**Question 1)**

Solve the following recurrence relations. For each one come up with a **precise** function of n in closed form (i.e., resolve all sigmas, recursive calls of function T, etc) using the substitution method. Note: An asymptotic answer is not acceptable for this question. Justify your solution and show all your work.

a) T(n)=T(n-1)+cn, T(0)=1,

b) T(n)=4T(n/2)+n , T(1)=1

c) T(n)= 2T(n/2)+1, T(1)=1

**Answer:**

a) T(n)=T(n-1)+cn, T(0)=1

Given, T(0) = 1 and T(n) = T(n - 1) + cn

T(n) = T(n - 1) + cn

Now, T(n - 1) = T(n - 2) + c(n - 1) [Since, T(m) = T(m - 1) + cm and m = n - 1 here]

Substitute T(n - 1) in above equation.

T(n) = T(n - 2) + c(n - 1) + cn

Now, T(n - 2) = T(n - 3) + c(n - 1) [Since, T(m) = T(m - 1) + cm and m = n - 2 here]

Substitute T(n - 2) in above equation.

T(n) = T(n - 3) + c(n - 2) + c(n - 1) + c(n)

And if we repeat it for k times, we get

T(n) = T(n - k - 1) + c(n - k) + ……. c(n - 1) + c(n)

We reach base case when n - k - 1 becomes 0

n - k - 1 = 0

== k = n - 1

and n - k = 1

At k = n - 1,

Now, T(n) = T(0) + c(1) + c(2) + c(3) + ……… c(n - 1) + c(n)

Given, T(0) = 1

So, T(n) = 1 + c[1 + 2 + 3 + ……. (n - 1) + n]

1 + 2 + ……(n - 1) + n = n * (n + 1) / 2 according to mathematics.

So, T(n) = c * n * (n + 1) / 2 + 1

**T(n) = (c / 2) * $n^2$ + (c / 2) * n + 1** is the function representing the time complexity of given function.

To Find the Big-O Notation of given function, we need to find n0, C where T(n) <= C * g(n) for all n >= n0

Consider a function G(n) by replacing each value in T(n) with c * $n^2$

Then, (c / 2) * $n^2$ <= c * $n^2$ for all n >= 1 since c > c / 2

(c / 2) * n <= c * $n^2$ for all n >= 1 since c > c / 2 and $n^2$ > n

1 <= c * $n^2$ for all n >= 1

If we add all three inequalities, then we get,

(c / 2) * $n^2$ + (c / 2) * n + 1 <= c * $n^2$ + c * $n^2$ + c * $n^2$ for all n >= 1

=== (c / 2) * $n^2$ + (c / 2) * n + 1 <= 3 * c * $n^2$ for all n >= 1

=== T(n) <= 3 * c * $n^2$ for all n >= 1

Therefore, T(n) = O( $n^2$ ) where n0 = 1 and C = 3 * c

b) T(n)=4T(n/2)+n , T(1)=1

Given, T(1)=1 and T(n)=4T(n/2)+n

$T(n)=4T(n/2) + n$

Now, $T(n / 2) = 4T(n / 4) + n / 2$ [Since, $T(m) = 4T(m / 2) + m$ and $m = n / 2$ here]

Substitute $T(n / 2)$ in above equation,

$T(n) = 4 * [4 * T(n / 4) + n / 2] + n$

$T(n) = 4 * 4 * T(n / 4) + 4 * n / 2 + n$

Now, $T(n / 4) = 4T(n / 8) + n / 4$ [Since, $T(m) = 4T(m / 2) + m$ and $m = n / 4$ here]

Substitute $T(n / 4)$ in above equation,

$T(n) = 4 * 4 * [4 * T(n / 8) + n / 4] + 4 * n / 2 + n$

$T(n) = 4 * 4 * 4 * T(n / 8) + 4 * 4 * n / 4 + 4 * n / 2 + n$

And if we repeat k - 1 times, we get

$T(n) = 4^k * T(n / 2^k) + 4^{(k - 1)} * n / 2^{(k - 1)} + \dots\dots\dots n$

We know, $4^{(k - 1)} / 2^{(k - 1)} = 2^{(k - 1)} * 2^{(k - 1)} / 2^{(k - 1)} = 2^{(k - 1)}$

So, $T(n) = 4^k * T(n / 2^k) + 2^{(k - 1)} * n + \dots\dots\dots n$

$T(n) = 4^k * T(n / 2^k) + n * [1 + 2 + 4 + \dots\dots 2^{(k - 1)}]$

$1 + 2 + 4 + \dots\dots 2^{(k - 1)} = 2^k - 1$ according to mathematics

So, $T(n) = 4^k * T(n / 2^k) + n * (2^k - 1)$

We reach base case when $n / 2^k = 1$

$n / 2^k = 1$

$n = 2^k$

$k = \log_2(n)$

At $k = \log_2(n)$

$T(n) = 4^{\log_2 n} * T(1) + n * (2^{\log_2 n} - 1)$

Substitute $T(1) = 1$

And $2^{\log_2 n} = n$ according to mathematics

And, $4^{\log_2 n} = 2^{2\log_2 n} = 2^{\log_2 n^\wedge 2} = n^2$

So, $T(n) = n^2 + n * (n - 1)$

$T(n) = 2n^2 - n$

**$T(n) = 2n^2 - n$** is the function representing the time complexity of given function.

To Find the Big-O Notation of given function, we need to find n0, C where $T(n) <= C * g(n)$ for all $n >= n0$

Consider a function $G(n)$ by replacing each value in $T(n)$ with $2 * n^2$

Then, $2 * n^2 = 2 * n^2$ for all $n >= 1$ since both are equal

$-1 * n <= 2 * n^2$ for all $n >= 1$ since left value is negative and right one is positive

If we add both inequalities, then we get,

$2 * n^2 - n <= 2 * n^2 + 2 * n^2$ for all $n >= 1$

=== $2 * n^2 - n <= 4 * n^2$ for all $n >= 1$

=== $T(n) <= 4 * n^2$ for all $n >= 1$

Therefore, $T(n) = O( n^2 )$ where $n0 = 1$ and $C = 4$

c) $T(n)= 2T(n/2)+1, T(1)=1$

Given, $T(n)= 2T(n/2)+1, T(1)=1$

$T(n) = 2 * T(n / 2) + 1$

Now, $T(n / 2) = 2 * T(n / 4) + 1$ [Since $T(m) = 2T(m / 2) + 1$ and $m = n / 2$ here]

Substitute $T(n / 2)$ in above equation

$T(n) = 2 * [2 * T(n / 4) + 1] + 1$

$T(n) = 2 * 2 * T(n / 4) + 2 + 1$

Now, $T(n / 4) = 2 * T(n / 8) + 1$ [Since $T(m) = 2T(m / 2) + 1$ and $m = n / 4$ here]

Substitute $T(n / 4)$ in above equation

$T(n) = 2 * 2 * [2 * T(n / 8) + 1] + 2 + 1$
$T(n) = 2 * 2 * 2 * T(n / 8) + 2^2 + 2 + 1$
And if we repeat k - 1 times, we get
$T(n) = 2^k * T(n / 2^k) + 2^{(k-1)} + \ldots\ldots 2 + 1$
$1 + 2 + 4 + \ldots\ldots 2^{(k - 1)} = 2^k - 1$ according to mathematics
$T(n) = 2^k * T(n / 2^k) + 2^k - 1$
We reach base case when $n / 2^k = 1$
$n / 2^k = 1$
$n = 2^k$
$k = \log_2(n)$
At $k = \log_2(n)$
$T(n) = 2^{\log_2 n} * T(1) + 2^{\log_2 n} - 1$
Substitute $T(1) = 1$
And $2^{\log_2 n} = n$ according to mathematics
$T(n) = n * 1 + n - 1$
$T(n) = 2n - 1$
$T(n) = 2n - 1$ is the function representing the time complexity of given function.
To Find the Big-O Notation of given function, we need to find n0, C where T(n) <= C * g(n) for all n >= n0
Consider a function G(n) by replacing each value in T(n) with 2 * n
Then, 2 * n = 2 * n for all n >= 1 since both are equal
-1 <= 2 * n for all n >= 1 since left value is negative and right value is positive
If we add both inequalities, then we get,
2 * n - 1 <= 2 * n + 2 * n for all n >= 1
=== T(n) <= 4 * n
Therefore, T(n) = O( n ) where n0 = 1 and C = 4

**Question 2)**
Consider Question 1 again. Apply Master Theorem if applicable for each case. Bound the recurrence relation in Big-O.

**Answer:**

Master's Theorem is used to easily find Big-O Notation for functions with patterns in their recurrence relation.
We can apply Master's theorem only if the recurrence relation is of form:
$T(n) = a * T(n / b) + f(n)$
Where a >= 1, b >= 2 and f(n) is an asymptotically positive function
It has 3 cases.
Case-I:
If $f(n) = \text{Theta}(n^d)$ where $d < \log_b a$
Then f(n) grows asymptotically slower than $\log_b a$
Therefore, $T(n) = \text{Theta}(n^{\log_b a})$
Case-II:
If $f(n) = \text{Theta}(n^d)$ where $d > \log_b a$
Then f(n) grows asymptotically faster than $\log_b a$
Therefore, $T(n) = \text{Theta}(n^d)$
Case-III:
If $f(n) = \text{Theta}(n^d \log^k n)$ where $d = \log_b a$
Then, $T(n) = \text{Theta}(n^d \log^{(k+1)} n)$

a) $T(n)=T(n-1)+cn$, $T(0)=1$

To Apply Master's Theorem, T(n) should be of form

$T(n) = a * T(n / b) + f(n)$

Since, given function $T(n)=T(n-1)+cn$ is not in that form. We can't apply Master's theorem on this function.

b) $T(n)=4T(n/2)+n$ , $T(1)=1$

To Apply Master's Theorem, T(n) should be of form

$T(n) = a * T(n / b) + f(n)$

Given function is of the same form where a = 4, b = 2 and f(n) = n

f(n) = Theta(n)

=== Theta($n^1$)

So, d = 1 in Master's Theorem

$\log_b a = \log_2 4 = 2$

Since, d = 1 < $\log_b a$ = 2, the given function follows Case-I pattern of Master's Theorem

According to Case-I of Master's Theorem,

If f(n) = Theta($n^d$) where d < $\log_b a$

Then f(n) grows asymptotically slower than $\log_b a$

Therefore, T(n) = Theta($n^{\log_b a}$)

Therefore, T(n) = Theta($n^{\log_b a}$) = Theta($n^2$)

Since, Theta represents a tight bound

$C1 * n^2 <= T(n) <= C2 * n^2$

If we consider, only $T(n) <= C2 * n^2$ it represents Big-O Notation

Therefore, T(n) = O($n^2$)

c) $T(n)= 2T(n/2)+1$, $T(1)=1$

To Apply Master's Theorem, T(n) should be of form

$T(n) = a * T(n / b) + f(n)$

Given function is of the same form where a = 2, b = 2 and f(n) = 1

f(n) = Theta(1)

=== Theta($n^0$)

So, d = 0 in Master's Theorem

$\log_b a = \log_2 2 = 1$

Since, d = 0 < $\log_b a$ = 1, the given function follows Case-I pattern of Master's Theorem

According to Case-I of Master's Theorem,

If f(n) = Theta($n^d$) where d < $\log_b a$

Then f(n) grows asymptotically slower than $\log_b a$

Therefore, T(n) = Theta($n^{\log_b a}$)

Therefore, T(n) = Theta($n^{\log_b a}$) = Theta($n^1$)

Since, Theta represents a tight bound

$C1 * n <= T(n) <= C2 * n$

If we consider, only $T(n) <= C2 * n$ it represents Big-O Notation

Therefore, T(n) = O(n)

**Question 3)**

A binary tree's "maximum depth" is the number of nodes along the longest path from the root node down to the farthest leaf node. Given the root of a binary tree, write a complete program in C++/Java that returns three's maximum depth. What is the time-complexity of your algorithm in the worst-case once you have n nodes in the tree. Analyze and clearly discuss your reasoning. Paste your complete program in the solution file.

**Answer:**

Java Code to Calculate maximum depth of a binary tree is

```
public static int maxDepth(Node root) {
    // Base Condition
    if(root == null) return 0;
    // Recursive Calls
    int left = maxDepth(root.left);
    int right = maxDepth(root.right);

    return Math.max(left, right) + 1;
}
```

Let T(n) is the time complexity of the above function.
Number of basic operations are:

| Code | Cost | Number of times it runs |
|---|---|---|
| if(root == null) return 0; | C1 | 1 |
| int left = maxDepth(root.left); | T(n / 2) | 1 |
| int right = maxDepth(root.right); | T(n / 2) | 1 |
| return Math.max(left, right) + 1; | C2 | 1 |

So, $T(n) = C1 + T(n / 2) + T(n / 2) + C2$
$T(n) = 2T(n / 2) + C1 + C2$

Master's Theorem is used to easily find Big-O Notation for functions with patterns in their recurrence relation.
We can apply Master's theorem only if the recurrence relation is of form:
$T(n) = a * T(n / b) + f(n)$
Where a >= 1, b >= 2 and f(n) is an asymptotically positive function
It has 3 cases.
Case-I:
If $f(n) = Theta(n^d)$ where $d < \log_b a$
Then f(n) grows asymptotically slower than $\log_b a$
Therefore, $T(n) = Theta(n^{\log_b a})$
Case-II:
If $f(n) = Theta(n^d)$ where $d > \log_b a$
Then f(n) grows asymptotically faster than $\log_b a$
Therefore, $T(n) = Theta(n^d)$
Case-III:
If $f(n) = Theta(n^d \log^k n)$ where $d = \log_b a$
Then, $T(n) = Theta(n^d \log^{(k+1)} n)$
Given function is of the same form where a = 2, b = 2 and f(n) = C1 + C2
$f(n) =$ Theta(1)
=== $Theta(n^0)$
So, d = 0 in Master's Theorem
$\log_b a = \log_2 2 = 1$
Since, $d = 0 < \log_b a = 1$, the given function follows Case-I pattern of Master's Theorem
According to Case-I of Master's Theorem,

If $f(n) = Theta(n^d)$ where $d < \log_b a$
Then f(n) grows asymptotically slower than $\log_b a$
Therefore, $T(n) = Theta(n^{\log_b a})$
Therefore, $T(n) = Theta(n^{\log_b a}) = Theta(n^1)$
Since, Theta represents a tight bound
C1 * n <= T(n) <= C2 * n
If we consider, only T(n) <= C2 * n it represents Big-O Notation
Therefore, T(n) = O(n)

**Question 4)**

Part (a) Write a **linear time divide and conquer algorithm** (i.e., $\theta(n)$ ) to calculate $x^n$ ( x is raised to the power n). Assume a and n are >=0.
Part (b) Analyze the time complexity of your algorithm in the worst-case by first writing its recurrence relation.
Part (c) Can you improve your algorithm to accomplish the end in O(log n) time complexity (we still look for a divide and conquer algorithm). If yes, write the corresponding algorithm, write the recurrence relation for its time complexity and analyze it. If no, justify your answer.

Answer:

a) Algorithm to find $x^n$ in linear time is

```
public static int linearXpowN(int x, int n){
        if(n == 0) return 1;
        return x * linearXpowN(x, n - 1);
}
```

b) Let T(n) is the time complexity of the above function. Number of basic operations are:

| Code | Cost | Number of times it runs |
|---|---|---|
| if(n == 0) return 1; | C1 | 1 |
| return x * linearXpowN(x, n - 1); | T(n - 1) | 1 |

So, T(n) = T(n - 1) + C1
Now, T(n - 1) = T(n - 2) + C1 [Since, T(m) = T(m - 1) + C1 and m = n - 1 here]
Substitute T(n - 1) in above equation,
T(n) = T(n - 2) + C1 + C1
Now, T(n - 2) = T(n - 3) + C1 [Since, T(m) = T(m - 1) + C1 and m = n - 2 here]
Substitute T(n - 2) in above equation,
T(n) = T(n - 3) + C1 + C1 + C1
If we repeat above process k times,
T(n) = T(n - k) + C1 + C1 + ….. (k - 1) times
We reach base case when n - k = 0 === k = n
At base case,
T(n) = T(0) + C1 + C1 + …… (n - 1) times
C1 + C1 + ……. n times
So, T(n) = C1 * n
To Find Big-O Notation we need to find g(n), C, n0 where T(n) <= C * g(n) for all n >= n0
Then, T(n) = O(g(n))
Let G(n) = 2 * C1 * n
We can say that,

$C1 * n <= 2 * C1 * n$ for all $n >= 1$ since C1 will always be <= 2 * C1

=== $T(n) <= 2 * C1 * n$ for all $n >= 1$

Therefore, T(n) = O(n) where n0 = 1 and C = 2 * C1

c) Algorithm to find $x^n$ in logarithmic time is

```
public static int logarithmicXpowN(int x, int n){
	if(n == 1) return x;
	int product = 1;
	if(n % 2 != 0){
		product = x;
		n -= 1;
	}
	int subProduct = logarithmicXpowN(x, n / 2);
	return product * subProduct * subProduct;
}
```

Let T(n) is the time complexity of the above function. Number of basic operations are:

| Code | Cost | Number of times it runs |
|---|---|---|
| if(n == 1) return x; | C1 | 1 |
| int product = 1; | C2 | 1 |
| if(n % 2 != 0) | C3 | 1 |
| product = x; | C4 | 1 |
| n -= 1; | C5 | 1 |
| int subProduct = logarithmicXpowN(x, n / 2); | T(n / 2) | 1 |
| return product * subProduct * subProduct; | C6 | 1 |

So, $T(n) = T(n / 2) + (C1 + C2 + C3 + C4 + C5 + C6)$

Let, $C = (C1 + C2 + C3 + C4 + C5 + C6)$

Then, $T(n) = T(n / 2) + C$

We can apply Master's theorem only if the recurrence relation is of form:

$T(n) = a * T(n / b) + f(n)$

Where $a >= 1$, $b >= 2$ and f(n) is an asymptotically positive function

It has 3 cases.

Case-I:

If $f(n) = \Theta(n^d)$ where $d < \log_b a$

Then f(n) grows asymptotically slower than $\log_b a$

Therefore, $T(n) = \Theta(n^{\log_b a})$

Case-II:

If $f(n) = \Theta(n^d)$ where $d > \log_b a$

Then f(n) grows asymptotically faster than $\log_b a$

Therefore, $T(n) = \Theta(n^d)$

Case-III:

If $f(n) = \Theta(n^d \log^k n)$ where $d = \log_b a$

Then, $T(n) = \Theta(n^d \log^{(k+1)} n)$

Given function is of the same form where a = 1, b = 2 and f(n) = C

$f(n) = \Theta(1)$

=== $\Theta(n^0)$

So, $d = 0$ in Master's Theorem

$\log_b a = \log_{21} = 0$

Since, $d = 0 == \log_b a = 0$, the given function follows Case-III pattern of Master's Theorem

According to Case-III of Master's Theorem,

If $f(n) = Theta(n^d \log^k n)$ where $d = \log_b a$

Then, $T(n) = Theta(n^d \log^{(k+1)} n)$

We can rewrite f(n) as

$f(n) =$ Theta(1)

$Theta(n^0 \log^0 n)$

So, $d = 0$ and $k = 0$ in Master's Theorem

Therefore, $T(n) = Theta(n^d \log^{(k+1)} n)$

=== $Theta(n^0 \log^{(0+1)} n)$

=== Theta(logn)

Therefore, $T(n) =$ Theta(logn)