CS 611: Theory of Computation

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Parenthesis Matching

Problem: Describe the the set of arithmetic expressions with correctly matched parenthesis.

Solution: Ignoring numbers and variables, and focussing only on parenthesis, correctly matched expressions can be defined as

- The ϵ is a valid expression
- A valid string ($\neq \epsilon$) must either be
  - The concatenation of two correctly matched expressions, or
  - It must begin with (and end with) and moreover, once the first and last symbols are removed, the resulting string must correspond to a valid expression.

Parenthesis Matching

Grammar

Taking E to be the set of correct expressions, the inductive definition can be succinctly written as

$$E \rightarrow \epsilon$$
$$E \rightarrow EE$$
$$E \rightarrow (E)$$

English Sentences

English sentences can be described as

$$
\begin{aligned}
\langle S \rangle &\rightarrow \langle NP \rangle \langle VP \rangle \\
\langle NP \rangle &\rightarrow \langle CN \rangle \mid \langle CN \rangle \langle PP \rangle \\
\langle VP \rangle &\rightarrow \langle CV \rangle \mid \langle CV \rangle \langle PP \rangle \\
\langle PP \rangle &\rightarrow \langle P \rangle \langle CN \rangle \\
\langle CN \rangle &\rightarrow \langle A \rangle \langle N \rangle \\
\langle CV \rangle &\rightarrow \langle V \rangle \mid \langle V \rangle \langle NP \rangle \\
\langle A \rangle &\rightarrow \text{a} \mid \text{the} \\
\langle N \rangle &\rightarrow \text{boy} \mid \text{girl} \mid \text{flower} \\
\langle V \rangle &\rightarrow \text{touches} \mid \text{likes} \mid \text{sees} \\
\langle P \rangle &\rightarrow \text{with}
\end{aligned}
$$

English Sentences

Examples

$$\overbrace{\underbrace{\text{a}}_{\text{article}}\ \underbrace{\text{boy}}_{\text{noun}}}^{\text{noun-phrs}}\ \overbrace{\underbrace{\text{sees}}_{\text{verb}}}^{\text{verb-phrs}}$$

English Sentences

Examples

$$\overbrace{\underbrace{\text{a}}_{\text{article}}\ \underbrace{\text{boy}}_{\text{noun}}}^{\text{noun-phrs}}\ \overbrace{\underbrace{\text{sees}}_{\text{verb}}}^{\text{verb-phrs}}$$

$$\overbrace{\underbrace{\text{the}}_{\text{article}}\ \underbrace{\text{boy}}_{\text{noun}}}^{\text{noun-phrs}}\ \overbrace{\underbrace{\text{sees}}_{\text{verb}}\ \underbrace{\text{a flower}}_{\text{noun-phrs}}}^{\text{verb-phrs}}$$

Applications

Such rules (or grammars) play a key role in

- Parsing programming languages and natural languages
- Markup Languages like HTML and XML.
- Modelling software

Context-Free Grammars

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- R is a finite set of rules or productions of the form $A \rightarrow \alpha$ where $A \in V$ and $\alpha \in (V \cup \Sigma)^*$
- $S \in V$ is the start symbol

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$$\begin{aligned} S &\to \epsilon \\ S &\to 0 \\ S &\to 1 \\ S &\to 0S0 \\ S &\to 1S1 \end{aligned}$$

Or more briefly, $R = \{S \to \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}$

Arithmetic Expressions

Consider the language of all arithmetic expressions (E) built out of integers (N) and identifiers (I), using only $+$ and $*$
$G_{\mathrm{exp}} = (\{E, I, N\}, \{a, b, 0, 1, (,), +, *, -\}, R, E)$ where R is

$$\begin{aligned}
&E \rightarrow I \mid N \mid E + E \mid E * E \mid (E) \\
&I \rightarrow a \mid b \mid Ia \mid Ib \\
&N \rightarrow 0 \mid 1 \mid N0 \mid N1 \mid -N \mid +N
\end{aligned}$$

Language of a CFG

Derivations

Expand the start symbol using one of its rules. Further expand the resulting string by expanding one of the variables in the string, by the RHS of one of its rules. Repeat until you get a string of terminals.

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$G_{\mathrm{pal}} = (\{S\}, \{0,1\}, \{S \rightarrow \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}, S)$ we have

$$S \Rightarrow 0S0 \Rightarrow 00S00 \Rightarrow 001S100 \Rightarrow 0010100$$

Formal Definition

Definition

Let $G = (V, \Sigma, R, S)$ be a CFG. We say $\alpha A \beta \Rightarrow_G \alpha\gamma\beta$, where $\alpha, \beta, \gamma \in (V \cup \Sigma)^*$ and $A \in V$ if $A \rightarrow \gamma$ is a rule of G.

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We say $\alpha \overset{*}{\Rightarrow}_G \beta$ if either $\alpha = \beta$ or there are $\alpha_0, \alpha_1, \ldots \alpha_n$ such that

$$\alpha = \alpha_0 \Rightarrow_G \alpha_1 \Rightarrow_G \alpha_2 \Rightarrow_G \cdots \Rightarrow_G \alpha_n = \beta$$

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Notation

When G is clear from the context, we will write $\Rightarrow$ and $\overset{*}{\Rightarrow}$ instead of $\Rightarrow_G$ and $\overset{*}{\Rightarrow}_G$.

Context-Free Language

Definition

The language of CFG $G = (V, \Sigma, R, S)$, denoted $L(G)$ is the collection of strings over the terminals derivable from S using the rules in R. In other words,

$$L(G) = \{w \in \Sigma^* \mid S \overset{*}{\Rightarrow} w\}$$

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Definition

A language L is said to be context-free if there is a CFG G such that $L = L(G)$.

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Recall, $L_{\mathrm{pal}} = \{w \in \{0,1\}^* \mid w = w^R\}$ is the language of palindromes.

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Proposition

$L(G_{\mathrm{pal}}) = L_{\mathrm{pal}}$

Proving Correctness of CFG

$L_{\mathrm{pal}} \subseteq L(G_{\mathrm{pal}})$

Proof.

Let $w \in L_{\mathrm{pal}}$. We prove that $S \stackrel{*}{\Rightarrow} w$

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- Base Cases: If $|w| = 0$ or $|w| = 1$ then $w = \epsilon$ or 0 or 1. And $S \rightarrow \epsilon \mid 0 \mid 1$.
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Proof (contd).

Let $w \in L(G)$, i.e., $S \overset{*}{\Rightarrow} w$. We will show $w \in L_{\mathrm{pal}}$

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Let $w \in L(G)$, i.e., $S \stackrel{*}{\Rightarrow} w$. We will show $w \in L_{\rm pal}$ by induction on the number of derivation steps.

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Let $w \in L(G)$, i.e., $S \overset{*}{\Rightarrow} w$. We will show $w \in L_{\text{pal}}$ by induction on the number of derivation steps.

- Base Case: If the derivation has only one step then the derivation must be $S \Rightarrow \epsilon$, $S \Rightarrow 0$ or $S \Rightarrow 1$. Thus $w = \epsilon$ or 0 or 1 and is in L_{Pal}.

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- Induction Step: Consider an $(n+1)$-step derivation of w. It must be of the form $S \Rightarrow 0S0 \stackrel{*}{\Rightarrow} 0x0 = w$ or $S \Rightarrow 1S1 \stackrel{*}{\Rightarrow} 1x1 = w$.

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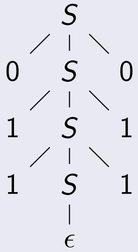
Parse Trees

For CFG $G = (V, \Sigma, R, S)$, a parse tree (or derivation tree) of G is a tree satisfying the following conditions:



Example Parse Tree with yield 011110

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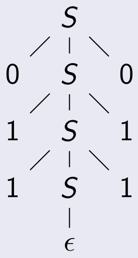
For CFG $G = (V, \Sigma, R, S)$, a parse tree (or derivation tree) of G is a tree satisfying the following conditions:

- Each interior node is labeled by a variable in V



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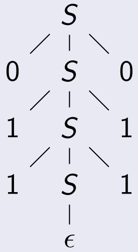

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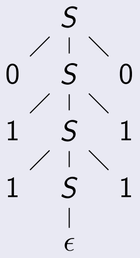
- If an interior node labeled by A with children labeled by $X_1, X_2, \ldots X_k$ (from the left), then $A \rightarrow X_1 X_2 \cdots X_k$ must be a rule.



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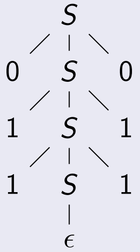

Example Parse Tree with yield 011110

Yield of a parse tree is the concatenation of leaf labels (left–right)

Parse Trees and Derivations

Proposition

Let $G = (V, \Sigma, R, S)$ be a CFG. For any $A \in V$ and $\alpha \in (V \cup \Sigma)^*$, $A \stackrel{*}{\Rightarrow} \alpha$ iff there is a parse tree with root labeled A and whose yield is α.

Parse Trees and Derivations

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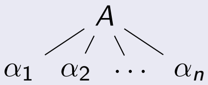
Proof.

($\Rightarrow$): Proof by induction on the number of steps in the derivation.

- Base Case: If $A \Rightarrow \alpha$ then $A \rightarrow \alpha$ is a rule in G. There is a tree of height 1, with root A and leaves the symbols in α. $\cdots\!\longrightarrow$



Parse Tree for Base Case

Parse Trees for Derivations

Proof (contd).

(⇒): Proof by induction on the number of steps in the derivation.

- Induction Step: Let $A \overset{*}{\Rightarrow} \alpha$ in $k+1$ steps.

Parse Trees for Derivations

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- Induction Step: Let $A \stackrel{*}{\Rightarrow} \alpha$ in $k + 1$ steps.
- Then $A \stackrel{*}{\Rightarrow} \alpha_1 X \alpha_2 \Rightarrow \alpha_1 \gamma \alpha_2 = \alpha$, where $X \rightarrow X_1 \cdots X_n = \gamma$ is a rule

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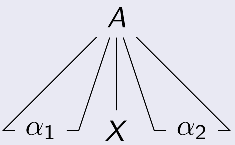

Parse Tree for Induction Step

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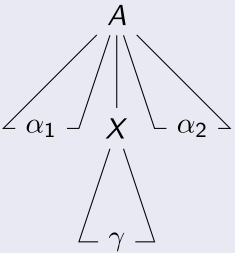
- Then $A \stackrel{*}{\Rightarrow} \alpha_1 X \alpha_2 \Rightarrow \alpha_1 \gamma \alpha_2 = \alpha$, where $X \rightarrow X_1 \cdots X_n = \gamma$ is a rule
- By ind. hyp., there is a tree with root A and yield $\alpha_1 X \alpha_2$.
- Add leaves $X_1, \ldots X_n$ and make them children of X. New tree is a parse tree with desired yield. $\cdots\rightarrow$



Parse Tree for Induction Step

Derivations for Parse Trees

Proof (contd).

($\Leftarrow$): Assume that there is a parse tree with root A and yield α. Need to show that $A \stackrel{*}{\Rightarrow} \alpha$.

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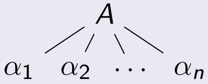
- Base Case: If tree has only one internal node, then it has the form as in picture



Parse Tree with one internal node

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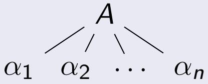
Derivations for Parse Trees

Proof (contd).

($\Leftarrow$): Assume that there is a parse tree with root A and yield α. Need to show that $A \stackrel{*}{\Rightarrow} \alpha$. Proof by induction on the number of internal nodes in the tree.

- Base Case: If tree has only one internal node, then it has the form as in picture
- Then, $\alpha = X_1 \cdots X_n$ and $A \rightarrow \alpha$ is a rule. Thus, $A \stackrel{*}{\Rightarrow} \alpha$.

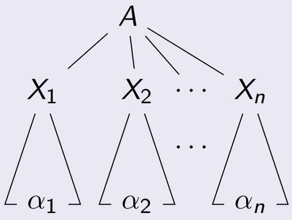


Parse Tree with one internal node

Derivations for Parse Trees

Proof (contd).

($\Leftarrow$) Induction Step: Suppose α is the yield of a tree with $k+1$ interior nodes. Let $X_1, X_2, \ldots X_n$ be the children of the root ordered from the left. Not all X_i are leaves, and $A \to X_1 X_2 \cdots X_n$ must be a rule.



Tree with $k+1$ internal nodes

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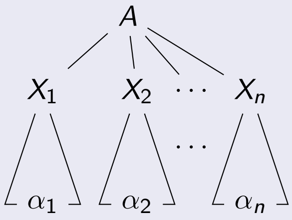
- Let α_i be the yield of the tree rooted at X_i; so X_i is a leaf $\alpha_i = X_i$



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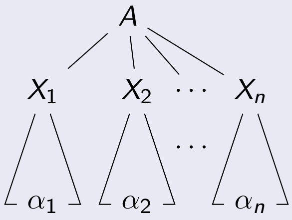
- Let α_i be the yield of the tree rooted at X_i; so X_i is a leaf $\alpha_i = X_i$
- Now if $j < i$ then all the descendents of X_j are to the left of the descendents of X_i. So $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$.

$\cdots\!\to$



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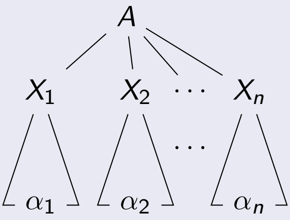

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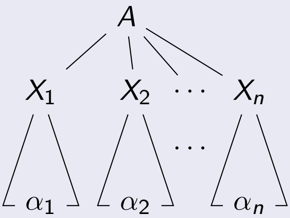

Derivations for Parse Trees

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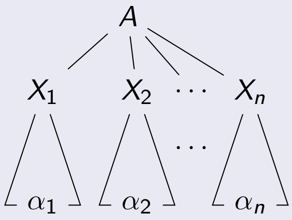
- Thus $A \Rightarrow X_1 X_2 \cdots X_n \overset{*}{\Rightarrow} \alpha_1 X_2 \cdots X_n \overset{*}{\Rightarrow} \alpha_1 \alpha_2 \cdots X_n \overset{*}{\Rightarrow} \alpha_1 \cdots \alpha_n = \alpha$ $\square$



Recap . . .

For a CFG G with variable A the following are equivalent

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Context-free-ness

CFGs have the property that if $X \stackrel{*}{\Rightarrow} \gamma$ then $\alpha X \beta \stackrel{*}{\Rightarrow} \alpha\gamma\beta$

Example: English Sentences

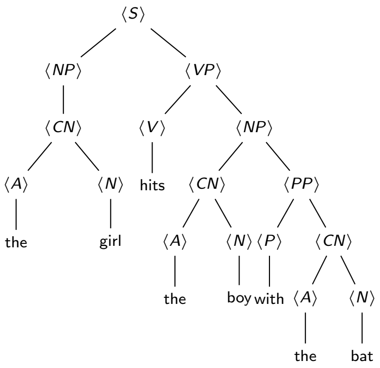
English sentences can be described as

$$
\begin{aligned}
\langle S\rangle &\rightarrow \langle NP\rangle\langle VP\rangle\\
\langle NP\rangle &\rightarrow \langle CN\rangle \mid \langle CN\rangle\langle PP\rangle\\
\langle VP\rangle &\rightarrow \langle CV\rangle \mid \langle CV\rangle\langle PP\rangle\\
\langle PP\rangle &\rightarrow \langle P\rangle\langle CN\rangle\\
\langle CN\rangle &\rightarrow \langle A\rangle\langle N\rangle\\
\langle CV\rangle &\rightarrow \langle V\rangle \mid \langle V\rangle\langle NP\rangle\\
\langle A\rangle &\rightarrow \text{a} \mid \text{the}\\
\langle N\rangle &\rightarrow \text{boy} \mid \text{girl} \mid \text{bat}\\
\langle V\rangle &\rightarrow \text{hits} \mid \text{likes} \mid \text{sees}\\
\langle P\rangle &\rightarrow \text{with}
\end{aligned}
$$

Multiple Parse Trees

Example 1

The sentence “the girl hits the boy with the bat” has the following parse tree



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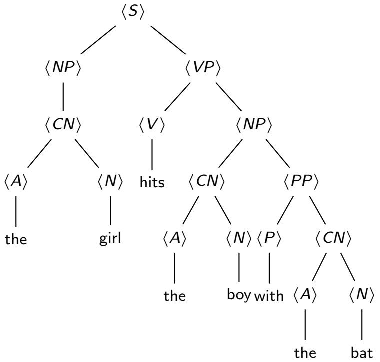

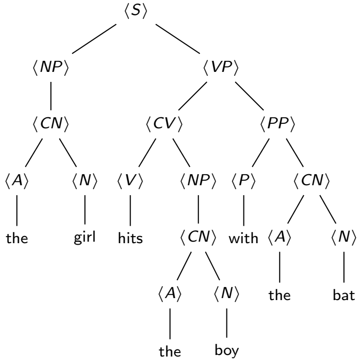

Example: Arithmetic Expressions

Consider the language of all arithmetic expressions (E) built out of integers (N) and identifiers (I), using only $+$ and $*$

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$G_{\rm exp} = (\{E, I, N\}, \{a, b, 0, 1, (,), +, *, -\}, R, E)$ where R is

$$\begin{aligned} E &\rightarrow I \mid N \mid -N \mid E + E \mid E * E \mid (E) \\ I &\rightarrow a \mid b \mid Ia \mid Ib \\ N &\rightarrow 0 \mid 1 \mid N0 \mid N1 \end{aligned}$$

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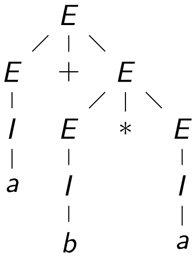
The parse tree for expression $a + b * a$ in the grammar G_{exp} is



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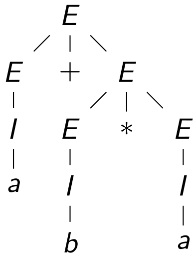

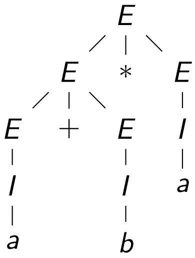

Ambiguity

Definition

A grammar $G = (V, \Sigma, R, S)$ is said to be ambiguous if there is $w \in \Sigma^*$ for which there are two different parse trees.

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Warning!

Existence of two derivations for a string does not mean the grammar is ambiguous!

Removing Ambiguity

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- Adding precedence to operators. For example, $*$ binds more tightly than $+$, or “else” binds with the innermost “if”.

An Example

Recall, G_{exp} has the following rules

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New CFG G'_{exp} has the rules

$$I \rightarrow a \mid b \mid Ia \mid Ib$$
$$N \rightarrow 0 \mid 1 \mid N0 \mid N1$$
$$F \rightarrow I \mid N \mid -N \mid (E)$$
$$T \rightarrow F \mid T*F$$
$$E \rightarrow T \mid E+T$$

Ambiguity: Computational Problems

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Problem: Given CFG G, find CFG G' such that $L(G) = L(G')$ and G' is unambiguous.

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Problem: Given CFG G, determine if G is ambiguous.
The problem is undecidable.

Problem: Is it the case that for every CFG G, there is a grammar G' such that $L(G) = L(G')$ and G' is unambiguous, *even if G' cannot be constructed algorithmically*?

Inherently Ambiguous Languages

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Definition

A context-free language L is said to be inherently ambiguous if every grammar G for L is ambiguous.

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Consider

$$L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$$

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