(14)

Periodic and Aperiodic Signals

Any signal will be said to be periodic if it's repeating itself after a specific time interval. For any continuous time interval, the signal $x(t)$ will be said periodic iff,

$$x(t+T) = x(t) \quad \text{for all 't'}.$$

where, T is positive and is the period of the signal.

The signal which does not maintain /obey the above mentioned property will be regarded as the aperiodic signal.

(*) Problem

i) Show that the complex exponential function is periodic. Determine the fundamental period.

The signal $x(t)$ if represents complex exponential function becomes,

$$x(t) = e^{jw_0 t}$$

It will be said periodic iff,

$$x(t) = e^{jw_0 t} = x(t+T)$$

$$\text{or,} \quad e^{jw_0 t} = e^{jw_0(t+T)} = e^{jw_0 t} \cdot e^{jw_0 T}$$

Thus, the required condition becomes, $e^{jw_0 T} = 1$.

Recall that, $e^{j2m\pi} = 1$ where m is positive integer.

$$\therefore \quad e^{jw_0 T} = 1 = e^{j2m\pi}$$

$$\text{or,} \quad w_0 T = 2m\pi$$

$$\therefore \quad T = \left[\frac{2\pi}{w_0}\right] m \quad (\text{and, } m = +\text{ve integer}).$$

Hence, the signal $x(t) = e^{jw_0 t}$ becomes a periodic signal and its fundamental period is $(2\pi/w_0)$. (Ans)

Properties of Periodic Signals

(i) Any periodic signal is an infinite length signal.

(ii) Area under $x(t)$ over any interval of duration 'T' is same, for any real number 'a' and 'b'. Mathematically,

$$\int_a^{a+T} x(t)\,dt = \int_b^{b+T} x(t)\,dt.$$

This is due to the fact that periodic signal takes the same values at the intervals of T.

(iii) The sum of 'M' periodic signals are not necessarily periodic signal. It is periodic if and only if (iff) the following condition are satisfied.

$$\frac{T}{T_i} = n_i \; ; \quad 1 \leq i \leq M$$

where T_i is the period of i^{th} signal in the sum and n_i is an integer.

Problem #1

Assume that, $x_1(t)$ and $x_2(t)$ are the two periodic signals with period T_1 and T_2, respectively. Determine under which condition the signal $x(t)$ will become periodic, if

$$x(t) = x_1(t) + x_2(t)$$

Also determine the period of the signal $x(t)$.

Soln.

Since, $x_1(t)$ is periodic with time period of T_1 and $x_2(t)$ is periodic with time period of T_2 then we can write

$$\left.\begin{aligned} x_1(t) &= x_1(t+T_1) = x_1(t+mT_1) \\ x_2(t) &= x_2(t+T_2) = x_2(t+nT_2) \end{aligned}\right\} \; m, n \text{ are integers.}$$

Now assume that, T_1 and T_2 are such that we can write,

$$T = mT_1 = nT_2$$

then,

$$x(t) = x_1(t) + x_2(t)$$

$$\text{or, } x(t+T) = x_1(t+T) + x_2(t+T)$$

$$= x_1(t+mT_1) + x_2(t+nT_2)$$

$$= x_1(t) + x_2(t).$$

So in that case, $x(t)$ becomes periodic.

Therefore the required condition for $x(t)$ becoming a periodic signal,

$$\frac{T_1}{T_2} = \frac{n}{m} = \text{a rational number.}$$

The time period of such signal will be,

$$\boxed{T = mT_1 = nT_2}$$

Problem #2

Calculate if the following signals are periodic or not:

i) $x_1(t) = j\,e^{j10t}$

ii) $x_2(t) = e^{(-1+j)t}$

Sol^n.

(i) $x_1(t) = j\,e^{j10t} = e^{j10t} \cdot e^{j\pi/2} = e^{j(10t+\pi/2)} = e^{j(\omega_0 t + \phi)}$

where, $\left.\begin{array}{l} \omega_0 = 10 \\ \phi = \pi/2 \end{array}\right\}$

We have already proved that $e^{j\omega_0 t}$ is a periodic signal. Thus $x_1(t)$ is periodic. Its period can be calculated as,

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{10} = \frac{\pi}{5} \quad \text{(Ans)}$$

(ii) $x_2(t) = e^{(-1+j)t}$

$= \underbrace{e^{-t}}_{\text{Decaying Fu}^n\text{.}} \cdot e^{jt}$

From the def^n. we can say that the signal is a complex decaying function.

Now, $e^{(-t+jt)} = e^{+(-1+j)t} = e^{(j^2+j)t} = e^{j(j+1)t} = e^{j\omega_0 t}$

$\therefore \omega_0 = (1+j) = \text{complex fu}^n$.

Since, frequency of any signal should be a real number, thus the signal cannot be a periodic signal. (Ans)

⊛ Note: The value of frequency can be negative but not an imaginary number.

* Energy and Power Signals

The energy and power of a continuous time signal are defined as,

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} |x(t)|^2 dt.$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt.$$

Note that power is the time average of energy. Any signal cann't be both energy signal or power signal simultaneously. However there are signals which are neither power signal nor energy signals.

Energy signal Defn. : $0 < E_x < \infty$ and $P_x = 0$.

Power signal Defn : $E_x = 0$ and $0 < P_x < \infty$.

* Important Parameters of any real/practical signal :

(i) All practical signals are energy signals as it have finite energies.

(ii) It is impossible to generate true power signal since such signals will have infinite duration.

(iii) All periodic signals (even the finite length periodic signals) are power signals but the opposite is not true. (Example - Unit step Signal)

Problem

Determine the values of P_x and E_x of the signal.

(i) $x_1(t) = e^{-2t} u(t)$

(ii) $x_2(t) = e^{j(2t+\pi/4)}$

Soln.

(i) $x_1(t) = e^{-2t} u(t)$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |e^{-2t} u(t)|^2 dt = \int_{-\infty}^{+\infty} e^{-4t} u(t)\, dt = \int_0^{\infty} e^{-4t} dt$$

$$= \frac{1}{4} \text{ (Ans)}$$

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{-4t} u(t)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_0^{T/2} e^{-4t} dt = \lim_{T\to\infty} \frac{1}{T} \left[\frac{e^{-4t}}{-4}\right]_0^{T/2}.$$

$$= \lim_{T\to\infty} \left\{ \left[\frac{1}{T}\right] \left[\frac{e^{-2T} - 1}{} \right] \right\} = 0$$

Since, $P_x = 0$ and E_x is finite, therefore the signal will be considered as a Energy signal.

(ii) $$x_2(t) = e^{j(2t+\pi/4)}$$

$$E_x = \int_{-\infty}^{+\infty} |x_2(t)|^2 \, dt$$

$$= \int_{-\infty}^{+\infty} |e^{j(2t+\pi/4)}|^2 dt = \int_{-\infty}^{+\infty} (1)\, dt = \underline{\infty} \quad \text{(Ans)}$$

Similarly,

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x_2(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |e^{j(2t+\pi/4)}|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt = \lim_{T\to\infty} \frac{1}{T} \left[T \right] = 1. \quad \text{(Ans)}.$$

Since, the signal has infinite energy but finite power, it will be referred as power signal.

* <u>Perseval's Theorem</u> :-

Perseval's theorem states that the total average power in a periodic signal equals to the sum of the average power in all of its harmonic components.

Mathematically we can write,

$$x(t) \xleftrightarrow{F.S.} X_n$$

then,
$$\frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |X_n|^2 = X_0^2 + \sum_{n=1}^{\infty} 2|X_n|^2 .$$

* <u>Proof</u>.

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = \frac{1}{T_0}\int_0^{T_0} x(t)\, x^*(t)\, dt = \frac{1}{T_0}\int_0^{T_0} x(t) \left(\sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t} \right)^* dt = \frac{1}{T_0}\int_0^{T_0} x(t) \left(\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t} \right) dt$$

$$= \sum_{n=-\infty}^{\infty} X_n^* \left(\frac{1}{T} \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt \right) = \sum_{n=-\infty}^{+\infty} X_n^* X_n = \sum_{n=-\infty}^{+\infty} |X_n|^2$$

(Only for interested students)

Energy Spectral Density

From the definition of energy of a signal $x(t)$ we can write,

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt \quad \cdots (1)$$

Now from Perseval's theorem we can write,

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega \quad \cdots (2)$$

Again,

$$\omega = 2\pi f.$$

$$\therefore \; d\omega = 2\pi \, df.$$

Hence from eq^n (1) and (2) we can write,

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega$$

$$= \int_{-\infty}^{+\infty} |X(\omega)|^2 \, df.$$

Perseval's theorem states that the total energy (E_x) of a signal can be estimated by computing energy per unit time ($|x(t)|^2$) and integrating it over all the time. Similarly, in frequency domain the total energy E_x may be calculated by determining the energy per unit frequency ($|X(\omega)|^2$) and integrating the same over all the frequencies.

Thus, $|X(\omega)|^2$ represents energy per unit bandwidth and it is refferred as energy spectral density or ESD of the signal $x(t)$. Mathematically, ESD is denoted as $\psi_x(\omega)$, and from the discussion we can write,

$$\psi_x(\omega) = |X(\omega)|^2$$

Generally, the unit of ESD are $(V/Hz)^2$ or $(V.s)^2$.

Power Spectral Density

Consider an infinite length signal $x(t)$. Now to determine the power spectral density of such signal, we need to assume that the signal is truncated after time 't' and we represent the truncated version of the signal as $x_1(t)$.

From the concept of Fourier transform we also assume that,

$$x_1(t) \longleftrightarrow X_1(\omega)$$

From the Perseval's theorem we can write,

$$E_{x_1} = \int_{-\infty}^{+\infty} |x_1(t)|^2 dt = \int_{-\infty}^{+\infty} |X_1(\omega)|^2 df. \quad \cdots (3)$$

Now assume that,

$$x_1(t) = \begin{cases} x(t) ; & -0.5T < t < 0.5T \\ 0 ; & \text{otherwise.} \end{cases}$$

then we can write,

$$E_{x_1} = \int_{-\infty}^{+\infty} |x_1(t)|^2 dt = \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

So we consider the replacement value in eq^n(3), and obtain

$$E_{x_1} = \int_{-T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X_1(\omega)|^2 df.$$

$$\text{or,} \quad \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{+\infty} |X_1(\omega)|^2 df$$

$$\text{or,} \quad P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{+\infty} |X_1(\omega)| df.$$

$$= \int_{-\infty}^{+\infty} \lim_{T\to\infty} \left[\frac{|X_1(\omega)|^2}{T} \right] df. = \int_{-\infty}^{+\infty} G_x(\omega)\, df.$$

where,

$$G_x(\omega) = \lim_{T\to\infty} \frac{|X_1(\omega)|^2}{T}$$

The term $G_x(\omega)$ is defined to be as power spectral density of the signal $x(t)$. Similar to the ESD, the units of PSD are V^2/Hz or A^2/Hz.