EE5609 Assignment 17

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1 Problem

Let $\mathbf{A}$ be a $n \times n$ real matrix with $\mathbf{A}^2 = \mathbf{A}$. Then

1) the eigenvalues of $\mathbf{A}$ are either 0 or 1
2) $\mathbf{A}$ is a diagonal matrix with diagonal entries 0 or 1
3) $rank(\mathbf{A}) = trace(\mathbf{A})$
4) if $rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A})$

2 Explanation

Since

$$\mathbf{A}^2 = \mathbf{A} \tag{2.0.1}$$
$$\implies \mathbf{A}^2 - \mathbf{A} = \mathbf{O} \tag{2.0.2}$$

From Cayley-Hamilton Theorem we have,

$$\lambda^2 - \lambda = 0 \tag{2.0.3}$$
$$\implies \lambda(\lambda - 1) = 0 \tag{2.0.4}$$
$$\implies \lambda = 0, 1 \tag{2.0.5}$$

Such a matrix with the property that $\mathbf{A}^2 = \mathbf{A}$ is an idempotent matrix with eigen values equal to 0 or 1. For such a matrix

$$rank(A) = trace(A) \tag{2.0.6}$$

Now for the matrix $\mathbf{I} - \mathbf{A}$ we have,

$$(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) \tag{2.0.7}$$
$$\implies \mathbf{I}^2 - \mathbf{IA} - \mathbf{AI} + \mathbf{A}^2 \tag{2.0.8}$$
$$\implies \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} \tag{2.0.9}$$
$$\implies \mathbf{I} - \mathbf{A} \tag{2.0.10}$$

Hence $\mathbf{I} - \mathbf{A}$ is an idempotent matrix. Therefore we conclude,

$$rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A}) \tag{2.0.11}$$

Thus, options (1),(3) and (4) are correct.