldsymbol{\Sigma}_t^{1/2} \mathbf{x}_t, \tag{1}$$

where $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ are the expected value and covariance matrix of the asset returns over the investment period, and $\mathbf{x}_t$ is a random vector with independent and identically distributed (i.i.d.) entries having mean zero and variance one. We are interested in the problem of optimal single-period (static) mean-variance portfolio selection, which can be mathematically formulated as the following quadratic optimization problem with linear constraints:

$$\begin{aligned} \min_{\mathbf{w}_t} \quad & \mathbf{w}_t^T \boldsymbol{\Sigma}_t \mathbf{w}_t \\ \text{s.t.} \quad & \mathbf{w}_t^T \boldsymbol{\mu}_t = \mu_d \\ & \mathbf{w}_t^T \mathbf{1}_M = 1, \end{aligned} \tag{2}$$

where μ_d represents the target or desired level of expected portfolio return[4], and $\mathbf{w}_t^T \mathbf{1}_M = 1$ is a budget constraint.

We shall assume without loss of generality that the forecasting sampling frequency coincides with the rebalancing frequency. In particular, mean vector and covariance matrix are forecasted with the return data over a prescribed estimation window up to the time of the investment decision. Since we only consider the case of a single-period investment horizon, in the sequel we will omit the subscript and let $\mathbf{w}_t = \mathbf{w}$ for notational convenience. The solution to (2) is straightforwardly given by

$$\mathbf{w}_{\mathsf{MV}} = \frac{C - \mu_d B}{AC - B^2} \boldsymbol{\Sigma}_t^{-1} \mathbf{1}_M + \frac{\mu_d A - B}{AC - B^2} \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\mu}_t, \tag{3}$$

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[3]**Notation**: All vectors are defined as column vectors and designated with bold lower case; all matrices are given in bold upper case; for both vectors and matrices a subscript will be added to emphasize dependence on dimension, though it will be occasionally dropped for the sake of clarity of presentation; $[\cdot]_j$ will be used for the jth entry of a vector; $(\cdot)^T$ denotes transpose; $\mathbf{I}_M$ denotes the $M \times M$ identity matrix; $\mathbf{1}_M$ denotes an M dimensional vector with all entries equal to one; $\mathrm{tr}\,[\cdot]$ denotes the matrix trace operator; $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields of dimension specified by a superscript; $\mathrm{Im}\,\{z\}$ denotes imaginary part of the complex argument; $\mathbb{R}^+ = \{z \in \mathbb{C} : \mathrm{Im}\,\{z\} > 0\}$; $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}\,\{z\} > 0\}$; $\mathbb{E}\,[\cdot]$ denotes expectation; given two quantites a, b, $a \asymp b$ will denote both quantities are asymptotic equivalents, i.e., $|a - b| \overset{a.s.}{\to} 0$, with $a.s.$ denoting almost sure convergence; K, K_p denote constant values not depending on any relevant quantity, apart from the latter on a parameter p; $|\cdot|$ denotes absolute value and $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced norm for matrices (i.e., spectral or strong norm), whereas $\|\cdot\|_F$ denotes Frobenius norm, i.e., for a matrix $\mathbf{A} \in \mathbb{C}^{M \times M}$ with eigenvalues denoted by $\lambda_m\,(\mathbf{A})$, $m = 1, \ldots, M$, such that $\lambda_M\,(\mathbf{A}) \leq \lambda_{M-1}\,(\mathbf{A}) \leq \ldots \leq \lambda_1\,(\mathbf{A})$, and spectral radius $\rho\,(\mathbf{A}) = \max_{1 \leq m \leq M}\,(|\lambda_m|)$, $\|\mathbf{A}\| = \left(\rho\,(\mathbf{A}^H \mathbf{A})\right)^{1/2}$, $\|\mathbf{A}\|_F = \left(\mathrm{Tr}\,[\mathbf{A}^H \mathbf{A}]\right)^{1/2}$, $\|\mathbf{A}\|_{\mathrm{tr}} = \mathrm{Tr}\left[\left(\mathbf{A}^H \mathbf{A}\right)^{1/2}\right]$.

[4]As conventionally, and for the sake of clarity of presentation, we will assume that logarithmic returns are well approximated by their linear counterparts, so that we can claim about the additivity of returns over both portfolio assets and intertemporally.

where $A = \mathbf{1}_M^T \boldsymbol{\Sigma}_t^{-1} \mathbf{1}_M$, $B = \mathbf{1}_M^T \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\mu}_t$ and $C = \boldsymbol{\mu}_t^T \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\mu}_t$. In particular, if the constraint on the level of return achieved is dropped, then we obtain the so-called *global minimum variance portfolio* (GMVP), which is given by

$$\mathbf{w}_{\mathsf{GMVP}} = \frac{\boldsymbol{\Sigma}_t^{-1} \mathbf{1}_M}{\mathbf{1}_M^T \boldsymbol{\Sigma}_t^{-1} \mathbf{1}_M}. \tag{4}$$

In fact, the latter is clearly also the solution to the general mean-variance problem if $\mu_t = 0$, as it is often assumed over short investment periods. Other special case of particular interest due to its implications in asset pricing theory is that of the *tangency portfolio* (TP), which is given by

$$\mathbf{w}_{\mathsf{TP}} = \frac{\boldsymbol{\Sigma}_t^{-1} \boldsymbol{\mu}_t}{\mathbf{1}_M^T \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\mu}_t}. \tag{5}$$

In practice, $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ are unknown and so they must be estimated from market data observations. Let $\hat{\boldsymbol{\mu}}_t$ and $\hat{\boldsymbol{\Sigma}}_t$ denote the forecasted values of the expected mean and the covariance matrix, respectively. Moreover, let $\hat{\mathbf{w}}_{\mathsf{GMVP}}$ and $\hat{\mathbf{w}}_{\mathsf{TP}}$ represent the sample construction of (4) and (5), respectively, based on the previous moment estimates. In the following, we briefly elaborate on the classical forecasting settings that are customarily applied to estimate the input parameters of the Markowitz portfolio optimization framework. Specifically, we consider in the first place the conventional assumption according to which the returns over consecutive investment periods are independent and identically distributed, and the two required moments are obtained by their respective unconditional estimators. Then, we turn our attention to conditional forecasting models based on linear and stationary stochastic processes; finally, we shortly comment on heteroscedastic models allowing for some time-variability of the multivariate volatility process.

Before proceeding further, let us introduce some useful notation. We will denote by $\{\mathcal{F}_{t-1}\}$ the information set of events up to the discrete-time instant $t-1$, i.e., the σ-field generated by the observed series $\{\mathbf{y}_l\}_{l<t}$. Conditional on the observation available up to the investment decision time, the covariance matrix of the stochastic process $\mathbf{y}_t$ is given by definition by $\boldsymbol{\Sigma}_t = \text{var}\left(\mathbf{y}_t | \mathcal{F}_{t-1}\right) = \text{var}\left(\boldsymbol{\varepsilon}_t | \mathcal{F}_{t-1}\right)$. Additionally, we let $\mathbf{Y}_N = [\mathbf{y}_{t-N}, \dots, \mathbf{y}_{t-1}]$ denote the sample data matrix with the N past return observations.

A. *The case of IID returns: weighted sampling and shrinkage estimation*

Under the classical assumption of i.i.d. returns, mean vector and covariance matrix are both modeled as constant over the entire estimation interval, i.e., $\boldsymbol{\mu}_l = \boldsymbol{\mu}$, $\boldsymbol{\Sigma}_l = \boldsymbol{\Sigma}$, $l = t - N, \dots, t - 1$. Hence, the standard forecasts of the moments are given in terms of a rolling-window by the (unconditional) sample

mean and sample covariance matrix, i.e., respectively,

$$\hat{\boldsymbol{\mu}} = \frac{1}{N} \sum_{n=t-N}^{t-1} \mathbf{y}_n = \frac{1}{N}\mathbf{Y}_N\mathbf{1}_N, \tag{6}$$

and

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{n=t-N}^{t-1} \left(\mathbf{y}_n - \hat{\mu}\right)\left(\mathbf{y}_n - \hat{\mu}\right)^T = \frac{1}{N}\mathbf{Y}_N \left(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right)\mathbf{Y}_N^T. \tag{7}$$

A classical extension of the standard estimators in (6) and (7) considers the effect of weighting the sample observations. Let $\mathbf{W}_{\boldsymbol{\mu},N} \in \mathbb{R}^{N \times N}$ and $\mathbf{W}_{\boldsymbol{\Sigma},N} \in \mathbb{R}^{N \times N}$ be two diagonal matrices with entries given by a set of nonnegative coefficients, respectively, $w_{\boldsymbol{\mu},n}$ and $w_{\boldsymbol{\Sigma},n}$, $n = 1, \ldots, N$. Specifically, the weighted sample mean and weighted sample covariance matrix are respectively defined as

$$\hat{\boldsymbol{\mu}}_{\mathsf{W}} = \frac{1}{N} \sum_{n=t-N}^{t-1} w_{\boldsymbol{\mu},n}\mathbf{y}_n = \frac{1}{N}\mathbf{Y}_N\mathbf{W}_{\boldsymbol{\mu},N}\mathbf{1}_N. \tag{8}$$

and

$$\hat{\boldsymbol{\Sigma}}_{\mathsf{W}} = \frac{1}{N} \sum_{n=t-N}^{t-1} w_{\boldsymbol{\Sigma},n} \left(\mathbf{y}_n - \hat{\boldsymbol{\mu}}_{\mathsf{W}}\right)\left(\mathbf{y}_n - \hat{\boldsymbol{\mu}}_{\mathsf{W}}\right)^T$$
$$= \frac{1}{N}\mathbf{Y}_N \left(\mathbf{I}_N - \frac{1}{N}\mathbf{W}_{\boldsymbol{\mu},N}\mathbf{1}_N\mathbf{1}_N^T\right)\mathbf{W}_{\boldsymbol{\Sigma},N}\left(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\mathbf{W}_{\boldsymbol{\mu},N}\right)\mathbf{Y}_N^T. \tag{9}$$

Weighted estimators are usually applied in order to reduce variability and improve the stability of parameter estimators, for instance by using stratified random sampling [33]. A related structure is the one obtained by the nonparametric bootstrap, for which the weights represent the number of times the corresponding observation appears in the bootstrap sample [34][5]. In the context of asset allocation, [37] (see also [17]) suggests averaging a sequence of portfolios obtained by resampling with replacement from the originally available sample. Regarded as bootstrap aggregating of *bagging*, such averages are used in statistics for variance reduction purposes as well as to stabilize the prediction out-of-sample performance as a remedy to overfitting (see, e.g., Chapter 10 in [38]). In particular, the bootstrap is typically used to provide small-sample corrections for possibly consistent but biased estimators. However, in high-dimensional settings, the standard application of the bootstrap generally yields inconsistent estimates of bias. An asymptotic refinement of the conventional bootstrap-based bias correction (see, e.g., [39] for standard methodology) is provided in [18] by resorting to random matrix theoretical results.

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[5]We assume that the choice of weights is given; possible weighting schemes range from the standard simple random sampling with replacement (i.e., uniform resampling following a multinomial distribution) to sampling from the empirical distribution of the asset returns with nonuniform weights by for instance assigning different resampling probabilities to the different observations using importance sampling (see, e.g., [35], [36] for more details)

A common further extension to (possibly weighted) sample estimation relies on the widespread family of Steinian (James-Stein-type) shrinkage estimators of the mean and covariance matrix of the observed samples. By means of regularizing or shrinking the estimators (8) and (9), we define:

$$\hat{\boldsymbol{\mu}}_{\mathsf{SHR}} = (1 - \delta)\, \hat{\boldsymbol{\mu}}_{\mathsf{W}} + \delta \boldsymbol{\mu}_0, \tag{10}$$

and

$$\hat{\boldsymbol{\Sigma}}_{\mathsf{SHR}} = (1 - \rho)\, \hat{\boldsymbol{\Sigma}}_{\mathsf{W}} + \rho \boldsymbol{\Sigma}_0, \tag{11}$$

where the nonrandom vector $\boldsymbol{\mu}_0$ and the positive matrix $\boldsymbol{\Sigma}_0$ are the shrinkage targets or, from a Bayesian perspective, the prior knowledge about the unknown $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, respectively, where δ are ρ are the shrinkage intensity parameters. Clearly, if the shrinkage intensity parameters are equal to 1 and $\mathbf{W}_{\boldsymbol{\mu},N} = \mathbf{W}_{\boldsymbol{\Sigma},N} = \mathbf{I}_N$, then the standard sample estimators are recovered. A typical example of shrinkage target for the covariance estimation is $\boldsymbol{\Sigma}_0 = \mathbf{I}_M$. Shinkage estimators in the context of portfolio optimization were first proposed in [40] (see also [10] for the covariance matrix, and [41] for a study of the combination of resampling and shrinkage).

As mentioned in the introduction, it has been recognized in the financial literature that, under severe estimation risk conditions, the estimated Markowitz's optimal portfolio rule and its various sophisticated extensions underperform out-of-the-sample the naive rule based on the *equally weighted portfolio* (EWP) choice. In an effort to incorporate this well-known fact into the portfolio selection process, some authors have considered optimizing a combination[6] of one or more sample portfolios, such as $\hat{\mathbf{w}}_{\mathsf{GMVP}}$ and $\hat{\mathbf{w}}_{\mathsf{TP}}$, and the uniformly weighted asset allocation given by $\mathbf{w}_{\mathsf{EWP}} = \mathbf{1}_M / M$ (see [3], [13], [16]).

B. Accounting for serial dependence: conditional models

The previous unconditional estimators of the moments of the asset returns are particularly well-suited for situations of static nature. Under a more general setting challenging the i.i.d. assumption, although a period-by-period computation of the sample statistics by means of a rolling window can indeed allow for some return predictability, the dynamic behavior of the input parameters is best modeled in practice by taking into account conditional information. For the sake of a more precise motivation, we first recall some empirically observed properties or attributes of time series of asset returns, the so-called stylized facts in

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[6]The rational behind this approach lies on the so-called fund-separation theorems in finance (see [42]).

the theory and practice of finance (see [43], and also [44] for a textbook exposition)[7]. Concerning their distributional properties, it has been observed that return series are leptokurtic or heavy-tailed (except for long time intervals, for which the log-normal assumption seems reasonable, at least for well-diversified portfolios), and extreme return values usually appear in clusters. Regarding their dynamics, conditional expected returns are usually negligible (at least relative to volatility values), and, more importantly, are not independent though exhibit little serial correlation. Conversely, squared returns, which are often used as a proxy of the unobserved covariance, show profound evidence of positive serial correlation with high persistence.

If we set aside the time variability of conditional covariances (i.e., particularly for long-term horizons), the dynamic dependence structure of the asset returns can be captured irrespectively of whether its origin is momentum, mean-reversion, or lead-lag relations by conditionally modeling the mean via a vector auto-regressive moving-average process (VARMA) with both orders equal one:

$$\mathbf{y}_t = \bar{\boldsymbol{\mu}} + \boldsymbol{\Phi}_M \mathbf{y}_{t-i} + \boldsymbol{\varepsilon}_t - \boldsymbol{\Pi}_M \boldsymbol{\varepsilon}_{t-j}, \tag{12}$$

where $\boldsymbol{\Phi}_M$ and $\boldsymbol{\Pi}_M$ are square fixed parameter matrices, and $\boldsymbol{\varepsilon}_t = \boldsymbol{\Sigma}^{1/2}\mathbf{x}_t$. The process is customarily assumed to be weakly (second-order or covariance) stationary and ergodic, as well as stable and invertible (see [45] for detailed characterization of multivariate time series models). VARMA processes of higher orders than the VARMA$(1, 1)$ in (12) are reported in the literature to be of less practical interest [46], and even further restrictions leading to first-order vector autoregressions (i.e., $\boldsymbol{\Theta}_M = \mathbf{0}$) are most often considered (see [47], and the more recent account in [48]). In the large dimensional portfolio setting, parsimony is crucial to maintain the efficiency and low complexity of the model estimation process, and so different simplifications based on structural restrictions are usually considered in practice. In particular, in the case of processes with scalar parameter matrices $\boldsymbol{\Phi}_M = \phi \mathbf{I}_M$ and $\boldsymbol{\Pi}_M = \pi \mathbf{I}_M$, then the population covariance matrix has a particular sparse structure, separable into cross-sectional $\boldsymbol{\Sigma}$ and temporal covariance components, which we will denote by $\boldsymbol{\Omega}$. Under the Gaussian assumption, the sample covariance matrix of members of this class of VARMA processes are doubly-correlated Wishart matrices. This matrix ensemble has been recently analyzed in the statistical physics literature in the

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[7]Although these facts approximately hold unchanged for different time intervals, some characteristics might arguably vary depending on the sampling frequency. According to the time elapsed between return observations, one might differentiate among long-term returns (e.g., weekly, monthly or yearly returns) and short-term returns (i.e., daily returns) - we have omitted purposely a further category including high-frequency data (i.e., intraday, tick data), as it requires different statistical methodologies which we will not consider in this work.

context of financial applications [49].

According to the VARMA model, the conditional covariance remains constant regardless of the data. Especially for short-term horizons (e.g., daily returns), the observed features of the volatility process are best accounted for by conditional heteroscedastic models, such as the class of specifications for the multivariate extension of generalized autoregressive conditionally heteroscedastic (GARCH) process [50], and the exponential weighted moving average (EWMA) scheme:

$$\boldsymbol{\Sigma}_t = \lambda \mathbf{y}_{t-1} \mathbf{y}_{t-1}^T + (1 - \lambda) \boldsymbol{\Sigma}_{t-1} = \lambda \sum_{n=t-N}^{t-1} (1 - \lambda)^{n-1} \mathbf{y}_n \mathbf{y}_n^T, \tag{13}$$

where λ is a smoothing prescribed parameter that characterizes the decay of the exponential memory[8]. Firstly proposed in [51], the EWMA model has been found very useful in estimating the market risk of portfolios, as well as in portfolio optimization [52].

C. Evaluating the performance of sample portfolios

The quality of a portfolio rule $\hat{\mathbf{w}}$ constructed based on *in-sample* forecasts of $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ can be measured by its achieved out-of-sample (realized) mean return $\mu_P(\hat{\mathbf{w}}) = \hat{\mathbf{w}}^T \boldsymbol{\mu}_t$ and risk $\sigma_P(\hat{\mathbf{w}}) = \sqrt{\hat{\mathbf{w}}^T \boldsymbol{\Sigma}_t \hat{\mathbf{w}}}$. In the study and practice of finance, measures of risk-adjusted achieved return are usually employed, being the Sharpe ratio a prominent one in portfolio management:

$$\mathsf{SR}(\hat{\mathbf{w}}) = \frac{\mu_P(\hat{\mathbf{w}})}{\sigma_P(\hat{\mathbf{w}})}. \tag{14}$$

In particular, notice that the tangency portfolio defined in (5) is the portfolio that maximizes (14) under the budget constraint.

As discussed in the introduction, for a small sample-size and relatively large universe of assets, the out-of-sample performance of standard portfolio constructions can be expected to considerably differ from the theoretical performance given by the true moments. In this paper, we extend existing analyses of the statistical properties of portfolio rules based on the standard sample mean and sample covariance matrix estimators, and characterize the performance deviations due to estimation risk in terms of nonrandom model and scenario parameters. We will concentrate on the case of the unconditional moment estimators (10) and (11) and conditional VARMA models with separable covariance structure. Specifically, we derive asymptotic deterministic equivalents of the out-of-sample performance of improved portfolio

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[8]Other possible and common choices of the weights for the past returns (adding up to 1) are equal weights (i.e. a rectangular window with equal weights), exponential weights (i.e. equivalent to an exponential moving average), weights following a power-law decay, or long memory weights (decaying logarithmically slowly).

implementations that are based on the previous estimators. We remark here that usual choices among practitioners of conditional heteroscedastic models, such as the EWMA model in (13), can also be fitted into our asymptotic framework by resorting to random matrix theoretical results dealing with general variance profiles (see [53]). Furthermore, we provide a mechanism to calibrate the set of weights and shrinkage parameters defining the improved portfolio constructions so as to optimize the achieved out-of-sample performance.

III. MAIN RESULTS: OUT-OF-SAMPLE ANALYSIS AND ASYMPTOTIC CORRECTIONS

In this section, we provide the main two results of the paper on the asymptotic characterization of the performance of sample portfolios and the proposed family of generalized consistent portfolio estimators are stated in Section III-A and Section III-B, respectively. We first summarize the technical hypotheses supporting our research and introduce some new definitions:

(As1) Let $\mathbf{R}_M \in \mathbb{R}^{M \times M}$ and $\mathbf{T}_N \in \mathbb{R}^{N \times N}$ be two deterministic nonnegative matrices having spectral norm bounded uniformly in M and N, i.e., $\|\mathbf{R}_M\|_{\mathrm{sup}} = \sup_{M \geq 1} \|\mathbf{R}_M\| < +\infty$ and $\|\mathbf{T}_N\|_{\mathrm{sup}} = \sup_{N \geq 1} \|\mathbf{T}_N\| < +\infty$, respectively; the matrix $\mathbf{T}_N$ is diagonal with entries denoted by t_n, $1 \leq n \leq N$.

(As2) Let $\mathbf{X}_M$ be an $M \times N$ matrix whose elements X_{ij}, $1 \leq i \leq M$, $1 \leq j \leq N$, are i.i.d. standardized Gaussian random variables.

(As3) We will consider the limiting regime defined by both dimensions M and N growing large without bound at the same rate, i.e., $N, M \to \infty$ such that $0 < \liminf c_M \leq \limsup c_M < \infty$, with $c_M = M/N$. Quantities that, under the previous double-limit regime, are asymptotically equivalent to a given a random variable, both depending on M and N, will be referred to as *asymptotic deterministic equivalents*, if only depend upon nonrandom model variables, and *generalized consistent estimators*, if they depend on observable random variables (e.g., sample data matrix).

Before proceeding with the out-of-sample performance characterization, we identify next the key quantities of study into which the Sharpe ratio performance measure in (14) can be decomposed. Let us first consider the unconditional model, where $\{\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\}$ are considered to be constant over the estimation window. Then, notice that the data observation matrix can be written as $\mathbf{Y}_N = \boldsymbol{\mu}\mathbf{1}_N^T + \boldsymbol{\Sigma}^{1/2}\mathbf{X}_N$, where $\mathbf{X}_N = [\mathbf{x}_{t-N}, \ldots, \mathbf{x}_{t-1}]$. For the sake of clarity of presentation, we will assume that the standard sample mean in (6) instead of its weighted version is applied in the definition of $\hat{\boldsymbol{\Sigma}}_{\mathrm{W}}$. Moreover, we also assume that the entries of $\mathbf{W}_{\boldsymbol{\mu},N}$ are chosen to be the eigenvalues of the matrix

$\left(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right)\mathbf{W}_{\Sigma,N}\left(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right)$, which will be denoted by $\mathbf{T}_N$. In particular, observe that, in this case,

$$\hat{\mathbf{\Sigma}}_{\mathsf{W}} = \frac{1}{N}\mathbf{\Sigma}^{1/2}\mathbf{X}_N\mathbf{T}_N\mathbf{X}_N^T\mathbf{\Sigma}^{1/2},$$

where we have used the fact that

$$\left(\boldsymbol{\mu}\mathbf{1}_N^T + \mathbf{\Sigma}^{1/2}\mathbf{X}_N\right)\left(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right) = \mathbf{\Sigma}^{1/2}\mathbf{X}_N\left(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right), \tag{15}$$

along with the invariance of the multivariate Gaussian distribution to orthogonal transformations. Notice that, under the Gaussian assumption, the matrix in 15 is matrix-variate normal distributed, i.e., $\mathbf{\Sigma}^{1/2}\mathbf{X}_N\left(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right) \sim \mathcal{MN}_{M\times N}\left(\mathbf{0}_{M\times N}, \mathbf{\Sigma}, \mathbf{I}_N\right)$, or equivalently, $\mathrm{vec}\left(\mathbf{\Sigma}^{1/2}\mathbf{X}_N\left(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right)\right) \sim \mathcal{N}_{MN}\left(\mathbf{0}_{MN}, \mathbf{\Sigma}\otimes\mathbf{I}_N\right)$; see [54, Section 3.3.2]. Consequently, $\hat{\mathbf{\Sigma}}_{\mathsf{W}}$ is a central quadratic forms (central Wishart distributed if $\mathbf{T}_N = \mathbf{I}_N$).

Now, let $\mathbf{R}_M = \mathbf{\Sigma}_0^{-1/2}\mathbf{\Sigma}\mathbf{\Sigma}_0^{-1/2}$, and also, with some abuse of notation, $\mathbf{R}_M^{1/2} = \mathbf{\Sigma}_0^{-1/2}\mathbf{\Sigma}^{1/2}$, and consider further the nonnegative scalars $\alpha_M = \rho/(1-\rho)$ and $\beta_M = \delta/(1-\delta)$. Moreover, we define $\tilde{\mathbf{Y}}_N = \mathbf{R}_M^{1/2}\mathbf{X}_N\mathbf{T}_N^{1/2}$, along with $\hat{\mathbf{\Sigma}}_M = \frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T + \alpha_M\mathbf{I}_M$, and $\hat{\boldsymbol{v}}_M = \frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\boldsymbol{v}}_N$, where $\tilde{\boldsymbol{v}}_N = \sqrt{N}\boldsymbol{v}_N$, with $\boldsymbol{v}_N$ being an N dimensional nonrandom vector with unit norm. Then, it is straightforward to see that the numerator and denominator of (14) can be written for the class of sample implementations of the optimal portfolio in (3) based on the unconditional estimators (10) and (11) in terms of the following random variables:

$$\hat{\xi}_M^{(1)} = \boldsymbol{v}_M^T\hat{\mathbf{\Sigma}}_M^{-1}\boldsymbol{v}_M, \tag{16}$$

$$\hat{\xi}_M^{(2)} = \boldsymbol{v}_M^T\hat{\mathbf{\Sigma}}_M^{-1}\hat{\boldsymbol{v}}_M, \tag{17}$$

$$\hat{\xi}_M^{(3)} = \hat{\boldsymbol{v}}_M^T\hat{\mathbf{\Sigma}}_M^{-1}\hat{\boldsymbol{v}}_M, \tag{18}$$

$$\hat{\xi}_M^{(4)} = \boldsymbol{v}_M^T\hat{\mathbf{\Sigma}}_M^{-1}\mathbf{R}_M\hat{\mathbf{\Sigma}}_M^{-1}\boldsymbol{v}_M, \tag{19}$$

$$\hat{\xi}_M^{(5)} = \boldsymbol{v}_M^T\hat{\mathbf{\Sigma}}_M^{-1}\mathbf{R}_M\hat{\mathbf{\Sigma}}_M^{-1}\hat{\boldsymbol{v}}_M, \tag{20}$$

$$\hat{\xi}_M^{(6)} = \hat{\boldsymbol{v}}_M^T\hat{\mathbf{\Sigma}}_M^{-1}\mathbf{R}_M\hat{\mathbf{\Sigma}}_M^{-1}\hat{\boldsymbol{v}}_M. \tag{21}$$

Notice that similar reasoning applies to the conditional model for the asset return in (12), since the sample covariance matrix of the process is a doubly-correlated Wishart matrix (cf. Section II-B), and the conditional mean estimator can be written from the VAR(1) model specification as $\hat{\boldsymbol{\mu}}_t = \tilde{\boldsymbol{\mu}} + \mathbf{\Sigma}^{1/2}\mathbf{X}_N\mathbf{1}_{\Lambda,N}$, where $\tilde{\boldsymbol{\mu}} = \sum_{n=1}^N\phi^{n-1}\bar{\boldsymbol{\mu}}$ and $\mathbf{1}_{\Lambda,N} = \Lambda_N\mathbf{1}_N$, with $\Lambda_N \in \mathbb{R}^{N\times N}$ being a diagonal matrix such that $[\Lambda_N]_n = \phi^n$. In general, the vector $\boldsymbol{v}_M$ takes values in $\left\{\mathbf{\Sigma}_0^{-1/2}\mathbf{1}_M/\sqrt{M}, \mathbf{\Sigma}_0^{-1/2}\boldsymbol{\mu}, \mathbf{\Sigma}_0^{-1/2}\boldsymbol{\mu}_0, \mathbf{\Sigma}_0^{-1/2}\tilde{\boldsymbol{\mu}}\right\}$, and $\boldsymbol{v}_N$ in $\left\{\mathbf{1}_N/\sqrt{N}, \mathbf{1}_{\Lambda,N}\right\}$.

By way of example, consider the estimation of the quantities $\{A, B, C\}$ defining the optimal mean-variance portfolio in (3), based on the unconditional estimators (10) and (11) with $\delta = 0$. Let us denote the estimators by $\left\{\hat{A}, \hat{B}, \hat{C}\right\}$. In particular, observe that $(1 - \rho)\,\hat{A} = \mathbf{1}_M^T \boldsymbol{\Sigma}_0^{-1/2} \left(\boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\Sigma}}_{\mathsf{W}}\boldsymbol{\Sigma}_0^{-1/2} + \alpha_M \mathbf{I}_M\right)^{-1} \boldsymbol{\Sigma}_0^{-1/2}\mathbf{1}_M$, and so we readily have $(1 - \rho)\,\hat{A} = M\hat{\xi}_M^{(1)}$, with $\boldsymbol{v}_M = \boldsymbol{\Sigma}_0^{-1/2}\mathbf{1}_M/\sqrt{M}$. Moreover, note that $(\mathbf{W}_{\boldsymbol{\mu},N}\mathbf{1}_N = \mathbf{T}_N^{1/2}\tilde{\mathbf{1}}_N$, where $\tilde{\mathbf{1}}_N = \mathbf{T}_N^{-1/2}\mathbf{W}_{\boldsymbol{\mu},N}\mathbf{1}_N)$ $(\frac{1}{N}\mathbf{1}_N^T\mathbf{W}_{\boldsymbol{\mu},N}\mathbf{1}_N = 1$ -in definition above-)

$$(1 - \rho)\,\hat{B} = \mathbf{1}_M^T \boldsymbol{\Sigma}_0^{-1/2} \left(\boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\Sigma}}_{\mathsf{W}}\boldsymbol{\Sigma}_0^{-1/2} + \alpha_M \mathbf{I}_M\right)^{-1} \boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\mu}}_{\mathsf{W}}$$

$$= \mathbf{1}_M^T \boldsymbol{\Sigma}_0^{-1/2} \left(\boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\Sigma}}_{\mathsf{W}}\boldsymbol{\Sigma}_0^{-1/2} + \alpha_M \mathbf{I}_M\right)^{-1} \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{\mu} + \frac{1}{N}\mathbf{1}_M^T \boldsymbol{\Sigma}_0^{-1/2} \left(\boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\Sigma}}_{\mathsf{W}}\boldsymbol{\Sigma}_0^{-1/2} + \alpha_M \mathbf{I}_M\right)^{-1} \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{\Sigma}$$

and therefore we have $(1 - \rho)\,\hat{B} = \sqrt{M}\hat{\xi}_M^{(1)} + \sqrt{M}\hat{\xi}_M^{(2)}$, where the vector $\boldsymbol{v}_M$ take values $\boldsymbol{v}_M = \boldsymbol{\Sigma}_0^{-1/2}\mathbf{1}_M/\sqrt{M}$ and $\boldsymbol{v}_M = \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{\mu}$, and $\tilde{\boldsymbol{v}}_N = \tilde{\mathbf{1}}_N$.

Furthermore, for the estimator of C, we have that $(\hat{\boldsymbol{\mu}}_{\mathsf{W}} = \frac{1}{N}\left(\boldsymbol{\mu}\mathbf{1}_N^T + \boldsymbol{\Sigma}^{1/2}\mathbf{X}_N\right)\mathbf{W}_{\boldsymbol{\mu},N}\mathbf{1}_N = \frac{1}{N}\boldsymbol{\mu}\mathbf{1}_N^T\mathbf{W}_{\boldsymbol{\mu},N}\mathbf{1}_N + \frac{1}{N}\boldsymbol{\Sigma}^{1/2}\mathbf{X}_N\mathbf{W}_{\boldsymbol{\mu},N}\mathbf{1}_N = \boldsymbol{\mu} + \frac{1}{N}\boldsymbol{\Sigma}^{1/2}\mathbf{X}_N\mathbf{T}_N^{1/2}\tilde{\mathbf{1}}_N)$ $(\|\boldsymbol{\mu}\|)$

$$(1 - \rho)\,\hat{C} = \hat{\boldsymbol{\mu}}_{\mathsf{W}}^T \boldsymbol{\Sigma}_0^{-1/2} \left(\boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\Sigma}}_{\mathsf{W}}\boldsymbol{\Sigma}_0^{-1/2} + \alpha_M \mathbf{I}_M\right)^{-1} \boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\mu}}_{\mathsf{W}}$$

$$= \boldsymbol{\mu}^T \boldsymbol{\Sigma}_0^{-1/2} \left(\boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\Sigma}}_{\mathsf{W}}\boldsymbol{\Sigma}_0^{-1/2} + \alpha_M \mathbf{I}_M\right)^{-1} \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{\mu}$$

$$+ 2\frac{1}{N}\boldsymbol{\mu}^T \boldsymbol{\Sigma}_0^{-1/2} \left(\boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\Sigma}}_{\mathsf{W}}\boldsymbol{\Sigma}_0^{-1/2} + \alpha_M \mathbf{I}_M\right)^{-1} \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{\Sigma}^{1/2}\mathbf{X}_N\mathbf{T}_N^{1/2}\tilde{\mathbf{1}}_N$$

$$+ \frac{1}{N^2}\tilde{\mathbf{1}}_N^T\mathbf{T}_N^{1/2}\mathbf{X}_N^T\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}_0^{-1/2} \left(\boldsymbol{\Sigma}_0^{-1/2}\hat{\boldsymbol{\Sigma}}_{\mathsf{W}}\boldsymbol{\Sigma}_0^{-1/2} + \alpha_M \mathbf{I}_M\right)^{-1} \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{\Sigma}^{1/2}\mathbf{X}_N\mathbf{T}_N^{1/2}\tilde{\mathbf{1}}_N$$

$$= \hat{\xi}_M^{(1)} + \hat{\xi}_M^{(2)} + \hat{\xi}_M^{(3)}.$$

Finally, notice that, additionally, the term is required to model the variance of the GMVP, and so is for modeling the return of the TP, but both terms can be straightforwardly represented similarly as $\left\{\hat{A}, \hat{B}, \hat{C}\right\}$.

$(1 - \rho)^2\,\mathbf{1}_M^T\hat{\boldsymbol{\Sigma}}_{\mathsf{SHR}}^{-1}\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}_{\mathsf{SHR}}^{-1}\mathbf{1}_M = M\hat{\xi}_M^{(4)}$,

$(1 - \rho)^2\,\hat{\boldsymbol{\mu}}_{\mathsf{W}}^T\hat{\boldsymbol{\Sigma}}_{\mathsf{SHR}}^{-1}\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}_{\mathsf{SHR}}^{-1}\hat{\boldsymbol{\mu}}_{\mathsf{W}} = \hat{\xi}_M^{(4)} + \hat{\xi}_M^{(5)} + \hat{\xi}_M^{(6)}$.

From above, the previous two assumptions on the weighting matrices clearly facilitate exposition and tractability. However, we remark that more general cases can be equivalently reduced to the above key quantities by algebraic manipulations essentially relying on the matrix inversion lemma (cf. identity (27) in Appendix A-A).

Now that the out-of-sample performance characterization problem has been reduced to the study of the behavior of the quantities (16) to (21), we proceed in the following two sections with their asymptotic analysis and consistent estimation.

A. Asymptotic performance analysis: a RMT approach

Define $\gamma = \gamma_M = \frac{1}{N} \operatorname{tr} \left[\mathbf{E}_M^2 \right]$ and $\tilde{\gamma} = \tilde{\gamma}_M = \frac{1}{N} \operatorname{tr} \left[\tilde{\mathbf{E}}_N^2 \right]$, where $\mathbf{E}_M = \mathbf{R}_M \left(\tilde{\delta}_M \mathbf{R}_M + \alpha \mathbf{I}_M \right)^{-1}$ and $\tilde{\mathbf{E}}_N = \mathbf{T}_N \left(\mathbf{I}_N + \delta_M \mathbf{T}_N \right)^{-1}$, with $\left\{ \tilde{\delta}_M, \delta_M \right\}$ being the unique positive solution to the following system of equations [55, Proposition 1]:

$$
\begin{cases}
\tilde{\delta}_M = \frac{1}{N} \operatorname{tr} \left[\mathbf{T}_N \left(\mathbf{I}_N + \delta_M \mathbf{T}_N \right)^{-1} \right] \\
\delta_M = \frac{1}{N} \operatorname{tr} \left[\mathbf{R}_M \left(\tilde{\delta}_M \mathbf{R}_M + \alpha \mathbf{I}_M \right)^{-1} \right].
\end{cases}
\tag{22}
$$

Then, we have the following result characterizing the asymptotic behavior of the random variables (16) to (21).

Theorem 1: (*Asymptotic Deterministic Equivalents*) Under Assumptions (**As1**) to (**As3**), the following asymptotic equivalences hold true:

$$
\hat{\xi}_M^{(1)} \asymp \boldsymbol{v}_M^T \left(\tilde{\delta}_M \mathbf{R}_M + \alpha_M \mathbf{I}_M \right)^{-1} \boldsymbol{v}_M,
$$

$$
\hat{\xi}_M^{(2)} \asymp 0,
$$

$$
\hat{\xi}_M^{(3)} \asymp \delta_M \boldsymbol{v}_N^T \mathbf{T}_N \left(\delta_M \mathbf{T}_N + \mathbf{I}_N \right)^{-1} \boldsymbol{v}_N,
$$

$$
\hat{\xi}_M^{(4)} \asymp \frac{1}{1 - \gamma_M \tilde{\gamma}_M} \boldsymbol{v}_M^T \mathbf{R}_M^{1/2} \left(\tilde{\delta}_M \mathbf{R}_M + \alpha_M \mathbf{I}_M \right)^{-2} \mathbf{R}_M^{1/2} \boldsymbol{v}_M,
$$

$$
\hat{\xi}_M^{(5)} \asymp 0,
$$

$$
\hat{\xi}_M^{(6)} \asymp \frac{\gamma_M}{1 - \gamma_M \tilde{\gamma}_M} \boldsymbol{v}_N^T \mathbf{T}_N \left(\delta_M \mathbf{T}_N + \mathbf{I}_N \right)^{-2} \boldsymbol{v}_N.
$$

Proof: See Appendix B. ∎

Using Theorem 1, estimates of the out-of-sample performance of optimal sample mean-variance portfolios based on the unconditional and conditional models in Section II are readily obtained. By means of the previous asymptotic approximations in a practically more meaningful and relevant double-limit regime (cf. Section IV), more accurate information about the underestimation and overestimation effects of the portfolio risk and return, respectively, can be provided.

The previous result is of interest on its own for characterization purposes as well as for scenario analysis in investment management. However, particularly for the calibration of unconditional models, one might well also be interested in estimates of the previous quantities that are given in terms of the available information, i.e., essentially, the data observation matrix. In the proposed asymptotic regime, it follows from Theorem 1 that both $\hat{\xi}_M^{(2)}$ and $\hat{\xi}_M^{(5)}$ are negligible and therefore can be discarded for analysis

and decision purposes. While $\hat{\xi}_M^{(1)}$ and $\hat{\xi}_M^{(3)}$ are already given in terms of only observable data[9], terms $\hat{\xi}_M^{(4)}$ and $\hat{\xi}_M^{(6)}$ happen to be defined in terms of the unknown $\mathbf{R}_M$. We next present a class of estimators of (19) and (21), or equivalently their asymptotic deterministic equivalents provided by Theorem 1, which are strongly consistent under the limiting regime in (**As3**).

B. Consistent estimation of optimal large dimensional portfolios

The parameters defining the estimators in (10) and (11), i.e., $\{\mathbf{W}_{\boldsymbol{\mu},N}, \mathbf{W}_{\boldsymbol{\Sigma},N}\}$ and $\{\delta, \rho\}$, effectively represent a set of degrees-of-freedom with respect to which the out-of-sample performance of a portfolio construction can be improved. For the calibration of unconditional models by means of optimizing the estimator parameterization, only the available sample data can be used in practice in order to select the previous set of parameters. To that effect, from the definition of the quantities (19) and (21) and the discussion above, the estimation of $\hat{\xi}_M^{(4)}$ and $\hat{\xi}_M^{(6)}$ are required. The naive approach is based on the *plug-in* or conventional estimator of $\hat{\xi}_M^{(4)}$, henceforth denoted with the subscript "cnv" by $\hat{\xi}_{cnv,M}^{(4)}$, which is given by replacing the unknown theoretical covariance matrix by the SCM, i.e.,

$$\hat{\xi}_{cnv,M}^{(4)} = \boldsymbol{v}_M^T \left(\tfrac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T + \alpha_M \mathbf{I}_M \right)^{-1} \tfrac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T \left(\tfrac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T + \alpha_M \mathbf{I}_M \right)^{-1} \boldsymbol{v}_M. \tag{23}$$

Additionally, let $\hat{\xi}_{cnv,M}^{(6)}$ denote the "plug-in" estimator of $\hat{\xi}_M^{(6)}$, and notice that

$$\hat{\xi}_{cnv,M}^{(6)} = \tfrac{1}{N}\boldsymbol{v}_N^T \tilde{\mathbf{Y}}_N^T \left(\tfrac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T + \alpha_M \mathbf{I}_M \right)^{-1} \tfrac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T \left(\tfrac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T + \alpha_M \mathbf{I}_M \right)^{-1} \tilde{\mathbf{Y}}_N \boldsymbol{v}_N$$

$$= \boldsymbol{v}_N^T \left(\tfrac{1}{N}\tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N \right)^2 \left(\tfrac{1}{N}\tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N + \alpha_M \mathbf{I}_N \right)^{-2} \boldsymbol{v}_N. \tag{24}$$

Before presenting the main result of this section, we provide an intermediate result that will be required for the statement of the improved estimators.

Proposition 1: Under Assumptions (**As1**) to (**As3**), a generalized consistent estimator of δ_M, denoted by $\hat{\delta}_M$, is given by the unique positive solution to the following equation:

$$\frac{1}{N} \text{tr} \left[\tfrac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T \left(\tfrac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T + \alpha_M \mathbf{I}_M \right)^{-1} \right] = \delta \frac{1}{N} \text{tr} \left[\mathbf{T}_N \left(\mathbf{I}_N + \delta \mathbf{T}_N \right)^{-1} \right].$$

Proof: The proof follows from the convergence result (40) in Proposition 2, for $\boldsymbol{\Theta}_M = \frac{1}{N}\mathbf{I}_M$ and $z = -1$. ∎

---

[9]This is not the case for $\boldsymbol{v}_M = \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{\mu}$, but still the consistent estimation of $\hat{\xi}_M^{(1)}$ can be handled straightforwarly by rearranging terms.

The following theorem provides estimators of $\hat{\xi}_M^{(4)}$ and $\hat{\xi}_M^{(6)}$, which are consistent in the double-limit regime in Assumption (**As3**).

Theorem 2: (*Generalized Consistent Estimators*) Under Assumptions (**As1**) to (**As3**), we have the following consistent estimators for $\hat{\xi}_M^{(4)}$ and $\hat{\xi}_M^{(6)}$:

$$\hat{\xi}_{gce,M}^{(4)} = a_M \hat{\xi}_{cnv,M}^{(4)}, \tag{25}$$

$$\hat{\xi}_{gce,M}^{(6)} = a_M \hat{\xi}_{cnv,M}^{(6)} + b_M, \tag{26}$$

where $\hat{\xi}_{cnv,M}^{(4)}$ and $\hat{\xi}_{cnv,M}^{(6)}$ are defined as in (23) and (24), respectively, and

$$a_M = \frac{1}{\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(\mathbf{I}_N + \hat{\delta}_M\mathbf{T}_N\right)^{-2}\right]},$$

$$b_M = -\frac{\hat{\delta}_M^2 \boldsymbol{v}_M^T \mathbf{T}_N^2 \left(\mathbf{I}_N + \hat{\delta}_M\mathbf{T}_N\right)^{-2} \boldsymbol{v}_M}{\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(\mathbf{I}_N + \hat{\delta}_M\mathbf{T}_N\right)^{-2}\right]},$$

with $\hat{\delta}_M$ being given by Proposition 1.

Proof: See Appendix C. ∎

Remark 1: The asymptotic equivalents and consistent estimators of $\hat{\xi}_M^{(1)}$ and $\hat{\xi}_M^{(4)}$ in Theorem and Theorem, respectively, generalize previous results on the characterization of quadratic forms depending on the eigenvalues and eigenvectors of the sample covariance matrix (see [56, Proposition 1],[57, Chapter 4] and [58, Theorem 1]).

Remark 2: We notice that Theorem 1, Proposition 1 and Theorem 2 hold verbatim if the vectors $\boldsymbol{v}_M$, $\boldsymbol{v}_N$ and the matrices $\mathbf{X}_N$, $\mathbf{R}_M$, $\boldsymbol{\Sigma}_M$, $\mathbf{T}_N$ have complex-valued entries.

IV. NUMERICAL VALIDATIONS

In this section, we provide the results of some simulations illustrating the power of the proposed analytical framework. In particular, we consider the construction of a GMVP based on synthetic data modeling a universe of $M = 50$ assets (e.g., Euro Stoxx 50) with annualized volatility (standard deviation) between 20% and 30%. For simple illustration purposes, we have assumed that the expected return is negligible compared to the asset covariance matrix, and so it has not been estimated. We run simulations considering estimation windows ranging from 20 to 200 return observations. Specifically, we measure the accuracy of approximating the out-of-sample (realized) variance of a GMVP by its asymptotic deterministic equivalent (ADE) given in terms of the investment scenario parameters (cf. Section III-A), the *conventional* (CNV) implementation based on the naive replacement of the unknown parameters by
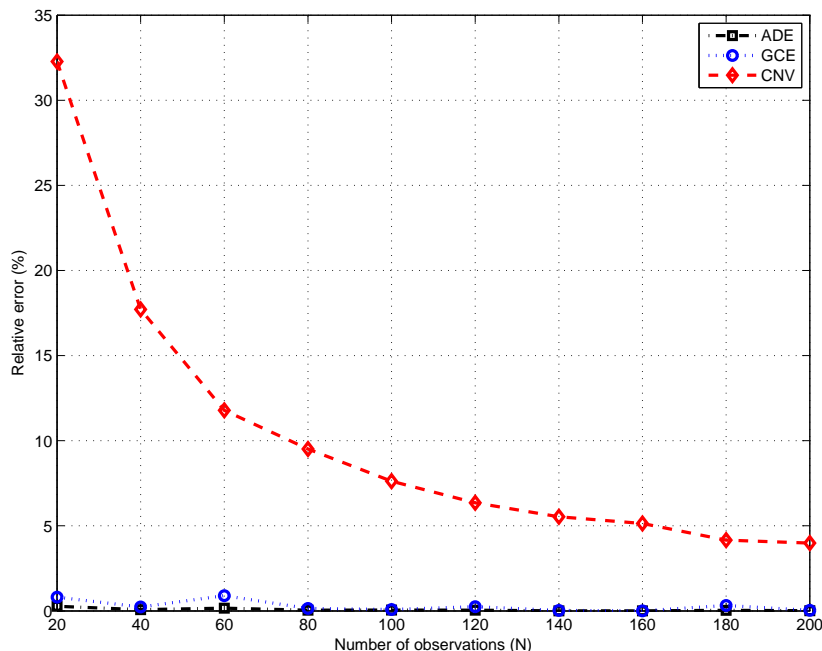
Fig. 1. Approximation of realized out-of-sample variance of a GMVP for fixed calibrating parameters

their sample counterparts, and its *generalized consistent estimator* (GCE) derived in Section III-B. Monte Carlo simulations (10^3 iterations) are run for three different scenarios, for which the approximation error in relative terms and in percentage is provided, i.e., $100 \times |\sigma_P(\hat{\mathbf{w}}_{\mathsf{GMVP}}) - \hat{\sigma}_P(\hat{\mathbf{w}}_{\mathsf{GMVP}})| / \sigma_P(\hat{\mathbf{w}}_{\mathsf{GMVP}})$, where $\hat{\sigma}_P(\hat{\mathbf{w}}_{\mathsf{GMVP}})$ denotes here any of the three approximations. Moreover, in all cases we have considered a covariance matrix shrinkage estimator with $\mathbf{\Sigma}_0 = \mathbf{I}_M$, and parameters ρ and $\mathbf{W}_{\mathbf{\Sigma},N}$ to be calibrated for optimal performance. In the first experiment, we consider fixed values of the calibrating parameters given by the coefficient $\rho = 0.05$ and a diagonal matrix $\mathbf{T} = \mathbf{W}_{\mathbf{\Sigma},N}$ given by half of its entries being equal to $t = 0.75$ and the other half equal to $2 - t$. Figure 1 shows the relative approximation error for each method. In the two other experiments, we consider the construction of GMVPs given by the calibration of the optimal (for minimum variance) parameters ρ or $\mathbf{W}_{\mathbf{\Sigma},N}$, respectively, where in each case the other parameter has been fixed to its value in the first experiment. Figures 2 and 3 show the results for the calibration of ρ and $\mathbf{W}_{\mathbf{\Sigma},N}$, respectively. In our simulations, we applied a naive optimization scheme to find the optimal parameters in these simple illustrative examples, as we do not pursue dealing with practical optimization issues in this work, but rather focus on a representative
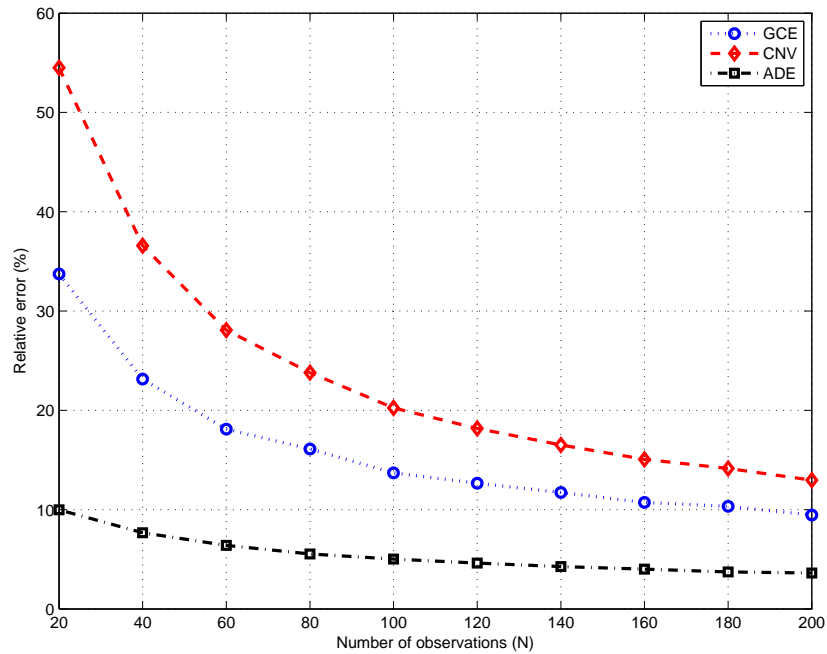
Fig. 2. Approximation of realized out-of-sample variance of a GMVP for fixed ρ and optimized $\mathbf{W}_{\mathbf{\Sigma},N}$

validation of the statistical results that we have derived; efficient optimization algorithms based, e.g., on successive convex approximation (see [59]), are left as future work and are now under investigation by the authors. From the simulation outputs, it is clear that the performance of the proposed consistent estimators is decreased whenever calibration of the parameters has to be performed, essentially due to the variability (fluctuations) of the estimators. An extensive simulation campaign is outside the scope of the section and the paper, but a reduction of this effect can be observed as expected by increasing for instance the number of assets in the universe (e.g., in the same illustrative line, $M = 300$ for the index Euro Stoxx 300). The use of information about the fluctuations of the estimators in order to improve the performance of the method is currently under investigation.

V. CONCLUSIONS

In this paper, we have provided a asymptotic framework for the analysis of the consistency of arbitrarily large sample mean-variance portfolios that are constructed on the basis of improved Bayesian or shrinkage estimation and weighted sampling. To that effect, we have resorted to recent contributions on the theory of the spectral analysis of large random matrices, based on a double-limit regime that is defined by both
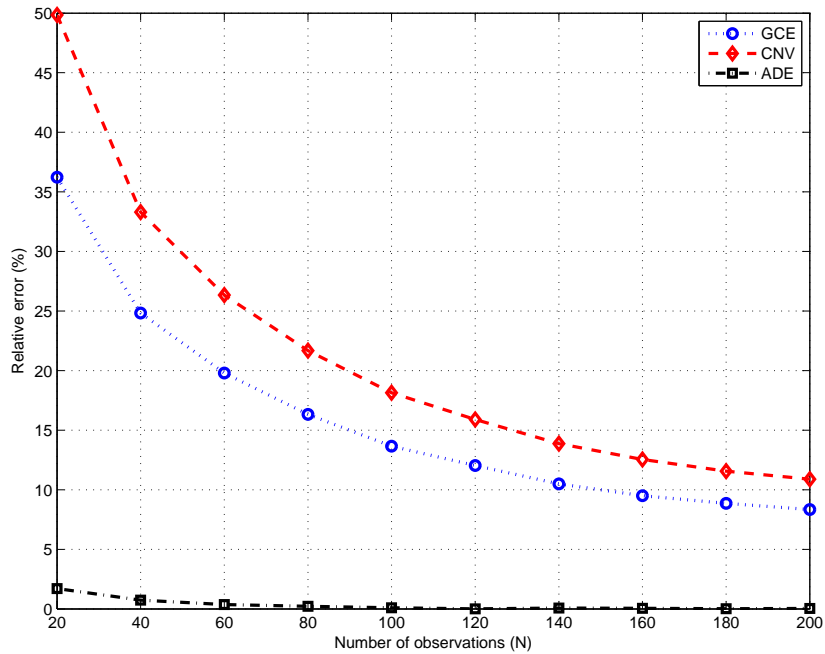
Fig. 3. Approximation of realized out-of-sample variance of a GMVP for fixed $\mathbf{W}_{\Sigma,N}$ and optimized ρ

the number of samples and the number of portfolio constituents going to infinity at the same rate. In spite of its asymptotic nature, by keeping both the return observation size and dimension to be of the same order of magnitude our results have proved to successfully describe the performance of sample portfolios under realistic, finite-size situations of interest. Furthermore, based on the previous characterization of the estimation risk, corrections of the level of risk underestimation and return overestimation of a specific portfolio constructions have been proposed so as to optimize the out-of-sample performance. Our proposed calibration rules represent a sensible portfolio choice improving on standard, usually overly optimistic Sharpe-based investment decisions.

APPENDIX A
TECHNICAL PRELIMINARIES

A. Further definitions and auxiliary relations

We first recall the Sherman–Morrison–Woodbury formula, or matrix inversion lemma, which will be used repeatedly in the sequel, i.e.,

$$(\mathbf{U}\boldsymbol{\Xi}\mathbf{V} + \boldsymbol{\Lambda})^{-1} = \boldsymbol{\Lambda}^{-1} - \boldsymbol{\Lambda}^{-1}\mathbf{U}\left(\boldsymbol{\Xi}^{-1} + \mathbf{V}\boldsymbol{\Lambda}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}\boldsymbol{\Lambda}^{-1}. \tag{27}$$

In particular, the next identity for rank-augmenting matrices follows from (27):

$$\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}. \tag{28}$$

Let $\mathbf{Q} = \left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-1}$, and also $\mathbf{Q}_{(n)} = \left(\frac{1}{N}\tilde{\mathbf{Y}}_{(n)}\tilde{\mathbf{Y}}_{(n)}^T - z\mathbf{I}_M\right)^{-1}$, where $\tilde{\mathbf{Y}}_{(n)} \in \mathbb{C}^{M \times N-1}$ is defined by extracting the nth column from the data matrix $\tilde{\mathbf{Y}}$. In particular, using (28), we get

$$\mathbf{Q} = \mathbf{Q}_{(n)} - \frac{\frac{1}{N}\mathbf{Q}_{(n)}\tilde{\mathbf{y}}_n\tilde{\mathbf{y}}_n^T\mathbf{Q}_{(n)}}{1 + \frac{1}{N}\tilde{\mathbf{y}}_n^T\mathbf{Q}_{(n)}\tilde{\mathbf{y}}_n}. \tag{29}$$

Using the definitions in Section III-A, we observe that

$$\delta_M - \tilde{\delta}_M\gamma_M = \alpha_M\frac{1}{N}\operatorname{tr}\left[\mathbf{R}_M\left(\tilde{\delta}_M\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\right] \tag{30}$$

$$\tilde{\delta}_M - \delta_M\tilde{\gamma}_M = \frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(\mathbf{I}_N + \delta_M\mathbf{T}_N\right)^{-2}\right]. \tag{31}$$

Additionally, the following definitions will be useful for our derivations:

$$\zeta_M = \frac{1}{1 - \gamma_M\tilde{\gamma}_M}\frac{1}{N}\operatorname{tr}\left[\mathbf{R}_M\left(\tilde{\delta}_M\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\right],$$

$$\tilde{\zeta}_M = -\frac{\tilde{\gamma}_M}{1 - \gamma_M\tilde{\gamma}_M}\frac{1}{N}\operatorname{tr}\left[\mathbf{R}_M\left(\tilde{\delta}_M\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\right].$$

In particular, notice that

$$\tilde{\zeta}_M = -\tilde{\gamma}_M\zeta_M. \tag{32}$$

Lemma 1: The following relations hold true:

$$\tilde{\delta}_M + \alpha_M\tilde{\zeta}_M = \frac{1}{1 - \gamma_M\tilde{\gamma}_M}\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(\mathbf{I}_N + \delta_M\mathbf{T}_N\right)^{-2}\right] \tag{33}$$

$$\delta_M - \alpha_M\zeta_M = \frac{\gamma_M}{1 - \gamma_M\tilde{\gamma}_M}\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(\mathbf{I}_N + \delta_M\mathbf{T}_N\right)^{-2}\right]. \tag{34}$$

Proof: We first show that $\delta_M - \alpha_M\zeta_M = \gamma_M\left(\tilde{\delta}_M + \alpha_M\tilde{\zeta}_M\right)$, and then prove that $\tilde{\delta}_M + \alpha_M\tilde{\zeta}_M = (1 - \gamma_M\tilde{\gamma}_M)^{-1}\left(\tilde{\delta}_M - \delta_M\tilde{\gamma}_M\right)$, so that the result follows finally by using (31). Let us handle the first equality. Using the definitions above, by simple partial fraction decomposition, we get

$$\delta_M - \alpha_M\zeta_M = \gamma_M\left(\tilde{\delta}_M - \alpha_M\tilde{\gamma}_M\zeta_M\right),$$

and the first equality follows by using (32). Regarding the second equality, using the definition of ζ_M along with (30) we notice that

$$\alpha_M\tilde{\zeta}_M = \frac{1}{1 - \gamma_M\tilde{\gamma}_M}\left(\tilde{\delta}_M\gamma_M - \delta_M\right),$$

and the equality follows by introducing the previous expression in $\tilde{\delta}_M + \alpha_M\tilde{\zeta}_M$ and finally rearranging terms. ∎

B. Some useful stochastic convergence results

The following two results will be useful to prove the vanishing characteristic of both $\hat{\xi}_M^{(2)}$ and $\hat{\xi}_M^{(5)}$.

Lemma 2: (Burkholder's inequality) Let $\{\mathcal{F}_l\}$ be a given filtration and $\{X_l\}$ a martingale difference sequence with respect to $\{\mathcal{F}_l\}$. Then, for any $p \in (1, \infty)$, there exist constants K_1 and K_2 depending only on p such that [60, Theorem 9]

$$K_1 \mathbb{E}\left[\left(\sum_{l=1}^{L} |X_l|^2\right)^{p/2}\right] \leq \mathbb{E}\left[\left|\sum_{l=1}^{L} X_l\right|^p\right] \leq K_2 \mathbb{E}\left[\left(\sum_{l=1}^{L} |X_l|^2\right)^{p/2}\right].$$

The result above as well as the next were originally proved for real variables. Extensions to the complex case are straightforward. The following result can be shown by using the martingale convergence theorem [61]. We provide a sketch of the proof, which essentially follows the exposition in [62, Theorem 20.10] (see also [63, Corollary 3] and references therein).

Theorem 3: Let $\{\mathcal{F}_l\}$ be a given filtration and $\{X_l\}$ a square-integrable martingale difference sequence with respect to $\{\mathcal{F}_l\}$. If

$$\sup_{L \geq 1} \frac{1}{L} \sum_{l=1}^{L} \mathbb{E}\left[|X_l|^2 \,|\mathcal{F}_{l-1}\right] < \infty,$$

then

$$\frac{1}{\sqrt{L}} \sum_{l=1}^{L} X_l \to 0,$$

almost surely, as $L \to \infty$.

Proof: Define $T_L = L^{-1/2} \sum_{l=1}^{L} X_l$ so that $\{T_L\}$ is a square-integrable martingale with respect to $\{\mathcal{F}_l\}$. In particular, we have

$$\sup_{L \geq 1} \mathbb{E}\left[|T_L|\right] \leq \sup_{L \geq 1} \mathbb{E}^{1/2}\left[|T_L|^2\right] < \infty,$$

the last inequality following from Burkholder's inequality in Lemma 2. Then, by the martingale convergence theorem we have that T_L converges almost surely as $L \to \infty$ to an integrable random variable, and the result follows by Kronecker's lemma (see, e.g., [61, pag. 31]). ∎

In the sequel, the matrix $\boldsymbol{\Theta}_M \in \mathbb{R}^{M \times M}$ will denote an arbitrary nonrandom matrix having trace norm bounded uniformly in M. Notice that $\|\boldsymbol{\Theta}_M\|_F \leq \|\boldsymbol{\Theta}_M\|_{\text{tr}}$, and so the Frobenius norm of $\boldsymbol{\Theta}_M$ is also uniformly bounded. For instance, if $\mathbf{Z}_M \in \mathbb{R}^{M \times M}$ is an arbitrary nonrandom matrix with uniformly bounded spectral (in M), then in the cases $\boldsymbol{\Theta}_M = \frac{1}{M}\mathbf{Z}_M$ and $\boldsymbol{\Theta}_M = \boldsymbol{v}_M \boldsymbol{v}_M^T$, we have $\left\|\frac{1}{M}\mathbf{Z}_M\right\|_F = \frac{1}{M^{1/2}}\left(\frac{1}{M} \text{tr}\left[\mathbf{Z}_M \mathbf{Z}_M^T\right]\right)^{1/2} = \mathcal{O}\left(M^{-1/2}\right)$ and $\left\|\boldsymbol{v}_M \boldsymbol{v}_M^T\right\|_F = \|\boldsymbol{v}_M\|^2 = \mathcal{O}(1)$, respectively. The following theorem will be instrumental in the proof of our results. The theorem is originally stated in a more general form for complex-valued matrices but applies verbatim for matrices with real-valued entries [64].

Theorem 4: Under Assumptions (**As1**) to (**As3**), for each $z \in \mathbb{C} - \mathbb{R}^+$,

$$\mathrm{tr}\left[\boldsymbol{\Theta}_M \left(\frac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-1}\right] \asymp \mathrm{tr}\left[\boldsymbol{\Theta}_M \left(x_M \mathbf{R} - z\mathbf{I}_M\right)^{-1}\right], \tag{35}$$

$$\mathrm{tr}\left[\boldsymbol{\Theta}_N \left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N - z\mathbf{I}_N\right)^{-1}\right] \asymp \mathrm{tr}\left[\boldsymbol{\Theta}_N \left(\mathbf{I}_N + e_M \mathbf{T}\right)^{-1}\right], \tag{36}$$

where $\{e_M = e_M(z), x_M = x_M(z)\}$ is the unique solution in $\mathbb{C}^+$ of the system of equations:

$$\begin{cases} e_M = \frac{1}{M}\,\mathrm{tr}\left[\mathbf{R}\left(x_M \mathbf{T} - z\mathbf{I}_M\right)^{-1}\right] \\ x_M = \frac{1}{N}\,\mathrm{tr}\left[\mathbf{T}\left(\mathbf{I}_N + e_M \mathbf{T}\right)^{-1}\right]. \end{cases}$$

Moreover, given a symmetric nonnegative definite matrix $\mathbf{A}_M \in \mathbb{C}^{M \times M}$, we also have, for each $z \in \mathbb{C} - \mathbb{R}^+$,

$$\mathrm{tr}\left[\boldsymbol{\Theta}_M \left(\mathbf{A}_M + \frac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-1}\right] \asymp \mathrm{tr}\left[\boldsymbol{\Theta}_M \left(\mathbf{A}_M + x_M \mathbf{R}_M - z\mathbf{I}_M\right)^{-1}\right]. \tag{37}$$

In particular, notice that $\left\{\tilde{\delta}_M, \delta_M\right\}$ coincides with $\{e_M = e_M(z), x_M = x_M(z)\}$ evaluated at $z = -\alpha_M$ (see [55, Proposition 1]). Moreover, we remark that where $\zeta_M = e'_M$ and $\tilde{\zeta}_M = x'_M$, where $e'_M = e'_M(z)$ and $x'_M = x'_M(z)$ are the derivatives wrt. z of, respectively, e_M and x_M, namely given by

$$e'_M = \frac{\frac{1}{M}\,\mathrm{tr}\left[\mathbf{R}_M \left(x_M \mathbf{R}_M - z\mathbf{I}_M\right)^{-2}\right]}{1 - \frac{1}{M}\,\mathrm{tr}\left[\mathbf{R}_M^2 \left(x_M \mathbf{R}_M - z\mathbf{I}_M\right)^{-2}\right] \frac{1}{N}\,\mathrm{tr}\left[\mathbf{T}_N^2 \left(\mathbf{I}_N + e_M \mathbf{T}_N\right)^{-2}\right]}, \tag{38}$$

$$x'_M = -\frac{\frac{1}{M}\,\mathrm{tr}\left[\mathbf{R}_M \left(x_M \mathbf{R}_M - z\mathbf{I}_M\right)^{-2}\right] \frac{1}{N}\,\mathrm{tr}\left[\mathbf{T}_N^2 \left(\mathbf{I}_N + e_M \mathbf{T}_N\right)^{-2}\right]}{1 - \frac{1}{M}\,\mathrm{tr}\left[\mathbf{R}_M^2 \left(x_M \mathbf{R}_M - z\mathbf{I}_M\right)^{-2}\right] \frac{1}{N}\,\mathrm{tr}\left[\mathbf{T}_N^2 \left(\mathbf{I}_N + e_M \mathbf{T}_N\right)^{-2}\right]}. \tag{39}$$

Along with Theorem 4, the following proposition will also be a key element in proving Theorem 1 and Theorem 2.

Proposition 2: Let the definitions and assumptions on the data model specified until now hold. Then, for each $z \in \mathbb{C} - \mathbb{R}^+$,

$$\mathrm{tr}\left[\boldsymbol{\Theta}_M \frac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T \left(\frac{1}{N}\tilde{\mathbf{Y}}_N \tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-1}\right] \asymp x_M \,\mathrm{tr}\left[\boldsymbol{\Theta}_M \mathbf{R} \left(x_M \mathbf{R} - z\mathbf{I}_M\right)^{-1}\right], \tag{40}$$

$$\mathrm{tr}\left[\boldsymbol{\Theta}_N \frac{1}{N}\tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N \left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N - z\mathbf{I}_N\right)^{-1}\right] \asymp e_M \,\mathrm{tr}\left[\boldsymbol{\Theta}_N \mathbf{T} \left(\mathbf{I}_N + e_M \mathbf{T}\right)^{-1}\right], \tag{41}$$

where $\{e_M = e_M(z), x_M = x_M(z)\}$ are defined as in Theorem 4. Moreover, we also have, for each

$z \in \mathbb{C} - \mathbb{R}^+$,

$$\mathrm{tr}\left[\boldsymbol{\Theta}_M \frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T\left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-2}\right] \asymp \left(x_M - zx_M'\right)\mathrm{tr}\left[\boldsymbol{\Theta}_M\mathbf{R}_M\left(x_M\mathbf{R}_M - z\mathbf{I}_M\right)^{-2}\right], \quad (42)$$

$$\mathrm{tr}\left[\boldsymbol{\Theta}_N \frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N - z\mathbf{I}_N\right)^{-2}\right] \asymp e_M'\,\mathrm{tr}\left[\boldsymbol{\Theta}_N\mathbf{T}_N\left(\mathbf{I}_N + e_M\mathbf{T}_N\right)^{-2}\right], \quad (43)$$

where e_M' and x_M' are given by (38) and (39), respectively.

Proof: The proof of (40) and (41) follow the same lines of reasoning. We show (41). First, notice that

$$\mathrm{tr}\left[\boldsymbol{\Theta}_N \frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N - z\mathbf{I}_N\right)^{-1}\right] = \mathrm{tr}\left[\boldsymbol{\Theta}_N \frac{1}{N}\tilde{\mathbf{Y}}^T\left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-1}\tilde{\mathbf{Y}}\right].$$

Moreover, using the matrix inversion lemma in (27) we get

$$\frac{1}{N}\tilde{\mathbf{Y}}^T\left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-1}\tilde{\mathbf{Y}} = \mathbf{I}_N - \left(\mathbf{I}_N - \frac{1}{z}\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\right)^{-1},$$

and, then, write

$$\mathrm{tr}\left[\boldsymbol{\Theta}_N \frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N - z\mathbf{I}_N\right)^{-1}\right] = \mathrm{tr}\left[\boldsymbol{\Theta}_N\right] + z\,\mathrm{tr}\left[\boldsymbol{\Theta}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N - z\mathbf{I}_N\right)^{-1}\right]. \quad (44)$$

Now, Theorem 4 yields

$$\mathrm{tr}\left[\boldsymbol{\Theta}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N - z\mathbf{I}_N\right)^{-1}\right] \asymp \mathrm{tr}\left[\boldsymbol{\Theta}_N\left(\mathbf{I}_N + e_M\mathbf{T}\right)^{-1}\right].$$

Then, from (44), we finally have that

$$\mathrm{tr}\left[\boldsymbol{\Theta}_N \frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N - z\mathbf{I}_N\right)^{-1}\right] \asymp \mathrm{tr}\left[\boldsymbol{\Theta}_N\left(\mathbf{I}_N - \left(\mathbf{I}_N + e_M\mathbf{T}\right)^{-1}\right)\right]$$

$$= e_M\,\mathrm{tr}\left[\boldsymbol{\Theta}_N\mathbf{T}\left(\mathbf{I}_N + e_M\mathbf{T}\right)^{-1}\right].$$

Regarding the proof of (42) and (43), we first notice that

$$\mathrm{tr}\left[\boldsymbol{\Theta}_M \frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T\left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-2}\right] = \frac{\partial}{\partial z}\left\{\mathrm{tr}\left[\boldsymbol{\Theta}_M \frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T\left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-1}\right]\right\},$$

$$\mathrm{tr}\left[\boldsymbol{\Theta} \frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N - z\mathbf{I}_N\right)^{-2}\right] = \frac{\partial}{\partial z}\left\{\mathrm{tr}\left[\boldsymbol{\Theta}_M \frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N - z\mathbf{I}_M\right)^{-1}\right]\right\}.$$

Moreover, the almost sure convergence stated in (40) and (41) is uniform on $\mathbb{C} - \mathbb{R}^+$, and therefore the convergence of the derivatives holds by the Weierstrass convergence theorem [65] (see alternatively argument in [66, Lemma 2.3] based on Vitali's theorem about the uniform convergence of sequences of uniformly bounded holomorphic functions towards a holomorphic function [67], [68]). ∎

APPENDIX B

PROOF OF THEOREM 1

Next, we separately proof the convergence of each term in the statement of the theorem.

A. *The terms $\hat{\xi}_M^{(1)}$ and $\hat{\xi}_M^{(3)}$*

In particular, the asymptotic deterministic equivalent of $\hat{\xi}_M^{(1)}$ follows readily by Theorem 4. Regarding $\hat{\xi}_M^{(3)}$, after observing first that

$$\hat{\boldsymbol{v}}_M^T \hat{\boldsymbol{\Sigma}}_M^{-1} \hat{\boldsymbol{v}}_M = \boldsymbol{v}_N^T \left(\frac{1}{N} \tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N + \alpha_M \mathbf{I}_M \right)^{-1} \frac{1}{N} \tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N \upsilon_N,$$

the result is obtained by applying (41) in Proposition 2.

B. *The term $\hat{\xi}_M^{(2)}$ and $\hat{\xi}_M^{(5)}$*

We recall that $\hat{\xi}_M^{(2)} = \boldsymbol{v}_M^T \hat{\boldsymbol{\Sigma}}_M^{-1} \hat{\boldsymbol{v}}_M = \frac{1}{N} \boldsymbol{v}_M^T \hat{\boldsymbol{\Sigma}}_M^{-1} \tilde{\mathbf{Y}}_N \tilde{\boldsymbol{v}}_N$ and write

$$\frac{1}{N} \boldsymbol{v}_M^T \hat{\boldsymbol{\Sigma}}_M^{-1} \tilde{\mathbf{Y}}_N \tilde{\boldsymbol{v}}_N = \frac{1}{N} \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \tilde{\mathbf{Y}}_N \tilde{\boldsymbol{v}}_N - q_n \frac{1}{N^2} \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \tilde{\mathbf{y}}_n \tilde{\mathbf{y}}_n^T \mathbf{Q}_{(n)} \tilde{\mathbf{Y}}_N \tilde{\boldsymbol{v}}_N \qquad (45)$$

where we have defined $q_n = \left(1 + \frac{1}{N} \mathbf{y}_n^T \mathbf{Q}_{(n)} \mathbf{y}_n \right)^{-1}$. Moreover, recall also that $\hat{\xi}_M^{(5)} = \boldsymbol{v}_M^T \hat{\boldsymbol{\Sigma}}_M^{-1} \mathbf{R}_M \hat{\boldsymbol{\Sigma}}_M^{-1} \hat{\boldsymbol{v}}_M = \frac{1}{N} \boldsymbol{v}_M^T \hat{\boldsymbol{\Sigma}}_M^{-1} \mathbf{R}_M \hat{\boldsymbol{\Sigma}}_M^{-1} \tilde{\mathbf{Y}}_N \tilde{\boldsymbol{v}}_N$, and consider first the following notations:

$$\chi_{j,n} = \frac{1}{N} \tilde{\mathbf{y}}_n^T \mathbf{Q}_{(n)} \mathbf{Z}_{j(n)} \tilde{\mathbf{y}}_n - \frac{1}{N} \operatorname{tr} \left[\mathbf{Q}_{(n)} \mathbf{Z}_{j(n)} \right],$$

for $j = 1, 2$, with $\mathbf{Z}_{j(n)}$ being arbitrary $M \times M$ dimensional matrices, possibly random but not depending on the $\mathbf{x}_n$, and such that $\sup_{M \geq 1} \left\| \mathbf{Z}_{j(n)} \right\| < +\infty$; in particular, $\mathbf{Z}_{1(n)} = \mathbf{I}_M$ and $\mathbf{Z}_{2(n)} = \mathbf{Q}_{(n)} \mathbf{R}_M$. Then, observe that we can write $\hat{\xi}_M^{(2)}$ and $\hat{\xi}_M^{(5)}$ as, respectively,

$$\frac{1}{N} \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \tilde{\mathbf{Y}}_N \tilde{\boldsymbol{v}}_N = \frac{1}{N} \sum_{n=1}^N \mathbf{z}_{1(n)}^T \mathbf{x}_n + \frac{1}{N} \sum_{n=1}^N \mathbf{z}_{2(n)}^T \mathbf{x}_n$$
$$+ \frac{1}{N} \sum_{n=1}^N \chi_{1,n} t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \tilde{v}_M^T \mathbf{Q}_{(n)} \mathbf{Z}_{1(n)} \tilde{\mathbf{y}}_n, \qquad (46)$$

where the first term on the RHS follows from $\frac{1}{N} \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \tilde{\mathbf{Y}}_N \tilde{\boldsymbol{v}}_N$, and, similarly,

$$\frac{1}{N} \boldsymbol{v}_M^T \hat{\boldsymbol{\Sigma}}_M^{-1} \mathbf{R}_M \hat{\boldsymbol{\Sigma}}_M^{-1} \tilde{\mathbf{Y}}_N \tilde{\boldsymbol{v}}_N = \frac{1}{N} \sum_{n=1}^N \mathbf{z}_{3(n)}^T \mathbf{x}_n - \frac{1}{N} \sum_{n=1}^N \mathbf{z}_{4(n)}^T \mathbf{x}_n - \frac{1}{N} \sum_{n=1}^N \mathbf{z}_{5(n)}^T \mathbf{x}_n$$
$$- \frac{1}{N} \sum_{n=1}^N \chi_{1,n} t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \mathbf{Z}_{2(n)} \tilde{\mathbf{y}}_n - \frac{1}{N} \sum_{n=1}^N \chi_{2,n} t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \mathbf{Z}_{1(n)} \tilde{\mathbf{y}}_n$$
$$+ \frac{1}{N} \sum_{n=1}^N t_n q_n^2 \left[\tilde{\boldsymbol{v}}_N \right]_n \frac{1}{N} \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \mathbf{Z}_{1(n)} \tilde{\mathbf{y}}_n \frac{1}{N} \tilde{\mathbf{y}}_n^T \mathbf{Q}_{(n)} \mathbf{Z}_{2(n)} \tilde{\mathbf{y}}_n \frac{1}{N} \tilde{\mathbf{y}}_n^T \mathbf{Q}_{(n)} \mathbf{Z}_{1(n)} \tilde{\mathbf{y}}_n,$$
$$(47)$$

where the following definitions apply:

$$\mathbf{z}_{1(n)} = t_n \left[\tilde{\boldsymbol{v}}_N \right]_n \mathbf{R}_M^{1/2} \mathbf{Z}_{1(n)} \mathbf{Q}_{(n)} \boldsymbol{v}_M,$$

$$\mathbf{z}_{2(n)} = t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \frac{1}{N} \operatorname{tr} \left[\mathbf{Q}_{(n)} \mathbf{Z}_{1(n)} \right] \mathbf{R}_M^{1/2} \mathbf{Z}_{1(n)} \mathbf{Q}_{(n)} \boldsymbol{v}_M,$$

$$\mathbf{z}_{3(n)} = t_n \left[\tilde{\boldsymbol{v}}_N \right]_n \mathbf{R}_M^{1/2} \mathbf{Z}_{2(n)} \mathbf{Q}_{(n)} \tilde{v}_M,$$

$$\mathbf{z}_{4(n)} = t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \frac{1}{N} \operatorname{tr} \left[\mathbf{Q}_{(n)} \right] \mathbf{R}_M^{1/2} \mathbf{Z}_{2(n)} \mathbf{Q}_{(n)} \boldsymbol{v}_M,$$

$$\mathbf{z}_{5(n)} = t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \frac{1}{N} \operatorname{tr} \left[\mathbf{Q}_{(n)} \mathbf{Z}_{2(n)} \right] \mathbf{R}_M^{1/2} \mathbf{Q}_{(n)} \boldsymbol{v}_M.$$

We now prove that the terms of the form $\frac{1}{N} \sum_{n=1}^{N} \mathbf{z}_{k(n)}^T \mathbf{x}_n$ vanish almost surely. To see this, we further define the following two $L = MN$ dimensional vectors, namely, $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T & \cdots & \mathbf{x}_N^T \end{bmatrix}$, and $\mathbf{z}_k = \begin{bmatrix} \mathbf{z}_{k(1)}^T & \cdots & \mathbf{z}_{k(N)}^T \end{bmatrix}$, $k = 1, 2, 3, 4, 5$. Then, we notice that

$$\frac{1}{N} \sum_{n=1}^{N} \mathbf{z}_{k(n)}^T \mathbf{x}_n = \frac{1}{N} \mathbf{z}_k^T \mathbf{x} \equiv \frac{1}{\sqrt{L}} \sum_{l=1}^{L} \eta_l, \tag{48}$$

where we have defined $\eta_{k,l} = Z_{k,l}^T X_l$, with $Z_{k,l}$ and X_l being the lth entries of $\mathbf{z}_k$ and $\mathbf{x}$, respectively. In particular, if $\mathcal{G}_l$ is the σ-field generated by the random variables $\{X_l\}$, then notice that $\{\eta_{k,l}\}$ forms a martingale difference sequence with respect to the filtration $\{\mathcal{G}_l\}$. Indeed, $\mathbb{E}\left[Z_{k,l}^T X_l \,\middle|\, \mathcal{G}_{l-1} \right] = 0$. Then, we notice that $\mathbb{E}\left[|\eta_{k,l}|^2 \,\middle|\, \mathcal{G}_{l-1} \right] = \mathbb{E}\left[|Z_{k,l}|^2 \,\middle|\, \mathcal{G}_{l-1} \right]$ is bounded by assumption and so by Theorem 3 we have that (48) vanishes almost surely.

We now handle the last term on the RHS of equation (46) together with the last three terms on the RHS of equation (47). From the developments in [64] it follows that, for a sufficiently large p,

$$\mathbb{E}\left[\left| t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \mathbf{Z}_{j(n)} \tilde{\mathbf{y}}_n \right|^{2p} \right] \leq \frac{K_p |z|^{2p}}{|\operatorname{Im}\{z\}|^{4p}}, \tag{49}$$

where K_p is a constant depending on p but not on M, N which may take different values at each appearance, and

$$\mathbb{E}\left[|\chi_{j,n}|^{2p} \right] \leq \frac{1}{N^p} \frac{K_p}{|\operatorname{Im}\{z\}|^{2p}}. \tag{50}$$

Then, using (49) and (50), and applying first Minkowski's and then the Cauchy-Schwarz inequalities, we get $(i, j = 1, 2)$

$$\mathbb{E}\left[\left| \frac{1}{N} \sum_{n=1}^{N} \chi_{i,n} t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \mathbf{Z}_{j(n)} \tilde{\mathbf{y}}_n \right|^p \right] \leq \frac{1}{N^p} \left(\sum_{n=1}^{N} \left(\mathbb{E}\left[\left| \chi_{i,n} t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \mathbf{Z}_{j(n)} \tilde{\mathbf{y}}_n \right|^p \right] \right)^{1/p} \right)^p$$

$$\leq \frac{1}{N^p} \left(\sum_{n=1}^{N} \mathbb{E}^{1/2p} \left[\left| t_n q_n \left[\tilde{\boldsymbol{v}}_N \right]_n \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \mathbf{Z}_{j(n)} \tilde{\mathbf{y}}_n \right|^{2p} \right] \mathbb{E}^{1/2p} \left[|\chi_{i,n}|^{2p} \right] \right)^p \leq \frac{1}{N^{p/2}} \frac{K_p |z|^p}{|\operatorname{Im}\{z\}|^{3p}}.$$

Furthermore, let $\mathcal{X}_n = \frac{1}{N} \sum_{n=1}^{N} t_n q_n^2 [\tilde{\boldsymbol{v}}_N]_n \frac{1}{N} \boldsymbol{v}_M^T \mathbf{Q}_{(n)} \mathbf{Z}_{1(n)} \tilde{\mathbf{y}}_n \frac{1}{N} \tilde{\mathbf{y}}_n^T \mathbf{Q}_{(n)} \mathbf{Z}_{2(n)} \tilde{\mathbf{y}}_n \frac{1}{N} \tilde{\mathbf{y}}_n^T \mathbf{Q}_{(n)} \mathbf{Z}_{1(n)} \tilde{\mathbf{y}}_n$ and, by using Jensen's inequality along with (49), observe that

$$\mathbb{E}\left[\left|\frac{1}{N} \sum_{n=1}^{N} \mathcal{X}_n\right|^p\right] \leq \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\left[|\mathcal{X}_n|^p\right] \leq \max_{1 \leq n \leq N} \mathbb{E}\left[|\mathcal{X}_n|^p\right] \leq \frac{1}{N^{1+\epsilon}} \frac{K_p |z|^q}{|\mathrm{Im}\{z\}|^r}, \quad \epsilon > 0,$$

for sufficiently large p and well-chosen q and r, on p but not on M, N. Then, the almost sure convergence to zero of the four terms for each $z \in \mathbb{C}^+$ follows then by Borel-Cantelli's lemma. Finally, convergence of the real nonnegative axis follows by an argument based on Montel's normal family theorem (see, e.g., Section 4 in [64]).

C. The term $\hat{\xi}_M^{(4)}$ and $\hat{\xi}_M^{(6)}$

Observe that

$$\boldsymbol{v}_M^T \hat{\boldsymbol{\Sigma}}_M^{-1} \mathbf{R}_M \hat{\boldsymbol{\Sigma}}_M^{-1} \boldsymbol{v}_M = \boldsymbol{v}_M^T \mathbf{R}_M^{-1/2} \left(\mathbf{X}_N \mathbf{T}_N \mathbf{X}_N^T + \alpha_M \mathbf{R}_M^{-1}\right)^{-2} \mathbf{R}_M^{-1/2} \boldsymbol{v}_M$$
$$= \frac{\partial}{\partial z} \left\{ \boldsymbol{v}_M^T \mathbf{R}_M^{-1/2} \left(\alpha_M \mathbf{R}_M^{-1} + \mathbf{X}_N \mathbf{T}_N \mathbf{X}_N^T - z \mathbf{I}_M\right)^{-1} \mathbf{R}_M^{-1/2} \boldsymbol{v}_M \right\} \bigg|_{z=0}.$$

Furthermore, using (37) in Theorem 4 we get

$$\boldsymbol{v}_M^T \mathbf{R}_M^{-1/2} \left(\alpha_M \mathbf{R}_M^{-1} + \mathbf{X}_N \mathbf{T}_N \mathbf{X}_N^T - z \mathbf{I}_M\right)^{-1} \mathbf{R}_M^{-1/2} \boldsymbol{v}_M \asymp \boldsymbol{v}_M^T \mathbf{R}_M^{-1/2} \left(\alpha_M \mathbf{R}_M^{-1} + \left(x_M^{(4)} - z\right) \mathbf{I}_M\right)^{-1} \mathbf{R}_M^{-1/2} \boldsymbol{v}_M,$$

where, for each z outside the real positive axis, $\left\{e_M^{(4)}(z), x_M^{(4)}(z)\right\}$ is the unique solution to the system:

$$e_M^{(4)}(z) = \frac{1}{N} \mathrm{tr}\left[\left(\alpha_M \mathbf{R}_M^{-1} + \left(x_M^{(4)}(z) - z\right) \mathbf{I}_M\right)^{-1}\right]$$
$$x_M^{(4)}(z) = \frac{1}{N} \mathrm{tr}\left[\mathbf{T}_N \left(\mathbf{I}_N + e_M^{(4)}(z) \mathbf{T}_N\right)^{-1}\right].$$

Then, using

$$x_M^{(4)\prime}(0) = -\left(1 - x_M^{(4)\prime}(0)\right) \frac{1}{N} \mathrm{tr}\left[\mathbf{R}_M^2 \left(x_M^{(4)}(0) \mathbf{R}_M + \alpha_M \mathbf{I}_M\right)^{-2}\right] \frac{1}{N} \mathrm{tr}\left[\mathbf{T}_N^2 \left(\mathbf{I}_N + e_M^{(4)}(0) \mathbf{T}_N\right)^{-2}\right],$$

we finally get

$$\frac{\partial}{\partial z} \left\{ \boldsymbol{v}_M^T \mathbf{R}_M^{-1/2} \left(\alpha_M \mathbf{R}_M^{-1} + \left(x_M^{(4)} - z\right) \mathbf{I}_M\right)^{-1} \mathbf{R}_M^{-1/2} \boldsymbol{v}_M \right\} \bigg|_{z=0}$$
$$= \left(1 - x_M^{(4)\prime}(0)\right) \boldsymbol{v}_M^T \mathbf{R}_M^{-1/2} \left(x_M^{(4)}(0) \mathbf{R}_M + \alpha_M \mathbf{I}_M\right)^{-2} \mathbf{R}_M^{-1/2} \boldsymbol{v}_M,$$

where

$$1 - x_M^{(4)\prime}(0) = 1 + e_M^{(4)\prime}(0)\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N^2\left(\mathbf{I}_N + e_M^{(4)}(0)\mathbf{T}_N\right)^{-2}\right]$$

$$= 1 + \left(1 - x_M^{(4)\prime}(0)\right)\frac{1}{N}\operatorname{tr}\left[\mathbf{R}_M^2\left(x_M^{(4)}(0)\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\right]\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N^2\left(\mathbf{I}_N + e_M^{(4)}(0)\mathbf{T}_N\right)^{-2}\right]$$

$$= \frac{1}{1 - \frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N^2\left(\mathbf{I}_N + e_M^{(4)}(0)\mathbf{T}_N\right)^{-2}\right]\frac{1}{N}\operatorname{tr}\left[\mathbf{R}_M^2\left(x_M^{(4)}(0)\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\right]}.$$

Let us now deal with $\hat{\xi}_M^{(6)}$. We recall that

$$\hat{\xi}_M^{(6)} = \hat{\boldsymbol{v}}_M^T\hat{\boldsymbol{\Sigma}}_M^{-1}\mathbf{R}_M\hat{\boldsymbol{\Sigma}}_M^{-1}\hat{\boldsymbol{v}}_M = \frac{1}{N}\boldsymbol{v}_N^T\mathbf{T}_N^{1/2}\mathbf{X}^T\left(\frac{1}{N}\mathbf{X}_N\mathbf{T}_N\mathbf{X}_N^T + \alpha_M\mathbf{R}_M^{-1}\right)^{-2}\mathbf{X}_N\mathbf{T}_N^{1/2}\boldsymbol{v}_N.$$

Let $\mathbf{A}_M = \mathbf{A}_M(t) = \alpha_M\mathbf{R}_M^{-1} + t\mathbf{I}_M$, with $t > 0$ being a real positive scalar, and observe that

$$\boldsymbol{v}_N^T\mathbf{T}_N^{1/2}\mathbf{X}^T\left(\frac{1}{N}\mathbf{X}_N\mathbf{T}_N\mathbf{X}_N^T + \alpha_M\mathbf{R}_M^{-1}\right)^{-2}\mathbf{X}_N\mathbf{T}_N^{1/2}\boldsymbol{v}_N$$

$$= -\frac{\partial}{\partial t}\left\{\boldsymbol{v}_N^T\mathbf{T}_N^{1/2}\mathbf{X}^T\left(\frac{1}{N}\mathbf{X}_N\mathbf{T}_N\mathbf{X}_N^T + \mathbf{A}_M(t)\right)^{-1}\mathbf{X}_N\mathbf{T}_N^{1/2}\boldsymbol{v}_N\right\}\Bigg|_{t=0}.$$

Furthermore, using the matrix inversion lemma in (27), we write

$$\mathbf{T}_N^{1/2}\mathbf{X}^T\left(\frac{1}{N}\mathbf{X}_N\mathbf{T}_N\mathbf{X}_N^T + \mathbf{A}_M(t)\right)^{-1}\mathbf{X}_N\mathbf{T}_N^{1/2} = \mathbf{I}_N - \left(\mathbf{I}_N + \mathbf{T}_N^{1/2}\mathbf{X}^T\left(\alpha_M\mathbf{R}_M^{-1} + t\mathbf{I}_M\right)^{-1}\mathbf{X}_N\mathbf{T}_N^{1/2}\right)^{-1},$$

and so we have

$$\boldsymbol{v}_N^T\mathbf{T}_N^{1/2}\mathbf{X}^T\left(\frac{1}{N}\mathbf{X}_N\mathbf{T}_N\mathbf{X}_N^T + \alpha_M\tilde{\boldsymbol{\Sigma}}\right)^{-2}\mathbf{X}_N\mathbf{T}_N^{1/2}\boldsymbol{v}_N$$

$$= \frac{\partial}{\partial t}\left\{\boldsymbol{v}_N^T\left(\mathbf{I}_N + \mathbf{T}_N^{1/2}\mathbf{X}^T\left(\alpha_M\mathbf{R}_M^{-1} + t\mathbf{I}_M\right)^{-1}\mathbf{X}_N\mathbf{T}_N^{1/2}\right)^{-1}\boldsymbol{v}_N\right\}\Bigg|_{t=0}.$$

Now, using Theorem 4 we get

$$\boldsymbol{v}_N^T\left(\frac{1}{N}\mathbf{T}_N^{1/2}\mathbf{X}^T\left(\alpha_M\mathbf{R}_M^{-1} + t\mathbf{I}_M\right)^{-1}\mathbf{X}_N\mathbf{T}_N^{1/2} + \mathbf{I}_N\right)^{-1}\boldsymbol{v}_N \asymp \boldsymbol{v}_N^T\left(x_M^{(6)}(-1)\mathbf{T}_N + \mathbf{I}_N\right)^{-1}\boldsymbol{v}_N,$$

where $\left\{x_M^{(6)}(-1) = x_M^{(6)}(t), e_M^{(6)}(-1) = e_M^{(6)}(t)\right\}$ is the solution to the following system of equations:

$$e_M^{(6)}(t) = \frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(x_M^{(6)}(t)\mathbf{T}_N + \mathbf{I}_M\right)^{-1}\right]$$

$$x_M^{(6)}(t) = \frac{1}{N}\operatorname{tr}\left[\left(\alpha_M\mathbf{R}_M^{-1} + \left(t + e_M^{(6)}(t)\right)\mathbf{I}_M\right)^{-1}\right].$$

Finally, notice that

$$\frac{\partial}{\partial t}\left\{\boldsymbol{v}_N^T\left(x_M^{(6)}(t)\mathbf{T}_N + \mathbf{I}_N\right)^{-1}\boldsymbol{v}_N\right\}\Bigg|_{t=0} = -x_M^{(6)\prime}(0)\boldsymbol{v}_N^T\left(x_M^{(6)}(0)\mathbf{T}_N + \mathbf{I}_N\right)^{-2}\boldsymbol{v}_N,$$

where $x_M^{(6)}(0) = \frac{1}{N} \operatorname{tr}\left[\left(\alpha_M \mathbf{R}_M^{-1} + e_M^{(6)}(0)\mathbf{I}_M\right)^{-1}\right]$, and

$$-x_M^{(6)\prime}(0) = \left(1 + e_M^{(6)\prime}(0)\right) \frac{1}{N}\operatorname{tr}\left[\boldsymbol{\Sigma}^2\left(e_M^{(6)}(0)\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\right]$$

$$= \left(1 - x_M^{(6)\prime}(0)\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(x_M^{(6)}(0)\mathbf{T}_N + \mathbf{I}_M\right)^{-2}\right]\right)\frac{1}{N}\operatorname{tr}\left[\mathbf{R}_M^2\left(e_M^{(6)}(0)\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\right]$$

$$= \frac{\frac{1}{N}\operatorname{tr}\left[\mathbf{R}_M^2\left(e_M^{(6)}(0)\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\right]}{1 - \frac{1}{N}\operatorname{tr}\left[\mathbf{R}_M^2\left(e_M^{(6)}(0)\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\right]\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(x_M^{(6)}(0)\mathbf{T}_N + \mathbf{I}_M\right)^{-2}\right]}.$$

APPENDIX C

PROOF OF THEOREM 2

We first show (25), i.e.,

$$\frac{1}{\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(\mathbf{I}_N + \hat{\delta}_M\mathbf{T}_N\right)^{-2}\right]}\boldsymbol{v}_M^T\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T\left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T + \alpha_M\mathbf{I}_M\right)^{-2}\boldsymbol{v}_M$$

$$\asymp \frac{1}{1 - \gamma_M\tilde{\gamma}_M}\boldsymbol{v}_M^T\mathbf{R}_M^{1/2}\left(\tilde{\delta}_M\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\mathbf{R}_M^{1/2}\boldsymbol{v}_M,$$

Using (42) in Proposition 2 with $\boldsymbol{\Theta}_M = \boldsymbol{v}_M\boldsymbol{v}_M^T$ and $z = -\alpha_M$, we have that (notice that $x_M - zx_M'|_{z=-\alpha_M} = \tilde{\delta}_M + \alpha_M\tilde{\zeta}_M$)

$$\left.\frac{1}{N}\boldsymbol{v}_M^T\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T\left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T - z\mathbf{I}_M\right)^{-2}\boldsymbol{v}_M\right|_{z=-\alpha_M} \asymp \left(\tilde{\delta}_M + \alpha_M\tilde{\zeta}_M\right)\boldsymbol{v}_M^T\mathbf{R}_M\left(\tilde{\delta}_M\mathbf{R}_M + \alpha_M\mathbf{I}_M\right)^{-2}\boldsymbol{v}_M,$$

and the proof follows by (33) in Lemma 1.

Let us now handle (26). We want to prove that, in effect,

$$\frac{1}{\frac{1}{N}\operatorname{tr}\left[\mathbf{T}_N\left(\mathbf{I}_N + \hat{\delta}_M\mathbf{T}_N\right)^{-2}\right]}\left(\boldsymbol{v}_M^T\left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T\right)^2\left(\frac{1}{N}\tilde{\mathbf{Y}}_N\tilde{\mathbf{Y}}_N^T + \alpha_M\mathbf{I}_M\right)^{-2}\boldsymbol{v}_M - \hat{\delta}_M^2\boldsymbol{v}_M^T\mathbf{T}^2\left(\mathbf{I}_N + \hat{\delta}_M\mathbf{T}\right)^{-2}\boldsymbol{v}_M\right)$$

$$\asymp \frac{\gamma_M}{1 - \gamma_M\tilde{\gamma}_M}\boldsymbol{v}_N^T\left(\delta_M\mathbf{T}_N + \mathbf{I}_N\right)^{-2}\boldsymbol{v}_N.$$

First, observe that

$$\boldsymbol{v}_N^T\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\right)^2\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N + \alpha_M\mathbf{I}_N\right)^{-2}\boldsymbol{v}_N$$

$$= \boldsymbol{v}_N^T\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N + \alpha_M\mathbf{I}_N\right)^{-1}\boldsymbol{v}_N - \alpha_M\frac{1}{N}\boldsymbol{v}_N^T\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N\left(\frac{1}{N}\tilde{\mathbf{Y}}_N^T\tilde{\mathbf{Y}}_N + \alpha_M\mathbf{I}_N\right)^{-2}\boldsymbol{v}_N.$$

Moreover, asymptotic deterministic equivalents of the two terms on the RHS can be found by (41) in Proposition 2 with $\Theta_N = \boldsymbol{v}_N \boldsymbol{v}_N^T$ and $z = -\alpha_M$ as

$$\boldsymbol{v}_N^T \frac{1}{N} \tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N \left(\frac{1}{N} \tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N + \alpha_M \mathbf{I}_N \right)^{-1} \boldsymbol{v}_N \asymp \delta_M \boldsymbol{v}_N \mathbf{T} \left(\mathbf{I}_N + \delta_M \mathbf{T} \right)^{-1} \boldsymbol{v}_N$$

and

$$\frac{1}{N} \boldsymbol{v}_N^T \tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N \left(\frac{1}{N} \tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N + \alpha_M \mathbf{I}_N \right)^{-2} \boldsymbol{v}_N \asymp \zeta_M \boldsymbol{v}_N^T \mathbf{T} \left(\mathbf{I}_N + \delta_M \mathbf{T} \right)^{-2} \boldsymbol{v}_N.$$

Then, by rearranging terms we can write

$$\boldsymbol{v}_N^T \left(\tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N \right)^2 \left(\tilde{\mathbf{Y}}_N^T \tilde{\mathbf{Y}}_N + \alpha_M \mathbf{I}_N \right)^{-2} \boldsymbol{v}_N \asymp \left(\delta_M - \alpha_M \zeta_M \right) \boldsymbol{v}_N^T \mathbf{T} \left(\mathbf{I}_N + \delta_M \mathbf{T} \right)^{-2} \boldsymbol{v}_N + \delta_M^2 \boldsymbol{v}_N^T \mathbf{T}^2 \left(\mathbf{I}_N + \delta_M \mathbf{T} \right)^{-2} \boldsymbol{v}_N,$$

and the result follows finally after straightforward algebraic manipulations by (34) in Lemma 1.

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