Problem 1.

1. Formulating Proposition $P(n)$

Let $P(n)$ denote $2^n < n!$. We need to show that $P(n)$ is true for all $n \geq 4$

2. Proving the base case, $n=4$

$P(n) : 2^n < n!$

LHS :- $P(4) : 2^4 = 16$

RHS :- $P(4) : 4! = 24$

Since, LHS < RHS we have proved $P(n)$ is true for base case.

3. Induction Hypothesis:

Let us assume that $P(4), P(5), \ldots P(K)$ is true

4. Proving $P(K+1)$ is true assuming the induction hypothesis is true

$P(K+1) : 2^{K+1} < (K+1)!$

LHS :- 2^{K+1}

$\Rightarrow 2^K \cdot 2$

According to induction hypothesis, $P(K)$ is true. So,

$P(K) : 2^K < K!$

Multiplying by 2 on both sides don't change inequality

so, $2^K \cdot 2 < 2 \cdot K!$

so, $2^{K+1} < 2 \cdot K!$ — ①

Since, $K \geq 4$ we can say for sure that

$2 < K+1$

Multiplying by $K!$ on both sides don't change inequality since $K! > 0$

$2 \cdot K! < (K+1) K!$

$2 \cdot K! < (K+1)!$ — ②

From 1 & 2 since, if $a < b$, $b < c$ then $a < c$

$$2^{K+1} < (K+1)!$$

which is $P(K+1)$

∴ We proved $P(K+1)$ is true assuming induction hypothesis is true

∴ According to Induction technique, since we showed that base case & induction step are true, we can say that $P(n)$ is true for all $n \geq 4$

Problem 3.

Given, $A = \{1, 2, 3, 4, 5\}$

$R \subseteq A \times A$ is a relation $\{(a,b) \mid a-b$ is a multiple of $2\}$

Now, $R = \{(1,1), (1,3), (1,5),$
$(2,2), (2,4),$
$(3,1), (3,3), (3,5),$
$(4,2), (4,4),$
$(5,1), (5,3), (5,5).$
$\}$

1. A binary relation $R \subseteq A \times A$ is an equivalence relation if it satisfies:
   - Reflexivity : for every $a \in A$, $(a,a) \in R$
   - Symmetricity : for every $a,b \in A$, if $(a,b) \in R$ then $(b,a) \in R$
   - Transitivity : for every $a,b,c \in A$, if $(a,b) \in R$ & $(b,c) \in R$, then $(a,c) \in R$

a) Reflexivity:

R is reflexive because $\forall a$, $(a,a) \in R$

Since $a - a = 0$ is a multiple of 2

b) Symmetricity:

Let $(a,b) \in R$. So, $a-b$ is a multiple of 2.

Let $a-b = 2k$ — ①

Now, consider $(b,a) \in R$. It is true if $b-a$ is a multiple of 2

$\Rightarrow b-a = -(a-b)$

From ①

$\Rightarrow b-a = -2k$

Since, $b-a$ is a multiple of 2, $(b,a) \in R$

$\therefore$ R is symmetric

c) Transitivity:

Let $(a,b) \in R$ & $(b,c) \in R$

So, assume

$a-b = 2k_1$ — ①

$b-c = 2k_2$ — ②

Add ① & ②

$a-b+b-c = 2k_1 + 2k_2$

$a-c = 2(k_1+k_2)$ — ③

Now, consider $(a,c) \in R$. It is true if $a-c$ is a multiple of 2.

From ③ $a-c = 2(k_1+k_2)$ which is a multiple of 2. So, $(a,c) \in R$

$\therefore$ R is transitive

Therefore, given relation R is an equivalence relation

2. $[1]\_R = \{1,3,5\}$

$[2]\_R = \{2,4\}$

$[3]\_R = \{1,3,5\}$

$[4]\_R = \{2,4\}$

$[5]\_R = \{1,3,5\}$

3. R has 2 distinct ~~ive~~ equivalence classes. They are : $\{1,3,5\}$ & $\{2,4\}$

4. If A is set of all natural numbers & R is defined as above, then R will still have 2 distinct equivalence classes : One with all even numbers & another with all odd numbers.