Chapter 1

Introduction: Some Representative Problems
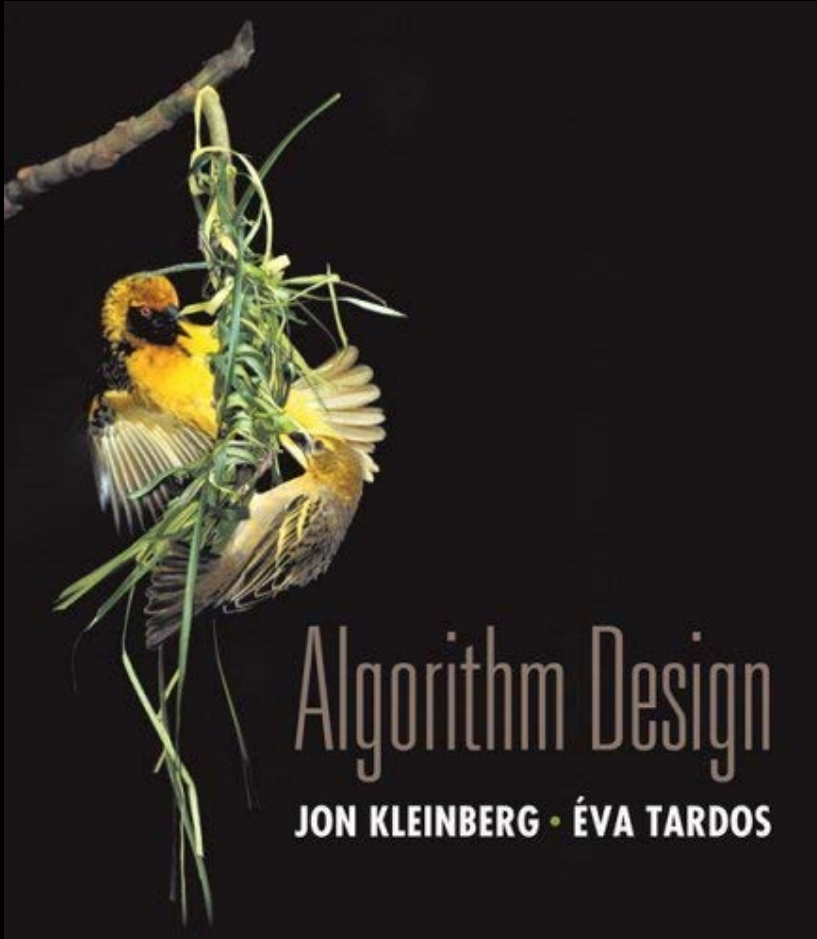
Algorithm Design

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1.1 A First Problem: Stable Matching

Matching Residents to Hospitals

Goal. Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

Unstable pair: applicant x and hospital y are unstable if:
- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.
- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.
- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

favorite →				least favorite →		

	1st	2nd	3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

Men's Preference Profile

	1st	2nd	3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Women's Preference Profile

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Q. Is assignment X-C, Y-B, Z-A stable?

	favorite →		least favorite →
	1st	**2nd**	**3rd**
X	A	B	C
Y	B	A	C
Z	A	B	C

Men's Preference Profile

	favorite →		least favorite →
	1st	**2nd**	**3rd**
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A. No. B and X will hook up.

	favorite → 1st	2nd	least favorite → 3rd
X	A	**B**	C
Y	B	A	C
Z	A	B	C

Men's Preference Profile

	favorite → 1st	2nd	least favorite → 3rd
A	Y	X	Z
B	**X**	Y	Z
C	X	Y	Z

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

Men's Preference Profile

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?
A. Yes.

favorite → ... ← least favorite

Men's Preference Profile

	1st	2nd	3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

Women's Preference Profile

	1st	2nd	3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

Stable roommate problem.

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D   ?

Stable Roommate Problem

Q. Do stable matchings always exist?
A. Not obvious a priori.

Stable roommate problem.
- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable

Stable Roommate Problem

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A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D   ?

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Doofus	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D $\Rightarrow$ A-B unstable

Stable Roommate Problem

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Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D $\Rightarrow$ A-B unstable
A-D, B-C ?

Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

Stable roommate problem.

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D $\Rightarrow$ A-B unstable
A-D, B-C $\Rightarrow$ A-C unstable

Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.
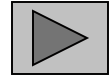
```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.
Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ∎

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▪

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

Q. How to start this proof?

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: A and Z prefer each other to their partner in the Gale-Shapley matching S*.

Q. How could this have happened?

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: A and Z prefer each other to their partner in the Gale-Shapley matching S*.

Q. How could this have happened?

Case 1: Z never proposed to A.

Case 2: Z proposed to A and A rejected/dumped Z

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: A and Z prefer each other to their partner in the Gale-Shapley matching S*.

Case 1: Z never proposed to A.   men propose in decreasing
                                         order of preference

S*

| Anna-Youp |
| Bertha-Zeger |
| . . . |

⇒ Z prefers his partner in S* to A.

⇒ A-Z is not an unstable pair.

Case 2: Z proposed to A.

⇒ A rejected Z (right away or later)

⇒ A prefers her partner in S* to Z.   ← women only trade up

⇒ A-Z is not an unstable pair.

In either case A-Z is not an unstable pair, a contradiction. ▪

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

- Assume men are named 1, …, n.
- Assume women are named 1', …, n'.

Engagements.

- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - set entry to `0` if unmatched
  - if m matched to w then `wife[m]=w` and `husband[w]=m`

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man `m`.

Efficient Implementation

Women rejecting/accepting.

- Does woman w prefer man m to man m'?
- For each woman, create inverse of preference list of men.
- Constant time access for each query after $O(n)$ preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

Amy prefers man 3 to 6
since inverse[3] < inverse[6]

```
for i = 1 to n
    inverse[pref[i]] = i
```

2   7

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!
- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

Q. What does this mean?

Contradiction! ▪

Man Optimality

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

In execution: first moment some man Y is rejected by best valid partner A in S.

... (idea: create another stable matching S' where Y is not rejected to derive contradiction)

Contradiction! ▪

Man Optimality

...-Youp
Anna-Zeger
...

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

In execution: first moment some man Y is rejected by best valid partner A in S.

When Y is rejected, A forms/stays engagement with a man, say Z, whom she prefers to Y.

... (idea: create another stable matching S' where Y is not rejected to derive contradiction)

Contradiction! ▪

	1st	2nd	3rd
Xander			
Youp	A		
Zeger			

	1st	2nd	3rd
Anna	Z	Y	
Bertha			
Clara			

Man Optimality

S

...-Youp
Anna-Zeger
· · ·

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

In execution: first moment some man Y is rejected by best valid partner A in S.

When Y is rejected, A forms/stays engagement with a man, say Z, whom she prefers to Y.

Stable S' with Y-A exists because Y-A is valid.

Let B be Z's partner in S'.

Q. Given what happened in S, does Z prefer A or B?

should exist:
S'

Anna-Youp
Bertha-Zeger
· · ·

Contradiction! ▪

S'

	1st	2nd	3rd
Xander			
Youp	A		
Zeger		B	

	1st	2nd	3rd
Anna	Z	Y	
Bertha		Z	
Clara			

Man Optimality

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

In execution: first moment some man Y is rejected by best valid partner A in S.

When Y is rejected, A forms/stays engagement with a man, say Z, whom she prefers to Y.

Stable S' with Y-A exists because Y-A is valid.

Let B be Z's partner in S'.

Z not rejected by any valid partner at the point when Y is rejected by A (in S). Thus, Z prefers A to B.

But A prefers Z to Y. Thus A-Z is unstable in S'.

Contradiction! ▪

S

...-Youp
Anna-Zeger
. . .

should exist:
S'

Anna-Youp
Bertha-Zeger
. . .

since Y was first rejected by a valid partner

S'

	1st	2nd	3rd
Xander			
Youp	A		
Zeger	A	B	

	1st	2nd	3rd
Anna	Z	Y	
Bertha		Z	
Clara			

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a **stable** matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)
Q. Which assumption to make?

Contradiction! ▪

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)
Suppose A-Z matched in S, but Z is not worst valid partner for A.

Idea: similar proof as man-optimal, and also use that fact!

Contradiction! ▪

S

Anna-Zeger
. . .
. . .

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)

Suppose A-Z matched in S, but Z is not worst valid partner for A.
There exists stable matching S' in which A is paired with a man, say Y,
whom she likes less than Z.
Let B be Z's partner in S'.

Q. Given what happened in S, does Z prefer A or B?

Contradiction! ▪

S

Anna-Zeger
. . .
. . .

S'

Anna-Youp
Bertha-Zeger
. . .

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)
Suppose A-Z matched in S, but Z is not worst valid partner for A.
There exists stable matching S' in which A is paired with a man, say Y,
whom she likes less than Z.
Let B be Z's partner in S'.
Z prefers A to B.   ← man-optimality by GS in S
Thus, A-Z is an unstable pair in S'.
Contradiction: S' was stable! ▪

S

Anna-Zeger
. . .
. . .

S'

Anna-Youp
Bertha-Zeger
. . .

Extensions: Matching Residents to Hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

Def. Matching S unstable if there is a hospital h and resident r such that:
- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Application: Matching Residents to Hospitals

NRMP. (National Resident Matching Program)

- Original use just after WWII. ← predates computer usage
- Ides of March, 23,000+ residents.

Rural hospital dilemma.

- Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
- Rural hospitals were under-subscribed in NRMP matching.
- How can we find stable matching that benefits "rural hospitals"?

Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

Lessons Learned

Powerful ideas learned in course.

- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

Potentially deep social ramifications. [legal disclaimer]

1.2 Five Representative Problems

Interval Scheduling

Input. Set of jobs with start times and finish times.
Goal. Find maximum cardinality subset of mutually compatible jobs.

↑
jobs don't overlap
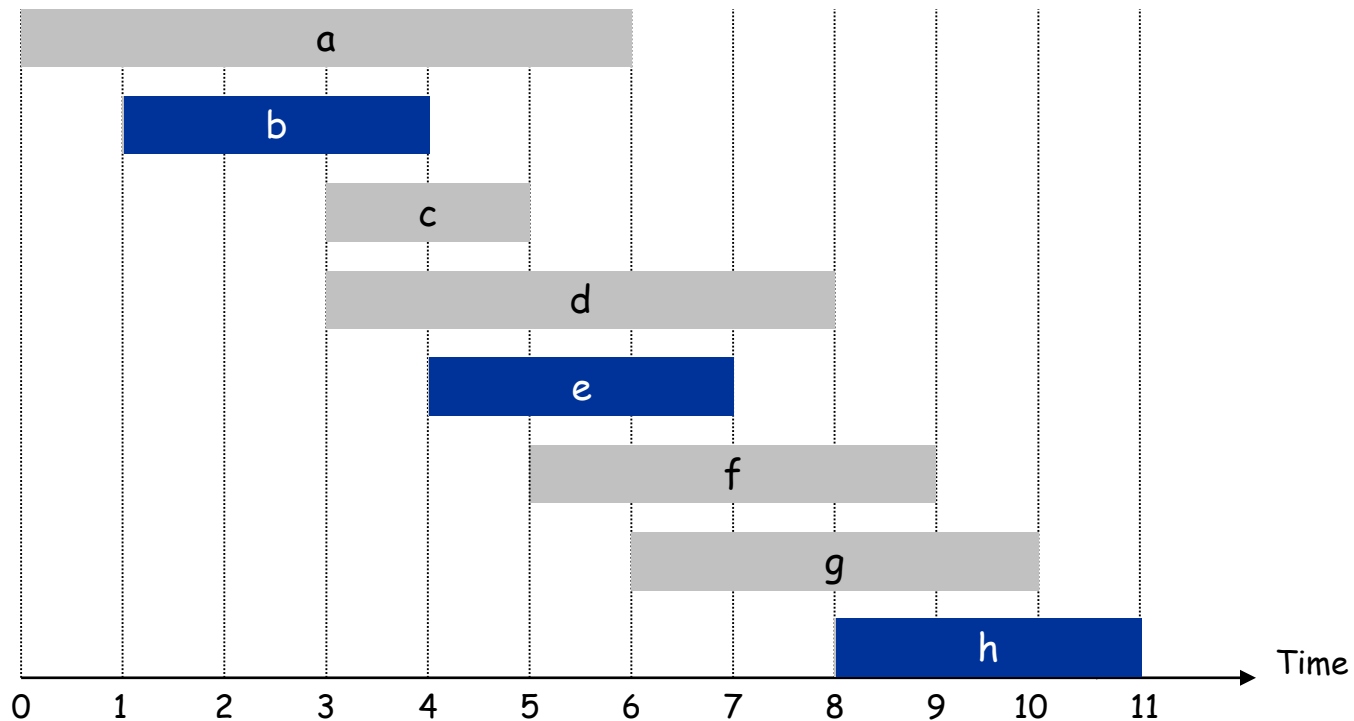
Weighted Interval Scheduling

Input. Set of jobs with start times, finish times, and weights.
Goal. Find maximum weight subset of mutually compatible jobs.
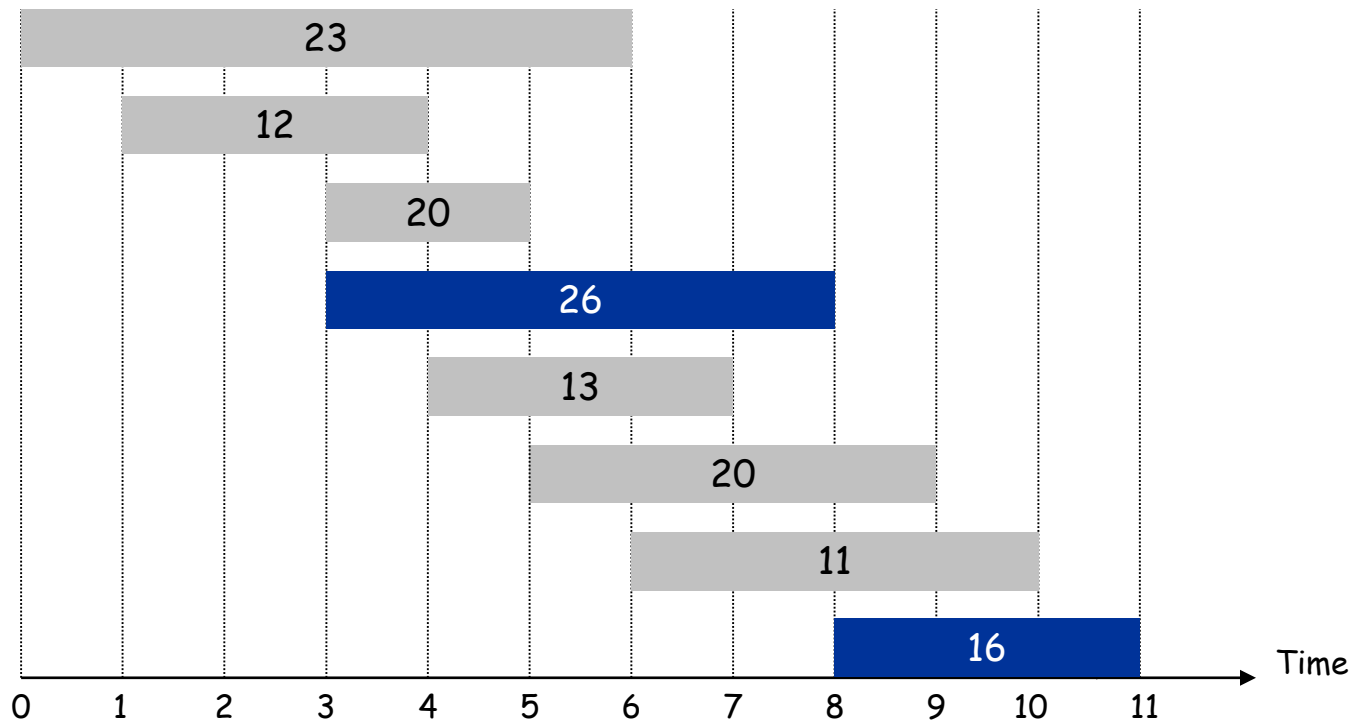
Bipartite Matching

Input. Bipartite graph.
Goal. Find maximum cardinality matching.
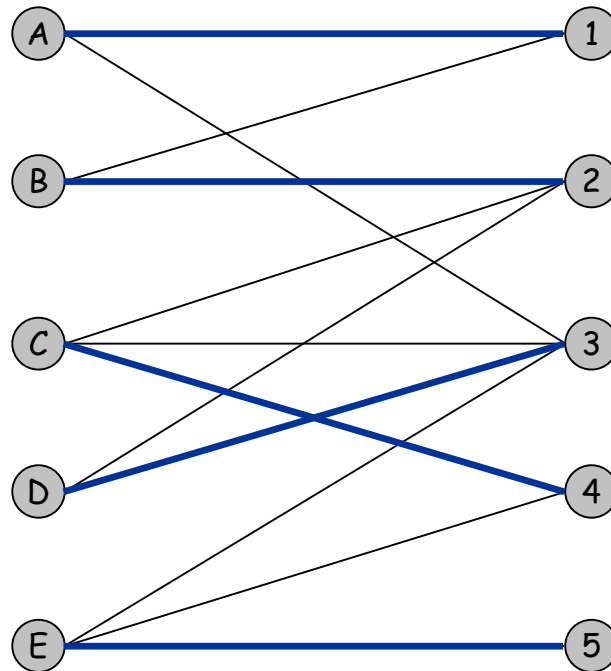
Independent Set

Input. Graph.

Goal. Find maximum cardinality independent set.

↑
subset of nodes such that no two
joined by an edge
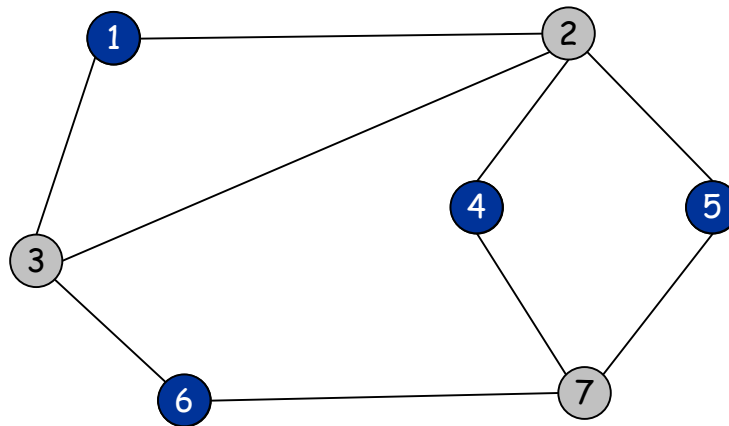
Competitive Facility Location

Input. Graph with weight on each each node.
Game. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a maximum weight subset of nodes.

| 10 | 1 | 5 | 15 | 5 | 1 | 5 | 1 | 15 | 10 |
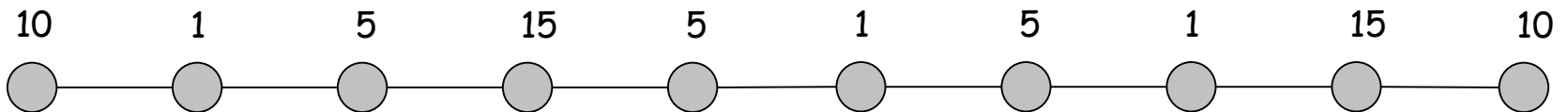
Second player can guarantee 20, but not 25.

Five Representative Problems

Variations on a theme: independent set.

Interval scheduling: n log n greedy algorithm.

Weighted interval scheduling: n log n dynamic programming algorithm.

Bipartite matching: n^k max-flow based algorithm.

Independent set: NP-complete.

Competitive facility location: PSPACE-complete.

Extra Slides

Stable Matching Problem

Goal: Given n men and n women, find a "suitable" matching.
- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

favorite →   least favorite →

	1st	2nd	3rd	4th	5th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Men's Preference List

Stable Matching Problem

Goal: Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	favorite → 1st	2nd	3rd	4th	least favorite → 5th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Women's Preference List

Understanding the Solution

Claim. The man-optimal stable matching is weakly Pareto optimal.

No other perfect matching (stable or unstable) where every man does strictly better

Pf.
- Let A be last woman in some execution of GS algorithm to receive a proposal.
- No man is rejected by A since algorithm terminates when last woman receives first proposal.
- No man matched to A will be strictly better off than in man-optimal stable matching. ▪

Deceit: Machiavelli Meets Gale-Shapley

Q. Can there be an incentive to misrepresent your preference profile?
- Assume you know men's propose-and-reject algorithm will be run.
- Assume that you know the preference profiles of all other participants.

Fact. No, for any man yes, for some women. No mechanism can guarantee a stable matching and be cheatproof.

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Women's True Preference Profile

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

Men's Preference List

	1st	2nd	3rd
Amy	Y	Z	X
Bertha	X	Y	Z
Clare	X	Y	Z

Amy Lies

Lessons Learned

Powerful ideas learned in course.

- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

Potentially deep social ramifications. [legal disclaimer]

- Historically, men propose to women. Why not vice versa?
- Men: propose early and often.
- Men: be more honest.
- Women: ask out the guys.
- Theory can be socially enriching and fun!
- CS majors get the best partners!

Man Optimality

Claim. GS matching S* is man-optimal.

Pf. (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference $\Rightarrow$ some man is rejected by valid partner.
- Let Y be first such man, and let A be first valid woman that rejects him.
- Let S be a stable matching where A and Y are matched.
- When Y is rejected, A forms (or reaffirms) engagement with a man, say Z, whom she prefers to Y.
- Let B be Z's partner in S.
- Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.
- But A prefers Z to Y.
- Thus A-Z is unstable in S. ▪

S

Amy-Yancey
Bertha-Zeus
. . .

↑
since this is first rejection
by a valid partner