Question 1)

Solve the following recurrence relations. For each one come up with a **precise** function of n in closed form (i.e., resolve all sigmas, recursive calls of function T, etc) using the substitution method. Note: An asymptotic answer is not acceptable for this question. Justify your solution and show all your work.

a) T(n)=T(n-1)+cn, T(0)=1,
b) T(n)=4T(n/2)+n , T(1)=1
c) T(n)= 2T(n/2)+1, T(1)=1

Answer:

a) T(n)=T(n-1)+cn, T(0)=1

Given, T(0) = 1 and T(n) = T(n - 1) + cn

T(n) = T(n - 1) + cn

Now, T(n - 1) = T(n - 2) + c(n - 1) [Since, T(m) = T(m - 1) + cm and m = n - 1 here]

Substitute T(n - 1) in above equation.

T(n) = T(n - 2) + c(n - 1) + cn

Now, T(n - 2) = T(n - 3) + c(n - 1) [Since, T(m) = T(m - 1) + cm and m = n - 2 here]

Substitute T(n - 2) in above equation.

T(n) = T(n - 3) + c(n - 2) + c(n - 1) + c(n)

And if we repeat it for k times, we get

T(n) = T(n - k - 1) + c(n - k) + ……. c(n - 1) + c(n)

We reach base case when n - k - 1 becomes 0

n - k - 1 = 0

== k = n - 1

and n - k = 1

At k = n - 1,

Now, T(n) = T(0) + c(1) + c(2) + c(3) + ……… c(n - 1) + c(n)

Given, T(0) = 1

So, T(n) = 1 + c[1 + 2 + 3 + ……. (n - 1) + n]

1 + 2 + ……(n - 1) + n = n * (n + 1) / 2 according to mathematics.

So, T(n) = c * n * (n + 1) / 2 + 1

T(n) = (c / 2) * n^2 + (c / 2) * n + 1 is the function representing the time complexity of given function.

To Find the Big-O Notation of given function, we need to find n0, C where T(n) <= C * g(n) for all n >= n0

Consider a function G(n) by replacing each value in T(n) with c * n^2

Then, (c / 2) * n^2 <= c * n^2 for all n >= 1 since c > c / 2

(c / 2) * n <= c * n^2 for all n >= 1 since c > c / 2 and n^2 > n

1 <= c * n^2 for all n >= 1

If we add all three inequalities, then we get,

(c / 2) * n^2 + (c / 2) * n + 1 <= c * n^2 + c * n^2 + c * n^2 for all n >= 1

=== (c / 2) * n^2 + (c / 2) * n + 1 <= 3 * c * n^2 for all n >= 1

=== T(n) <= 3 * c * n^2 for all n >= 1

Therefore, T(n) = O(n^2) where n0 = 1 and C = 3 * c

b) T(n)=4T(n/2)+n , T(1)=1

Given, T(1)=1 and T(n)=4T(n/2)+n

$T(n)=4T(n/2) + n$

Now, $T(n / 2) = 4T(n / 4) + n / 2$ [Since, $T(m) = 4T(m / 2) + m$ and $m = n / 2$ here]

Substitute $T(n / 2)$ in above equation,

$T(n) = 4 * [4 * T(n / 4) + n / 2] + n$

$T(n) = 4 * 4 * T(n / 4) + 4 * n / 2 + n$

Now, $T(n / 4) = 4T(n / 8) + n / 4$ [Since, $T(m) = 4T(m / 2) + m$ and $m = n / 4$ here]

Substitute $T(n / 4)$ in above equation,

$T(n) = 4 * 4 * [4 * T(n / 8) + n / 4] + 4 * n / 2 + n$

$T(n) = 4 * 4 * 4 * T(n / 8) + 4 * 4 * n / 4 + 4 * n / 2 + n$

And if we repeat k - 1 times, we get

$T(n) = 4^k * T(n / 2^k) + 4^{(k - 1)} * n / 2^{(k - 1)} + \ldots\ldots\ldots n$

We know, $4^{(k - 1)} / 2^{(k - 1)} = 2^{(k - 1)} * 2^{(k - 1)} / 2^{(k - 1)} = 2^{(k - 1)}$

So, $T(n) = 4^k * T(n / 2^k) + 2^{(k - 1)} * n + \ldots\ldots\ldots n$

$T(n) = 4^k * T(n / 2^k) + n * [1 + 2 + 4 + \ldots.. 2^{(k - 1)}]$

$1 + 2 + 4 + \ldots.. 2^{(k - 1)} = 2^k - 1$ according to mathematics

So, $T(n) = 4^k * T(n / 2^k) + n * (2^k - 1)$

We reach base case when $n / 2^k = 1$

$n / 2^k = 1$

$n = 2^k$

$k = \log_2(n)$

At $k = \log_2(n)$

$T(n) = 4^{\log_2 n} * T(1) + n * (2^{\log_2 n} - 1)$

Substitute $T(1) = 1$

And $2^{\log_2 n} = n$ according to mathematics

And, $4^{\log_2 n} = 2^{2\log_2 n} = 2^{\log_2 n^{\wedge}2} = n^2$

So, $T(n) = n^2 + n * (n - 1)$

$T(n) = 2n^2 - n$

$T(n) = 2n^2 - n$ is the function representing the time complexity of given function.

To Find the Big-O Notation of given function, we need to find n0, C where $T(n) <= C * g(n)$ for all $n >= n0$

Consider a function $G(n)$ by replacing each value in $T(n)$ with $2 * n^2$

Then, $2 * n^2 = 2 * n^2$ for all $n >= 1$ since both are equal

$-1 * n <= 2 * n^2$ for all $n >= 1$ since left value is negative and right one is positive

If we add both inequalities, then we get,

$2 * n^2 - n <= 2 * n^2 + 2 * n^2$ for all $n >= 1$

=== $2 * n^2 - n <= 4 * n^2$ for all $n >= 1$

=== $T(n) <= 4 * n^2$ for all $n >= 1$

Therefore, $T(n) = O(n^2)$ where $n0 = 1$ and $C = 4$

c) $T(n)= 2T(n/2)+1, T(1)=1$

Given, $T(n)= 2T(n/2)+1, T(1)=1$

$T(n) = 2 * T(n / 2) + 1$

Now, $T(n / 2) = 2 * T(n / 4) + 1$ [Since $T(m) = 2T(m / 2) + 1$ and $m = n / 2$ here]

Substitute $T(n / 2)$ in above equation

$T(n) = 2 * [2 * T(n / 4) + 1] + 1$

$T(n) = 2 * 2 * T(n / 4) + 2 + 1$

Now, $T(n / 4) = 2 * T(n / 8) + 1$ [Since $T(m) = 2T(m / 2) + 1$ and $m = n / 4$ here]

Substitute $T(n / 4)$ in above equation

$T(n) = 2 * 2 * [2 * T(n / 8) + 1] + 2 + 1$
$T(n) = 2 * 2 * 2 * T(n / 8) + 2^2 + 2 + 1$
And if we repeat k - 1 times, we get
$T(n) = 2^k * T(n / 2^k) + 2^{(k-1)} + \ldots.. 2 + 1$
$1 + 2 + 4 + \ldots.. 2^{(k - 1)} = 2^k - 1$ according to mathematics
$T(n) = 2^k * T(n / 2^k) + 2^k - 1$
We reach base case when $n / 2^k = 1$
$n / 2^k = 1$
$n = 2^k$
$k = \log_2(n)$
At $k = \log_2(n)$
$T(n) = 2^{\log_2 n} * T(1) + 2^{\log_2 n} - 1$
Substitute $T(1) = 1$
And $2^{\log_2 n} = n$ according to mathematics
$T(n) = n * 1 + n - 1$
$T(n) = 2n - 1$
T(n) = 2n - 1 is the function representing the time complexity of given function.
To Find the Big-O Notation of given function, we need to find n0, C where T(n) <= C * g(n) for all n >= n0
Consider a function G(n) by replacing each value in T(n) with 2 * n
Then, 2 * n = 2 * n for all n >= 1 since both are equal
-1 <= 2 * n for all n >= 1 since left value is negative and right value is positive
If we add both inequalities, then we get,
2 * n - 1 <= 2 * n + 2 * n for all n >= 1
=== T(n) <= 4 * n
Therefore, T(n) = O(n) where n0 = 1 and C = 4

Question 2)
Consider Question 1 again. Apply Master Theorem if applicable for each case. Bound the recurrence relation in Big-O.

Answer:

Master's Theorem is used to easily find Big-O Notation for functions with patterns in their recurrence relation.
We can apply Master's theorem only if the recurrence relation is of form:
$T(n) = a * T(n / b) + f(n)$
Where a >= 1, b >= 2 and f(n) is an asymptotically positive function
It has 3 cases.
Case-I:
If $f(n) = \text{Theta}(n^d)$ where $d < \log_b a$
Then f(n) grows asymptotically slower than $\log_b a$
Therefore, $T(n) = \text{Theta}(n^{\log_b a})$
Case-II:
If $f(n) = \text{Theta}(n^d)$ where $d > \log_b a$
Then f(n) grows asymptotically faster than $\log_b a$
Therefore, $T(n) = \text{Theta}(n^d)$
Case-III:
If $f(n) = \text{Theta}(n^d \log^k n)$ where $d = \log_b a$
Then, $T(n) = \text{Theta}(n^d \log^{(k+1)} n)$

a) $T(n)=T(n-1)+cn$, $T(0)=1$

To Apply Master's Theorem, T(n) should be of form

$T(n) = a * T(n / b) + f(n)$

Since, given function $T(n)=T(n-1)+cn$ is not in that form. We can't apply Master's theorem on this function.

b) $T(n)=4T(n/2)+n$, $T(1)=1$

To Apply Master's Theorem, T(n) should be of form

$T(n) = a * T(n / b) + f(n)$

Given function is of the same form where $a = 4$, $b = 2$ and $f(n) = n$

f(n) = Theta(n)

=== Theta(n^1)

So, d = 1 in Master's Theorem

$\log_b a = \log_2 4 = 2$

Since, $d = 1 < \log_b a = 2$, the given function follows Case-I pattern of Master's Theorem

According to Case-I of Master's Theorem,

If f(n) = Theta(n^d) where $d < \log_b a$

Then f(n) grows asymptotically slower than $\log_b a$

Therefore, T(n) = Theta($n^{\log_b a}$)

Therefore, T(n) = Theta($n^{\log_b a}$) = Theta(n^2)

Since, Theta represents a tight bound

$C1 * n^2 <= T(n) <= C2 * n^2$

If we consider, only $T(n) <= C2 * n^2$ it represents Big-O Notation

Therefore, $T(n) = O(n^2)$

c) $T(n)= 2T(n/2)+1$, $T(1)=1$

To Apply Master's Theorem, T(n) should be of form

$T(n) = a * T(n / b) + f(n)$

Given function is of the same form where $a = 2$, $b = 2$ and $f(n) = 1$

f(n) = Theta(1)

=== Theta(n^0)

So, d = 0 in Master's Theorem

$\log_b a = \log_2 2 = 1$

Since, $d = 0 < \log_b a = 1$, the given function follows Case-I pattern of Master's Theorem

According to Case-I of Master's Theorem,

If f(n) = Theta(n^d) where $d < \log_b a$

Then f(n) grows asymptotically slower than $\log_b a$

Therefore, T(n) = Theta($n^{\log_b a}$)

Therefore, T(n) = Theta($n^{\log_b a}$) = Theta(n^1)

Since, Theta represents a tight bound

$C1 * n <= T(n) <= C2 * n$

If we consider, only $T(n) <= C2 * n$ it represents Big-O Notation

Therefore, $T(n) = O(n)$

Question 3)

A binary tree's "maximum depth" is the number of nodes along the longest path from the root node down to the farthest leaf node. Given the root of a binary tree, write a complete program in C++/ Java that returns three's maximum depth. What is the time-complexity of your algorithm in the worst-case once you have n nodes in the tree. Analyze and clearly discuss your reasoning. Paste your complete program in the solution file.

Answer:

Java Code to Calculate maximum depth of a binary tree is

```
public static int maxDepth(Node root) {
        // Base Condition
        if(root == null) return 0;
        // Recursive Calls
        int left = maxDepth(root.left);
        int right = maxDepth(root.right);

        return Math.max(left, right) + 1;
}
```

Let T(n) is the time complexity of the above function.
Number of basic operations are:

Code	Cost	Number of times it runs
Base Condition		
if(root == null) return 0;	C1	1
Recursive Calls		
int left = maxDepth(root.left);	T(n / 2)	1
int right = maxDepth(root.right);	T(n / 2)	1
return Math.max(left, right) + 1;	C2	1

So, $T(n) = T(n / 2) + T(n / 2) + C2$ and $T(1) = C1$
$T(n) = 2T(n / 2) + C2$ and $T(1) = C1$

Given, $T(n)= 2T(n/2)+C2, T(0)=C1$

$T(n) = 2 * T(n / 2) + C2$

Now, $T(n / 2) = 2 * T(n / 4) + C2$ [Since $T(m) = 2T(m / 2) + C2$ and $m = n / 2$ here]

Substitute T(n / 2) in above equation

$T(n) = 2 * [2 * T(n / 4) + C2] + C2$

$T(n) = 2 * 2 * T(n / 4) + 2C2 + C2$

Now, $T(n / 4) = 2 * T(n / 8) + C2$ [Since $T(m) = 2T(m / 2) + C2$ and $m = n / 4$ here]

Substitute T(n / 4) in above equation

$T(n) = 2 * 2 * [2 * T(n / 8) + 1C2] + 2C2 + C2$

$T(n) = 2 * 2 * 2 * T(n / 8) + 2^2C2+ 2C2 + 1C2$

And if we repeat k - 1 times, we get

$T(n) = 2^k * T(n / 2^k) + [2^{(k-1)} + \ldots.. 2 + 1]C2$

$1 + 2 + 4 + \ldots.. 2^{(k - 1)} = 2^k - 1$ according to mathematics

$T(n) = 2^k * T(n / 2^k) + [2^k - 1]C2$

We reach base case when $n / 2^k = 1$

$n / 2^k = 1$

$n = 2^k$

$k = \log_2(n)$

At $k = \log_2(n)$

$T(n) = 2^{\log_2 n} * T(1) + [2^{\log_2 n} - 1]C2$

Substitute $T(1) = C1$
And $2^{\log_2 n} = n$ according to mathematics
$T(n) = n * C1 + [n - 1]C2$
$T(n) = [C1 + C2]n - C2$
T(n) = [C1 + C2]n - C2 is the function representing the time complexity of given function.
To Find the Big-O Notation of given function, we need to find n0, C where T(n) <= C * g(n) for all n >= n0
Consider a function G(n) by replacing each value in T(n) with [C1 + C2] * n
Then, [C1 + C2] * n = [C1 + C2] * n for all n >= 1 since both are equal
-C2 <= [C1 + C2] * n for all n >= 1 since left value is negative and right value is positive
If we add both inequalities, then we get,
[C1 + C2]n - C2 <= [C1 + C2]n + [C1 + C2]n for all n >= 1
=== T(n) <= 2[C1 + C2]n
Therefore, T(n) = O(n) where n0 = 1 and C = 2[C1 + C2]
We can also solve the recurrence relation Using Master's theorem.
Master's Theorem is used to easily find Big-O Notation for functions with patterns in their recurrence relation.
We can apply Master's theorem only if the recurrence relation is of form:
T(n) = a * T(n / b) + f(n)
Where a >= 1, b >= 2 and f(n) is an asymptotically positive function
It has 3 cases.
Case-I:
If f(n) = Theta(n^d) where d < $\log_b a$
Then f(n) grows asymptotically slower than $\log_b a$
Therefore, T(n) = Theta($n^{\log_b a}$)
Case-II:
If f(n) = Theta(n^d) where d > $\log_b a$
Then f(n) grows asymptotically faster than $\log_b a$
Therefore, T(n) = Theta(n^d)
Case-III:
If f(n) = Theta($n^d \log^k n$) where d = $\log_b a$
Then, T(n) = Theta($n^d \log^{(k+1)} n$)
Given function is of the same form where a = 2, b = 2 and f(n) = C2
f(n) = Theta(1)
=== Theta(n^0)
So, d = 0 in Master's Theorem
$\log_b a = \log_2 2 = 1$
Since, d = 0 < $\log_b a$ = 1, the given function follows Case-I pattern of Master's Theorem
According to Case-I of Master's Theorem,
If f(n) = Theta(n^d) where d < $\log_b a$
Then f(n) grows asymptotically slower than $\log_b a$
Therefore, T(n) = Theta($n^{\log_b a}$)
Therefore, T(n) = Theta($n^{\log_b a}$) = Theta(n^1)
Since, Theta represents a tight bound
C1 * n <= T(n) <= C2 * n
If we consider, only T(n) <= C2 * n it represents Big-O Notation
Therefore, T(n) = O(n)

Question 4) Task given in video lectures

Answer:

Given, $T(n) = T(n - 1) + T(n - 2) + O(1)$, $T(0) = 1$ and $T(0) = T(1) = O(1)$

$T(n) = T(n - 1) + T(n - 2) + O(1)$

We know that $T(n - 1) > T(n - 2)$ since $T(n - 1) = T(n - 2) + T(n - 3) + O(1)$ i.e; Time Complexity of $T(n - 1)$ has $T(n - 2)$ Time Complexity + Some more.

=== $T(n - 2) < T(n - 1)$

=== $T(n - 2) + T(n - 1) < T(n - 1) + T(n - 1)$

=== $T(n - 2) + T(n - 1) + O(1) < T(n - 1) + T(n - 1) + O(1)$

=== $T(n) < 2 * T(n - 1) + O(1)$

So, $T(n) < 2 * T(n - 1) + O(1)$

$T(n) < 2 * [2 * T(n - 2) + O(1)] + O(1)$

=== $T(n) < 4 * T(n - 2) + 2 * O(1) + O(1)$

$T(n) < 4 * [2 * T(n - 3) + O(1)] + 2 * O(1) + O(1)$

$T(n) < 8 * T(n - 3) + 4 * O(1) + 2 * O(1) + O(1)$

If we repeat for k steps

$T(n) < 2^k * T(n - k) + 2^{(k-1)} * O(1) + 2^{(k-2)} * O(1) + \ldots\ldots 2 * O(1) + O(1)$

$T(n) < 2^k * T(n - k) + O(1) * [2^{(k-1)} + 2^{(k-2)} + \ldots\ldots 2 + 1]$

$2^{(k-1)} + 2^{(k-2)} + \ldots\ldots 2 + 1 = 2^k - 1$ according to mathematics

$T(n) < 2^k * T(n - k) + O(1) * (2^k - 1)$

We reach base case when $n - k = 0$ === $k = n$

At base case,

$T(n) < 2^n * T(0) + O(1) * (2^n - 1)$

$T(n) < 2^n * O(1) + O(1) * (2^n - 1)$

=== $T(n) < O(1) * [2^n + 2^n - 1]$

=== $T(n) < O(1) * [2* 2^n - 1]$

=== $T(n) < 2O(1) * 2^n - O(1)$

We know that $- O(1) < 2O(1) * 2^n$ since left hand value is -ve and right hand value is +ve.

So, $2O(1) * 2^n - O(1) < 2O(1) * 2^n + 2O(1) * 2^n$

=== $T(n) < 4O(1) * 2^n$

Therefore, $T(n) = O(2^n)$