A Appendix

A.1 Proof for Thm. 1

Proof. We note that $\text{int}(\mathcal{C}) = \text{int}(\mathcal{K}) \times \mathbb{R}$. It is easy to show that $\mathcal{C}$ is $\text{hypo}_{\mathcal{B}}(f)$-free, since $\text{int}(\mathcal{C}) \cap \text{hypo}_{\mathcal{B}}(f) = \varnothing$. Assume that there exists a $\text{hypo}_{\mathcal{B}}(f)$-free set $\mathcal{C}'$ including $\mathcal{C}$. Then the recession cone of $\mathcal{C}'$ must include that of $\mathcal{C}$, so $\mathcal{C}' = \mathcal{K}' \times \mathbb{R}$ for some closed convex set $\mathcal{K}'$ including $\mathcal{K}$. Moreover, $\mathcal{K}'$ must be a $\mathcal{B}$-free set, otherwise, there exists a point $x \in \mathcal{B} \cap \text{int}(\mathcal{K}')$ such that $(x, f(x)) \in \text{int}(\mathcal{K}') \times \mathbb{R} = \text{int}(\mathcal{C}')$. However, since $\mathcal{K}$ is maximally $\mathcal{B}$-free, this implies that $\mathcal{K} = \mathcal{K}'$. As a result, $\mathcal{C} = \mathcal{C}'$, so $\mathcal{C}$ is maximal. □

Proof for Prop. 1

Proof.

$$\sigma(\pi)v^i(\pi) = \sum_{j \in \mathcal{N}_i} \sigma(\pi)_{\pi(j)} = \sum_{j \in \mathcal{N}_i} \Big(f(v^j(\pi)) - f(v^j(\pi)) \Big) = f^i(v^i(\pi)) - f(0) = f(v^i(\pi)),$$

where the first equation follows from Defn. 2, the second equation follows from Lemma 2, and the last two equations follow from the expansion of the sum. □

A.2 Proof for Prop. 2

Proof. As $\mathsf{F}(x) = \max_{s \in \text{ext}(EPM_f)} sx$, F is the maximum of a set of linear functions. This implies that it is positive-homogeneous of degree-1 and convex, and it is easy to show the other results. □

A.3 Proof for Prop. 3

Proof. Let π^* be the permutation found by the sorting algorithm. By Lemma 2, $\sigma(\pi^*)$ is in $\text{ext}(EPM_f)$ and hence a feasible solution to (3). Next, we prove the optimality of $\sigma(\pi^*)$. Let $d = \min_{i \in \mathcal{N}} \tilde{x}_i$, then $\tilde{x} + d\mathbf{1} \geq 0$. The following inequalities hold:

$$\sigma(\pi^*)\tilde{x} \leq \max_{s \in \text{ext}(EPM_f)} s(\tilde{x} + d\mathbf{1} - d\mathbf{1}) \leq \max_{s \in \text{ext}(EPM_f)} s(\tilde{x} + d\mathbf{1}) + \max_{s \in \text{ext}(EPM_f)} s(-d\mathbf{1}). \tag{7}$$

It is easy to show that $(\tilde{x} + d\mathbf{1})_{\pi^*(1)} \geq \cdots \geq (\tilde{x} + d\mathbf{1})_{\pi^*(n)}$, as $\tilde{x} + d\mathbf{1} \geq 0$, by the sorting algorithm, $\sigma(\pi^*)(\tilde{x} + d\mathbf{1}) = \max_{s \in \text{ext}(EPM_f)} s(\tilde{x} + d\mathbf{1})$. As the entries of $-d\mathbf{1}$ are identical, it follows from Prop. 1 that for any $\pi \in \Pi$, $\sigma(\pi)(-d\mathbf{1}) = -df(\mathbf{1})$. Therefore, for any $s \in \text{ext}(EPM_f)$, $s(-d\mathbf{1}) = -df(\mathbf{1})$, so $\sigma(\pi^*)(-d\mathbf{1}) = \max_{s \in \text{ext}(EPM_f)} s(-d\mathbf{1})$. Looking at the inequalities (7), they become equations, because

$$\sigma(\pi^*)\tilde{x} \leq \max_{s \in \text{ext}(EPM_f)} s\tilde{x} \leq \sigma(\pi^*)(\tilde{x} + d\mathbf{1}) + \sigma(\pi^*)(-d\mathbf{1}) = \sigma(\pi^*)\tilde{x}.$$

Therefore, $\sigma(\pi^*)$ is an optimal solution to (3). □

A.4 Proof for Thm. 2

Proof. We consider a point $v \in \mathcal{B}$ and look at the line $\ell = \{(v,t) : t \in \mathbb{R}\}$. It can be separated into the restricted epigraph $\ell_+ := \{(v,t) : f(v) \leq t\}$ and the restricted hypograph $\ell_- := \{(v,t) : f(v) \geq t\}$, as $\ell_+ \cap \ell_- = (v, f(v))$ and $\ell = \ell_+ \cup \ell_-$. First, we know that, by definition of Q_f and Lemma 1, $\ell_+ \subseteq Q_f \subseteq EE_f$. Second, by Prop. 1, the point $(v, f(v))$ supports some facets of EE_f, so the point (v,t) with $t < f(v)$ is separated by these facets from EE_f. Thereby, we know that $\ell_- \cap EE_f = \{(v, f(v))\}$. To summarize, we know that $EE_f \cap \ell = \ell_+$ and $(v, f(v)) \in \mathrm{bd}(EE_f)$. As $\mathrm{gra}_{\mathcal{B}}(f) = \cup_{v \in B}\{(v, f(v))\}$, we have that $\mathrm{gra}_{\mathcal{B}}(f) \subseteq \mathrm{bd}(EE_f)$. As the hypograph $\mathrm{hypo}_{\mathcal{B}}(f) = \cup_{v \in \mathcal{B}}\{(v,t) : f(v) \geq t\}$ (union of restricted hypographs), we have that $EE_f \cap \mathrm{hypo}_{\mathcal{B}}(f) = \mathrm{gra}_{\mathcal{B}}(f)$. □

A.5 Proof for Cor. 2

Proof. Since $\mathrm{gra}_{\mathcal{B}}(f) \subseteq \mathrm{bd}(EE_f)$, we conclude that $EE_f \cap \mathrm{hypo}_{\mathcal{B}}(f) \subseteq \mathrm{bd}(EE_f)$ and hence $\mathrm{int}(EE_f) \cap \mathrm{hypo}_{\mathcal{B}}(f) = \varnothing$. Additionally, EE_f is convex and hence $\mathrm{hypo}_{\mathcal{B}}(f)$-free. As $Q_f \subseteq EE_f$, Q_f is $\mathrm{hypo}_{\mathcal{B}}(f)$-free set. □

A.6 Proof for Prop. 4

Proof. By Prop. 1, since $\varnothing = \pi(\mathcal{N}_0) \subsetneq \cdots \subsetneq \pi(\mathcal{N}_n) = \mathcal{N}$, by Defn. 2, $0 = v^0(\pi) < \cdots < v^n(\pi) = 1$, so $V(\pi)$ is a monotone chain. Conversely, given a monotone chain $(x^0, \cdots, x^n)$, its inverse map π exists and satisfies that $\pi(0) = 0$; and for all $i \in \mathcal{N}$, $\pi(i)$ is the index of the non-zero entry of $x^i - x^{i-1}$. □

A.7 Proof for Cor. 3

Proof. First, we note that $C(\Pi')$, as a relaxation of EE_f includes $\mathrm{gra}_{\mathcal{B}}(f)$. Next, we assume that Π' is a cover. Then points of $\mathrm{gra}_{\mathcal{B}}(f)$ support facets of $C(\Pi')$. By Thm. 3, $C(\Pi')$ is $\mathrm{hypo}_{\mathcal{B}}(f)$-free. Finally, if Π' is a minimal cover, then each facet of $C(\Pi')$ has a point of $\mathrm{gra}_{\mathcal{B}}(f)$ in its interior. By Thm. 3, $C(\Pi')$ is maximally $\mathrm{hypo}_{\mathcal{B}}(f)$-free. □

A.8 Proof for Prop. 5

Proof. As $\bar{f}$ is concave overestimator of f over X and F is convex underestimator of f over X, $\bar{f} \geq \mathsf{F}$ over X. Suppose, to aim at a contradiction, that $\bar{f}(x^*) = \mathsf{F}(x^*)$. Define a concave function $g := \bar{f} - \mathsf{F}$, then for all $x \in X$, $g(x) \geq 0$, and $g(x^*) = 0$. By its concavity, there exists an affine overestimating function a of g, such that $g(x^*) = a(x^*) = 0$, and for all $x \in X$, $0 \leq g(x) \leq a(x)$. As $x^* \in \mathrm{relint}(X)$, the affinity of a implies that $a = g = 0$ over X, i.e., $\bar{f} = \mathsf{F}$ over X. So f is both concave and convex over X and thus affine over X, which is a contradiction. □

A.9 Proof for Prop. 8

Proof. Since the extended envelope G is the maximum of linear functions, it is convex and piece-wise linear, and so ζ^j is concave and piece-wise linear. Since $\zeta^j(0) = \ell\tilde{t} - \mathsf{G}(\tilde{x})$, it follows from the assumption $z^* \in \text{int}(\mathcal{C})$ that $\tilde{t} > \mathsf{G}(\tilde{x})$ and thus $\zeta^j(0) > 0$. Since $\mathcal{C}$ is closed and convex, $\eta'_j = \eta^*_j$ if and only if $z^* + \eta'_j r^j \in \text{bd}(\mathcal{C})$. That is $\mathsf{G}(r^j_x \eta_j + \tilde{x}) = \mathsf{G}(\tilde{x}) + r^j_t \eta'_j$, i.e., $\zeta^j(\eta'_j) = 0$. Since $s^* \in \partial\mathsf{G}(\tilde{x} + r^j_x \eta_j)$, by the chain rule, $\ell r^j_t - s^* r^j_x$ is a subgradient of ζ^j. By the concavity of ζ^j, its subgradients are non-increasing. $\square$