N-th Fibonacci Number

$$F_n = F_{n-1} + F_{n-2}$$

$$F_0 = 0, F_1 = 1$$

Algorithms for Computation of N-th Fibonacci Number

Since Fibonacci Number is defined in a recursive fashion, can we write a recursive algorithm for it?

What is recursion function?

Algorithms for Computation of N-th Fibonacci Number

```
Algorithm 1:

function xyz1(n)
     if (n<2) then return n
     else return xyz1(n-1)+
                 xyz1(n-2);
```

Time complexity?

Algorithms for Computation of N-th Fibonacci Number

Can we do better? How?

Can we do better?

What if we keep two pointers and keep moving the pointers?

```
Algorithm 2:

function xyz2(n)
    i:=1; j:=0;
    for k:=1 to n do
           j:=i+j;
           i:=j-i;
    return j;
```

What is the complexity of this algorithm?

Algorithms for Computation of N-th Fibonacci Number

Can we do even better?

How?

Can we do even better?

```
Algorithm 1:

function xyz1(n)
     if (n<2) then return n
     else return xyz1(n-1)+
                 xyz1(n-2);
```

```
Algorithm 2:

function xyz2(n)
    i:=1; j:=0;
    for k:=1 to n do
           j:=i+j;
           i:=j-i;
    return j;
```

```
Algorithm 3:

function xyz3(n)
   i:=1; j:=0;
   k:=0; h:=1;
   while n>0 do
      if (n is odd) then
         t:=jh;
         j:=ih + jk + t;
         i:=ik + t;
      t:=square(h);
      h:=2kh + t;
      k:=square(k) + t;
      n:= n div 2;
   return j;
```

How to Compute Fibonacci Number in O(log n) time?

- Transform Fibonacci number computation problem to a matrix chain multiplication problem.
- Matrix Chain Multiplication Problem
  - $P = A_1 * A_2 * A_3 * \ldots * A_p$, where A_i is an n x n matrix.
  - $A_1 * A_2$, where A_i is an n x n matrix, can be done in $O(n^3)$ complexity.
  - If dimensions of A_1 & A_2 are constant, the product $A_1 * A_2$ can be done in constant time.
  - The matrix chain $A_1 * A_2 * \ldots * A_n$, where $A_1 = A_2 = \ldots = A_n$, can be computed in O(log n) time.

Computation of the n-th Fibonacci Number, F_n

$$\begin{bmatrix} F_{n-1} & F_n \end{bmatrix}$$

$$= \begin{bmatrix} F_{n-1} & F_{n-1} + F_{n-2} \end{bmatrix}$$

$$= \begin{bmatrix} F_{n-2} & F_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} F_{n-2} & F_{n-1} \end{bmatrix} * A, \text{ where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} F_{n-3} & F_{n-2} \end{bmatrix} * A * A$$

$$= \begin{bmatrix} F_{n-3} & F_{n-2} \end{bmatrix} * A^2$$

$$= [F_{n-4} \quad F_{n-3}] * A^3$$

....

....

$$= [F_0 \quad F_1] * A^{n-1}$$

$$= [F_0 \quad F_1] * A^{-1} * A * A^{n-1}$$

$$= [0 \quad 1] \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} A^n$$

$$= [0 \quad 1] \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} A^n$$

$$= [1 \quad 0] * A^n$$

Hence,

$[F_{n-1} \quad F_n] = x * A^n$,

where $x = [1 \quad 0]$

and $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^4 = A^2.A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^8 = A^4.A^4 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix}$$

Product is of the form $\begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 + x_2 \end{bmatrix}$

In the algorithm $x_1 = k,\ x_2 = h$

$$\begin{bmatrix} k & h \\ h & k+h \end{bmatrix}\begin{bmatrix} k & h \\ h & k+h \end{bmatrix} = \begin{bmatrix} k' & h' \\ h' & k'+h' \end{bmatrix}$$

where, $k' = k^2 + h^2$

$$h' = kh + kh + h^2 = 2kh + h^2$$

$$\begin{bmatrix} i & j \end{bmatrix}\begin{bmatrix} k & h \\ h & k+h \end{bmatrix} = \begin{bmatrix} i' & j' \end{bmatrix}$$

$i' = ik + jh$

$j' = ih + jk + jh$

Example 1: Computation of F_7

$[F_6 \quad F_7] = x * A^7$

$n = 7$	$n = 3$	$n = 1$
$x' = x\,A$ $A' = A^2$	$x'' = x' \,.\, A'$ $= x\,A \,.\, A^2$ $= x\,A^3$ $A'' = A'^{\,2}$ $= A^4$	$x''' = x'' \,.\, A''$ $= x\,A^3 \,.\, A^4$ $= x\,A^7$ $A''' = A''^{\,2}$ $= A^8$

$$x''' = x\,A^7$$

Example 2: Computation of F_8

$$[\,F_7 \quad F_8] = x \,.\, A^8$$

n = 8	n = 4	n = 2	n = 1
$A' = A^2$	$A'' = A'^{\,2}$ $= A^4$	$A''' = A''^{\,2}$ $= A^8$	$x' = x \,.\, A'''$ $= x \,.\, A^8$ $A'''' = A'''^{\,2}$

$$x' = x\, A^8$$