CS 611: Theory of Computation

Hongmin Li

Department of Computer Science
California State University, East Bay

Expressive Power of NFAs and DFAs

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In other words:

- *For any DFA D, there is an NFA N such that $L(N) = L(D)$, and*
- *for any NFA N, there is a DFA D such that $L(D) = L(N)$.*

Converting DFAs to NFAs

Proposition

For any DFA D, there is an NFA N such that $L(N) = L(D)$.

Proof.

Is a DFA an NFA? Essentially yes! Syntactically, not quite. The formal definition of DFA has $\delta_{\mathsf{DFA}} : Q \times \Sigma \to Q$ whereas $\delta_{\mathsf{NFA}} : Q \times (\Sigma \cup \{\epsilon\}) \to \mathcal{P}(Q)$.

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For DFA $D = (Q, \Sigma, \delta_D, q_0, F)$, define an "equivalent" NFA $N = (Q, \Sigma, \delta_N, q_0, F)$ that has the exact same set of states, initial state and final states. Only difference is in the transition function.

$$\delta_N(q, a) = \{\delta_D(q, a)\}$$

for $a \in \Sigma$ and $\delta_N(q, \epsilon) = \emptyset$ for all $q \in Q$. □

Simulating an NFA on Your Computer

NFA Acceptance Problem

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How do we write a computer program to solve the NFA Acceptance problem?

Two Views of Nondeterminsm

Guessing View

At each step, the NFA “guesses” one of the choices available; the NFA will guess an “accepting sequence of choices” if such a one exists.

Parallel View

At each step the machine “forks” threads corresponding to each of the possible next states.

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Parallel View

At each step the machine "forks" threads corresponding to each of the possible next states.

Very useful in simulating/running NFA on inputs.

Algorithm for Simulating an NFA

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Keep track of the current state of each of the active threads.

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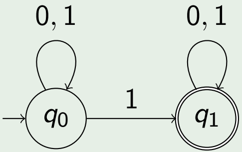
Example



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Consider the input $w = 111$. The execution (listing only the states of currently active threads) is

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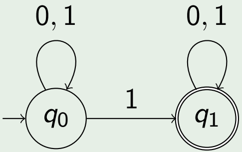

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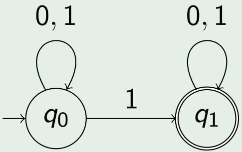

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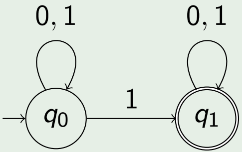

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Observations

- Exponentially growing memory: more threads for longer inputs. Can we do better?

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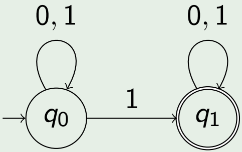
- If two threads are in the same state, then we can ignore one of the threads
  - Threads in the same state will "behave" identically; either one of the descendent threads of both will reach a final state, or none of the descendent threads of both will reach a final state

Parsimonious Algorithm in Action

Example



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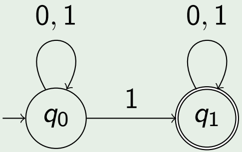

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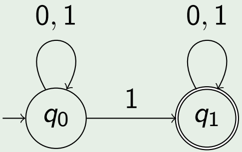

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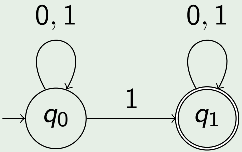

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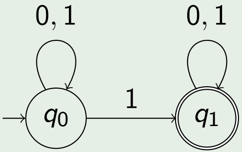

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Revisiting NFA Simulation Algorithm

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Revisiting NFA Simulation Algorithm

- Need to keep track of the states of the active threads
  - Unordered: Without worrying about exactly which thread is in what state
  - No Duplicates: Keeping only one copy if there are multiple threads in same state
- How much memory is needed?
  - If Q is the set of states of the NFA N, then we need to keep a subset of Q!
  - Can be done in $|Q|$ bits of memory (i.e., $2^{|Q|}$ states), which is finite!!

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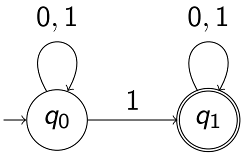
- The DFA runs the simulation algorithm
- DFA remembers the current states of active threads without duplicates, i.e., maintains a subset of states of the NFA
- When a new symbol is read, it updates the states of the active threads
- Accepts whenever one of the threads is in a final state

Example of Equivalent DFA



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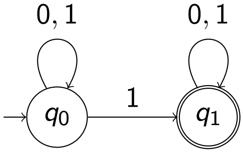

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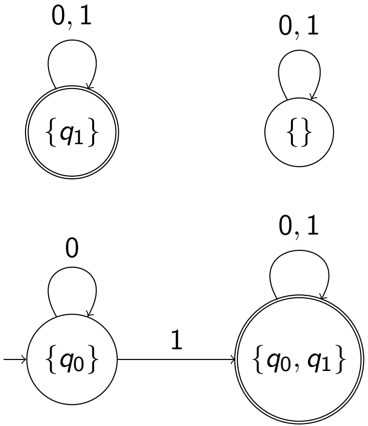

DFA D equivalent to N

Recall . . .

Definition

For an NFA $M = (Q, \Sigma, \delta, q_0, F)$, string w, and state $q_1 \in Q$, we say $\hat{\Delta}(q_1, w)$ to denote states of all the active threads of computation on input w from q_1.

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$$\hat{\Delta}(q_1, w) = \{q \in Q \mid q_1 \xrightarrow{w}_M q\}$$

Formal Construction

Given NFA $N = (Q, \Sigma, \delta, q_0, F)$, construct DFA $\det(N) = (Q', \Sigma, \delta', q_0', F')$ as follows.

- $Q' =$
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$$\delta'(A, a) = \bigcup_{q \in A} \hat{\Delta}(q, a)$$

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$$\forall w \in \Sigma^*.\ \mathrm{det}(N) \text{ accepts } w \text{ iff } N \text{ accepts } w$$
$$\forall w \in \Sigma^*. \hat{\delta}(q_0', w) \in F' \text{ iff } \hat{\Delta}(q_0, w) \cap F \neq \emptyset$$

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$$\forall w \in \Sigma^*.\ \text{for } A = \hat{\delta}(q_0', w),\ A \cap F \neq \emptyset \text{ iff } \hat{\Delta}(q_0, w) \cap F \neq \emptyset$$

We will instead prove the stronger claim $\forall w \in \Sigma^*.\ \hat{\delta}(q_0', w) = A$ iff $\hat{\Delta}(q_0, w) = A$.

Correctness Proof

Lemma

$\forall w \in \Sigma^*.\ \hat{\delta}(q_0', w) = A$ *iff* $\hat{\Delta}(q_0, w) = A$.

Proof.

By induction on $|w|$

- Base Case $|w| = 0$: Then $w = \epsilon$. Now

$$\hat{\delta}(q_0', \epsilon) = q_0' \qquad \text{defn. of } \hat{\delta}$$

- Induction Hypothesis: Assume inductively that the statement holds $\forall w.\ |w| = n$

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$$\begin{aligned} \hat{\delta}(q_0', \epsilon) &= q_0' && \text{defn. of } \hat{\delta} \\ &= \hat{\Delta}(q_0, \epsilon) && \text{defn. of } q_0' \end{aligned}$$

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Correctness Proof

Induction Step

Proof (contd).

- Induction Step: If $|w| = n+1$ then $w = ua$ with $|u| = n$ and $a \in \Sigma$.

$$\hat{\delta}(q_0', ua) = \delta(\hat{\delta}(q_0', u), a) \qquad \text{defn. of } \hat{\delta}$$

$\square$

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&= \bigcup_{q \in \hat{\Delta}(q_0, u)} \hat{\Delta}(q, a) && \text{defn. of } \delta
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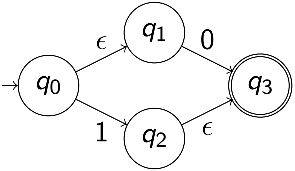
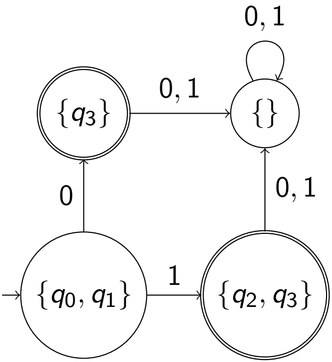
&= \bigcup_{q \in \hat{\Delta}(q_0, u)} \hat{\Delta}(q, a) && \text{defn. of } \delta \\
&= \hat{\Delta}(q_0, ua) && \text{prop. about } \hat{\Delta}
\end{aligned}
$$

□

Another Example



Example NFA N_ϵ



DFA D'_ϵ for N_ϵ (only relevant states)