## Fourier Transform

Many signals are not periodic and thus we cann't use F.S. directly to represent in freq$^n$. domain. For such case we need some transformation to time-to-frequency transformation.

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)\,\underbrace{e^{-j\omega t}}_{-ve\ sign\ !!}\,dt$$

Other Representation

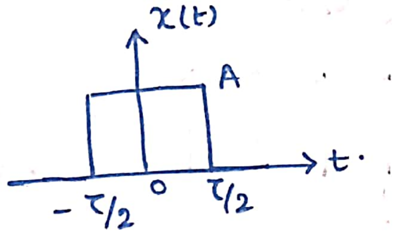
$x(t) \longleftrightarrow X(\omega)$

For Inverse F.T.

$$\tilde{x}(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{+\infty} X(\omega)\,\underbrace{e^{j\omega t}}_{+ve\ sign\ !!}\,d\omega$$

**Problem**

$$\tilde{x}(t) = \begin{cases} A \ ; & |t| < \tau/2 \\ 0 \ ; & |t| > \tau/2 . \end{cases} \longrightarrow$$




$$X(\omega) = \int_{t=-\infty}^{+\infty} \tilde{x}(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\tau/2}^{\tau/2} A e^{-j\omega t}\, dt = A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2}$$

$$= A\left[\frac{e^{-j\omega\tau/2} - e^{j\omega\tau/2}}{-j\omega}\right]$$

$$= \frac{A}{\omega}\left[\frac{\cos(\omega\tau/2) - j\sin(\omega\tau/2) - \cos(\omega\tau/2) - j\sin\left(\frac{\omega\tau}{2}\right)}{-j}\right]$$

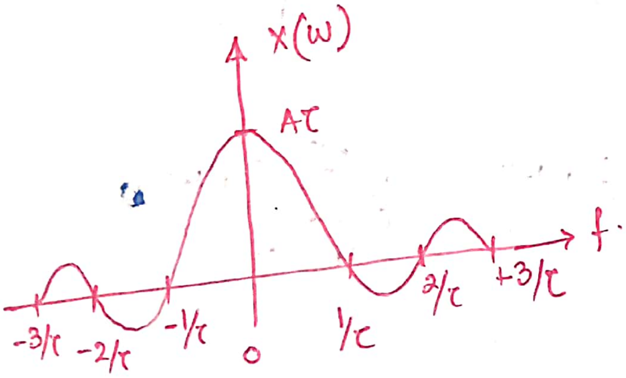
$$\left.\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta . \end{aligned}\right\}$$

$$= \frac{A}{\omega}\left[2\sin(\omega\tau/2)\right]$$

$$= \frac{A}{2\pi f}\left[2\sin\left(\frac{2\pi f\tau}{2}\right)\right] \quad (\because \omega = 2\pi f)$$

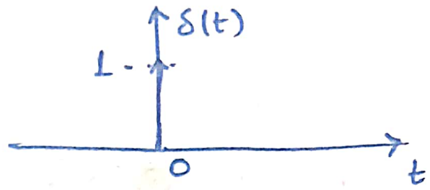
$$= \frac{A}{\pi f}\left[\sin(\pi f\tau)\right]$$

$$= A\tau\left[\frac{\sin(\pi f\tau)}{\pi f\tau}\right] = A\tau\,\mathrm{Sinc}(\pi f\tau)$$



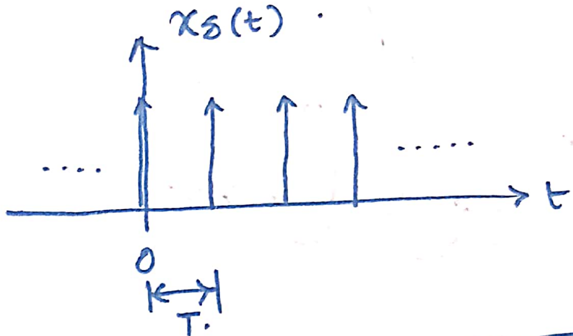
## * Concept of Impulse Function to Impulse Train Function

$$\delta(t) = \begin{cases} 1; & t=0 \\ 0; & \text{otherwise.} \end{cases}$$




Impulse Train $\longrightarrow$ Periodic sequence of impulse trains

$[x_\delta(t)]$ ← Representation.




$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$T$ ↳ Period.

H.W. Problem.

$\boxed{\text{⊛ Find Fourier Transform of } x_\delta(t)}$

As $x_\delta(t)$ is periodic we can write

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$

where, $\omega_0 = \frac{2\pi}{T}$.

$$\therefore \; c_n = \frac{1}{T}\int_{-\infty}^{+\infty} \delta(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_{-\infty}^{+\infty} e^{-jn\omega_0 t}\, dt. = \frac{1}{T}.$$

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{+\infty} \frac{1}{T} e^{jn\omega_0 t} = \frac{1}{T}\sum_{n=-\infty}^{+\infty} e^{jn\omega_0 t}$$

## Properties of F.T.



(i) If $x(t)$ is real and even ; $X(\omega)$ is also real and even.

$$\because \quad x(t) = x(-t)$$
$$X(\omega) = X(-\omega)$$

(ii) If $x(t)$ is real and odd; $X(\omega)$ is also imaginary and odd.

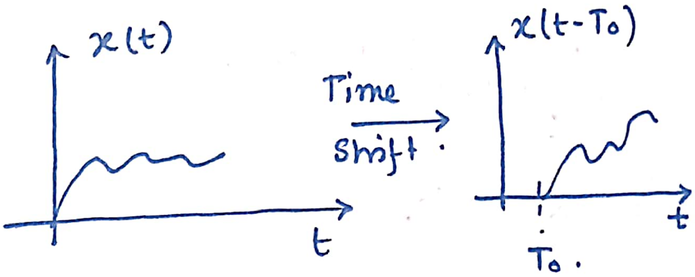
(iii) If $x(t) = a\,x_1(t) + b\,x_2(t)$ then

$$X(\omega) = F[x(t)] = F[a x_1(t) + b x_2(t)] = a X_1(\omega) + b X_2(\omega)$$

Linearity Property

(iv) Time Shifting Property

$$x(t) \rightarrow x(t - T_0)$$




$$\text{F.T.}\left[x(t-T_0)\right] = X(\omega)\,e^{-j\omega t_0}$$

(v) Multiplication Property

$$x(t) \longrightarrow \otimes \longrightarrow x(t)e^{j\omega_0 t}$$

$$\uparrow e^{j\omega_0 t}$$

$$F.\left[x(t)e^{j\omega_0 t}\right] = X(\omega - \omega_0)$$

(vi) Expansion in time domain means compression is spectral domain

$$x(at) \rightarrow F\left[x(at)\right] = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Time dilation.

(vii) Concealation Property

$$F\left[x_1(t)\,x_2(t)\right] = \left[X_1(\omega) * X_2(\omega)\right]\frac{1}{2\pi}$$

**Property 1. (Linearity)**

$$F[a(x_1(t)) + b\,x_2(t)]$$

$$= \int_{-\infty}^{+\infty} [a\,x_1(t) + b\,x_2(t)]\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{+\infty} a\,x_1(t)\,e^{-j\omega t}\,dt + \int_{-\infty}^{+\infty} b\,x_2(t)\,e^{-j\omega t}\,dt$$

$$= a\,X_1(\omega) + b\,X_2(\omega)$$

**Property 2 (Time Shifting)**

$$x(t) \leftrightarrow X(\omega)$$

$$x(t-t_0) \leftrightarrow X(\omega)\,e^{-j\omega t_0}$$

$$F[x(t-t_0)] = \int_{-\infty}^{+\infty} x(t-t_0)\,e^{-j\omega_0 t_0}\,dt$$

Replace,

$t - t_0 = \tau$ $\qquad dt = d\tau$

$\therefore t = (t_0 + \tau) \rightarrow$

$$\rightarrow F[x(\tau)\,e^{-j\omega_0(t_0+\tau)}\,dt]$$

$$= F[x(\tau)\,e^{-j\omega\tau}\,d\tau \cdot e^{-j\omega t_0}]$$

$$= \left(\int_{-\infty}^{+\infty} x(\tau)\,e^{-j\omega\tau}\,d\tau\right) e^{-j\omega t_0}$$

$$= X(\omega)\,e^{-j\omega t_0}$$

**Property 3 (Frequency Shifting)**

$$x(t) \leftrightarrow X(\omega)$$

$$x(t)\,e^{+j\omega_0 t_0} \leftrightarrow X(\omega-\omega_0)$$

$$F[x(t)\,e^{j\omega_0 t_0}] = \int_{-\infty}^{+\infty} (x(t)\,e^{j\omega_0 t_0})\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)\,e^{-j(\omega-\omega_0)t}\,dt$$

$$= X(\omega-\omega_0)$$

**Property 4 (Time and Frequency Scaling Property)**

$$x(t) \leftrightarrow X(\omega)$$

$$x(at) \leftrightarrow \frac{1}{|a|}\,X\left(\frac{\omega}{a}\right)$$

$$F[x(at)] = \int_{-\infty}^{+\infty} x(at)\,e^{-j\omega t}\,dt$$

$$\left.\begin{matrix} t \rightarrow \infty \rightarrow \tau \rightarrow -\infty \\ t \rightarrow -\infty \rightarrow \tau \rightarrow +\infty \end{matrix}\right\}$$

**Case 1: 'a' is real constant and positive.**

consider $\tau = at$.

$d\tau = a\,dt$.

$$F[x(at)] = \int_{-\infty}^{+\infty} x(\tau)\,e^{-j\omega\frac{\tau}{a}}\,\frac{d\tau}{a}.$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)\,e^{-j\left(\frac{\omega}{a}\right)\tau}\,d\tau = \frac{1}{a}\,X\left(\frac{\omega}{a}\right).$$

**Case 2: 'a' is real constant but -ve.**

Again,

$\tau = -at$

$$F[x(-at)] = \int_{-\infty}^{+\infty} x(-at)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{+\infty} x(-at)\,e^{-j\omega t}\,dt$$

$$= \int_{+\infty}^{-\infty} x(\tau)\,e^{-j\left(-\frac{\omega}{a}\right)\tau}\left(-\frac{d\tau}{a}\right)$$

$$= -\frac{1}{a}\int_{+\infty}^{-\infty} x(\tau)\,e^{-j\left(-\frac{\omega}{a}\right)\tau}\,d\tau$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)\,e^{-j\left(-\frac{\omega}{a}\right)\tau}\,d\tau = \frac{1}{a}\,X\left(-\frac{\omega}{a}\right)$$

## Property 5 (Multiplication or Modulation Property)

$x_1(t) \leftrightarrow X_1(\omega)$

$x_2(t) \leftrightarrow X_2(\omega)$

$$x_1(t)\, x_2(t) \leftrightarrow \left[X_1(\omega) \circledast X_2(\omega)\right] \frac{1}{2\pi}.$$

$$F\left[x_1(t)x_2(t)\right] = \int_{-\infty}^{+\infty} \left[x_1(t)\, x_2(t)\right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} x_1(t) \left(x_2(t)\, e^{-j\omega t}\, dt\right)$$

$$= \int_{-\infty}^{+\infty} \underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} x_1(\theta)\, e^{j\theta t}\, d\theta\right)}\, x_2(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_1(\theta) \left( \underbrace{\int_{-\infty}^{+\infty} \left(x_2(t)\, e^{j\theta t}\right) e^{-j\omega t}\, dt}_{\text{Freq Shifting Property } (**)} \right) d\theta.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(\theta)\, X_2(\omega-\theta)\, d\theta.$$

$$= \frac{1}{2\pi}\left[X_1(\omega) \circledast X_2(\omega)\right]$$

By defn.

Convolution means

$$x(t) \circledast h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

Defn. from Inverse fourier transfor

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Replace '$\omega$' by '$\theta$' for representation purpose.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\theta)\, e^{j\theta t}\, d\theta$$

## ** Convolution Integral

The o/p of a system can be represented by the weighted sum of the present and past input values.

$$y(t) = x(0)h(t) + x(1)\,h(t-1) + x(2)\,h(t-2) + \cdots$$

$$= \sum_{n=0}^{\infty} x(n)\, h(t-n)$$

(**) Recall that, $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)$

$$\int_{-\infty}^{+\infty} x(t)\, e^{j\omega_0 t} \cdot e^{-j\omega t}\, dt \leftrightarrow \int_{-\infty}^{+\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt = X(\omega-\omega_0)$$

Similarly,

$$\int_{-\infty}^{+\infty} x_2(t)\, e^{j\theta t} \cdot e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} x_2(t)\, e^{-j(\omega-\theta)t}\, d\theta = X_2(\omega-\theta)$$