# Homework 1
## CS611: Theory of Computation

**Instructions:** This homework has **2 required problems and 1 optional problem** that must be solved **individually**.

**Recommended Reading:** Chapter 0 of Introduction to the theory of computation

**Strong Induction**

To prove that $P(n)$ is true for all positive integers, where $P(n)$ is a propositional statement, we prove the following:
**Base Case:** We prove that the proposition $P(1)$ is true.
**Induction Hypothesis:** We assume that $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)]$ is true.
**Induction Step:** We show that $P(k+1)$ is true.
In strong induction, you can assume $P(1), P(2), \ldots, P(k)$ are all true, while proving $P(k+1)$ is true. This is in contrast to the proof in class where you only assume $P(k)$ is true.

**Problem 1**. [Category: Proof] Use mathematical induction to prove that $2^n < n!$ for every integer $n$ with $n \geq 4$. (Note that this inequality is false for $n = 1$, 2, and 3.)
We will use induction to prove the above statement:

1. Formulate the problem in terms of proposition $P(n)$;
2. Prove the base case, $P(4)$ is true;
3. Write the induction hypothesis;
4. Prove that $P(k+1)$ is true assuming the induction hypothesis is true.

**Problem 2**. [Category: OPTIONAL] Show that if $n$ is an integer greater than 1, then $n$ can be written as the product of primes.
We will use strong induction to prove the above statement:

1. Formulate the problem in terms of proposition $P(n)$;
2. Prove the base case, $P(2)$ is true;
3. Write the strong induction hypothesis;
4. Prove that $P(k+1)$ is true assuming the induction hypothesis is true.

**Problem 3**. [Category: Comprehensive] Let $A = \{1, 2, 3, 4, 5\}$, $R \subseteq A \times A$ be the relation $\{(a, b) | a - b$ is a multiple of $2\}$.

1. Show that $R$ is an equivalence relation. Recall that $R$ is an equivalence relation if it is reflexive ($\forall a \in A, (a, a) \in R$), symmetric ($\forall a, b \in A$, if $(a, b) \in R$, then $(b, a) \in R$), and transitive ($\forall a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$).

2. Given $a \in A$, let $[a]\_R$, called an equivalence class, be the set of all elements related to $a$ through $R$, that is, $[a]\_R = \{b|(a, b) \in R\}$. What is $[1]\_R, [2]\_R, [3]\_R, [4]\_R, [5]\_R$?

3. How many distinct equivalence classes are there for $R$?

4. If $A$ is the set of all natural numbers and $R$ is defined as above, then how many distinct equivalence classes does it have?