EE5609 Assignment 5

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The python solution code is available at

https://github.com/Shantanu2508/Matrix_Theory/blob/master/Assignment_5/assignment5part2.py

1 Problem

Prove that the equation

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$$

represents two straight lines and find the angle between the lines.

2 Explanation

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

(2.0.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.5}$$

The lines intercept if

$$\left|\mathbf{V}\right| < 0 \tag{2.0.6}$$

3 Solution

From (2.0.3) and (2.0.4)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \tag{3.0.2}$$

$$f = -35 \tag{3.0.3}$$

From (2.0.1) and (2.0.5)

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \tag{3.0.4}$$

$$\implies 12\begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2}\begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ -\frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2}\begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \tag{3.0.5}$$

$$\left|\mathbf{V}\right| = -\frac{529}{4} < 0 \tag{3.0.6}$$

The given equation therefore represents two intersecting lines. Since $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}\mathbf{V}\mathbf{P}^T = \mathbf{D} = diag\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \tag{3.0.7}$$

or equivalently

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{3.0.8}$$

Eigen vectors of real symmetric matrix $\mathbf{V}$ are orthogonal. The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 12 & -\frac{7}{2} \\ -\frac{7}{2} & \lambda + 10 \end{vmatrix} = 0 \tag{3.0.9}$$

$$\implies \lambda^2 - 2\lambda - \frac{529}{4} = 0 \tag{3.0.10}$$

$$\implies \lambda_1 = -\frac{527}{50}, \quad \lambda_2 = \frac{627}{50} \tag{3.0.11}$$

The eigen vector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{3.0.12}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{3.0.13}$$

The orthogonal eigen-vector matrix

$$\mathbf{P} = \frac{1}{\|\mathbf{p}\|}\begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \begin{pmatrix}\frac{3}{20} & -\frac{49}{50}\\ -\frac{49}{50} & -\frac{3}{20}\end{pmatrix} \tag{3.0.14}$$

$$\mathbf{D} = \begin{pmatrix}-\frac{527}{50} & 0\\ 0 & \frac{627}{50}\end{pmatrix} \tag{3.0.15}$$

Let $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$ with $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$. Substituting in (2.0.2)

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{3.0.16}$$

$$\implies \quad \mathbf{y}^T\mathbf{D}\mathbf{y} = 0 \tag{3.0.17}$$

$$\implies \quad \begin{pmatrix}\pm\sqrt{\frac{\lambda_1}{\lambda_2}} & 1\end{pmatrix}\mathbf{y} = 0 \tag{3.0.18}$$

Substituting $\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c}$ in (3.0.18)

$$\implies \quad \begin{pmatrix}\pm\sqrt{\frac{10}{12}} & 1\end{pmatrix}(\mathbf{P^T}\mathbf{x} - \mathbf{P^T}\mathbf{c}) = 0 \tag{3.0.19}$$

The equations of lines are

$$\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 5 \tag{3.0.20}$$

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7 \tag{3.0.21}$$

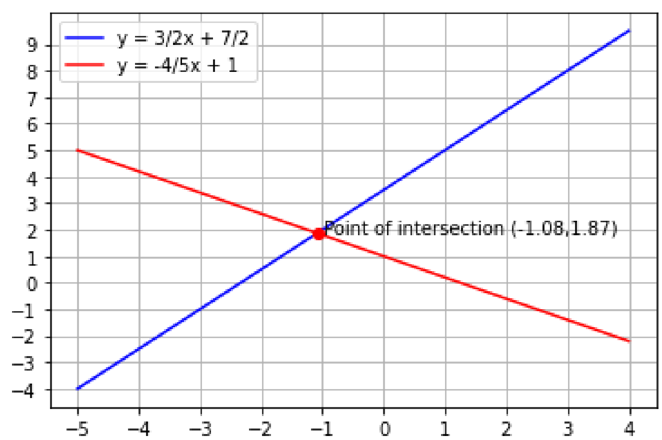
The angle between the lines can be expressed in-terms of normal vectors

$$\mathbf{n_1} = \begin{pmatrix}4\\5\end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix}3\\-2\end{pmatrix} \tag{3.0.22}$$

as

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{3.0.23}$$

$$\implies \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \tag{3.0.24}$$



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