

Given Preferences Lists are:

Men:

	I	II	III	IV	V
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

Women:

	I	II	III	IV	V
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

GS Algorithm is:

Initialize each person to be free.

```

while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged and m' to be free
    else
        w rejects m
}

```

Algorithm Tracing:

numberOfProposals: 0

Pairs Made: []

Free Men: [V, W, X, Y, Z]

Yet to Propose List of Men in Order of Preference:

```

{
    V: [A, B, C, D, E],
    W: [B, C, D, A, E],
    X: [C, D, A, B, E],
    Y: [D, A, B, C, E],
    Z: [A, B, C, D, E]
}

```

Step-I:

Choose a Man m who is free and hasn't proposed to every woman

m = V

w = 1st woman on m's list to whom m has not yet proposed

w = A

V proposes A, A is free. So, V and A are paired.

Increase numberOfProposals by 1.

Remove V from Free Men List and A from yet to propose list of V.

Output after Step-I:

numberOfProposals: 1

Pairs Made: [V: A]

Free Men: [W, X, Y, Z]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [B, C, D, E],
    W: [B, C, D, A, E],
    X: [C, D, A, B, E],
    Y: [D, A, B, C, E],
    Z: [A, B, C, D, E]
}
```

Step-II:

Choose a Man m who is free and hasn't proposed to every woman

m = W

w = 1st woman on m's list to whom m has not yet proposed

w = B

W proposes B, B is free. So, W and B are paired.

Increase numberOfProposals by 1.

Remove W from Free Men List and B from yet to propose list of W.

Output after Step-II:

numberOfProposals: 2

Pairs Made: [V: A, W: B]

Free Men: [X, Y, Z]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [B, C, D, E],
    W: [C, D, A, E],
    X: [C, D, A, B, E],
    Y: [D, A, B, C, E],
    Z: [A, B, C, D, E]
}
```

Step-III:

Choose a Man m who is free and hasn't proposed to every woman

m = X

w = 1st woman on m's list to whom m has not yet proposed

w = C

X proposes C, C is free. So, X and C are paired.

Increase numberOfProposals by 1.

Remove X from Free Men List and C from yet to propose list of X.

Output after Step-III:

numberOfProposals: 3

Pairs Made: [V: A, W: B, X: C]

Free Men: [Y, Z]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [B, C, D, E],
    W: [C, D, A, E],
    X: [D, A, B, E],
    Y: [D, A, B, C, E],
    Z: [A, B, C, D, E]
}
```

Step-IV:

Choose a Man m who is free and hasn't proposed to every woman

m = Y

w = 1st woman on m's list to whom m has not yet proposed

w = D

Y proposes D, D is free. So, Y and D are paired.

Increase numberOfProposals by 1.

Remove Y from Free Men List and D from yet to propose list of Y.

Output after Step-IV:

numberOfProposals: 4

Pairs Made: [V: A, W: B, X: C, Y: D]

Free Men: [Z]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [B, C, D, E],
    W: [C, D, A, E],
    X: [D, A, B, E],
    Y: [A, B, C, E],
    Z: [A, B, C, D, E]
}
```

Step-V:

Choose a Man m who is free and hasn't proposed to every woman

m = Z

w = 1st woman on m's list to whom m has not yet proposed

w = A

Z proposes A. A is paired with V but she prefers Z over V.

Increase numberOfProposals by 1.

A trades up from V to Z.

Remove Z from Free Men List and A from yet to propose list of Z. Add V to Free Men List

Output after Step-V:

numberOfProposals: 5

Pairs Made: [W: B, X: C, Y: D, Z: A]

Free Men: [V]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [B, C, D, E],
    W: [C, D, A, E],
    X: [D, A, B, E],
    Y: [A, B, C, E],
    Z: [B, C, D, E]
}
```

Step-VI:

Choose a Man m who is free and hasn't proposed to every woman

m = V

w = 1st woman on m's list to whom m has not yet proposed

w = B

V proposes B. B is paired with W but prefers V over W.

Increase numberOfProposals by 1.

B trades up from W to V

Remove V from Free Men List and B from yet to propose list of V. Add W to Free Men List

Output after Step-VI:

numberOfProposals: 6

Pairs Made: [X: C, Y: D, Z: A, V: B]

Free Men: [W]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [C, D, E],
    W: [C, D, A, E],
    X: [D, A, B, E],
    Y: [A, B, C, E],
    Z: [B, C, D, E]
}
```

Step-VII:

Choose a Man m who is free and hasn't proposed to every woman

m = W

w = 1st woman on m's list to whom m has not yet proposed

w = C

W proposes C. C is paired with X but prefers W over X.

Increase numberOfProposals by 1.

C trades up from X to W

Remove W from Free Men List and C from yet to propose list of W. Add X to Free Men List

Output after Step-VII:

numberOfProposals: 7

Pairs Made: [Y: D, Z: A, V: B, W: C]

Free Men: [X]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [C, D, E],
    W: [D, A, E],
}
```

```

        X: [D, A, B, E],
        Y: [A, B, C, E],
        Z: [B, C, D, E]
    }

```

Step-VIII:

Choose a Man m who is free and hasn't proposed to every woman

m = X

w = 1st woman on m's list to whom m has not yet proposed

w = D

X proposes D. D is paired with Y but prefers X over Y.

Increase numberOfProposals by 1.

D trades up from Y to X

Remove X from Free Men List and D from yet to propose list of X. Add Y to Free Men List

Output after Step-VIII:

numberOfProposals: 8

Pairs Made: [Z: A, V: B, W: C, X: D]

Free Men: [Y]

Yet to Propose List of Men in Order of Preference:

```

    {
        V: [C, D, E],
        W: [D, A, E],
        X: [A, B, E],
        Y: [A, B, C, E],
        Z: [B, C, D, E]
    }

```

Step-IX:

Choose a Man m who is free and hasn't proposed to every woman

m = Y

w = 1st woman on m's list to whom m has not yet proposed

w = A

Y proposes A. A is paired with Z but prefers Y over Z.

Increase numberOfProposals by 1.

A trades up from Z to Y

Remove Y from Free Men List and A from yet to propose list of Y. Add Z to Free Men List

Output after Step-IX:

numberOfProposals: 9

Pairs Made: [V: B, W: C, X: D, Y: A]

Free Men: [Z]

Yet to Propose List of Men in Order of Preference:

```

    {
        V: [C, D, E],
        W: [D, A, E],
        X: [A, B, E],
        Y: [B, C, E],
        Z: [B, C, D, E]
    }

```

Step-X:

Choose a Man m who is free and hasn't proposed to every woman

m = Z

w = 1st woman on m's list to whom m has not yet proposed

w = B

Z proposes B. B is paired with V but prefers Z over V.

Increase numberOfProposals by 1.

B trades up from V to Z

Remove Z from Free Men List and B from yet to propose list of Z. Add V to Free Men List

Output after Step-X:

numberOfProposals: 10

Pairs Made: [W: C, X: D, Y: A, Z: B]

Free Men: [V]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [C, D, E],
    W: [D, A, E],
    X: [A, B, E],
    Y: [B, C, E],
    Z: [C, D, E]
}
```

Step-XI:

Choose a Man m who is free and hasn't proposed to every woman

m = V

w = 1st woman on m's list to whom m has not yet proposed

w = C

V proposes C. C is paired with W but prefers V over W.

Increase numberOfProposals by 1.

C trades up from W to V

Remove V from Free Men List and C from yet to propose list of V. Add W to Free Men List

Output after Step-XI:

numberOfProposals: 11

Pairs Made: [X: D, Y: A, Z: B, V: C]

Free Men: [W]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [D, E],
    W: [D, A, E],
    X: [A, B, E],
    Y: [B, C, E],
    Z: [C, D, E]
}
```

Step-XII:

Choose a Man m who is free and hasn't proposed to every woman

m = W

w = 1st woman on m's list to whom m has not yet proposed

w = D

W proposes D. D is paired with X but prefers W over X.

Increase numberOfProposals by 1.

D trades up from X to W

Remove W from Free Men List and D from yet to propose list of V. Add X to Free Men List

Output after Step-XII:

numberOfProposals: 12

Pairs Made: [Y: A, Z: B, V: C, W: D]

Free Men: [X]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [D, E],
    W: [A, E],
    X: [A, B, E],
    Y: [B, C, E],
    Z: [C, D, E]
}
```

Step-XIII:

Choose a Man m who is free and hasn't proposed to every woman

m = X

w = 1st woman on m's list to whom m has not yet proposed

w = A

X proposes A. A is paired with Y but prefers X over Y.

Increase numberOfProposals by 1.

A trades up from Y to X

Remove X from Free Men List and A from yet to propose list of X. Add Y to Free Men List

Output after Step-XIII:

numberOfProposals: 13

Pairs Made: [Z: B, V: C, W: D, X: A]

Free Men: [Y]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [D, E],
    W: [A, E],
    X: [B, E],
    Y: [B, C, E],
    Z: [C, D, E]
}
```

Step-XIV:

Choose a Man m who is free and hasn't proposed to every woman

m = Y

w = 1st woman on m's list to whom m has not yet proposed

w = B

Y proposes B. B is paired with Z but prefers Y over Z.

Increase numberOfProposals by 1.

B trades up from Z to Y

Remove Y from Free Men List and B from yet to propose list of Y. Add Z to Free Men List

Output after Step-XIV:

numberOfProposals: 14

Pairs Made: [V: C, W: D, X: A, Y: B]

Free Men: [Z]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [D, E],
    W: [A, E],
    X: [B, E],
    Y: [C, E],
    Z: [C, D, E]
}
```

Step-XV:

Choose a Man m who is free and hasn't proposed to every woman

m = Z

w = 1st woman on m's list to whom m has not yet proposed

w = C

Z proposes C. C is paired with V but prefers Z over V.

Increase numberOfProposals by 1.

C trades up from V to Z

Remove Z from Free Men List and C from yet to propose list of Z. Add V to Free Men List

Output after Step-XV:

numberOfProposals: 15

Pairs Made: [W: D, X: A, Y: B, Z: C]

Free Men: [V]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [D, E],
    W: [A, E],
    X: [B, E],
    Y: [C, E],
    Z: [D, E]
}
```

Step-XVI:

Choose a Man m who is free and hasn't proposed to every woman

m = V

w = 1st woman on m's list to whom m has not yet proposed

w = D

V proposes D. D is paired with W but prefers V over W.

Increase numberOfProposals by 1.

D trades up from W to V

Remove V from Free Men List and D from yet to propose list of V. Add W to Free Men List

Output after Step-XVI:

numberOfProposals: 16

Pairs Made: [X: A, Y: B, Z: C, V: D]

Free Men: [W]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [E],
    W: [A, E],
    X: [B, E],
    Y: [C, E],
    Z: [D, E]
}
```

Step-XVII:

Choose a Man m who is free and hasn't proposed to every woman

m = W

w = 1st woman on m's list to whom m has not yet proposed

w = A

W proposes A. A is paired with X but prefers W over X.

Increase numberOfProposals by 1.

A trades up from X to W

Remove W from Free Men List and A from yet to propose list of W. Add X to Free Men List

Output after Step-XVII:

numberOfProposals: 17

Pairs Made: [Y: B, Z: C, V: D, W: A]

Free Men: [X]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [E],
    W: [E],
    X: [B, E],
    Y: [C, E],
    Z: [D, E]
}
```

Step-XVIII:

Choose a Man m who is free and hasn't proposed to every woman

m = X

w = 1st woman on m's list to whom m has not yet proposed

w = B

X proposes B. B is paired with Y but prefers X over Y.

Increase numberOfProposals by 1.

B trades up from Y to X

Remove X from Free Men List and B from yet to propose list of X. Add Y to Free Men List

Output after Step-XVIII:

numberOfProposals: 18

Pairs Made: [Z: C, V: D, W: A, X: B]

Free Men: [Y]

Yet to Propose List of Men in Order of Preference:

```
{
    V: [E],
    W: [E],
    X: [E],

```

```

        Y: [C, E],
        Z: [D, E]
    }

```

Step-XIX:

Choose a Man m who is free and hasn't proposed to every woman

m = Y

w = 1st woman on m's list to whom m has not yet proposed

w = C

Y proposes C. C is paired with Z but prefers Y over Z.

Increase numberOfProposals by 1.

C trades up from Z to Y

Remove Y from Free Men List and C from yet to propose list of Y. Add Z to Free Men List

Output after Step-XIX:

numberOfProposals: 19

Pairs Made: [V: D, W: A, X: B, Y: C]

Free Men: [Z]

Yet to Propose List of Men in Order of Preference:

```

    {
        V: [E],
        W: [E],
        X: [E],
        Y: [E],
        Z: [D, E]
    }

```

Step-XX:

Choose a Man m who is free and hasn't proposed to every woman

m = Z

w = 1st woman on m's list to whom m has not yet proposed

w = D

Z proposes D. D is paired with V but prefers Z over V.

Increase numberOfProposals by 1.

D trades up from V to Z

Remove Z from Free Men List and D from yet to propose list of Z. Add V to Free Men List

Output after Step-XX:

numberOfProposals: 20

Pairs Made: [W: A, X: B, Y: C, Z: D]

Free Men: [V]

Yet to Propose List of Men in Order of Preference:

```

    {
        V: [E],
        W: [E],
        X: [E],
        Y: [E],
        Z: [E]
    }

```

Step-XXI:

Choose a Man m who is free and hasn't proposed to every woman

m = V

w = 1st woman on m's list to whom m has not yet proposed

w = E

V proposes E. E is free. So, V & E are paired.

Increase numberOfProposals by 1.

Remove V from Free Men List and E from yet to propose list of V.

Output after Step-XXI:

numberOfProposals: 21

Pairs Made: [W: A, X: B, Y: C, Z: D, V: E]

Free Men: []

Yet to Propose List of Men in Order of Preference:

```
{  
    V: [],  
    W: [E],  
    X: [E],  
    Y: [E],  
    Z: [E]  
}
```

Algorithm ends because free men list is empty.

The Stable Matching pairs formed by GS Algorithm are [W: A, X: B, Y: C, Z: D, V: E]

The Number of proposal occurred are 21

Algorithm Analysis:

Step	Cost	Number of times it runs
Initialize each person to be free.	C1	1
while(some man is free and hasn't proposed to every woman)	C2	$n * (n - 1)$
Choose such a man m	C3	$n * (n - 1)$
w = 1 st woman on m's list to whom m has not yet proposed	C4	$n * (n - 1)$
if (w is free) assign m and w to be engaged	C5	$n * (n - 1)$
else if (w prefers m to her fiancé m') assign m and w to be engaged and m' to be free	C6	$n * (n - 1)$
else w rejects m	C7	$n * (n - 1)$

$$f(n) = (C2 + C3 + C4 + C5 + C6 + C7) * n^2 - (C2 + C3 + C4 + C5 + C6 + C7) * n + C1$$

$$\begin{aligned} \text{Let } g(n) &= 3 * (C1 + C2 + C3 + C4 + C5 + C6 + C7) * n^2 \\ &= (C1 + C2 + C3 + C4 + C5 + C6 + C7) * n^2 + (C1 + C2 + C3 + C4 + C5 + C6 + C7) * \\ &n^2 + (C1 + C2 + C3 + C4 + C5 + C6 + C7) * n^2 \end{aligned}$$

We can clearly say that

- 1) $(C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n^2 \leq (C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n^2$ for all $n \geq 1$ since right hand side has C_1 which is always ≥ 1
- 2) $-(C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n \leq (C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n^2$ for all $n \geq 1$ since left hand side is a negative value whereas right hand side is a positive value
- 3) $C_1 \leq (C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n^2$ for all $n \geq 1$ since left hand side is a constant and right hand side varies with n which is ≥ 1 .

If we add all the above 3, the invariant won't change. So,

$$(C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n^2 - (C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n + C_1 \leq (C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n^2 + (C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n^2 + (C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n^2$$

$$\implies f(n) \leq 3 * (C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7) * n^2 \text{ for all } n \geq 1$$

According to Upper Bound Definition, $f(n) = O(g(n))$ if there exists a positive constants C, n_0

where $f(n) \leq C * g(n)$ for all $n \geq n_0$

Therefore, $f(n) = O(n^2)$ where $C = 3 * (C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7)$ and $n_0 = 1$