Chapter 13

Bayesian Games

We have so far studied strategic form games with complete information, where the the entire game is common knowledge to the players. We will now study games with incomplete information, where at least one player has *private information* about the game which the other players may not know. While complete information games provide a convenient and useful abstraction for strategic situations, incomplete information games are more realistic. Incomplete information games are central to the theory of mechanism design. In this chapter, we study a particular form of these games called *Bayesian games* and introduce the important notion of *Bayesian Nash equilibrium*.

13.1 Games with Incomplete Information

A game with *incomplete information* is one in which, when the players are ready to make a move, at least one player has *private information* about the game which the other players may not know. The initial private information that a player has, just before making a move in the game, is called the *type* of the player. For example, in an auction involving a single indivisible item, each player has a valuation for the item, and typically this player would know this valuation deterministically while the other players may only have probabilistic information about how much this player values the item.

Definition 13.1 (Strategic Form Game with Incomplete Information). *A strategic form game with incomplete information is defined as a tuple* $\Gamma = \langle N, (\Theta_i), (S_i), (p_i), (u_i) \rangle$ *where*

- $N = \{1, 2, \ldots, n\}$ *is the set of players.*
- Θ_i *is the set of types of player* i *where* $i = 1, 2, \ldots, n$.
- S_i *is the set of actions or pure strategies of player* i *where* $i = 1, 2, \ldots, n$.
- *The belief function* p_i *is a mapping from* Θ_i *into* $\Delta(\Theta_{-i})$*, the set of probability distributions over* Θ_{-i}*. That is, for any possible type* $\theta_i \in \Theta_i$*,* p_i *specifies a probability distribution* $p_i(.|\theta_i)$ *over the set* Θ_{-i} *representing player* i*'s beliefs about the types of the other players if his own type were* θ_i*;*

- *The payoff function* $u_i : \Theta_1 \times \ldots \times \Theta_n \times S_1 \times \ldots \times S_n \to \mathbb{R}$ *assigns to each profile of types and each profile of actions, a payoff that player* i *would get.*

When we study such a game, we assume that

(1) Each player i knows the entire structure of the game as defined above.
(2) Each player i knows his own type $\theta_i \in \Theta_i$. The player learns his type through some signals and each element in his type set is a summary of the information gleaned from the signals.
(3) The above facts are common knowledge among all the players in N.
(4) The exact type of a player is not known deterministically to the other players who however have a probabilistic guess of what this type is. The belief functions p_i describe these conditional probabilities. Note that the belief functions p_i are also common knowledge among the players.

Bayesian Games

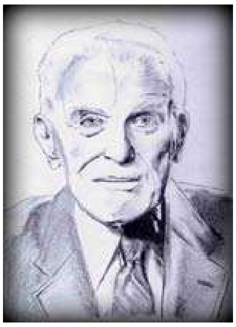
John Harsanyi (joint winner of the Nobel Prize in Economic Sciences in 1994 with John Nash and Reinhard Selten) proposed in 1968, *Bayesian* games to represent games with incomplete information. We first define the notion of consistency which is a natural and reasonable assumption to make in games with incomplete information.

Definition 13.2 (Consistency of Beliefs). *We say beliefs* $(p_i)_{i \in N}$ *are* consistent *if there is some common prior distribution over the set of type profiles* Θ *such that each player's beliefs given his type are just the conditional probability distributions that can be computed from the prior distribution.*

If the game is finite, beliefs are consistent if there exists some probability distribution $\mathbb{P} \in \Delta(\Theta)$ such that

$$p_i(\theta_{-i}|\theta_i) = \frac{\mathbb{P}(\theta_i, \theta_{-i})}{\sum\limits_{t_{-i} \in \Theta_{-i}} \mathbb{P}(\theta_i, t_{-i})} \; \forall \theta_i \in \Theta_i; \; \forall \theta_{-i} \in \Theta_{-i}; \; \forall i \in N.$$

In a consistent model, differences in beliefs among players can be logically explained by differences in information. If the model is not consistent, differences in beliefs among players can only be explained by differences of opinion that cannot be derived from any differences in information and must be simply assumed a priori [1]. When consistency of beliefs is satisfied, we refer to the games as *Bayesian games*.



In 1994, **John Charles Harsanyi** was awarded the Nobel Prize in Economic Sciences, jointly with Professor John Nash and Professor Reinhard Selten, for their pioneering analysis of equilibria in non-cooperative games. Harsanyi is best known for his work on games with incomplete information and in particular Bayesian games, which he published as a series of three celebrated papers titled *Games with incomplete information played by Bayesian players* in the Management Science journal in 1967 and 1968.

His work on analysis of Bayesian games is of foundational value to mechanism design since mechanisms crucially use the framework of games with incomplete information. Harsanyi is also acclaimed for his intriguing work on *utilitarian ethics*, where he applied game theory and economic reasoning in political and moral philosophy. Harsanyi's collaboration with Reinhard Selten on the topic of equilibrium analysis resulted in a celebrated book entitled *A General Theory of Equilibrium Selection in Games* (MIT Press, 1988).

John Harsanyi was born in Budapest, Hungary, on May 29, 1920. He got two doctoral degrees – the first one in philosophy from the University of Budapest in 1947 and the second one in economics from Stanford University in 1959. His adviser at Stanford University was Professor Kenneth Arrow, who got the Economics Nobel Prize in 1972. Harsanyi worked at the University of California, Berkeley, from 1964 to 1990 when he retired. He died on August 9, 2000, in Berkeley, California.

Reinhard Selten, a joint winner of the Nobel prize in economic sciences along with John Nash and John Harsanyi, is a key contributor to the theory of incomplete information games besides John Harsanyi. In fact Harsanyi refers to the type agent representation of Bayesian games as the *Selten game*. Selten is best known for his fundamental work on extensive form games and their transformation to strategic form through a representation called the agent normal form. Selten is also widely known for his deep work on bounded rationality.

Selten is also widely regarded as a pioneer of experimental economics. Harsanyi and Selten, in their remarkable book *A General Theory of Equilibrium Selection in Games* (MIT Press, 1988), develop a general framework to identify a unique equilibrium as the solution of a given finite strategic form game. Their solution can be thought of as the limit of an evolutionary process.

Selten was born in Breslau (currently in Poland but formerly in Germany) on October 5, 1930. He earned a doctorate in Mathematics from Frankfurt University, working with Professor Ewald Burger and Professor Wolfgang Franz. He is currently a Professor Emeritus at the University of Bonn, Germany.

The phrases *actions* and *strategies* are used differently in the Bayesian game context. A strategy for a player i in Bayesian games is defined as a mapping from

Θ_i to S_i. A strategy s_i of a player i, therefore, specifies a pure action for each type of player i; $s_i(\theta_i)$ for a given $\theta_i \in \Theta_i$ would specify the pure action that player i would play if his type were θ_i. The notation $s_i(.)$ is used to refer to the pure action of player i corresponding to an arbitrary type from his type set. When it is convenient, we use $a_i \in S_i$ to represent a typical action of player i.

13.2 Examples of Bayesian Games

Example 13.1 (A Two Player Bargaining Game). This example is taken from the book by Myerson [1]. There are two players, player 1 (seller) and player 2 (buyer). Player 1 wishes to sell an indivisible item and player 2 is interested in buying this item. Each player knows what the object is worth to himself but thinks that its value to the other player may be any integer from 1 to 100 with probability $\frac{1}{100}$. The type of the seller has the natural interpretation of being the willingness to sell (minimum price at which the seller is prepared to sell the item), and the type of the buyer has the natural interpretation of being the willingness to pay (maximum price the buyer is prepared to pay for the item). Assume that each player will simultaneously announce a bid between 0 and 100 for trading the object. If the buyer's bid is greater than or equal to the seller's bid they will trade the object at a price equal to the average of their bids; otherwise no trade occurs. For this game:

$$\begin{aligned}
N &= \{1,2\} \\
\Theta_1 &= \Theta_2 = \{1,2,\ \ldots\ ,100\} \\
S_1 &= S_2 = \{0,1,2,\ \ldots\ ,100\} \\
p_1(\theta_2|\theta_1) &= \frac{1}{100}\ \forall \theta_1 \in \Theta_1;\ \forall \theta_2 \in \Theta_2 \\
p_2(\theta_1|\theta_2) &= \frac{1}{100}\ \forall \theta_1 \in \Theta_1;\ \forall \theta_2 \in \Theta_2 \\
u_1(\theta_1,\theta_2,s_1,s_2) &= \frac{s_1+s_2}{2} - \theta_1 \quad \text{if } s_2 \geq s_1 \\
&= 0 \qquad\qquad\quad \text{if } s_2 < s_1 \\
u_2(\theta_1,\theta_2,s_1,s_2) &= \theta_2 - \frac{s_1+s_2}{2} \quad \text{if } s_2 \geq s_1 \\
&= 0 \qquad\qquad\quad \text{if } s_2 < s_1.
\end{aligned}$$

Note that the beliefs p_1 and p_2 are consistent with the prior:

$$\mathbb{P}(\theta_1,\theta_2) = \frac{1}{10000}\ \forall \theta_1 \in \Theta_1\ \forall \theta_2 \in \Theta_2$$

where

$$\Theta_1 \times \Theta_2 = \{1,\ \ldots\ ,100\} \times \{1,\ \ldots\ ,100\}.$$

□

Example 13.2 (A Sealed Bid Auction). Consider a seller who wishes to sell an indivisible item through an auction. Let there be two prospective buyers who bid for this item. The buyers have their individual valuations for this item. These valuations could be considered as the types of the buyers. Here the game consists of the two bidders, namely the buyers, so $N = \{1, 2\}$. The two bidders submit bids, say b_1 and b_2 for the item. Let us say that the one who bids higher is awarded the item with a tie resolved in favor of bidder 1. The winner determination function therefore is:

$$\begin{aligned} f_1(b_1, b_2) &= 1 \text{ if } b_1 \geq b_2 \\ &= 0 \text{ if } b_1 < b_2 \end{aligned}$$

$$\begin{aligned} f_2(b_1, b_2) &= 1 \text{ if } b_1 < b_2 \\ &= 0 \text{ if } b_1 \geq b_2. \end{aligned}$$

Assume that the valuation set for each buyer is the real interval $[0, 1]$ and also that the strategy set for each buyer is again $[0, 1]$. This means $\Theta_1 = \Theta_2 = [0, 1]$ and $S_1 = S_2 = [0, 1]$. If we assume that each player believes that the other player's valuation is chosen according to an independent uniform distribution, then note that

$$p_1([x, y]|\theta_1) = y - x \ \ \forall\, 0 \leq x \leq y \leq 1; \ \forall \theta_1 \in \Theta_1.$$

$$p_2([x, y]|\theta_2) = y - x \ \ \forall\, 0 \leq x \leq y \leq 1; \ \forall \theta_2 \in \Theta_2.$$

In a first price auction, the winner will pay what is bid by her, and therefore the utility function of the players is given by

$$u_i(\theta_1, \theta_2, b_1, b_2) = f_i(b_1, b_2)(\theta_i - b_i); \ i = 1, 2.$$

This completes the definition of the Bayesian game underlying a first price auction involving two bidders. One can similarly develop the Bayesian game for the second price sealed bid auction. Note that only u_1 and u_2 would be different in the case of second price auction. □

13.3 Type Agent Representation and the Selten Game

This is a representation of Bayesian games that enables a Bayesian game to be transformed to a strategic form game (with complete information). Given a Bayesian game

$$\Gamma = \langle N, (\Theta_i), (S_i), (p_i), (u_i) \rangle$$

the Selten game is an equivalent strategic form game

$$\Gamma^s = \langle N^s, (S_{\theta_i})_{\substack{\theta_i \in \Theta_i \\ i \in N}}, (U_{\theta_i})_{\substack{\theta_i \in \Theta_i \\ i \in N}} \rangle.$$

The idea used in formulating a Selten game is to have *type agents*. Each player in the original Bayesian game is now replaced with a number of type agents; in fact, a player is replaced by exactly as many type agents as the number of types in the type set of that player. We can safely assume that the type sets of the players are mutually disjoint. The set of players in the Selten game is given by:

$$N^s = \bigcup_{i \in N} \Theta_i.$$

Note that each type agent of a particular player can play precisely the same actions as the player himself. This means that for every $\theta_i \in \Theta_i$,

$$S_{\theta_i} = S_i.$$

The payoff function U_{θ_i} for each $\theta_i \in \Theta_i$ is the conditional expected utility to player i in the Bayesian game given that θ_i is his actual type. It is a mapping with the following domain and co-domain:

$$U_{\theta_i} : \left(\underset{i \in N}{\times} \ \underset{\theta_i \in \Theta_i}{\times} S_i \right) \to \mathbb{R}.$$

We will explain the way U_{θ_i} is derived using an example. This example is developed, based on the illustration in the book by Myerson [1].

Example 13.3 (Selten Game for a Bayesian Pricing Game). Consider two firms, company 1 and company 2. Company 1 produces a product x_1 whereas company 2 produces either product x_2 or product y_2. The product x_2 is somewhat similar to product x_1 while the product y_2 is a different line of product. The product to be produced by company 2 is a closely guarded secret, so it can be taken as private information of company 2. We thus have $N = \{1, 2\}$, $\Theta_1 = \{x_1\}$, and $\Theta_2 = \{x_2, y_2\}$. Each firm has to choose a price for the product it produces, and this is the strategic decision to be taken by the company. Company 1 has the choice of choosing a low price a_1 or a high price b_1 whereas company 2 has the choice of choosing a low price a_2 or a high price b_2. We therefore have $S_1 = \{a_1, b_1\}$ and $S_2 = \{a_2, b_2\}$. The type of company 1 is common knowledge since Θ_1 is a singleton. Therefore, the belief probabilities of company 2 about company 1 are given by $p_2(x_1|x_2) = 1$ and $p_2(x_1|y_2) = 1$. Let us assume the belief probabilities of company 1 about company 2 to be $p_1(x_2|x_1) = 0.6$ and $p_1(y_2|x_1) = 0.4$. Let the utility functions for the two possible type profiles $(\theta_1 = x_1,\ \theta_2 = x_2)$ and $(\theta_1 = x_1,\ \theta_2 = y_2)$ be given as in Tables 13.1 and 13.2.

	2	
1	a_2	b_2
a_1	1, 2	0, 1
b_1	0, 4	1, 3

Table 13.1: u_1 and u_2 for $\theta_1 = x_1; \theta_2 = x_2$

This completes the description of the Bayesian game. We now derive the equivalent Selten game. We have

$$N^s = \Theta_1 \cup \Theta_2 = \{x_1, x_2, y_2\}$$
$$S_{x_1} = S_1 = \{a_1, b_1\}$$
$$S_{x_2} = S_{y_2} = S_2 = \{a_2, b_2\}.$$

1	2	
	a_2	b_2
a_1	$1,3$	$0,4$
b_1	$0,1$	$1,2$

Table 13.2: u_1 and u_2 for $\theta_1 = x_1; \theta_2 = y_2$

Note that

$$U_{\theta_i} : S_1 \times S_2 \times S_2 \to \mathbb{R} \;\; \forall \theta_i \in \Theta_i, \forall i \in N$$

$$\begin{aligned} S_1 \times S_2 \times S_2 = \{ & (a_1, a_2, a_2), (a_1, a_2, b_2), (a_1, b_2, a_2), (a_1, b_2, b_2), \\ & (b_1, a_2, a_2), (b_1, a_2, b_2), (b_1, b_2, a_2), (b_1, b_2, b_2)\}. \end{aligned}$$

The above gives the set of all strategy profiles of all the type agents. A typical strategy profile can be represented as $(s_{x_1}, s_{x_2}, s_{y_2})$. This could also be represented as $(s_1(.), s_2(.))$ where the strategy s_1 is a mapping from Θ_1 to S_1, and the strategy s_2 is a mapping from Θ_2 to S_2. In general, for an n player Bayesian game, a pure strategy profile is of the form

$$((s_{\theta_1})_{\theta_1 \in \Theta_1}, (s_{\theta_2})_{\theta_2 \in \Theta_2}, \ldots, (s_{\theta_n})_{\theta_n \in \Theta_n}).$$

An equivalent way to write this would be $(s_1(.), s_2(.), \ldots, s_n(.))$, where s_i is a mapping from Θ_i to S_i for $i = 1, 2, \ldots, n$. The payoffs for type agents (in the Selten game) are obtained as conditional expectations over the type profiles of the rest of the agents. For example, let us compute the payoff $U_{x_1}(a_1, a_2, b_2)$, which is the expected payoff obtained by type agent x_1 (belonging to player 1) when this type agent plays action a_1 and the type agents x_2 and y_2 of player 2 play the actions a_2 and b_2 respectively. In this case, the type of player 1 is known, but the type of player could be x_2 or y_2 with probabilities given by the belief function $p_1(.|x_1)$. The following conditional expectation gives the required payoff.

$$\begin{aligned} U_{x_1}(a_1, a_2, b_2) &= p_1(x_2|x_1)\; u_1(x_1, x_2, a_1, a_2) \\ &\quad + p_1(y_2|x_1)\; u_1(x_1, y_2, a_1, b_2) \\ &= (0.6)(1) + (0.4)(0) \\ &= 0.6. \end{aligned}$$

It can be similarly shown that

$$\begin{aligned} U_{x_1}(b_1, a_2, b_2) &= 0.4 \\ U_{x_2}(a_1, a_2, b_2) &= 2 \\ U_{x_2}(a_1, b_2, b_2) &= 1 \\ U_{y_2}(a_1, a_2, b_2) &= 4 \\ U_{y_2}(a_1, a_2, a_2) &= 3. \end{aligned}$$

From the above, we see that

$$\begin{aligned}U_{x_1}(a_1,a_2,b_2) &> U_{x_1}(b_1,a_2,b_2)\\U_{x_2}(a_1,a_2,b_2) &> U_{x_2}(a_1,b_2,b_2)\\U_{y_2}(a_1,a_2,b_2) &> U_{y_2}(a_1,a_2,a_2).\end{aligned}$$

We can immediately conclude that the action profile (a_1, a_2, b_2) is a Nash equilibrium of the Selten game. Another way of representing this profile would be (s_1^*, s_2^*) where $s_1^*(x_1) = a_1$ and $s_2^*(x_2) = a_2$; $s_2^*(y_2) = b_2$. □

Payoff Computation in Selten Game

From now on, when there is no confusion, we will use u instead of U. In general, given a Bayesian game $\Gamma = \langle N, (\Theta_i), (S_i), (p_i), (u_i)\rangle$, suppose $(s_1, \ldots, s_n)$ is a strategy profile where for $i = 1, \ldots, n$, s_i is a mapping from Θ_i to S_i. Assume the current type of player i to be θ_i. Then the expected utility to player i is given by

$$u_i((s_i, s_{-i})|\theta_i) = \mathbb{E}_{\theta_{-i}}\left[(u_i(\theta_i, \theta_{-i}, s_i(\theta_i), s_{-i}(\theta_{-i}))\right]$$

For a finite Bayesian game, the above immediately translates to

$$u_i((s_i, s_{-i})|\theta_i) = \sum_{t_{-i}\in\Theta_{-i}} p_i(t_{-i}|\theta_i)(u_i(\theta_i, t_{-i}, s_i(\theta_i), s_{-i}(t_{-i}))$$

With this setup, we now define the notion of Bayesian Nash equilibrium.

13.4 Bayesian Nash Equilibrium

Definition 13.3 (Pure Strategy Bayesian Nash Equilibrium). *A pure strategy Bayesian Nash equilibrium in a Bayesian game*

$$\Gamma = \langle N, (\Theta_i), (S_i), (p_i), (u_i)\rangle$$

can be defined in a natural way as a pure strategy Nash equilibrium of the equivalent Selten game. That is, a profile of strategies $(s_1^*, \ldots, s_n^*)$ is a pure strategy Bayesian Nash equilibrium if $\forall i \in N$; $\forall s_i : \Theta_i \to S_i$; $\forall \theta_i \in \Theta_i$,

$$u_i((s_i^*, s_{-i}^*) \mid \theta_i) \geq u_i((s_i, s_{-i}^*) \mid \theta_i)$$

That is, $\forall i \in N$; $\forall a_i \in S_i$; $\forall \theta_i \in \Theta_i$,

$$\mathbb{E}_{\theta_{-i}}\left[u_i(\theta_i, \theta_{-i}, s_i^*(\theta_i), s_{-i}^*(\theta_{-i}))\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i(\theta_i, \theta_{-i}, a_i, s_{-i}^*(\theta_{-i}))\right]$$

Example 13.4 (Bayesian Pricing Game). Consider the Bayesian pricing game being discussed. We make the following observations.

- When $\theta_2 = x_2$, the strategy b_2 is strongly dominated by a_2. Thus player 2 chooses a_2 when $\theta_2 = x_2$.

- When $\theta_2 = y_2$, the strategy a_2 is strongly dominated by b_2 and therefore player 2 chooses b_2 when $\theta_2 = y_2$.
- When the action profiles are (a_1, a_2) or (b_1, b_2), player 1 has payoff 1 regardless of the type of player 2. In all other profiles, payoff of player 1 is zero.
- Since $p_1(x_2|x_1) = 0.6$ and $p_1(y_2|x_1) = 0.4$, player 1 thinks that the type x_2 of player 2 is more likely than type y_2.

The above arguments imply that a pure strategy Bayesian Nash equilibrium in the above example is given by:

$$(s^*_{x_1} = a_1, s^*_{x_2} = a_2, s^*_{y_2} = b_2)$$

thus validating what we have already shown. In the above equilibrium, the strategy of company 1 is to price the product low whereas the strategy of company 2 is to price the product low if it produces x_2 and to price the product high if it produces y_2. It can be seen that the above is the unique pure strategy Bayesian Nash equilibrium for this game.

The above example clearly illustrates the ramification of analyzing each matrix separately. If it is common knowledge that player 2's type is x_2, then the unique Nash equilibrium is (a_1, a_2). If it is common knowledge that player 2 has type y_2, then we get (b_1, b_2) as the unique Nash equilibrium. However, in a Bayesian game, the type of player 2 is not common knowledge, and hence the above prediction based on analyzing the matrices separately would be wrong. □

Example 13.5 (First Price Sealed Bid Auction). Consider again the example of first price sealed bid auction with two prospective buyers. Here the two buyers are the players. Each buyer submits a sealed bid, $b_i \geq 0 \quad (i = 1, 2)$. The sealed bids are looked at, and the buyer with the higher bid is declared the winner. If there is a tie, buyer 1 is declared the winner. The winning buyer pays to the seller an amount equal to his bid. The losing bidder does not pay anything.

Let us make the following assumptions:

(1) θ_1, θ_2 are independently drawn from the uniform distribution on $[0, 1]$.
(2) The sealed bid of buyer i takes the form $b_i(\theta_i) = \alpha_i \theta_i$, where $\alpha_i \in (0, 1]$. This assumption implies that player i bids a fraction α_i of his value; this is a reasonable assumption that implies a linear relationship between the bid and the value. Each buyer knows that the bids are of the above form. Buyer 1 (buyer 2) seeks to compute an appropriate value for α_1 (α_2).

Buyer 1's problem is now to bid in order to maximize his expected payoff:

$$\max_{b_1 \geq 0} (\theta_1 - b_1) P\{b_2(\theta_2) \leq b_1\}.$$

Since the bid of player 2 is $b_2(\theta_2) = \alpha_2 \theta_2$ and $\theta_2 \in [0, 1]$, the maximum bid of buyer 2 is α_2. Buyer 1 knows this and therefore $b_1 \in [0, \alpha_2]$. Also,

$$\begin{aligned} P\{b_2(\theta_2) \leq b_1\} &= P\{\alpha_2 \theta_2 \leq b_1\} \\ &= P\{\theta_2 \leq \frac{b_1}{\alpha_2}\} \\ &= \frac{b_1}{\alpha_2} \text{ (since } \theta_2 \text{ is uniform over } [0, 1]). \end{aligned}$$

Thus buyer 1's problem is:

$$\max_{b_1 \in [0, \alpha_2]} (\theta_1 - b_1)\frac{b_1}{\alpha_2}.$$

The solution to this problem is

$$b_1(\theta_1) = \begin{cases} \frac{\theta_1}{2} \text{ if } \frac{\theta_1}{2} \leq \alpha_2 \\ \alpha_2 \text{ if } \frac{\theta_1}{2} > \alpha_2. \end{cases}$$

We can show on similar lines that

$$b_2(\theta_2) = \begin{cases} \frac{\theta_2}{2} \text{ if } \frac{\theta_2}{2} \leq \alpha_1 \\ \alpha_1 \text{ if } \frac{\theta_2}{2} > \alpha_1. \end{cases}$$

Let $\alpha_1 = \alpha_2 = \frac{1}{2}$. Then we get

$$b_1(\theta_1) = \frac{\theta_1}{2} \;\; \forall \, \theta_1 \in \Theta_1 = [0, 1]$$
$$b_2(\theta_2) = \frac{\theta_2}{2} \;\; \forall \, \theta_2 \in \Theta_2 = [0, 1].$$

If $b_2(\theta_2) = \frac{\theta_2}{2}$, the best response of buyer 1 is $b_1(\theta_1) = \frac{\theta_1}{2}$ since $\alpha_1 = \frac{1}{2}$. Similarly, if $b_1(\theta_1) = \frac{\theta_1}{2}$, the best response of buyer 2 is $b_2(\theta_2) = \frac{\theta_2}{2}$ since $\alpha_2 = \frac{1}{2}$. This implies that the profile $\left(\frac{\theta_1}{2}, \frac{\theta_2}{2}\right)$ $\forall \theta_1 \in \Theta_1$, $\forall \theta_2 \in \Theta_2$ is a Bayesian Nash equilibrium of the Bayesian game underlying the first price auction (under the setting that we have considered). □

13.5 Dominant Strategy Equilibria

Dominant strategy equilibria of Bayesian games can again be defined using the Selten game representation. We only define the notion of *very weakly dominant strategy equilibrium* and leave the definitions of *weakly dominant strategy equilibrium* and *strongly dominant strategy equilibrium* to the reader.

Definition 13.4 (Very Weakly Dominant Strategy Equilibrium). *Given a Bayesian game*

$$\Gamma = \langle N, (\Theta_i), (S_i), (p_i), (u_i) \rangle$$

a profile of strategies $(s_1^*, \ldots, s_n^*)$ is called a very weakly dominant strategy equilibrium if $\forall i \in N$; $\forall s_i : \Theta_i \to S_i$; $\forall s_{-i} : \Theta_{-i} \to S_{-i}$; $\forall \theta_i \in \Theta_i$

$$u_i((s_i^*, s_{-i}) \mid \theta_i) \geq u_i((s_i, s_{-i}) \mid \theta_i)$$

That is, $\forall i \in N$; $\forall a_i \in S_i$; $\forall \theta_i \in \Theta_i$; $\forall s_{-i} : \Theta_{-i} \to S_{-i}$;,

$$\mathbb{E}_{\theta_{-i}} \left[u_i(\theta_i, \theta_{-i}, s_i^*(\theta_i), s_{-i}(\theta_{-i})) \right] \geq \mathbb{E}_{\theta_{-i}} \left[u_i(\theta_i, \theta_{-i}, a_i, s_{-i}(\theta_{-i})) \right]$$

A close examination of the above definition (note the presence of s_{-i} on the left hand side as well as the right hand side) shows that the notion of dominant strategy equilibrium is independent of the belief functions, and this is what makes it a very powerful notion and a very strong property. The notion of dominant strategy equilibrium is used extensively in mechanism design theory to define *dominant strategy implementation*. Often very weakly dominant strategy equilibrium is used in these settings.

Example 13.6 (Second Price Auction). We have shown above that the first price sealed bid auction has a Bayesian Nash equilibrium. Now we consider the second price sealed bid auction with two bidders and show that it has a weakly dominant strategy equilibrium. Let us say buyer 2 announces his bid as b_2. There are two cases.

(1) $\theta_1 \geq b_2$
(2) $\theta_1 < b_2$

Case 1: $\theta_1 \geq b_2$

Let b_1 be the bid of buyer 1. Here there are two cases.

- If $b_1 \geq b_2$, then the payoff for buyer 1 is $\theta_1 - b_2 \geq 0$.
- If $b_1 < b_2$, then the payoff for buyer 1 is 0.
- Thus in this case, the maximum payoff possible is $\theta_1 - b_2 \geq 0$.

If $b_1 = \theta_1$ (that is, buyer 1 announces his true valuation), then payoff for buyer 1 is $\theta_1 - b_2$, which happens to be the maximum possible payoff as shown above. Thus announcing θ_1 is a best response to buyer 1 whatever the announcement of buyer 2.

Case 2: $\theta_1 < b_2$

Here again there are two cases: $b_1 \geq b_2$ and $b_1 < b_2$.

- If $b_1 \geq b_2$, then the payoff for buyer 1 is $\theta_1 - b_2$, which is negative.
- If $b_1 < b_2$, then buyer 1 does not win and payoff for him is zero.
- Thus in this case, the maximum payoff possible is 0.

If $b_1 = \theta_1$, payoff for buyer 1 is 0. By announcing $b_1 = \theta_1$, his true valuation, buyer 1 gets zero payoff, which in this case is a best response.

We can now make the following observations about this example.

- Bidding his true valuation is optimal for buyer 1 regardless of the bid of buyer 2.
- Similarly bidding his true valuation is optimal for buyer 2 whatever the bid of buyer 1.
- This means truth revelation is a very weakly dominant strategy for each player, and $(s_1^*(\theta_1) = \theta_1, s_2^*(\theta_2) = \theta_2)$ is a very weakly dominant strategy equilibrium.

We leave it to the reader to show that the equilibrium is in fact a weakly dominant strategy equilibrium as well. □

13.6 Summary and References

In this chapter, we have introduced strategic form games with incomplete information. We summarize the main points of this chapter below.

- In a game with incomplete information, every player, in addition to strategies (actions), also has private information which is called the type of the player. Each player has a type set. Also, each player has a probabilistic guess about the types of the rest of the players. The utilities depend not only on the actions chosen by the players but also the types of the players. Bayesian games provide a common way of representing strategic form games with incomplete information.
- Harsanyi and Selten have developed a theory of Bayesian games. The central idea of their theory is to transform a Bayesian game into a strategic form game with complete information using the so called type agent representation. The resulting strategic form games with complete information are called Selten games.
- A strategy of a player in a Bayesian game is a mapping from the player's type set to his action set. Using these mappings, different notions of equilibrium can be defined. Bayesian Nash equilibrium is a natural extension of pure strategy Nash equilibrium to the case of Bayesian games.
- We have illustrated the computation of Bayesian Nash equilibrium and dominant strategy equilibria using the familiar examples of first price auction and second price auction, respectively.

The material discussed in this chapter is mainly drawn from the the book by Myerson [1]. John Harsanyi wrote a series of three classic papers introducing, formalizing, and elaborating upon Bayesian games. These papers [2, 3, 4] appeared in 1967 and 1968.

References

[1] Roger B. Myerson. *Game Theory: Analysis of Conflict.* Harvard University Press, Cambridge, Massachusetts, USA, 1997.

[2] John C. Harsanyi. "Games with incomplete information played by Bayesian players. Part I: The basic model". In: *Management Science* **14** (1967), pp. 159–182.

[3] John C. Harsanyi. "Games with incomplete information played by Bayesian players. Part II: Bayesian equilibrium points". In: *Management Science* **14** (1968), pp. 320–334.

[4] John C. Harsanyi. "Games with incomplete information played by Bayesian players. Part III: The basic probability distribution of the game". In: *Management Science* **14** (1968), pp. 486–502.

13.7 Exercises

(1) Write down the definitions of weakly dominant strategy equilibrium and strongly dominant strategy equilibrium for Bayesian games.

(2) Consider the example of first price sealed bid auction with two buyers which we discussed in this chapter. What is the common prior with respect to which the beliefs of the two players are consistent?

(3) We have shown for the second price auction that bidding true valuations is a

very weakly dominant strategy equilibrium. Show that this equilibrium is weakly dominant as well. Also, show that this equilibrium is not strongly dominant.

(4) Consider two agents 1 and 2 where agent 1 is the seller of an indivisible item and agent 2 is a prospective buyer of the item. The type θ_1 of agent 1 (seller) can be interpreted as the willingness to sell of the agent (minimum price at which agent 1 is willing to sell). The type θ_2 of agent 2 (buyer) has the natural interpretation of willingness to pay (maximum price the buyer is willing to pay). Assume that $\Theta_1 = \Theta_2 = [0, 1]$ and that each agent thinks that the type of the other agent is uniformly distributed over the real interval $[0, 1]$. Define the following protocol. The seller and the buyer are asked to submit their bids b_1 and b_2 respectively. Trade happens if $b_1 \leq b_2$ and trade does not happen otherwise. If trade happens, the buyer gets the item and pays the seller an amount $\frac{(b_1+b_2)}{2}$. Compute a Bayesian Nash equilibrium of the Bayesian game here.

(5) **Programming Assignment**. Given a finite Bayesian game, write a program to transform it into a Selten game and compute all Bayesian Nash equilibria (if they exist) and all dominant strategy equilibria (if they exist).