* Fourier Series (FS):-

The Fourier Series is a mathematical tool that allows the representation of any periodic signal as the sum of harmonically related sinusoids. Any periodic signal, i.e. one for which $x(t) = x(t+T)$, can be expressed by a Fourier series provided that,

i) if it is discontinuous, there are a finite number of discontinuous in the period T;

ii) it has a finite average value over the period T;

iii) it has a finite number of positive and negative maxima in the period T

When these conditions (also known as Dirichlet conditions) are satisfied, the Fourier series exist. The Fourier series is of two types: (a) Trigonometric Fourier Series (b) exponential Fourier Series.

* Trigonometric Fourier Series

The trigonometric Fourier series is expressed as,

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right) \quad \cdots (1)$$

Where, a_0, a_n and b_n are the trigometric Fourier series co-efficients, and these are obtained from $x(t)$ using the following relation,

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt \quad \text{(DC Term)}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt \quad \text{(AC Term)}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt \quad \text{(AC Term)}$$

Note: these conditions for integration is also flexible. Thus, the integrations can be carried out from $-T/2$ to $T/2$ or over any other full period that may simplify the calculation.

* Polar Form Representation

There are two ways to represent the Fourier Series in polar form:

Case I: Assume, $\left.\begin{aligned} a_n &= C_n \cos(\theta_n) \\ b_n &= -C_n \sin(\theta_n) \end{aligned}\right\}$

where, c_n and θ_n are related as,

$$c_0 = a_0$$
$$c_n = \sqrt{a_n^2 + b_n^2} \; ; \text{ for } n \geq 1$$
$$\theta_n = \tan^{-1}(-b_n/a_n)$$

Replacing the new variables in eq^n. (1) we get,

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(c_n \cos\theta_n \cos n\omega_0 t - c_n \sin\theta_n \sin(n\omega_0 t) \right)$$
$$= a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$$

$$\therefore \quad \boxed{x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)}$$

Case II

Assume, $\left.\begin{aligned} a_n &= c_n \sin(\phi_n) \\ b_n &= c_n \sin(\phi_n) \end{aligned}\right\}$

where, c_n and ϕ_n are related to a_n and b_n as,

$$c_0 = a_0$$
$$c_n = \sqrt{a_n^2 + b_n^2} \; ; \text{ for } n \geq 1$$
$$\phi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right)$$

Similarly, replacing the new variables in eq^n (1) we get,

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[c_n \sin(\phi_n) \cos(n\omega_0 t) + c_n \cos(\phi_n) \sin(n\omega_0 t) \right]$$
$$= a_0 + \sum_{n=1}^{\infty} c_n \sin(n\omega_0 t + \phi_w)$$

Thus,

$$\boxed{x(t) = c_0 + \sum_{n=1}^{\infty} c_n \sin(n\omega_0 t + \phi_n)}$$

* Some important properties:

a. $\int_0^T \sin(m\omega_o t)\,dt = 0$; for all values 'm'

b. $\int_0^T \cos(n\omega_o t)\,dt = 0$; for all $n \neq 0$

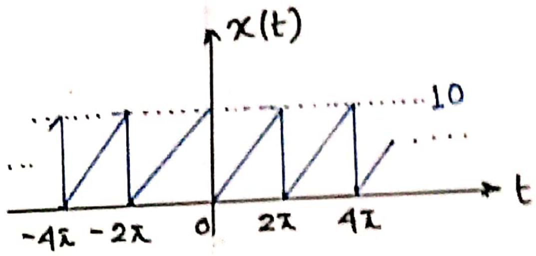
c. $\int_0^T \sin(m\omega_o t)\cos(n\omega_o t)\,dt = 0$; for all m, n

d. $\int_0^T \sin(m\omega_o t)\sin(n\omega_o t)\,dt = \begin{cases} 0 \;; & m \neq n \\ T/2 \;; & m = n \end{cases}$

e. $\int_0^T \cos(m\omega_o t)\cos(n\omega_o t)\,dt = \begin{cases} 0 \;; & m \neq n \\ T/2 \;; & m = n. \end{cases}$

* Example 1

Find the trigonometric Fourier Series for the shown waveform.



The waveform is periodic $T = 2\pi$, and the fundamental frequency $\omega_0 = 2\pi/T = 1$.

Mathematically, the waveform is given by,

$$x(t) = \frac{10}{2\pi} t \;; \quad 0 < t < 2\pi$$

The Fourier Series of the signal $x(t)$ is,

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right)$$

The Fourier coefficients are,

$$\therefore \quad a_0 = \frac{1}{T} \int_0^T x(t)\,dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{10}{2\pi} t \right) dt = \frac{10}{4\pi^2} \left[\frac{t^2}{2} \right]_0^{2\pi} = 5.$$

Similarly,

$$a_n = \frac{2}{T} \int_0^T x(t)\cos(n\omega_o t)\,dt = \frac{2}{2\pi} \int_0^{2\pi} \left(\frac{10}{2\pi} t \right) \cos(nt)\,dt$$

$$= \frac{10}{2\pi^2} \left[(t/n)\sin(nt) + \frac{1}{n^2}\cos(nt) \right]_0^{2\pi} = \frac{10}{2n^2\pi^2} \left[\cos(2n\pi) - \cos(0) \right] = 0$$

Similarly,

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{10}{2\pi}t\right)\sin(nt)\,dt = \frac{10}{2\pi^2}\left[-\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_0^{2\pi}$$

$$= -\frac{10}{n\pi}$$

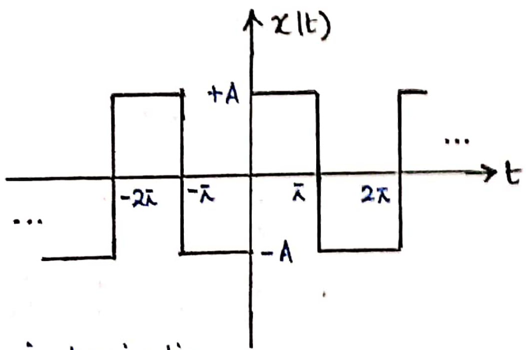
Final expression for the Fourier Series of the $x(t)$, becomes

$$x(t) = 5 - \sum_{n=1}^{\infty}\left(0 + \frac{10}{n\pi}\sin(nt)\right)$$

$$\boxed{x(t) = 5 - \frac{10}{\pi}\sin(t) - \frac{10}{2\pi}\sin(2t) - \frac{10}{3\pi}\sin(3t) - \cdots}$$ (Ans)

Example #2

Determine the Fourier Series expression of the given square wave signal below.



The waveform is periodic with the time period $T = 2\pi$. Hence, the fundamental frequency becomes,

$\omega_0 = T/2\pi = (2\pi/2\pi) = 1$

For the given signal, the signal def^n. will be,

$$x(t) = \begin{cases} A; & 0 < t < \pi \\ -A; & \pi < t < 2\pi \end{cases}$$

Note that, the signal is odd i.e. $x(-t) = -x(t)$. Hence we can write that,

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi} x(t)\,dt + \int_{\pi}^{2\pi} x(t)\,dt\right]$$

$$= \frac{1}{2\pi}\left[A\int_0^{\pi} dt - A\int_{\pi}^{2\pi} dt\right] = 0$$

Similarly,

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T}\int_0^{2\pi} x(t)\cos(n\omega_0 t)\,dt = \frac{2}{T}\left[A\int_0^{\pi}\cos(n\omega_0 t)\,dt - A\int_{\pi}^{2\pi}\cos(n\omega_0 t)\,dt\right]$$

$$= 0$$

and,

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt = \frac{4}{2\pi}\int_0^{\pi} A\sin(nt)\,dt$$

$$= \frac{2A}{\pi}\left[-\frac{1}{n}\cos(nt)\right]_0^{\pi} = \frac{2A}{n\pi}\left(1-\cos(n\pi)\right)$$

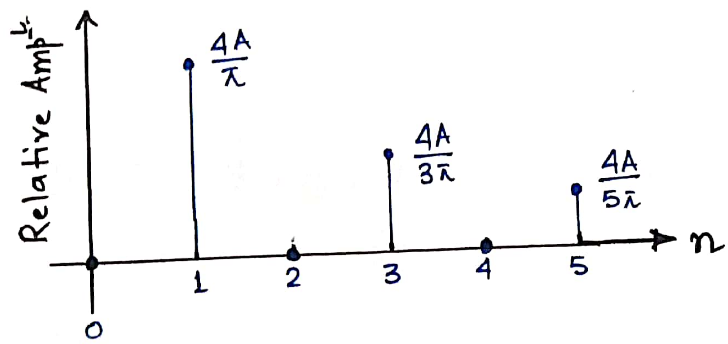
$$\therefore\ b_n = \begin{cases} \frac{4A}{n\pi} & ;\ \text{if } n = 1, 3, 5, \dots \text{ odd values.} \\ 0 & ;\ \text{if } n = 2, 4, 6, \dots \text{ even values.} \end{cases}$$

Hence, the trigonometric fourier Series, of the given signal $x(t)$ is,

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

$$= \frac{4A}{\pi}\sin(t) + \frac{4A}{3\pi}\sin(3t) + \frac{4A}{5\pi}\sin(5t) + \dots$$

Hence, the Fourier Series Line Spectrum will be,



* **Exponential Fourier Series :-**

The exponential Fourier series of the signal $x(t)$ is expressed as,

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}$$

where,

$$X_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt$$

are the exponential Fourier series coefficients of the signal $x(t)$.

Relation between trigometric Fourier Series and Exponential Fourier Series :-

Recall that,

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)\right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \left[\frac{e^{jn\omega_o t} + e^{-jn\omega_o t}}{2}\right] + b_n \left[\frac{e^{jn\omega_o t} - e^{-jn\omega_o t}}{2}\right]\right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \left[\frac{e^{jn\omega_o t} + e^{-jn\omega_o t}}{2}\right] + (-jb_n)\left[\frac{e^{jn\omega_o t} - e^{-jn\omega_o t}}{2}\right]\right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_o t} + \left(\frac{a_n + jb_n}{2}\right) e^{-jn\omega_o t}\right]$$

Assume,
$$X_n = \frac{a_n - jb_n}{2}$$

Hence,

$$X_n = \frac{1}{2}\left(\frac{2}{T}\int_0^T x(t)\cos(n\omega_o t)\,dt - j\frac{2}{T}\int_0^T x(t)\sin n\omega_o t\,dt\right)$$

$$= \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_o t}\,dt$$

Thus, the exponential Fourier series is just another way of representation of Trigonometric Fourier Series or vice versa. The two form carry exactly identical information.

Example #3

Determine the Fourier series expression for the signal

$$x(t) = \cos(3\omega_o t) + \cos^2(2\omega_o t)$$

We have, $\left(\because \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \text{ and } e^{j\omega_o t} = \cos(\omega_o t) + j\sin(\omega_o t)\right)$

$$x(t) = \cos(3\omega_o t) + \cos^2(2\omega_o t)$$

$$= \cos(3\omega_o t) + \frac{1}{2} + \frac{1}{2}\cos(4\omega_o t)$$

$$= \frac{1}{2}\left[e^{+j3\omega_o t} + e^{-j3\omega_o t}\right] + \frac{1}{2} + \frac{1}{4}\left[e^{j4\omega_o t} - e^{-j4\omega_o t}\right]$$

$$= \underbrace{\frac{1}{2}}_{\text{DC}} + \underbrace{\frac{1}{2}\left(e^{j3\omega_o t} + e^{-j3\omega_o t}\right)}_{\text{3rd Harmonic}} + \underbrace{\frac{1}{4}\left(e^{j4\omega_o t} - e^{-j4\omega_o t}\right)}_{\text{4th Harmonic}} \quad \cdots (1)$$

Recall that,

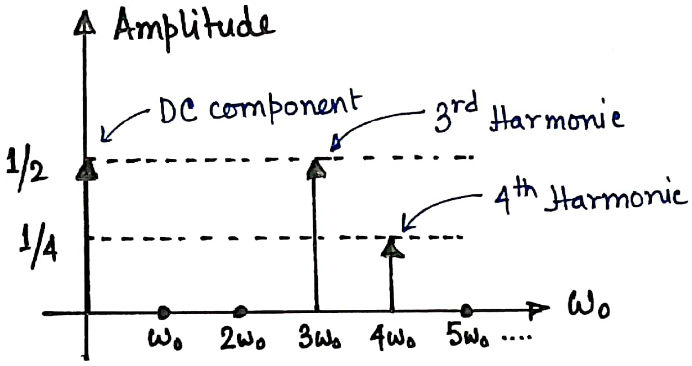
$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$$

$$= C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + C_3 e^{j3\omega_0 t} + C_4 e^{j4\omega_0 t} + \cdots \quad \cdots (2)$$

Now compare eq^n. (1) and (2) and we write,

$$\left.\begin{array}{l} C_0 = \frac{1}{2} \\ C_1 = 0 \\ C_2 = 0 \end{array}\right\} \quad \left.\begin{array}{l} C_3 = \frac{1}{2} \\ C_4 = \frac{1}{4} \\ C_5 = C_6 = \cdots = 0 \end{array}\right\}$$

Hence, the line spectrum becomes,



Review Problems

1. Find the exponential Fourier Series and sketch the corresponding Fourier spectrum X_n versus ω for the full-wave spectrum of sine wave signal. Mathematical expression of such signal is,

$$x(t) = A \sin(t) \,;\ 0 < t < 2\pi$$

2. Find the trigonometric Fourier series of the signal $x(t)$, and draw the line spectrum.

$$x(t) = \begin{cases} A \sin(t) \,; & 0 < t < \pi \\ 0 \,; & \pi < t < 2\pi \end{cases}$$

3. Determine the line spectrum of the signal, $x(t)$ which is defined as,

$$x(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2. \end{cases}$$