CS 611: Theory of Computation

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Regular Expressions and Regular Languages

Why do they have such similar names?

Theorem

L is a regular language if and only if there is a regular expression R such that $L(R) = L$

i.e., Regular expressions have the same "expressive power" as finite automata.

Proof.

- Given regular expression R, can construct NFA N such that $L(N) = L(R)$
- Given DFA M, will construct regular expression R such that $L(M) = L(R)$ □

DFA to Regular Expression

- Given DFA M, will construct regular expression R such that $L(M) = L(R)$. In two steps:
  - Construct a "Generalized NFA" (GNFA) G from the DFA M
  - And then convert G to a regex R

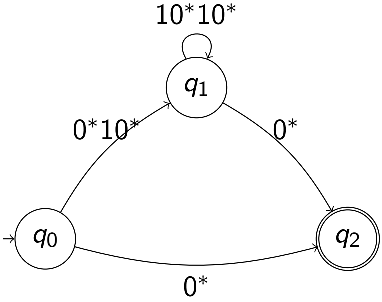
Generalized NFA

- A GNFA is similar to an NFA, but:
  - There is a single accept state.
  - The start state has no incoming transitions, and the accept state has no outgoing transitions.
    - These are "cosmetic changes": Any NFA can be converted to an equivalent NFA of this kind.
  - The transitions are labeled not by characters in the alphabet, but by regular expressions.
    - For every pair of states (q_1, q_2), the transition from q_1 to q_2 is labeled by a regular expression $\rho(q_1, q_2)$.
  - "Generalized NFA" because a normal NFA has transitions labeled by ϵ, elements in Σ (a union of elements, if multiple edges between a pair of states) and $\emptyset$ (missing edges).

Generalized NFA

- Transition: GNFA non-deterministically reads a block of characters from the input, chooses an edge from the current state q_1 to another state q_2, and if the block of symbols matches the regex $\rho(q_1, q_2)$, then moves to q_2.
- Acceptance: G accepts w if there exists some sequence of valid transitions such that on starting from the start state, and after finishing the entire input, G is in the accept state.

Generalized NFA: Example



Example GNFA G

Accepting run of G on 11110100 is

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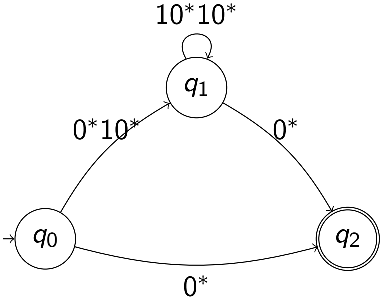

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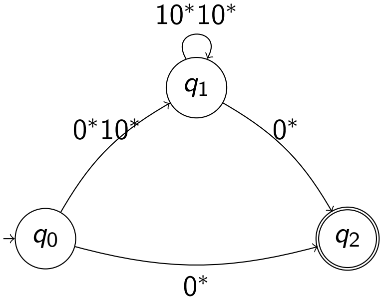
Accepting run of G on 11110100 is

$q_0 \xrightarrow{1}_G q_1$

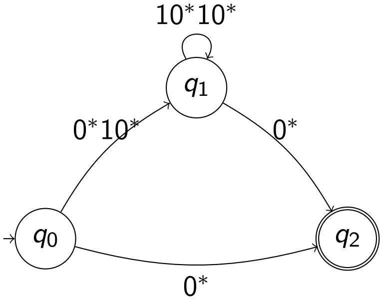
Generalized NFA: Example



Example GNFA G

Accepting run of G on 11110100 is
$q_0 \xrightarrow{1}_G q_1 \xrightarrow{11}_G q_1$

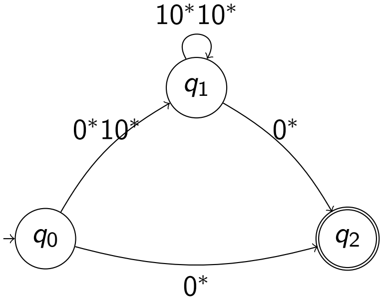
Generalized NFA: Example



Example GNFA G

Accepting run of G on 11110100 is
$q_0 \xrightarrow{1}_G q_1 \xrightarrow{11}_G q_1 \xrightarrow{101}_G q_1$

Generalized NFA: Example



Example GNFA G

Accepting run of G on 11110100 is
$q_0 \xrightarrow{1}_G q_1 \xrightarrow{11}_G q_1 \xrightarrow{101}_G q_1 \xrightarrow{00}_G q_2$

Generalized NFA: Definition

Definition

A generalized nondeterministic finite automaton (GNFA) is $G = (Q, \Sigma, q_0, q_F, \rho)$, where

- Q is the finite set of states
- Σ is the finite alphabet
- $q_0 \in Q$ initial state

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- $q_0 \in Q$ initial state
- $q_F \in Q$, a single accepting state
- $\rho : (Q \setminus \{q_F\}) \times (Q \setminus \{q_0\}) \to \mathcal{R}_\Sigma$, where $\mathcal{R}_\Sigma$ is the set of all regular expressions over the alphabet Σ

Generalized NFA: Definition

Definition

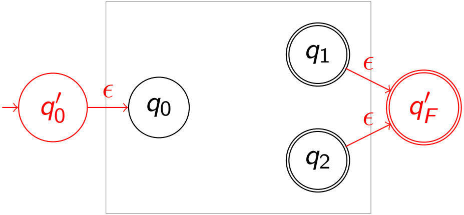
For a GNFA $M = (Q, \Sigma, q_0, q_F, \rho)$ and string $w \in \Sigma^*$, we say M **accepts** w iff there exist $x_1, \ldots, x_t \in \Sigma^*$ and states $r_0, \ldots, r_t$ such that

- $w = x_1 x_2 x_3 \cdots x_t$
- $r_0 = q_0$ and $r_t = q_F$
- for each $i \in [1, t]$, $x_i \in L(\rho(r_{i-1}, r_i))$,

Converting DFA to GNFA

A DFA $M = (Q, \Sigma, \delta, q_0, F)$ can be easily converted to an equivalent GNFA $G = (Q', \Sigma, q'_0, q'_F, \rho)$:

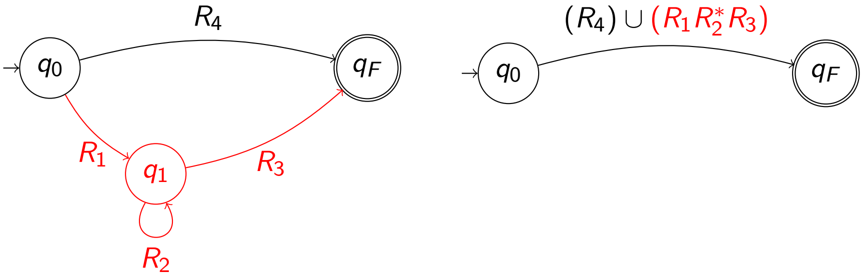
- $Q' = Q \cup \{q'_0, q'_F\}$ where $Q \cap \{q'_0, q'_F\} = \emptyset$
- $$\rho(q_1, q_2) = \begin{cases} \epsilon, & \text{if } q_1 = q'_0 \text{ and } q_2 = q_0 \\ \epsilon, & \text{if } q_1 \in F \text{ and } q_2 = q'_F \\ \bigcup_{\{a \mid \delta(q_1, a) = q_2\}} a & \text{otherwise} \end{cases}$$



Prove: $L(G) = L(M)$.

GNFA to Regex

- Suppose G is a GNFA with only two states, q_0 and q_F.
- Then $L(R) = L(G)$ where $R = \rho(q_0, q_F)$.
- How about G with three states?



- Plan: Reduce any GNFA G with $k > 2$ states to an equivalent GFA with $k - 1$ states.

GNFA to Regex: From k states to $k-1$ states

Definition (Deleting a GNFA State)

Given GNFA $G = (Q, \Sigma, q_0, q_F, \rho)$ with $|Q| > 2$, and any state $q^* \in Q \setminus \{q_0, q_F\}$, define GNFA $\mathrm{rip}(G, q^*) = (Q', \Sigma, q_0, q_F, \rho')$ as follows:

- $Q' = Q \setminus \{q^*\}$.

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- $Q' = Q \setminus \{q^*\}$.
- For any $(q_1, q_2) \in Q' \setminus \{q_F\} \times Q' \setminus \{q_0\}$ (possibly $q_1 = q_2$), let

$$\rho'(q_1, q_2) = (R_1 R_2^* R_3) \cup R_4,$$

  where $R_1 = \rho(q_1, q^*)$, $R_2 = \rho(q^*, q^*)$, $R_3 = \rho(q^*, q_2)$ and $R_4 = \rho(q_1, q_2)$.

Claim. For any $q^* \in Q \setminus \{q_0, q_F\}$, G and $\text{rip}(G, q^*)$ are equivalent.

GNFA to Regex: From k states to $k-1$ states

$w \in L(G) \implies w \in L(G')$

Proof.

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- $w \in L(G) \implies w = x_1 x_2 x_3 \cdots x_t$, and a sequence of states $q_0 = r_0, r_1, \ldots, r_t = q_F$ s.t. $x_i \in L(\rho(r_{i-1}, r_i))$.

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  - Case $a = b$. $(s_{j-1}, s_j) = (r_{b-1}, r_b)$ and $x_{[a,b]} = x_b \in L(R_4)$.

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  - Case $a = b$. $(s_{j-1}, s_j) = (r_{b-1}, r_b)$ and $x_{[a,b]} = x_b \in L(R_4)$.
  - Case $a = b + 1 + u$. $x_a \in L(R_1)$, $x_{a+1}, \ldots, x_{b-1} \in L(R_2)$ and $x_b \in L(R_3)$. So $x_{[a,b]} \in L(R_1 R_2^u R_3)$.

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i.e., $w \in L(G) \implies w \in L(G')$.

GNFA to Regex: From k states to $k-1$ states

$w \in L(G') \implies w \in L(G)$

Proof (contd).

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- To build a sequence of blocks $x_1, \ldots, x_t$ and a sequence of states $q_0 = r_0, \ldots, r_t = q_F$ to show $w \in L(G)$:

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  - Case $y_j \in L(R_1 R_2^* R_3)$. $y_j = z_0 \cdots z_{u+1}$ where $z_0 \in L(R_1)$, $z_1, \ldots, z_u \in L(R_2)$ and $z_{u+1} = L(R_3)$ (for some finite u). Insert $u+1$ copies of q^* between s_{j-1} and s_j. Divide y_j into $u+2$ blocks $(z_0, \ldots, z_{u+1})$. □

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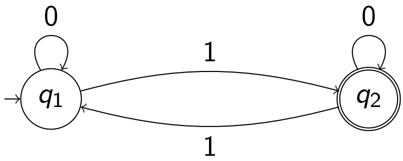
- For any GNFA $G = (Q, \Sigma, q_0, q_F, \rho)$ with $|Q| > 2$, for any $q^* \in Q \setminus \{q_0, q_F\}$, G and $\text{rip}(G, q^*)$ are equivalent. $\text{rip}(G, q^*)$ has one fewer state than G.
- So given G, by applying rip repeatedly (choosing q^* arbitrarily each time), we can get a GNFA G' with two states s.t. $L(G) = L(G')$. Formally, by induction on the number of states in G.
- For a 2-state GNFA G', $L(G') = L(R)$, where $R = \rho(q_0, q_F)$.

DFA to Regex: Example



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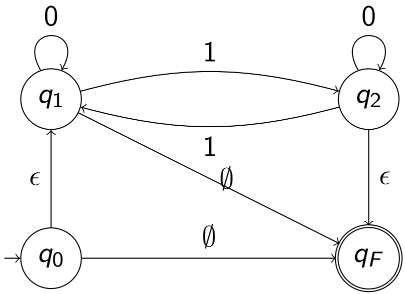

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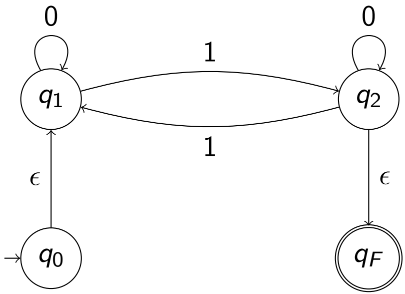

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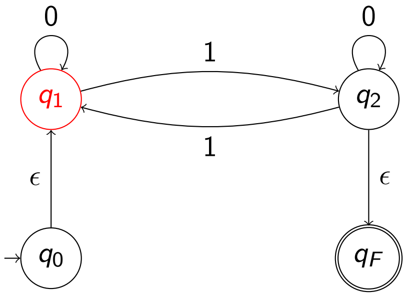
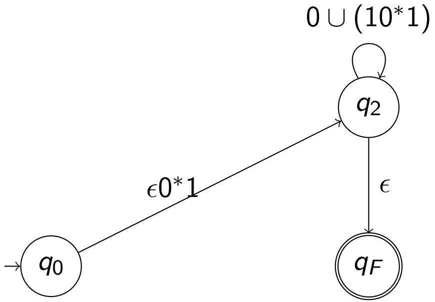
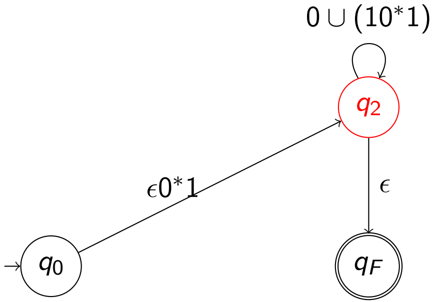


DFA to Regex: Example



DFA to Regex: Example



DFA to Regex: Example