Question 1.

We are given as input a set of n jobs, where job j has a processing time p_j, a deadline d_j. Given a schedule (i.e., an ordering of the jobs), consider that each job j has the completion time C_j; we define the lateness l_j of job j as the amount of time $C_j - d_j$ after its deadline that the job completes, or as 0 if $C_j \leq d_j$. Our goal is to minimize the maximum lateness, $\max l_j$. Consider the following greedy rules for producing an ordering that minimizes the maximum lateness. For each rule, please explain why it gives the optimal ordering or give a counterexample. You can assume that all processing times and deadlines are distinct.

(a) Schedule the requests in increasing order of processing time p_j.

(b) Schedule the requests in increasing order of the product $d_j \times p_j$.

(c) Schedule the requests in increasing order of deadline d_j.

Answer:

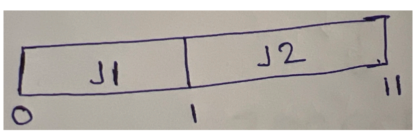
We are given a set of n jobs where job j has a processing time p_j and deadline d_j. If a job j starts at s_j it finishes at $f_j = s_j + t_j$. We define lateness l_j of a job j as the max$\{0, f_j - d_j\}$. Our goal is to minimize the maximum lateness i.e; minimize max(l_1, l_2 …. l_n).

a) Schedule the requests in increasing order of processing time p_j.

This is not an optimal algorithm. Consider the below counterexample,

Jobs	pj	dj
J1	1	100
J2	10	10

If we follow this rule and schedule requests in increasing order of p_j we get,



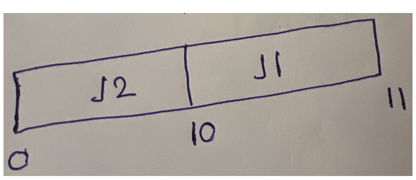
Since J1 has processing time (1) < J2 processing time (10)

Lateness of J1 = max(0, 1 - 100) = max(0, -99) = 0

Lateness of J2 = max(0, 11 - 10) = max(0, 1) = 1

Maximum Lateness = max(0, 1) = 1

Let's consider another scheduling order:



Lateness of J1 = max(0, 11 - 100) = max(0, -89) = 0

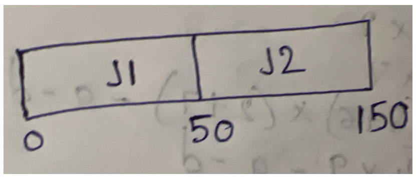
Lateness of J2 = max(0, 10 - 10) = max(0, 0) = 0

Maximum Lateness = max(0, 0) = 0

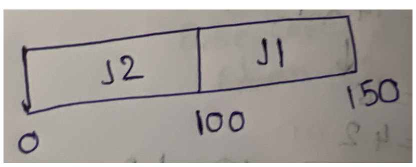
Since we are able to get better maximum lateness using another rule than the given rule, this rule is not an optimal solution for the given problem.
b) Schedule the requests in increasing order of product dj x pj.
This is not an optimal algorithm. Consider the below counterexample,

Jobs	pj	dj
J1	50	125
J2	100	100

If we follow this rule and schedule requests in increasing order of p_j x dj we get,



Since J1 has pj x dj (50 x 125 = 6250) < J2 pj x dj (100 x 100 = 10000)
Lateness of J1 = max(0, 50 - 125) = max(0, -75) = 0
Lateness of J2 = max(0, 150 - 100) = max(0, 50) = 50
Maximum Lateness = max(0, 50) = 50
Let's consider another scheduling order:



Lateness of J1 = max(0, 150 - 125) = max(0, 25) = 25
Lateness of J2 = max(0, 100 - 100) = max(0, 0) = 0
Maximum Lateness = max(0, 25) = 25
Since we are able to get better maximum lateness using another rule than the given rule, this rule is not an optimal solution for the given problem.
c) Schedule the requests in increasing order of processing time d_j.
This is the optimal algorithm for this problem. This algorithm is a greedy algorithm.
Observation1: There exists an optimal solution with no idle time.
Observation2: This greedy algorithm has no idle time.
Definition1: An inversion in a Schedule S is a pair of jobs i and j such that deadline of i < deadline of j but j scheduled before i.
Observation3: This greedy algorithm has no inversions.
Observation4: If a schedule with no idle time has an inversion, then it has at least one pair of inverted jobs scheduled consecutively.
Observation5: Swapping 2 adjacent, inverted jobs reduces the number of inversions by one and doesn't increase the max lateness.
Claim: This greedy algorithm is optimal.
Proof by Contradiction:

Let S’ is an optimal schedule with fewer number of inversions than the schedule S given by greedy algorithm.

We can assume S’ has no idle time.

If S’ has no inversions, then S’ = S.

If S’ has an inversion, let i-j be an adjacent inverted pair.

Swapping i and j doesn’t increase the max lateness and strictly decreases number of inversions.

This contradicts definition of S’

Therefore by contradiction, schedule S given by the greedy algorithm is the optimal solution.

Algorithm:

```
Sort Jobs by dj.
t = 0
for j = 1 to n{
    Assign job j to interval [t, t + tj]
    sj = t
    fj = t + tj
    t = t + tj
    Output interval [sj, fj]
}
```

Question 2.

Explain Karatsuba’s algorithm first. Then use it to solve the following multiplication problem. Show all your work. Apply recursive calls of the multiplication function until the base case, where each number has only one digit.

X= 86

Y= 27

XY=?

Answer:

The Karatsuba algorithm is a fast multiplication algorithm that uses a divide and conquer approach to multiply two numbers. Being able to multiply numbers quickly is very important since computer scientists often consider multiplication to be a constant time O(1) operation. But for large numbers the actual time complexity using naive algorithm is $O(n^2)$. The Karatsuba’s Algorithm reduces the number of subproblems to three as opposed to four in naive algorithm. In naive algorithm,

$$x * y = xH * yH * 10^n + (xH * yL + xL * yH) * 10^{(n/2)} + xL * yL$$

Karatsuba reduced number of subproblems from 4 to 3 by changing the middle part in the above equation i.e; (xH * yL + xL * yH)

Let a = xH * yH

d = xL * yL

Then we can rewrite (xH * yL + xL * yH) as (xH + xL) * (yH + yL) − a − d

Let e = (xH + xL) * (yH + yL) − a − d

Now,

$$x * y = a * 10^n + e * 10^{(n/2)} + d$$

Here the recursive calls are, xH * yH, xL * yL and (xH + xL) * (yH + yL)

The recurrence relation is, T(n) = 3 * T(n / 2) + O(n) and T(1) = O(1)

$a = 3$, $b = 2$ and $f(n) = O(n)$
$\log_b a = \log_2 3 = 1.585$
$f(n) = O(n) = O(n^1)$ ===> $d = 1$
Since $d < \log_b a$ the given recurrence relation follows Case-I of Master's Theorem
According to Case-I of Master's Theorem, if $d < \log_b a$, the given function grows slower than $n^{\log_b a}$. Therefore, $T(n) = \text{Theta}(n^{\log_b a})$

So, $T(n) = \text{Theta}(n^{1.585})$

Given, $X = 86$, $Y = 27$
Algorithm Tracing:

86×27
↓
$a = 8 \times 2$
$d = 6 \times 7$
$e = (8+6) \times (2+7) - a - d$
$= 14 \times 9 - a - d$

8×2
Base Case
Return 16

6×7
Base Case
Return 42

14×9
↓
$a = 1 \times 0$
$d = 4 \times 9$
$e = (1+4) \times (0+9) - a - d$
$= 5 \times 9 - a - d$

1×0
Base Case
Return 0

4×9
Base Case
Return 36

5×9
Base Case
Return 45

14×9

$\downarrow$

$a = 1 \times 0$
$d = 4 \times 9$
$e = (1+4) \times (0+9) - a - d$
$= 5 \times 9 - a - d$

$\downarrow$

$a = 0$
$d = 36$
$e = 45 - 0 - 36 = 9$

$\downarrow$

Return $0 \times 10^2 + 9 \times 10^1 + 36$
$= 126$

86×27

$\downarrow$

$a = 8 \times 2$

$d = 6 \times 7$

$e = (8+6) \times (2+7) - a - d$

$= 14 \times 9 - a - d$

$\downarrow$

$a = 16$

$d = 42$

$e = 126 - 16 - 42$

$= 68$

$\downarrow$

Return $16 \times 10^2 + 68 \times 10^1 + 42$

$= 2322$

Question 3.
Use Strassen Matrix Multiplication algorithm to multiply the following matrices:

$$A=\begin{bmatrix}1 & 2 & 1 & 4\\ 2 & 1 & 3 & 2\\ 3 & 1 & 2 & 4\\ 1 & 3 & 1 & 2\end{bmatrix},\ B=\begin{bmatrix}2 & 6 & 3 & 4\\ 2 & 1 & 1 & 2\\ 3 & 1 & 2 & 1\\ 2 & 2 & 1 & 2\end{bmatrix}$$

Answer:
Naive Divide and Conquer method to multiply 2 matrices involves 8 multiplications. Strassen reduced those multiplications from 8 to 7. The multiplications are

$$\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix} \times \begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}$$

$$\begin{aligned}
P_1 &= A_{11} \times (B_{12} - B_{22})\\
P_2 &= (A_{11} + A_{12}) \times B_{22}\\
P_3 &= (A_{21} + A_{22}) \times B_{11}\\
P_4 &= A_{22} \times (B_{21} - B_{11})\\
P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22})\\
P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22})\\
P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12})
\end{aligned}$$

$$\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6\\
C_{12} &= P_1 + P_2\\
C_{21} &= P_3 + P_4\\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}$$

Algorithm is

- Divide: partition A and B into n/2-by-n/2 blocks.
- Compute: 14 n/2-by-n/2 matrices via 10 matrix additions.
- Conquer: multiply 7 n/2-by-n/2 matrices recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

We reach base case when each matrix is of size 1-by-1. The recurrence relation for Strassen's Matrix Multiplication Algorithm is

$T(n) = 7 * T(n / 2) + O(n^2)$
$a = 7$, $b = 2$ and $f(n) = O(n^2)$
$\log_b a = \log_2 7 = 2.807$
$f(n) = O(n) = O(n2)$ ===> $d = 2$
Since $d < \log_b a$ the given recurrence relation follows Case-I of Master's Theorem
According to Case-I of Master's Theorem, if $d < \log_b a$, the given function grows slower than $n^{\log_b a}$. Therefore, $T(n) = \Theta(n^{\log_b a})$

So, $T(n) = \Theta(n^{2.807})$

Given,

$$A=\begin{bmatrix}1 & 2 & 1 & 4\\ 2 & 1 & 3 & 2\\ 3 & 1 & 2 & 4\\ 1 & 3 & 1 & 2\end{bmatrix},\ B=\begin{bmatrix}2 & 6 & 3 & 4\\ 2 & 1 & 1 & 2\\ 3 & 1 & 2 & 1\\ 2 & 2 & 1 & 2\end{bmatrix}$$

$$A=\begin{bmatrix}1 & 2 & 1 & 4\\ 2 & 1 & 3 & 2\\ 3 & 1 & 2 & 4\\ 1 & 3 & 1 & 2\end{bmatrix},\quad B=\begin{bmatrix}2 & 6 & 3 & 4\\ 2 & 1 & 1 & 2\\ 3 & 1 & 2 & 1\\ 2 & 2 & 1 & 2\end{bmatrix}$$

$$A_{11}=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\qquad B_{11}=\begin{bmatrix}2 & 6\\ 2 & 1\end{bmatrix}$$

$$A_{12}=\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}\qquad B_{12}=\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}$$

$$A_{21}=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\qquad B_{21}=\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}$$

$$A_{22}=\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}\qquad B_{22}=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$P_1=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\times\left(\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}-\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\times\begin{bmatrix}1 & 3\\ 0 & 0\end{bmatrix}$$

$$P_2=\left(\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}+\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}\right)\times\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}2 & 6\\ 5 & 3\end{bmatrix}\times\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$P_3=\left(\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}+\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}\right)\times\begin{bmatrix}2 & 6\\ 2 & 1\end{bmatrix}=\begin{bmatrix}5 & 5\\ 2 & 5\end{bmatrix}\times\begin{bmatrix}2 & 6\\ 2 & 1\end{bmatrix}$$

$$P_4=\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}\times\left(\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}-\begin{bmatrix}2 & 6\\ 2 & 1\end{bmatrix}\right)=\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}\times\begin{bmatrix}1 & -5\\ 0 & 1\end{bmatrix}$$

$$P_5=\left(\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}+\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}\right)\times\left(\begin{bmatrix}2 & 6\\ 2 & 1\end{bmatrix}+\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}3 & 6\\ 3 & 3\end{bmatrix}\times\begin{bmatrix}4 & 7\\ 3 & 3\end{bmatrix}$$

$$P_6=\left(\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}-\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}\right)\times\left(\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}+\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}-1 & 0\\ 2 & 0\end{bmatrix}\times\begin{bmatrix}5 & 2\\ 3 & 4\end{bmatrix}$$

$$P_7=\left(\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}-\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\right)\times\left(\begin{bmatrix}2 & 6\\ 2 & 1\end{bmatrix}+\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\times\begin{bmatrix}5 & 10\\ 3 & 3\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} \times \begin{bmatrix}1 & 3\\ 0 & 0\end{bmatrix}$$

$A_{11} = [1]$ $\quad B_{11} = [1]$
$A_{12} = [2]$ $\quad B_{12} = [3]$
$A_{21} = [2]$ $\quad B_{21} = [0]$
$A_{22} = [1]$ $\quad B_{22} = [0]$

$$P_1 = [1] \times ([3] - [0]) = [1] \times [3] = [3]$$
$$P_2 = ([1] + [2]) \times [0] = [3] \times [0] = [0]$$
$$P_3 = ([2] + [1]) \times [1] = [3] \times [1] = [3]$$
$$P_4 = [1] \times ([0] - [1]) = [1] \times [-1] = [-1]$$
$$P_5 = ([1] + [1]) \times ([1] + [0]) = [2] \times [1] = [2]$$
$$P_6 = ([2] - [1]) \times ([0] + [0]) = [1] \times [0] = [0]$$
$$P_7 = ([1] - [2]) \times ([1] + [3]) = [-1] \times [4] = [-4]$$

$\downarrow$

$$C_{11} = [2] + [-1] - [0] + [0] = [1]$$
$$C_{12} = [3] + [0] = [3]$$
$$C_{21} = [3] + [-1] = [2]$$
$$C_{22} = [2] + [3] - [3] - [-4] = [6]$$

$\downarrow$

$$\begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix}$$

$$\begin{bmatrix}2 & 6\\ 5 & 3\end{bmatrix} \times \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$A_{11} = [2]$ $\quad B_{11} = [2]$
$A_{12} = [6]$ $\quad B_{12} = [1]$
$A_{21} = [5]$ $\quad B_{21} = [1]$
$A_{22} = [3]$ $\quad B_{22} = [2]$

$$P_1 = [2] \times ([1] - [2]) = [2] \times [-1] = [-2]$$
$$P_2 = ([2] + [6]) \times [2] = [8] \times [2] = [16]$$
$$P_3 = ([5] + [3]) \times [2] = [8] \times [2] = [16]$$
$$P_4 = [3] \times ([1] - [2]) = [3] \times [-1] = [-3]$$
$$P_5 = ([2] + [3]) \times ([2] + [2]) = [5] \times [4] = [20]$$
$$P_6 = ([6] - [3]) \times ([1] + [2]) = [3] \times [3] = [9]$$
$$P_7 = ([2] - [5]) \times ([2] + [1]) = [-3] \times [3] = [-9]$$

$\downarrow$

$$C_{11} = [20] + [-3] - [16] + [9] = [10]$$
$$C_{12} = [-2] + [16] = [14]$$
$$C_{21} = [16] + [-3] = [13]$$
$$C_{22} = [20] + [-2] - [16] - [-9] = [11]$$

$\downarrow$

$$\begin{bmatrix}10 & 14\\ 13 & 11\end{bmatrix}$$

$$\begin{bmatrix}5 & 5\\ 2 & 5\end{bmatrix} \times \begin{bmatrix}2 & 6\\ 2 & 1\end{bmatrix}$$

$A_{11} = [5]$ $\quad B_{11} = [2]$
$A_{12} = [5]$ $\quad B_{12} = [6]$
$A_{21} = [2]$ $\quad B_{21} = [2]$
$A_{22} = [5]$ $\quad B_{22} = [1]$

$$P_1 = [5] \times ([6] - [1]) = [5] \times [5] = [25]$$
$$P_2 = ([5] + [5]) \times [1] = [10] \times [1] = [10]$$
$$P_3 = ([2] + [5]) \times [2] = [7] \times [2] = [14]$$
$$P_4 = [5] \times ([2] - [2]) = [5] \times [0] = [0]$$
$$P_5 = ([5] + [5]) \times ([2] + [1]) = [10] \times [3] = [30]$$
$$P_6 = ([5] - [5]) \times ([2] + [1]) = [0] \times [3] = [0]$$
$$P_7 = ([5] - [2]) \times ([2] + [6]) = [3] \times [8] = [24]$$

$\downarrow$

$$C_{11} = [30] + [0] - [10] + [0] = [20]$$
$$C_{12} = [25] + [10] = [35]$$
$$C_{21} = [14] + [0] = [14]$$
$$C_{22} = [30] + [25] - [14] - [24] = [17]$$

$\downarrow$

$$\begin{bmatrix}20 & 35\\ 14 & 17\end{bmatrix}$$

$$\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix} \times \begin{bmatrix}1 & -5\\ 0 & 1\end{bmatrix}$$

$A_{11} = [2]$ $\quad B_{11} = [1]$
$A_{12} = [4]$ $\quad B_{12} = [-5]$
$A_{21} = [1]$ $\quad B_{21} = [0]$
$A_{22} = [2]$ $\quad B_{22} = [1]$

$$P_1 = [2] \times ([-5] - [1]) = [2] \times [-6] = [-12]$$
$$P_2 = ([2] + [4]) \times [1] = [6] \times [1] = [6]$$
$$P_3 = ([1] + [2]) \times [1] = [3] \times [1] = [3]$$
$$P_4 = [2] \times ([0] - [1]) = [2] \times [-1] = [-2]$$
$$P_5 = ([2] + [2]) \times ([1] + [1]) = [4] \times [2] = [8]$$
$$P_6 = ([4] - [2]) \times ([0] + [1]) = [2] \times [1] = [2]$$
$$P_7 = ([2] - [1]) \times ([1] + [-5]) = [1] \times [-4] = [-4]$$

$\downarrow$

$$C_{11} = [8] + [-2] - [6] + [2] = [2]$$
$$C_{12} = [-12] + [6] = [-6]$$
$$C_{21} = [3] + [-2] = [1]$$
$$C_{22} = [8] + [-12] - [3] - [-4] = [-3]$$

$\downarrow$

$$\begin{bmatrix}2 & -6\\ 1 & -3\end{bmatrix}$$

$$\begin{bmatrix}3 & 6\\ 3 & 3\end{bmatrix} \times \begin{bmatrix}4 & 7\\ 3 & 3\end{bmatrix}$$

$A_{11} = [3]$ $B_{11} = [4]$

$A_{12} = [6]$ $B_{12} = [7]$

$A_{21} = [3]$ $B_{21} = [3]$

$A_{22} = [3]$ $B_{22} = [3]$

$P_1 = [3] \times ([7] - [3]) = [3] \times [4] = [12]$

$P_2 = ([3] + [6]) \times [3] = [9] \times [3] = [27]$

$P_3 = ([3] + [3]) \times [4] = [6] \times [4] = [24]$

$P_4 = [3] \times ([3] - [4]) = [3] \times [-1] = [-3]$

$P_5 = ([3] + [3]) \times ([4] + [3]) = [6] \times [7] = [42]$

$P_6 = ([6] - [3]) \times ([3] + [3]) = [3] \times [6] = [18]$

$P_7 = ([3] - [3]) \times ([4] + [7]) = [0] \times [11] = [0]$

$\downarrow$

$C_{11} = [42] + [-3] - [27] + [18] = [30]$

$C_{12} = [12] + [27] = [39]$

$C_{21} = [24] + [-3] = [21]$

$C_{22} = [42] + [12] - [24] - [0] = [30]$

$\downarrow$

$$\begin{bmatrix}30 & 39\\ 21 & 30\end{bmatrix}$$

$$\begin{bmatrix}-1 & 0\\ 2 & 0\end{bmatrix} \times \begin{bmatrix}5 & 2\\ 3 & 4\end{bmatrix}$$

$A_{11} = [-1]$ $B_{11} = [5]$

$A_{12} = [0]$ $B_{12} = [2]$

$A_{21} = [2]$ $B_{21} = [3]$

$A_{22} = [0]$ $B_{22} = [4]$

$P_1 = [-1] \times ([2] - [4]) = [-1] \times [-2] = [2]$

$P_2 = ([-1] + [0]) \times [4] = [-1] \times [4] = [-4]$

$P_3 = ([2] + [0]) \times ([5]) = [2] \times [5] = [10]$

$P_4 = [0] \times ([3] - [5]) = [0] \times [-2] = [0]$

$P_5 = ([-1] + [0]) \times ([5] + [4]) = [-1] \times [9] = [-9]$

$P_6 = ([0] - [0]) \times ([3] + [4]) = [0] \times [7] = [0]$

$P_7 = ([-1] - [2]) \times ([5] + [2]) = [-3] \times [7] = [-21]$

$\downarrow$

$C_{11} = [-9] + [0] - [-4] + [0] = [-5]$

$C_{12} = [2] + [-4] = [-2]$

$C_{21} = [10] + [0] = [10]$

$C_{22} = [-9] + [2] - [10] - [-21] = [4]$

$\downarrow$

$$\begin{bmatrix}-5 & -2\\ 10 & 4\end{bmatrix}$$

$$\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix} \times \begin{bmatrix}5 & 10\\ 3 & 3\end{bmatrix}$$

$A_{11} = [-2]$ $B_{11} = [5]$

$A_{12} = [1]$ $B_{12} = [10]$

$A_{21} = [1]$ $B_{21} = [3]$

$A_{22} = [-2]$ $B_{22} = [3]$

$P_1 = [-2] \times ([10] - [3]) = [-2] \times [7] = [-14]$

$P_2 = ([-2] + [1]) \times [3] = [-1] \times [3] = [-3]$

$P_3 = ([1] + [-2]) \times [5] = [-1] \times [5] = [-5]$

$P_4 = [-2] \times ([3] - [5]) = [-2] \times [-2] = [4]$

$P_5 = ([-2] + [-2]) \times ([5] + [3]) = [-4] \times [8] = [-32]$

$P_6 = ([1] - [-2]) \times ([3] + [3]) = [3] \times [6] = [18]$

$P_7 = ([-2] - [1]) \times ([5] + [10]) = [-3] \times [15] = [-45]$

$\downarrow$

$C_{11} = [-32] + [4] = [-3] + [18] = [-7]$

$C_{12} = [-14] + [-3] = [-17]$

$C_{21} = [-5] + [4] = [-1]$

$C_{22} = [-32] + [-14] - [-5] - [-45] = [4]$

$\downarrow$

$$\begin{bmatrix}-7 & -17\\ -1 & 4\end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 3 & 2 \\ 3 & 1 & 2 & 4 \\ 1 & 3 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 6 & 3 & 4 \\ 2 & 1 & 1 & 2 \\ 3 & 1 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad B_{11} = \begin{bmatrix} 2 & 6 \\ 2 & 1 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \qquad B_{12} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B_{21} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \qquad B_{22} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \times \left(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$P_2 = \left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \right) \times \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$P_3 = \left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \right) \times \begin{bmatrix} 2 & 6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 2 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 6 \\ 2 & 1 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \times \left(\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 2 & 1 \end{bmatrix} \right) = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

$$P_5 = \left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \right) \times \left(\begin{bmatrix} 2 & 6 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) = \begin{bmatrix} 3 & 6 \\ 3 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 7 \\ 3 & 3 \end{bmatrix}$$

$$P_6 = \left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \right) \times \left(\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} \times \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$$

$$P_7 = \left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \right) \times \left(\begin{bmatrix} 2 & 6 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \right) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \times \begin{bmatrix} 5 & 10 \\ 3 & 3 \end{bmatrix}$$

$$\downarrow$$

$$P_1 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \qquad P_5 = \begin{bmatrix} 30 & 39 \\ 21 & 30 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 10 & 14 \\ 13 & 11 \end{bmatrix} \qquad P_6 = \begin{bmatrix} -5 & -2 \\ 10 & 4 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 20 & 35 \\ 14 & 17 \end{bmatrix} \qquad P_7 = \begin{bmatrix} -7 & -17 \\ -1 & 4 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$$

$$\downarrow$$

$$C_{11} = \begin{bmatrix} 30 & 39 \\ 21 & 30 \end{bmatrix} + \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} 10 & 14 \\ 13 & 11 \end{bmatrix} + \begin{bmatrix} -5 & -2 \\ 10 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 17 \\ 19 & 20 \end{bmatrix}$$

$$C_{12} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 10 & 14 \\ 13 & 11 \end{bmatrix} = \begin{bmatrix} 11 & 17 \\ 15 & 17 \end{bmatrix}$$

$$C_{21} = \begin{bmatrix} 20 & 35 \\ 14 & 17 \end{bmatrix} + \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 22 & 29 \\ 15 & 14 \end{bmatrix}$$

$$C_{22} = \begin{bmatrix} 30 & 39 \\ 21 & 30 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 20 & 35 \\ 14 & 17 \end{bmatrix} - \begin{bmatrix} -7 & -17 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 24 \\ 10 & 15 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 17 & 17 & 11 & 17 \\ 19 & 20 & 15 & 17 \\ 22 & 29 & 18 & 24 \\ 15 & 14 & 10 & 15 \end{bmatrix}$$