TY_B4_48_EXTC_Fredrick Ambrose

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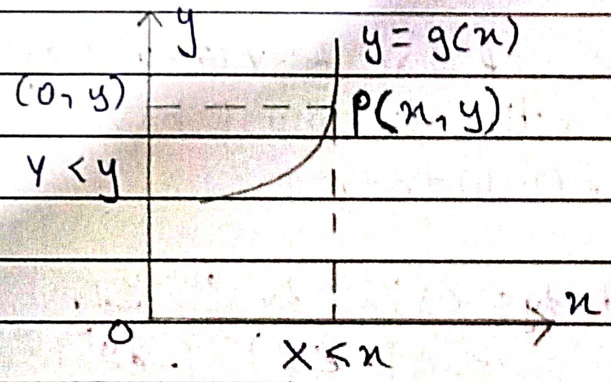
RSA Assignment-1

Q1] If X is a continuos random variable & $Y = g(x)$ is a strictly monotonic function of x then prove that

$$f_Y(y) = f_X(x) \cdot \frac{dx}{dy}$$

→ When $g(x)$ is strictly increasing function of x

· Let $F_Y(y)$ be cummulative density function of Y



← It is a non-linear function

When x increases y also increases

Let Two points: $(x_1, y_1) < (x_2, y_2)$

if $y_2 < y_1$ & $x_2 < x_1$

$$Y < y \qquad X < x$$

$$P(Y<y) = P(X<x)$$

$$F_Y(y) = F_X(x)$$

$$F_Y(y) = F_X(g^{-1}(y))$$

$$\left\{ \begin{array}{l} y = g(x) \\ x = g^{-1}(y) \end{array} \right.$$

By differentiating it we get,

$$f_Y(y) = f_X(x) \cdot \frac{dx}{dy}$$

Q2] The joint probability density function of two continuos random variables X & Y is given by

$$f_{XY}(x,y) = Ce^{-x}.e^{-y} \quad 0<x<\infty, \; 0<y<\infty$$
$$= 0 \quad \text{elsewhere}$$

a) find value of normalisation constant (c).

b) $f_X(x)$, $F_Y(y)$, $F_{X/Y}(x/y)$, $f_{Y/X}(y/x)$
$E(Y/X=x)$, $E(X/Y=y)$

→ To find the constant c, so that $f_{XY}(x,y)$ represents a probability density function, we must have

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = 1$$

But $\int_0^\infty \int_0^\infty ce^{-x}.e^{-y}dx\,dy = c\int_0^\infty e^{-x}dx \int_0^\infty e^{-y}dy$

$$= c\left[\frac{-e^{-x}}{-1}\right]_0^\infty \left[\frac{-e^{y}}{1}\right]_0^\infty = c$$

Since this is equal to 1, c = 1.

(i) Now, $f_X(x) = \int_0^\infty f_{XY}(x,y)dy = \int_0^\infty e^{-x}.e^{-y}dy = e^{-x}[-e^{y}]_0^\infty$

$= e^{-x}$

Similarly, $f_Y(y) = e^{-y}$.

(ii) $F_Y(y) = \int_0^y f_y\,dy = \int_0^y e^{-y}.dy = [-e^{-y}]_0^y = 1-e^{-y}$

(iii) $f_{X/Y}(x/y) = \dfrac{f(x,y)}{f(y)} = \dfrac{e^{-x}.e^{-y}}{e^{-y}} = e^{-x}$

(iv) $f_{Y/X}(y/x) = \dfrac{f(x,y)}{f(x)} = \dfrac{e^{-x}.e^{-y}}{e^{-x}} = e^{-y}$

(v) $F_{X/Y}(x/y) = \int_0^x f_{X/Y}(x/y)\,dx = \int_0^x e^{-x}dx = [-e^{-x}]_0^x = 1-e^{x}$

(vi) $E(Y/X=x) = \int_0^\infty y \cdot f_{Y/X}(y/x)dy = \int_0^\infty y \cdot e^{-y}\, dy$

$= \left[y(-e^{-y}) - \int -e^{-y} \cdot 1 \cdot dy\right]_0^\infty = \left[-ye^{-y} - e^{-y}\right]_0^\infty = 1$

(vii) Similarly, $E(X/Y=y) = 1$

Q3] The joint probability function of two dimensional random variable (X,Y) is given by

$$f_{XY}(x,y) = ce^{-3x-5y}, \quad x \geq 0; \quad y \geq 0$$

(i) find value of c, find probability that $x<2$, $y>0.2$
(ii) find marginal probability density function of x & y
(iii) Are X & Y independent
(iv) Find $E[X/Y]$ & $E[Y/X]$

→ i) To find the constant c, so that $f_{XY}(x,y)$ represents a probability density function, we must have

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = 1$$

$$\int_0^\infty \int_0^\infty c e^{-3x-5y}\, dx\, dy = 1$$

$$\Rightarrow c\int_0^\infty e^{-3x}\,dx \int_0^\infty e^{-5y}\,dy = 1$$

$$\Rightarrow c\left[\frac{-e^{-3x}}{3}\right]_0^\infty \left[\frac{-e^{-5x}}{5}\right]_0^\infty = 1$$

$$\Rightarrow c\left[0 - \frac{1}{3}\right]\left[0 - \frac{1}{5}\right] = 1$$

$$\Rightarrow \frac{1}{15}c = 1$$

$$\Rightarrow c = 15$$

So, $f_{X,Y}(x,y) = 15e^{-3x-5y}$, $x \geq 0$, $y \geq 0$

$$P(X<2, Y>0.2) = 15\int_0^2 e^{-3x}\,dx \int_2^\infty e^{-5y}\,dy$$

$$= 15\left[\frac{-e^{-3x}}{3}\right]_0^2 \left[\frac{-e^{-5y}}{5}\right]_{0.2}^\infty$$

$$= (1-e^{-6})(e^{-1})$$

$$= 0.367$$

(ii) $f_X(x) = \int_0^\infty 15e^{-3x}\cdot(e^{-5y})\,dy$

$$= 15e^{-3x}\left[\frac{-e^{-5y}}{5}\right]_0^\infty = 3e^{-3x} \begin{cases} x>0 \\ 0 & \text{elsewhere} \end{cases}$$

$f_Y(y) = \int_0^\infty 15e^{-5y}\cdot(e^{-3x})\,dx$

$$= 15e^{-5y}\left[\frac{-e^{-3x}}{3}\right]_0^\infty = 5e^{-5y} \begin{cases} y>0 \\ 0 & \text{elsewhere} \end{cases}$$

(iii) Now X, Y are independent if $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$

Since, $f_X(x)\cdot f_Y(y) = (3\cdot e^{-3x})(5e^{-5y})$

$$= 15e^{-3x-5y} = f_{X,Y}(x,y)$$

X & Y are independent

(iv) Since (X, Y) are independent if $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$

$$f_X(x)\cdot f_Y(y) = f_X(x) = 3e^{-3x},\ x>0$$

$$\therefore E[X/Y=y] = \int_0^\infty x f_{X/Y}(x/y)\,dx = \int_0^\infty x f_X(x)\,dx = \int_0^\infty x\cdot 3\cdot e^{-3x}\,dx$$

$$= 3\left[x\cdot\frac{e^{-3x}}{3} - \int\frac{e^{-3x}}{-3}\cdot 1\,dx\right]_0^\infty = 3\left[x\cdot\frac{e^{-3x}}{3} - \frac{e^{-3x}}{9}\right]_0^\infty = 3\left[(0-0)-\left(0-\frac{1}{9}\right)\right] = \frac{1}{3}$$

Similarly, we can obtain

$$E[Y/X=x] = \int_0^\infty y\cdot 5\cdot e^{-5y}\,dy = \frac{1}{5}$$

Q4] A two dimensional random variable has following pdf:-

$$f_{X,Y}(x,y) = k \cdot xy e^{-(x^2+y^2)} \qquad x \geq 0,\ y \geq 0$$

find i) value of k , ii) Marginal density of X & Y

iii) Conditional densities of X & Y iv) Are X & Y independent

→ (i) To find value of k

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} kxy e^{-(x^2+y^2)}\,dx\,dy = 1$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} kxy e^{-x^2-y^2}\,dx\,dy = 1$$

$$\Rightarrow k\int_{-\infty}^{\infty} y e^{-y^2}dy \cdot \int_{-\infty}^{\infty} x e^{-x^2}dx = 1 \text{ or } k\int_{0}^{\infty} y e^{-y^2}dy \cdot \int_{0}^{\infty} x e^{-x^2}dx = 1$$

$$\Rightarrow \int_0^{\infty} x e^{-x^2}dx = \int_0^{\infty} \cancel{\sqrt{t}}\, e^{-t}\frac{1}{2\cancel{\sqrt{t}}}\,dt$$

$$\left\{ \begin{aligned} &\text{Put } x^2 = t \\ &x = \sqrt{t} \\ &dx = \frac{1}{2\sqrt{t}}\,dt \end{aligned} \right.$$

$$= \frac{1}{2}\int_0^{\infty} e^{-t}dt = \frac{1}{2}\left[-e^{-t}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[e^{-\infty} + e^{0}\right] = \frac{1}{2}$$

$\Rightarrow$ Similarly,

$$\int_0^{\infty} y e^{-y^2}dy = \frac{1}{2}$$

$$\Rightarrow \text{So, } k \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{k}{4}$$

$$\Rightarrow \frac{k}{4} = 1$$

$$\Rightarrow k = 4$$

(ii) To find marginal density functions :-

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)\,dy$$

$$= \int_0^\infty 4xy e^{-(x^2+y^2)}\,dy$$

$$= 4xe^{-x^2}\int_0^\infty e^{-y^2}\cdot y\,dy$$

Put $y^2 = t$

$y = \sqrt{t}$

$dy = \frac{1}{2\sqrt{t}}\,dt$

$$= 4xe^{-x^2}\int_0^\infty e^{-t}\left(\frac{dt}{2}\right) = \frac{4}{2}xe^{-x^2}\left[\frac{e^{-t}}{-1}\right]_0^\infty$$

$$= 2xe^{-x^2};\ x \geq 0$$

Similarly, $f_Y(y) = 2ye^{-y^2};\ y \geq 0$

(iii) The conditional density of Y is

$$f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{4xy\,e^{-(x^2+y^2)}}{2xe^{-x^2}}$$

$$= 2ye^{-y^2};\ y \geq 0$$

The conditional density of X is

$$f_{X/Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{4xy\,e^{-(x^2+y^2)}}{2ye^{-y^2}}$$

$$= 2xe^{-x^2};\ x \geq 0$$

(iv) $f_X(x) = 2xe^{-x^2}$ & $f_Y(y) = 2ye^{-y^2}$ for $x \geq 0$ & $y \geq 0$

$$f_X(x)\cdot f_Y(y) = 2xe^{-x^2}\cdot 2ye^{-y^2}$$

$$= 4xy\,e^{-(x^2+y^2)};\ x \geq 0,\ y \geq 0$$

$$= f_{X,Y}(x,y)$$

Hence, X & Y are independent random variables.

Q5] If joint pdf of (X, Y) is given as $f_{X,Y}(x,y) = e^{-(x+y)}$; $x \geq 0$, $y \geq 0$. find probability density function of (U, V). Where $U = \frac{X}{X+Y}$ & $V = X+Y$. Are U & V independent.

→ Since $u = \frac{x}{x+y}$ & $v = x+y$, we get $u = \frac{x}{v}$

$\therefore x = uv$ & $y = v - x$; $\therefore y = v - uv = v(1-u)$

$$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v - uv + uv = v$$

- The joint probability density function of (U, V) is

$f_{UV}(u,v) = f_{XY}(x,y)\,|J| = e^{-(x+y)} \cdot v = e^{-v} \cdot v$

- Since $x \geq 0$, $y \geq 0$, $uv \geq 0$, $v > 0$ & $v(1-u) \geq 0$ $\therefore v \geq 0$ & $1-u \geq 0$

$\therefore 1 \geq u$ $\therefore v \geq 0$ & $0 \leq u \leq 1$.

$\therefore f_{UV}(u,v) = e^{-v} \cdot v$, $v \geq 0$, $0 \leq u \leq 1$

Now, the pdf of U is given by

$$f_U(u) = \int_0^\infty e^{-v} v\,dv = \left[v e^{-v} - \int -e^{-v} \cdot 1 \cdot dv \right]_0^\infty$$

$$= \left[v e^{-v} - e^{-v} \right]_0^\infty = 1, \quad 0 \leq u \leq 1$$

- The pdf of V is given by

$$f_V(v) = \int_0^1 e^{-v} v\,du = e^{-v} v \int_0^1 du = e^{-v} \cdot v, \quad v \geq 0$$

- Since, $f_{UV}(u,v) = f_U(u) \cdot f_V(v)$, the random variables U & V are independent.

Q 6] State central limit theorem & give its significance.

→ We discuss central limit theorem in two forms i.e.

1. Central Limit Theorem (Liapounoff form)

→ If $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables with $E(X_i) = \mu_i$ & $Var[X_i] = \sigma_i^2$; $i = 1, 2 \ldots n$ then under certain general condition, $S_n = X_1 + X_2 + \cdots + X_n$ is a normal variate with mean $\mu = \sum_{i=1}^{n} \mu_i$ & variance $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$ as n tends to infinity.

· A pariticular form of the above central limit theorem known as Lindeberg - Levy's theorem is as follows :-

2. Central Limit Theorem (Lindeberg-Levy's form)

→ If $X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed random variables with $E(X_i) = \mu_i$ & $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots n$ & if $S_n = X_1 + X_2 + \cdots + X_n$ then under certain general conditions S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$ as n tends to infinity.

Significance of Central Limit Theorem :-

1) The central limit theorem is regarded by some statistician as the most important theorem in probability theory from both theoretical & practical point of view.

2) The central limit theorem is applicable for any distribution of X_i's.

3) The central limit theorem deals with convergence in distribution, which works better near the centre. So, it is called the central limit theorem.

Q7] The joint probability density function of (X, Y) is

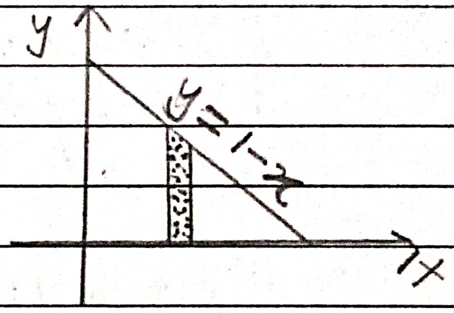
$f_{XY}(x,y) = 24xy \quad x>0,\ y>0,\ X+Y \le 1$

$= 0 \quad$ elsewhere

• find conditional mean & variance of Y given X.

→ By definition $E(Y|X=x) = \int_0^{1-x} y \cdot f_{Y|X}(y|x)\,dy$

Now, $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$



• The marginal probability density function of X i.e. $f_X(x)$ is given by

$f_X(x) = \int_0^{1-x} f_{X,Y}(x,y)\,dy$

$= \int_0^{1-x} 24xy\,dy = 24x \cdot \left[\dfrac{y^2}{2}\right]_0^{1-x} = 12(1-x)^2, \quad 0<x\le 1$

∴ Conditional probability density function of Y for given X, $f_{Y/X}(y/x)$ is given by

∴ $f_{Y/X}(y/x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{24xy}{12x(1-x)^2} = \dfrac{2y}{(1-x)^2}$,

$0<y<1-x$

Conditional mean,

$E(Y|X=x) = \int_0^{1-x} y \cdot f_{Y|X}(y|x)\,dy = \int_0^{1-x} \dfrac{2y^2}{(1-x)^2}\,dy$

$= \dfrac{2}{(1-x)^2}\left[\dfrac{y^3}{3}\right]_0^{1-x} = \dfrac{2}{3}(1-x)$

$E(Y^2|X=x) = \int_0^{1-x} y^2 \cdot f_{Y|X}(y|x)\,dy = \int_0^{1-x} \dfrac{2y^3}{(1-x)^2}\,dx$

$= \dfrac{2}{(1-x)^2}\left[\dfrac{y^4}{4}\right]_0^{1-x} = \dfrac{1}{2}(1-x)^2$

$\therefore$ Conditional variance,

$$\therefore V(Y|X=x) = E[Y^2|X=x] - [E(Y|X=x)]^2$$
$$= \frac{1}{2}(1-x)^2 - \frac{4}{9}(1-x)^2 = \frac{1}{18}(1-x)^2$$

Q8] Random variables X & Y have joint pdf

$$f_{XY}(x,y) = \begin{cases} \frac{(x-y)^2}{40} & -1<x<1,\ -3<y<3 \\ 0 & \text{elsewhere} \end{cases}$$

find i) $f_X(x)$ $f_Y(y)$ ii) Mean & variance of X & Y iii) Second order moment of X & Y. iv) Correlation coeffecient of X & Y.

$\rightarrow$ (i) Marginal probability density functions are obtained as follows:-

$$f_X(x) = \int_{-3}^{3} f_{X,Y}(x,y)\,dy = \int_{-3}^{3} \frac{x^2-2xy+y^2}{40}\,dy$$

$$= \frac{1}{40}\left[x^2y - xy^2 + \frac{y^3}{3}\right]_{-3}^{3} = \frac{1}{40}\left[\left(3x^2-9x+\frac{27}{3}\right) - \left(-3x^2-9x-\frac{27}{3}\right)\right]$$

$$= \frac{1}{40}(6x^2+18) = \frac{3x^2+9}{20}$$

$$f_Y(y) = \int_{-1}^{1} f_{X,Y}(x,y)\,dx = \int_{-1}^{1}\left[\frac{x^2-2xy+y^2}{40}\right]dx$$

$$f_Y(y) = \frac{1}{40}\left[\frac{x^3}{3} - x^2y + xy^2\right]_{-1}^{1} = \frac{1}{40}\left[\left[\frac{1}{3}-y+y^2\right] - \left[-\frac{1}{3}-y-y^2\right]\right]$$

$$= \frac{1}{40}\left[\frac{2}{3}+2y^2\right] = \frac{1+3y^2}{60}$$

(ii) Mean & variances are obtained as follows:-

$$m_X = \int_{-1}^{1} x \cdot f_X(x)\,dx = \int_{-1}^{1}\left[\frac{3x^3+9x}{20}\right]dx$$

$$= \frac{1}{20}\left[\frac{3x^4}{4}+\frac{9x^2}{2}\right]_{-1}^{1} = \frac{1}{20}\left[\left(\frac{3}{4}+\frac{9}{2}\right) - \left(\frac{3}{4}+\frac{9}{2}\right)\right] = 0$$

$$m_Y = \int_{-3}^{3} y \cdot f_Y(y)\,dy = \int_{-3}^{3} \left[\frac{y + 3y^3}{60}\right] dy$$

$$= \frac{1}{60}\left[\frac{y^2}{2} + \frac{3y^4}{4}\right]_{-3}^{3} = \frac{1}{60}\left[\left[\frac{9}{2} + \frac{243}{4}\right] - \left[\frac{9}{2} + \frac{243}{4}\right]\right] = 0$$

$$E[X^2] = \int_{-1}^{1} x^2 \cdot f_X(x)\,dx = \int_{-1}^{1} \left[\frac{3x^4 + 9x^2}{20}\right] dx = \frac{1}{20}\left[\frac{3x^5}{5} + 3x^3\right]_{-1}^{1}$$

$$= \frac{1}{20}\left[\left(\frac{3}{5} + 3\right) - \left(-\frac{3}{5} - 3\right)\right] = \frac{1}{20}\left[\frac{6}{5} + 6\right]$$

$$= \frac{36}{100} = 0.36$$

$$\therefore \sigma_X^2 = E(X^2) - [E(X)]^2 = 0.36$$

$$E[Y^2] = \int_{-3}^{3} y^2 \cdot f_Y(y)\,dy = \int_{-3}^{3} \left[\frac{y^2 + 3y^4}{60}\right] dy$$

$$= \frac{1}{60}\left[\frac{y^3}{3} + \frac{3y^5}{5}\right]_{-3}^{3} = \frac{1}{60}\left[\left[9 + \frac{729}{5}\right] - \left[-9 - \frac{729}{5}\right]\right]$$

$$= \frac{1}{30}\left[9 + \frac{729}{5}\right] = \frac{258}{50} = \frac{129}{25}$$

$$\therefore \sigma_Y^2 = E[Y^2] - [E(Y)]^2 = \frac{129}{25}$$

(iii) Second order moment of X & Y is given by

$$E(XY) = \int_{-1}^{1}\int_{-3}^{3} xy \cdot \frac{(x^2 - 2xy + y^2)}{40}\,dx\,dy$$

$$= \frac{1}{40}\int_{-1}^{1}\int_{-3}^{3} (x^3y - 2x^2y^2 + xy^3)\,dx\,dy$$

$$= \frac{1}{40}\int_{-1}^{1} \left[\frac{x^3y^2}{2} - 2x^2\cdot\frac{y^3}{3} + \frac{xy^4}{4}\right]_{-3}^{3} dx$$

$$= \frac{1}{40}\int_{-1}^{1} \left[x^3\cdot\frac{9}{2} - 2x^2\cdot 9 + x\cdot\frac{81}{4}\right] - \left[x^3\cdot\frac{9}{2} + 2x^2\cdot 9 + x\frac{81}{4}\right] dx$$

$$\therefore E[XY] = \frac{1}{40}\int_{-1}^{1} -36x^2\,dx = \frac{-36}{40}\left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{-12}{40}[1-(-1)] = \frac{-12}{20} = -3/5$$

(iv) The correlation coeffecient is given by

$$\rho_{X,Y} = \frac{E(XY) - E(X)\cdot E(Y)}{\sigma_X \cdot \sigma_Y} = -\frac{(3/5) - 0.0}{\sqrt{(36/100)}\sqrt{(129/25)}}$$

$$= -3/5 \cdot 10/6 \cdot 5/\sqrt{129} = \frac{-5}{\sqrt{129}}$$

Q9] If $X = \cos\theta$, $Y = \sin\theta$ where θ is uniformly distributed over $(0, 2\pi)$ prove that (i) X & Y are uncorrelated (ii) X & Y are not independent.

→ (i) We have $f_\Theta(\theta) = \begin{cases} 1/2\pi, & 0 < \theta < 2\pi \\ 0, & \text{elsewhere} \end{cases}$

$$\therefore E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^{2\pi} \cos\theta \cdot \frac{1}{2\pi}\cdot d\theta = \frac{1}{2\pi}[\sin\theta]_0^{2\pi} = 0$$

Similarly $\therefore E(Y) = 0$

Also $E(XY) = \int_0^{2\pi} \cos\theta \sin\theta \frac{1}{2\pi} d\theta = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\theta\, d\theta$

$$= \frac{1}{4\pi}\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi} = 0$$

$$\therefore \text{cov}(X, Y) = E(XY) - E(X)\cdot E(Y) = 0$$

But $\rho = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$ $\therefore \rho = 0$

$\therefore$ X & Y are uncorrelated

(ii) Now, $E[X^2] = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{1}{4\pi}\int_0^{2\pi}(1+\cos 2\theta)$

$$= \frac{1}{4\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{1}{4\pi}\cdot 2\pi = \frac{1}{2}$$

Similarly, $E[Y^2] = \frac{1}{2}$

$$E[X^2Y^2] = \frac{1}{2\pi}\int_0^{2\pi}(\sin\theta\cos\theta)^2 d\theta = \frac{1}{8\pi}\int_0^{2\pi}\sin^2\theta\, d\theta$$

$$= \frac{1}{8\pi}\int_0^{2\pi}\left[\frac{1-\cos 4\theta}{2}\right]d\theta = \frac{1}{16\pi}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{1}{16\pi}[2\pi] = 8$$

$\therefore E[X^2]\cdot E[Y^2] \neq E[X^2Y^2]$

Hence X, Y are not Independent

Q10] A two dimensional random variable (X, Y) has following distribution

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)} & 0 < y < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Find (i) $E(XY)$ (ii) $\text{cov}(X,Y)$ (iii) $\rho_{X,Y}$

$\rightarrow f_X(x) = 2\int_0^x e^{-x}\cdot e^{-y}\,dy = 2e^{-x}\left[-e^{-y}\right]_0^x = -2e^{-x}\left[e^{-x}-1\right]$

$= 2(e^{-x} - e^{-2x}),\ x > 0$ ($\because y$ varies from 0 to x)

& $f_Y(y) = 2\int_y^\infty e^{-x}\cdot e^{-y}\,dx = 2e^{-y}\left[-e^{-x}\right]_y^\infty = -2e^{-y}\left[-e^{-y}\right]$

$= 2e^{-2y},\ y > 0$

$$E(X) = \int_0^\infty x\cdot f_X(x)\,dx = \int_0^\infty x\cdot 2(e^{-x} - e^{-2x})\,dx$$

$$= 2\left[x\left[-e^{-x} + \frac{e^{-2x}}{2}\right] - 1\left[e^{-x} - \frac{e^{-2x}}{4}\right]\right]_0^\infty$$

$$= 2\left[0+\left[1-\frac{1}{4}\right]\right] = 2\cdot\frac{3}{4} = \frac{3}{2}$$

$$E[Y] = \int_0^\infty y \cdot f_Y(y)\,dy = \int_0^\infty y\cdot 2e^{-2y}\,dy$$

$$= 2\left[y\left[\frac{-e^{-2y}}{2}\right] - 1\cdot\left[\frac{e^{-2y}}{4}\right]\right]_0^\infty = 2\left[0+\frac{1}{4}\right] = \frac{1}{2}$$

$$E[X^2] = \int_0^\infty x^2 f_X(x)\,dx = \int_0^\infty x^2\cdot 2(e^{-x}-e^{-2x})\,dx$$

$$E[X^2] = 2\left[x^2\left[-e^{-x}+\frac{e^{-2x}}{2}\right] - 2x\left[e^{-x}-\frac{e^{-2x}}{4}\right] + 2\left[-e^{-x}+\frac{e^{-2x}}{8}\right]\right]$$

$$= 2\left[0-2\left(-1+\frac{1}{8}\right)\right] = 2\left[\frac{7}{4}\right] = \frac{7}{2}$$

$$E[Y^2] = \int_0^\infty y^2\cdot f_Y(y)\,dy = \int_0^\infty y^2\cdot 2e^{-2y}\,dy$$

$$= 2\left[y^2\left[\frac{-e^{-2y}}{2}\right] - (2y)\left[\frac{e^{2y}}{4}\right] + 2\left[\frac{-e^{2y}}{8}\right]\right]_0^\infty$$

$$= 2\left[0+\frac{1}{4}\right] = \frac{1}{2}$$

$$E[XY] = \int_{y=0}^\infty \int_{x=y}^\infty xy\, f_{X,Y}(x,y)\,dx\,dy = \int_{y=0}^\infty \int_{x=y}^\infty xy\cdot 2e^{-x}\cdot e^{-y}\,dx\,dy$$

$$= 2\int_{y=0}^\infty ye^{-y}\left[\int_{x=y}^\infty xe^{-x}\,dx\right]dy = 2\int_{y=0}^\infty ye^{-y}\left[x\cdot\frac{e^{-x}}{-1} - \frac{e^{-x}}{+1}\right]_y^\infty dy$$

$$= 2\int_0^\infty ye^{-y}(ye^{-y}+e^{-y})\,dy = 2\int_0^\infty (y^2e^{-2y}+ye^{-2y})\,dy$$

$$= 2\left[\left\{y^2\cdot\frac{e^{-2y}}{-2} - (2y)\frac{e^{-2y}}{4} + 2\frac{e^{-2y}}{-8}\right\} + \left\{y\cdot\frac{e^{-2y}}{-2} - 1\cdot\frac{e^{-2y}}{4}\right\}\right]_0^\infty$$

$$= 2\left[0-\left[-\frac{1}{4}-\frac{1}{4}\right]\right] = 2\left(\frac{1}{2}\right) = 1$$

$\therefore E(XY) = 1$

$\therefore cov(X, Y) = E(XY) - E(X) \cdot E(Y) = 1 - \frac{3}{2} \cdot \frac{1}{2} = 1 - \frac{3}{4} = \frac{1}{4}$

$\therefore cov(X, Y) = \frac{1}{4}$

Now, $Var(X) = E(X^2) - [E(X)]^2 = \frac{7}{2} - \left(\frac{3}{2}\right)^2 = \frac{7}{2} - \frac{9}{4} = \frac{5}{4}$

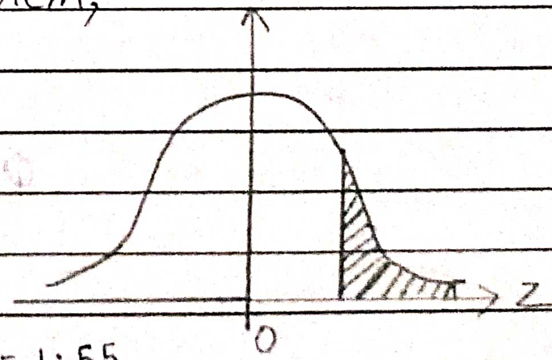
$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\therefore \rho = \frac{cov(X, Y)}{\sigma_X \sigma_Y} = \frac{1/4}{(\sqrt{5}/2)(1/2)} = \frac{1}{4} \cdot \frac{4}{\sqrt{5}} = \frac{1}{\sqrt{5}}$

Q11] The lifetime of a certain brand of electric bulb may be considered a random variable with mean 1200 hours & standard deviation 250 hours. Using the Central limit theorem find the probability that the average lifetime of 60 bulbs exceeds 1250 hours

→ If $\bar{X}$ denotes the average (mean) lifetime of 60 bulbs then by the Central Limit Theorem,

$$z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$



is a SNV with $\mu = 1200$, $\sigma = 250$ & $n = 60$

if $\bar{X} = 1250$, $\therefore z = \frac{1250 - 1200}{250/\sqrt{60}} = 1.55$

$\therefore P(z > 1.55) =$ area to the right of 1.55

$= 0.5 -$ area between $z = 0$ & $z = 1.55$

$= 0.5 - 0.4394 = 0.0606$

Q12] A distribution has unknown mean μ & variance 1·5. Using central limit theorem find the size of the sample such that the probability that difference between sample mean & the population mean will be less than 0·5 is 0·95.

→ We have $E(X_i) = \mu$ & $Var(X_i) = 1.5$

If $\bar{X}$ is the sample mean then $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ is a S.N.V

We have $|\bar{X} - \mu| = 0.5$,

$$\therefore |Z| = \left|\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right| = \left|\frac{0.5}{\sqrt{1.5}/\sqrt{n}}\right| = |0.4082\sqrt{n}|$$

We know from then table that $P(|Z|) > 0.95$ when $Z = 1.96$

$$\therefore 0.4082\sqrt{n} = 1.96$$

$$\therefore \sqrt{n} = \frac{1.96}{0.4082}$$

$$\therefore n = 23.05$$

Hence, n must be atleast 24

Q13] Define characteristic function of a random variable X state & prove properties of characteristic function. Find mean & variance for characteristic function

→ The characteristic function of a random variable X denoted by $\phi_X(\omega)$ is defined by

$$\phi_X(\omega) = E(e^{i\omega X})$$ where ω is an auxillary variable.

$$\phi_X(\omega) = \sum e^{i\omega x} p(x)$$ (For discrete probability distribution)

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$ (For continuos probability distribution)

* Properties of Characteristic Function

1. $\mu_r' =$ coeff of $\frac{i^r \omega^r}{r!}$ in the expansion of $\phi_X(\omega)$

Proof - By definition

$$\phi_X(\omega) = E(e^{i\omega X}) = E\left[1 + \frac{i\omega X}{1!} + \frac{i^2\omega^2 X^2}{2!} + \dots + \frac{i^n \omega^n X^n}{n!} + \dots\right]$$

$$= 1 + i\omega E(X) + \frac{i^2\omega^2}{2!} E(X^2) + \dots + \frac{i^n\omega^n}{n!} E(X^n) + \dots$$

But $\mu_r' = E(X^r)$

$\therefore$ $\mu_r' =$ Coeff of $\frac{i^r\omega^r}{r!}$ in the expansion of $\phi_X(\omega)$

2. $\mu_r' = \frac{1}{i^r}\left[\frac{d^r}{d\omega^r}\phi(\omega)\right]_{\omega=0}$

→ Proof - Differentiating both sides of (1) with respect to ω, r times & then putting $\omega = 0$,

$$\left[\frac{d^r \phi_X(\omega)}{d\omega^r}\right]_{\omega=0} = i^r \cdot E(X^r) \quad \therefore \mu_r' = \frac{1}{i^r}\left[\frac{d^r}{d\omega^r}\phi_X(\omega)\right]_{\omega=0}$$

3. If $Y = aX + b$ then $\phi_Y(\omega) = e^{ib\omega}\phi_X(a\omega)$

Proof - By definition

$$\phi_Y(\omega) = E(e^{i\omega Y}) = E(e^{i\omega(aX+b)}) = E(e^{ia\omega} \cdot e^{ib\omega})$$

$$= e^{ib\omega} E(e^{ia\omega X}) = e^{ib\omega}\phi_X(a\omega)$$

Putting $b = 0$, we get of $Y = aX$ then

$$\phi_Y(\omega) = E(e^{ia\omega X}) = \phi_X(a\omega)$$

4. If X & Y are independent random variates then
$$\Phi_{X+Y}(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$$

Proof - By definition $\Phi_{X+Y}(\omega) = E[e^{i\omega(X+Y)}]$
$$= E[e^{i\omega X} \cdot e^{i\omega Y}]$$
$$= E(e^{i\omega X}) \cdot E[e^{i\omega Y}]$$
[$\because$ X, Y are independent]
$$= \Phi_X(\omega) \cdot \Phi_Y(\omega)$$

5. If $\Phi(\omega)$ is the characteristic function of a continuos random variable X, whose probability density function is $f_X(x)$ then
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\omega) e^{-ix\omega} d\omega$$

Proof - By definition $\Phi_X(\omega) = \int_{-\infty}^{\infty} f(X) e^{i\omega X} dx$

- But the r.h.s is the Fourier Transform of $f(X)$. Hence, $f(X)$ is the Inverse Fourier Transform of $\Phi(\omega)$

i.e. $f(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega) e^{i\omega X} dx$

6. If the density function of X is known then the density function of $Y = g(X)$ can be found from the characteristic function of X if $Y = g(X)$ is a one-one correspondence.

Proof - Let the probability density function of X be $f_X(x)$

Then $\Phi_{g(x)}(\omega) = \int_{-\infty}^{\infty} e^{ig(x)\omega} \cdot f_X(x) dx$

Putting $g(X) = Y$ & $g(x) = y$, suppose $f_X(x)dx$ changes to say $h(y)\,dy$

$$\therefore \Phi_Y(\omega) = \int_{-\infty}^{\infty} e^{iy\omega} h(y) dy$$

$\therefore$ $h(y)$ is the probability density function of y.

7. Uniqueness Theorem :- Characteristic function uniquely determines the distribution i.e. if two probability density functions $f_1(x)$ & $f_2(x)$ have the characteristic functions $\Phi_1(t)$ & $\Phi_2(t)$ they are identical if $\Phi_1(t) = \Phi_2(t)$ & conversely.

Proof - (i) Let $f_1(x) = f_2(x)$

Now $\Phi_1(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_1(x)\,dx$ &

$$\Phi_2(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_2(x)\,dx$$

Since $f_1(x) = f_2(x)$; $\Phi_1(\omega) = \Phi_2(\omega)$.

* Characteristic Function of standard distribution

- Binomial distribution

$\rightarrow$ Mean $= \mu_1' = np$

Variance $= \mu_2' - \mu_1'^2 = n^2p^2 - np^2 + np - n^2p^2$

$= np(1-p) = npq$

- Poisson distribution

$\rightarrow$ Mean $= \mu_1' = m$

Variance $= \mu_2' - \mu_1'^2 = m$

- Geometric distribution

$\rightarrow$ Mean $= \mu_1' = q/p$

Variance $= \mu_2' - \mu_1'^2 = q/p^2 + q^2/p^2 - q^2/p^2 = q/p^2$

- Standard Normal distribution

$\rightarrow$ Mean $= b+a/2$

Variance $= (b-a)^2/12$

- Exponential distribution

$\rightarrow$ Mean $= 1/m = \mu_1'$

Variance $= \mu_2' - \mu_1'^2 = 2/m^2 - 1/m^2 = 1/m^2$

Q14] Define correlation, covariance, correlation coeffecient. Whean are X, Y uncorrelated? When are they orthogonal?
prove that $C_{XY}(x,y) = R_{XY}(x,y) - E(X)\cdot E(Y)$
prove that $|C_{XY}| \leq \sigma_X \sigma_Y$. Reduce that $-1 \leq \rho \leq 1$

- Correlation - If $Var(X)>0$ & $Var(Y)>0$ then the correlation of X & Y is defined by

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

- Covariance - If X & Y are two random variables then the covariance between them denoted by $Cov(X,Y)$ or C_{XY} & is defined by

$$C_{XY} = E[\{X - E[X]\}\{Y - E(Y)\}]$$

On expanding the r.h.s we get

$$C_{XY} = E[XY] - E[X]\cdot E[Y]$$

- Correlation coeffecient - If X & Y are two random variables then the correlation coeffecient between them is denoted by $r_{X,Y}$ or $P_{X,Y}$ & is defined by

$$P_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

- Where σ_X & σ_Y are the standard deviation of X & Y.

☆ • If $E[XY] = E[X]\cdot E[Y]$ then the variables are uncorrelated.

• If $E[XY] = 0$ the variables are orthogonal.

a) • $C_{XY} = E[\{X - E[X]\}\{Y - E[Y]\}]$

On expanding the r.h.s

$$C_{XY} = E[XY - XE[Y] - YE[X] + E[X]\cdot E[Y]]$$
$$= E[XY] - E[X]\cdot E[Y] - E[Y]\cdot E[X] + E[X]\cdot E[Y]]$$

$$C_{XY} = E[XY] - E[X]\cdot E[Y]$$
$$C_{X,Y}(x,y) = R_{X,Y}(x,y) - E[X]\cdot E[Y]$$
$$\because R_{X,Y}(x,y) = E[XY]$$
$$\therefore C_{XY} = C_{X,Y}(x,y) = Cov(X,Y)$$

b) • $|C_{XY}| \leq \sigma_X \cdot \sigma_Y$

→ Let a be any real number & consider

$$E[\{a[X - E[X]] + [Y - E[Y]]\}^2]$$

On expansion,

$$E[\{a[X - E[X]] + [Y - E[Y]]\}^2]$$
$$= E[a^2[X - E[X]]^2 + [Y - E[Y]]^2 + 2a[X - E[X]][Y - E[Y]]]$$
$$= E[a^2[X - E[X]]^2 + E[Y - E[Y]]^2] + E[2a[X - E[X]][Y - E[Y]]]$$
$$= a^2E[X - E[X]]^2 + E[Y - E[Y]]^2 + 2aE[[X - E[X]][Y - E[Y]]]$$
$$= a^2\sigma^2_X + \sigma^2_Y + 2aC_{XY}$$
$$= a^2\sigma^2_X + 2aC_{XY} + \sigma^2_Y$$

Since the LHS is positive, the RHS must also be positive. Hence, its discriminant must be negative

$$\Delta = \sqrt{4C_{XY}^2 - 4\sigma_X^2\sigma_Y^2} \leq 0$$
$$C_{XY}^2 \leq \sigma_X^2\sigma^2_Y$$
$$|C_{XY}| \leq \sigma_X\sigma_Y$$

Since $\rho = \dfrac{C_{XY}}{\sigma_X\sigma_Y}$ & $|C_{XY}| \leq \sigma_X\sigma_Y$

$$|\rho| \leq 1$$

i.e. $-1 \leq \rho \leq 1$