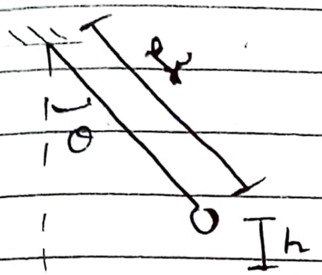
Langragian Equations

* PENDULUM

y is the state vector of variable θ.



$$\dot{y} = \frac{dy}{dt} = y_1$$
$$\ddot{y} = \frac{d^2y}{dt^2} = y_2$$

Potential Energy $P.E. = mgh = mgL(1-\cos\theta)$

Kinetic Energy $K.E. = \frac{1}{2}mv^2 = \frac{1}{2}I\omega^2$

$$K.E. = \frac{1}{2}mL^2(\dot{\theta})^2 \qquad \left(\because \omega = \frac{d\theta}{dt} = \dot{\theta}\right)$$

Langragian Function $L = K.E. - P.E.$

$$= \frac{1}{2}mL^2(\dot{\theta})^2 - mgL(1-\cos\theta)$$

$$\therefore L = \frac{1}{2}mL^2(\dot{\theta})^2 - mgL + mgL\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mL^2(\dot{\theta}) \qquad \frac{\partial L}{\partial \theta} = -mgL\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = mL^2(\ddot{\theta}) + mgL\sin\theta = u$$

(u is the external input)

$$\therefore \ddot{\theta} + \frac{g\sin\theta}{L} = \frac{u}{mL^2}$$

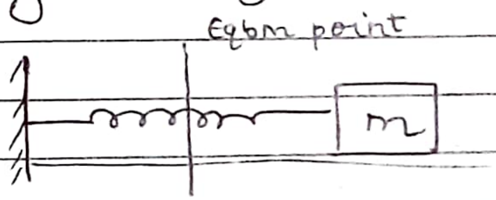
$$\ddot{\theta} = \frac{u}{mL^2} - \frac{g\sin\theta}{L}$$

$$\therefore \boxed{\ddot{\theta} = -\frac{g \sin y}{L} + \frac{u}{mL^2}}$$

Jacobian w.r.t. y $\quad A = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix}$

Jacobian w.r.t. u $\quad B = \begin{bmatrix} 0 \\ 1/mL^2 \end{bmatrix}$

* Spring mass System



State vector - y

$\dot{y} = y_1$

$\ddot{y} = y_2$

Potential Energy $\quad P.E. = \frac{1}{2} kx^2$

Kinetic Energy $\quad K.E. = \frac{1}{2} mv^2 = \frac{1}{2} m(\dot{y})^2$

Langragian function (L) $= K.E. - P.E.$

$$= \frac{1}{2} m(\dot{y})^2 - \frac{1}{2} k y^2$$

$$\frac{\partial L}{\partial \dot{y}} = m(\dot{y}) \qquad \frac{\partial L}{\partial y} = -ky$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) = m(\ddot{y})$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = u$$

(u is the external input)

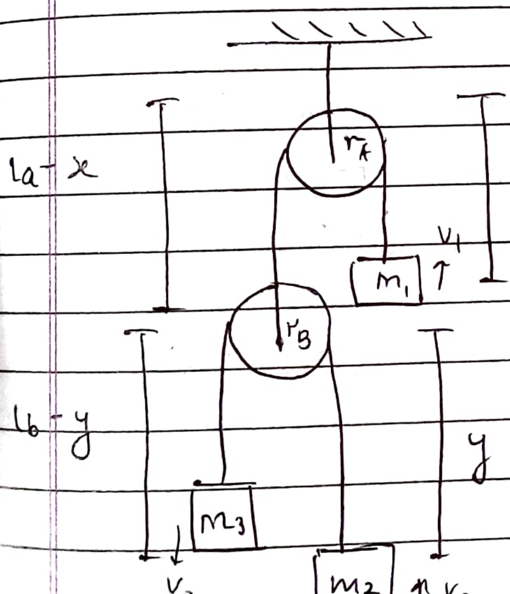
$$m\ddot{y} + ky = u$$

$$\therefore \boxed{\ddot{y} = -\frac{ky}{m} + \frac{u}{m}}$$

Jacobian w.r.t. y $\quad A = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}$

Jacobian w.r.t. u $\quad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$

* Complexe Pulley



Assume

$m_1 < m_2 + m_3$

$m_3 > m_2$

$\dot{x} = y_2 \qquad x = y_1$

$\dot{y} = y_4 \qquad y = \dot{y}_3$

Dist travelled by $m_1 = x$

$m_2 = l_a - x + y$

$m_3 = l_a - x + l_b - y$

$\therefore$

$v_1 = -\dot{x}$

$v_2 = +\dot{y} + \dot{x}$

$v_3 = -\dot{y} + \dot{x}$

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Potential Energy P.E. =

$$m_1 g(-x) + m_2 g(-l_a + x - y) + m_3 g(-l_a + x - l_b + y)$$

$$= -m_1 g x + m_2 g x + m_3 g x$$
$$- m_2 g (l_a) - m_3 g (l_a) - m_3 g (l_b)$$
$$- m_2 g y + m_3 g y$$

$$= x(-m_1 g + m_2 g + m_3 g) + y(-m_2 g + m_3 g)$$
$$+ l_a(-m_2 g - m_3 g) + l_b(-m_3 g)$$

Kinetic Energy

$$K.E. = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2$$

$$= \frac{1}{2} m_1 (\dot{x})^2 + \frac{1}{2} m_2 (\dot{y} + \dot{x})^2 + \frac{1}{2} m_3 (-\dot{y} + \dot{x})^2$$

Langragian function $L = K.E. - P.E.$

$$\frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} + m_2(\dot{y} + \dot{x}) + m_3(-\dot{y} + \dot{x})$$
$$= m_1 \dot{x} + m_2 \dot{y} + m_2 \dot{x} + m_3 \dot{x} - m_3 \dot{y}$$
$$= \dot{x}(m_1 + m_2 + m_3) + \dot{y}(m_2 - m_3)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1 + m_2 + m_3)\ddot{x} + (m_2 - m_3)\ddot{y}$$

$$\frac{\partial L}{\partial x} = (-m_1 + m_2 + m_3) g$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{u_1}{r_a}$$

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$$\therefore (m_1+m_2+m_3)\ddot{x} + (m_2-m_3)\ddot{y} - (-m_1+m_2+m_3)g = \frac{u_1}{r_a}$$

$$\therefore \ddot{x} = \frac{-(m_2-m_3)}{m_1+m_2+m_3}\ddot{y} + \frac{(-m_1+m_2+m_3)g}{m_1+m_2+m_3} + \frac{u_1}{r_a(m_1+m_2+m_3)} \quad -(1)$$

$$\begin{aligned}\frac{\partial L}{\partial \dot{y}} &= m_2(\dot{y}+\dot{x}) - m_3(-\dot{y}+\dot{x}) \\ &= m_2\dot{y} + m_2\dot{x} + m_3\dot{y} - m_3\dot{x} \\ &= (m_2-m_3)\dot{x} + (m_2+m_3)\dot{y}\end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = (m_2-m_3)\ddot{x} + (m_2+m_3)\ddot{y}$$

$$\frac{\partial L}{\partial y} = (-m_2+m_3)g$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial \dot{y}} = \frac{u_2}{r_b}$$

$$\therefore (m_2-m_3)\ddot{x} + (m_2+m_3)\ddot{y} - (-m_2+m_3)g = \frac{u_2}{r_b}$$

$$\therefore \ddot{y} = \frac{-(m_2-m_3)}{m_2+m_3}\ddot{x} + \frac{(-m_2+m_3)g}{m_2-m_3} + \frac{u_2}{r_b} \quad -(2)$$

Substituting ② in ①,

$$\ddot{x} = \frac{-(m_2-m_3)^2 g + \frac{u_2(m_2-m_3)}{r_b} + (m_2-m_3)(-m_1+m_2+m_3)g + u_1(m_2-m_3)}{m_2 r_a (m_1+5m_3)}$$

Jacobian of eqn w.r.t. y gives.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Jacobian of eqn w.r.t. $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ gives :

$$B = \begin{bmatrix} 0 & 0 \\ \frac{m_2-m_3}{m_2 r_a (m_1+5m_3)} & \frac{m_2-m_3}{m_2 r_a \cdot r_b (m_1+5m_3)} \\ 0 & 0 \\ \frac{\frac{(m_2-m_3)^2}{m_2 r_a (m_1-5m_3)}}{(m_2+m_3)} & \frac{\frac{1}{r_b} + \frac{(m_2-m_3)^2}{m_2 r_a r_b (m_1+5m_3)}}{m_2+m_3} \end{bmatrix}$$