CS 611: Theory of Computation

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Finite Languages

Definition

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$\{0, 1, 00, 10\}$ is a finite language, however, $(00 \cup 11)^*$ is not.

Finiteness and Regularity

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Proof.

Let $L = \{w_1, w_2, \ldots w_n\}$. Then $R = w_1 \cup w_2 \cup \cdots \cup w_n$ is a regular expression defining L. $\square$

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Above is a weak argument because $E = \{w \in \{0,1\}^* \mid w \text{ has an equal number of 01 and 10 substrings}\}$ is regular!

Proving Non-Regularity

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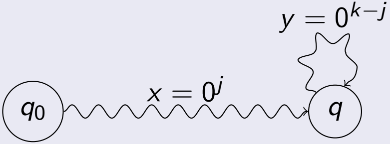
- There must be $j < k \leq n$ such that $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k)$ $(= q$ say).
- Let $x = 0^j$, $y = 0^{k-j}$, and $z = 0^{n-k}1^n$; so $xyz = 0^n1^n$. $\cdots\!\longrightarrow$

Proving Non-Regularity

Proof (contd).



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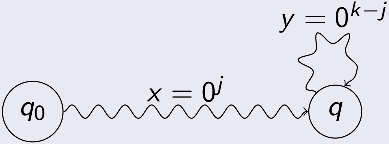

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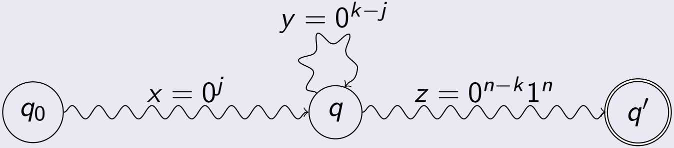

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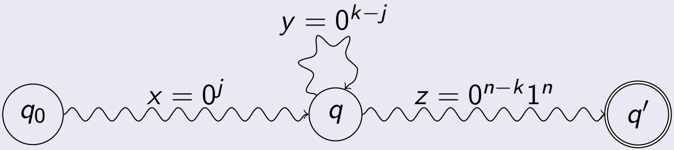

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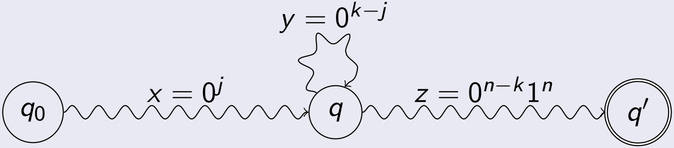

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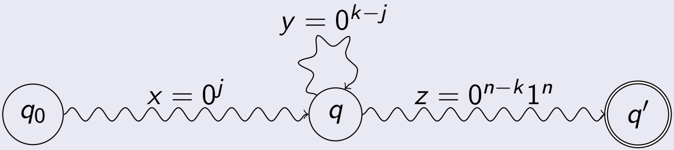

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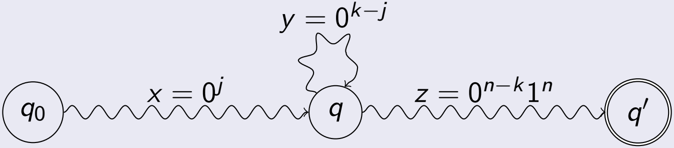

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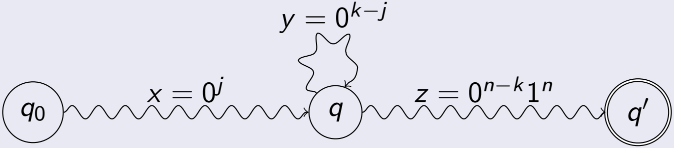

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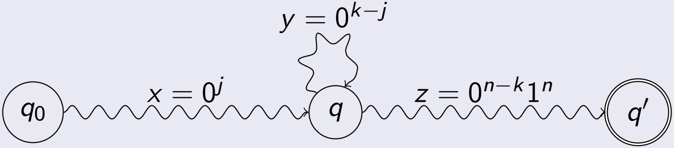

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- So M accepts $0^{n-k+j}1^n$ as well.

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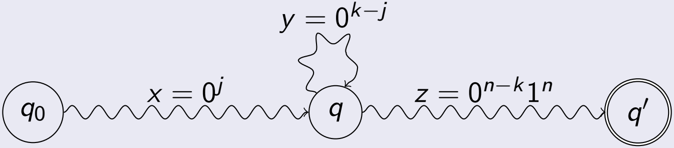

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\end{aligned}$$

- So M accepts $0^{n-k+j}1^n$ as well. But, $0^{n-k+j}1^n \notin L_{\text{eq}}$! $\square$

Pumping Lemma: Overview

Pumping Lemma

The lemma generalizes this argument. Gives the template of an argument that can be used to easily prove that many languages are non-regular.

Pumping Lemma

The Statement

Lemma

If L is regular then there is a number p (the pumping length) such that $\forall w \in L$ with $|w| \geq p$, $\exists x, y, z \in \Sigma^*$ such that $w = xyz$ and

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$\forall w \in L$ with $|w| \geq p$,
$\exists x, y, z \in \Sigma^*$ such that $w = xyz$, $|y| > 0$ and $|xy| \leq p$,
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Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA such that $L(M) = L$ and let $p = |Q|$.

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- Since $s_0, s_1, \ldots, s_i, \ldots s_p$ are $p + 1$ states, there must be j, k, $0 \leq j < k \leq p$ such that $s_j = s_k$ $(= q$ say$)$.

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- Take $x = w_1 \cdots w_j$, $y = w_{j+1} \cdots w_k$, and $z = w_{k+1} \cdots w_n$
- Observe that since $j < k \leq p$, we have $|xy| \leq p$ and $|y| > 0$.

$\cdots\!\longrightarrow$

Proof . . .

Technical Claim

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For all $i \geq 1$, $\hat{\delta}(q_0, xy^i) = \hat{\delta}(q_0, x)$.

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- Base Case: By our assumption that $s_j = s_k$ and the definition of x and y, we have $\hat{\delta}(q_0, xy) = s_k = s_j = \hat{\delta}(q_0, x)$.

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$$\hat{\delta}(q_0, xy^{\ell+1}) = \hat{\delta}(\hat{\delta}(q_0, xy^{\ell}), y)$$

□

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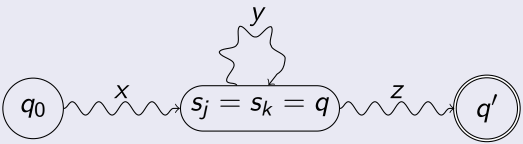
□

Completing the Proof

Proof (contd).



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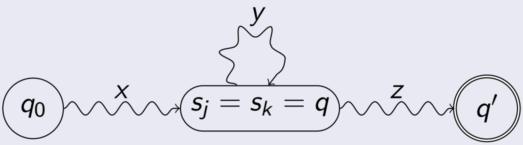

- We have $\hat{\delta}(q_0, xy^i) = \hat{\delta}(q_0, x)$ for all $i \geq 1$

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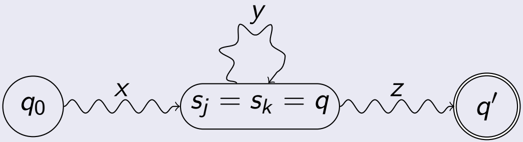

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Completing the Proof

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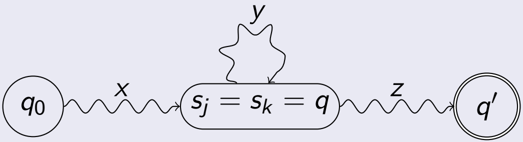

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- Observe,
  $\hat{\delta}(q_0, xz) = \hat{\delta}(\hat{\delta}(q_0, x), z) = \hat{\delta}(\hat{\delta}(q_0, xy), z) = \hat{\delta}(q_0, w)$. So $xz \in L$

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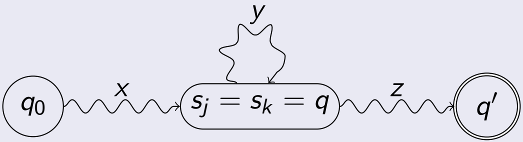

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- Observe, $\hat{\delta}(q_0, xz) = \hat{\delta}(\hat{\delta}(q_0, x), z) = \hat{\delta}(\hat{\delta}(q_0, xy), z) = \hat{\delta}(q_0, w)$. So $xz \in L$
- Similarly, $\hat{\delta}(q_0, xy^iz) = \hat{\delta}(q_0, xyz) \in F$ and so $xy^iz \in L$ $\square$

Finite Languages and Pumping Lemma

Question

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Recall Pumping Lemma: If L is regular then there is a number p (the pumping length) such that $\forall w \in L$ with $|w| \geq p$, $\exists x, y, z \in \Sigma^*$ such that $w = xyz$ and

1. $|y| > 0$
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Question

Do finite languages really satisfy the condition in the pumping lemma?

Recall Pumping Lemma: If L is regular then there is a number p (the pumping length) such that $\forall w \in L$ with $|w| \geq p$, $\exists x, y, z \in \Sigma^*$ such that $w = xyz$ and

1. $|y| > 0$
2. $|xy| \leq p$
3. $\forall i \geq 0.\ xy^i z \in L$

Answer

Yes, they do. Let p be larger than the longest string in the language. Then the condition "$\forall w \in L$ with $|w| \geq p, \ldots$" is vaccuously satisfied as there are no strings in the language longer than p!

Using the Pumping Lemma

L regular implies that L satisfies the condition in the pumping lemma.If L regular $\Rightarrow$ pumping Lemma.Then, Not pumping Lemma $\Rightarrow$ L not regular.

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Pumping Lemma, in contrapositive

If L does not satisfy the pumping condition, then L not regular.

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Pumping Condition

$$\exists p. \quad \forall w \in L.\ \text{with } |w| \geq p \quad \exists x, y, z \in \Sigma^*.\ w = xyz \quad \left.\begin{array}{ll} (1) & |y| > 0 \\ (2) & |xy| \leq p \\ (3) & \forall i \geq 0.\ xy^i z \in L \end{array}\right\}$$

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Negation of the Pumping Condition

$$\nexists p. \quad \forall w \in L. \text{ with } |w| \geq p \quad \exists x, y, z \in \Sigma^*. \ w = xyz$$

$$\left.\begin{array}{ll} (1) & |y| > 0 \\ (2) & |xy| \leq p \\ (3) & \forall i \geq 0. \ xy^iz \in L \end{array}\right\}$$

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Equivalent to showing that if (1), (2) then (3) does not. In other words, we can find i such that $xy^iz \notin L$

Using the Pumping Lemma

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Pumping Lemma, in contrapositive

$\forall p \geq 0$,
$\exists w \in L$ with $|w| \geq p$,
$\forall x, y, z \in \Sigma^*$ such that $w = xyz$, $|y| > 0$ and $|xy| \leq p$,
$\exists i \geq 0.\ xy^i z \notin L$.

Proving something like $\forall$ bal $\exists$ bla $\forall$ bla $\exists$ bla means winning a game.
Example: Two players:You, Adversary. Rules:First Adversary says a number. Then You say a number. You win if your number is bigger.

Game View

Think of using the Pumping Lemma as a game between you and an opponent.

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$\forall x, y, z$	Opponent divides w into x, y, and z such that $\lvert y\rvert > 0$, and $\lvert xy\rvert \leq p$

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Pumping Lemma: If L is regular, opponent has a winning strategy (no matter what you do).
Contrapositive: If you can beat the opponent, L not regular.
Your strategy should work for any p and any subdivision that the opponent may come up with.

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- Since $|w| > p$, there are x, y, z such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^i z \in L_{0n1n}$, for all i.

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- Since $|xy| \leq p$, $x = 0^r$, $y = 0^s$ and $z = 0^t 1^p$. Further, as $|y| > 0$, we have $s > 0$.

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$$xy^0z = 0^r\epsilon 0^t1^p = 0^{r+t}1^p$$

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$$xy^0 z = 0^r \epsilon 0^t 1^p = 0^{r+t} 1^p$$

Since $r + t < p$, $xy^0 z \notin L_{0n1n}$. Contradiction! $\square$

Example II

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$L_{\mathrm{eq}} = \{w \in \{0,1\}^* \mid w$ *has an equal number of 0s and 1s*$\}$ *is not regular.*

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- Consider $w = 0^p 1^p$

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$L_{\rm eq} = \{w \in \{0,1\}^* \mid w \textit{ has an equal number of 0s and 1s}\}$ *is not regular.*

Proof.

Suppose $L_{\rm eq}$ is regular. Let p be the pumping length for $L_{\rm eq}$.

- Consider $w = 0^p1^p$
- Since $|w| > p$, there are x, y, z such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_{\rm eq}$, for all i.

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$$xy^0z = 0^r\epsilon 0^t1^p = 0^{r+t}1^p$$

Since $r + t < p$, $xy^0z \not\in L_{\rm eq}$. Contradiction! □

A Tale of two Proofs

Non Pumping Lemma

Suppose L_{eq} is recognized by DFA M with p states. Consider the input 0^p1^p.

Pumping Lemma

Suppose L_{eq} is regular. Let p be pumping length for L_{eq}. Consider $w = 0^p1^p$.

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- $j < k$ and $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k) = q$

Pumping Lemma

Suppose L_{eq} is regular. Let p be pumping length for L_{eq}. Consider $w = 0^p1^p$. There exist x, y, z such that

- $w = xyz$, $|xy| \leq p$, $|y| > 0$: so for some r, s, t, $x = 0^r$, $y = 0^s$ and $z = 0^t1^p$, with $s > 0$.

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- Since $0^p1^p \in L_{\text{eq}}$, $0^k0^{(p-k)}1^p$ is accepted by M and so is $0^j0^{(p-k)}1^p$.

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- $xy^iz \in L_{\text{eq}}$ for all i: so $xy^0z \in L_{\text{eq}}$.

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- But $0^j0^{(p-k)}1^p \notin L_{\rm eq}$.

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- $xy^iz \in L_{\rm eq}$ for all i: so $xy^0z \in L_{\rm eq}$.
- But $xy^0z = 0^{p-s}1^p \notin L_{\rm eq}$

Example III

Proposition

$L_p = \{0^i \mid i \textit{ prime}\}$ *is not regular*

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Suppose L_p is regular. Let p be the pumping length for L_p.

- Consider $w = 0^m$, where $m \geq p + 2$ and m is prime.

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Proof.

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- Consider $w = 0^m$, where $m \geq p + 2$ and m is prime.
- Since $|w| > p$, there are x, y, z such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_p$, for all i.

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- Thus, $x = 0^r$, $y = 0^s$ and $z = 0^t$. Further, as $|y| > 0$, we have $s > 0$.

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- Thus, $x = 0^r$, $y = 0^s$ and $z = 0^t$. Further, as $|y| > 0$, we have $s > 0$. $xy^{r+t}z = 0^r(0^s)^{(r+t)}0^t = 0^{r+s(r+t)+t}$.

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- Thus, $x = 0^r$, $y = 0^s$ and $z = 0^t$. Further, as $|y| > 0$, we have $s > 0$. $xy^{r+t}z = 0^r(0^s)^{(r+t)}0^t = 0^{r+s(r+t)+t}$. Now $r + s(r + t) + t = (r + t)(s + 1)$. Further $m = r + s + t \geq p + 2$ and $s > 0$ mean that $t \geq 2$ and $s + 1 \geq 2$. Thus, $xy^{r+t}z \notin L_p$. Contradiction! $\square$

Example IV

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Is $L_{xx} = \{xx \mid x \in \{0,1\}^*\}$ regular?

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- Consider $w = 0^p 0^p \in L$.

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Suppose L_{xx} is regular, and let p be the pumping length of L_{xx}.

- Consider $w = 0^p0^p \in L$.
- Can we find substrings x, y, z satisfying the conditions in the pumping lemma?

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Suppose L_{xx} is regular, and let p be the pumping length of L_{xx}.

- Consider $w = 0^p0^p \in L$.
- Can we find substrings x, y, z satisfying the conditions in the pumping lemma? Yes! Consider $x = \epsilon, y = 00, z = 0^{2p-2}$.

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- Does this mean L_{xx} satisfies the pumping lemma? Does it mean it is regular?

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  - No! We have chosen a bad w. To prove that the pumping lemma is violated, we only need to exhibit some w that cannot be pumped.

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- Does this mean L_{xx} satisfies the pumping lemma? Does it mean it is regular?
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- Another bad choice $(01)^p(01)^p$.

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Suppose L_{xx} is regular. Let p be the pumping length for L_{xx}.

- Consider $w = 0^p10^p1$.
- Since $|w| > p$, there are x, y, z such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_p$, for all i.
- Since $|xy| \leq p$, $x = 0^r$, $y = 0^s$ and $z = 0^t10^p1$. Further, as $|y| > 0$, we have $s > 0$.

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$$xy^0z = 0^r\epsilon 0^t10^p1 = 0^{r+t}10^p1$$

Since $r + t < p$, $xy^0z \notin L_{xx}$. Contradiction! $\square$

Lessons on Expressivity

Limits of Finite Memory

Finite automata cannot

- "keep track of counts": e.g., L_{0n1n} not regular.
- "compare far apart pieces" of the input: e.g. L_{xx} not regular.
- do "computations that require it to look at global properties" of the input. e.g. L_{prime} not regular.

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- "keep track of counts": e.g., L_{0n1n} not regular.
- "compare far apart pieces" of the input: e.g. L_{xx} not regular.
- do "computations that require it to look at global properties" of the input. e.g. L_{prime} not regular.

...and pumping lemma provides one way to find out some of these limitations.