EE5609 Assignment 18

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1 Problem

Let $\mathbf{M}$ be a $n \times n$ Hermitian matrix of rank $k, k \neq n$. If $\lambda \neq 0$ is an eigenvalue of $\mathbf{M}$ with corresponding unit column vector $\mathbf{u}$, with $\mathbf{Mu} = \lambda\mathbf{u}$ then which of the following are true?

1) $rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) = k - 1$
2) $rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) = k$
3) $rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) = k + 1$
4) $(\mathbf{M} - \lambda\mathbf{uu}^*)^n = \mathbf{M}^n - \lambda^n\mathbf{uu}^*$

2 Explanation

Objective	Explanation
Rank of $\mathbf{M} - \lambda\mathbf{uu}^*$	Since $$rank\,(\mathbf{A} - \mathbf{B}) \geq rank\,(\mathbf{A}) - rank\,(\mathbf{B}) \quad (2.0.1)$$ $$\implies rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) \geq rank\,(\mathbf{M}) - rank\,(\mathbf{uu}^*) \quad (2.0.2)$$ $$\implies rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) \geq k - rank\,(\mathbf{uu}^*) \quad (2.0.3)$$ If $\mathbf{A}$ is a non-zero column vector of order $m \times 1$ and $\mathbf{B}$ is a non-zero row vector of order $1 \times n$ then $rank\,(AB) = 1$. So, $$rank\,(\mathbf{uu}^*) = 1 \quad (2.0.4)$$ $$\implies rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) \geq k - 1 \quad (2.0.5)$$ Also since, $$\mathbf{M} - \lambda\mathbf{uu}^* = \mathbf{M} - \mathbf{Muu}^* = \mathbf{M}\,(I - \mathbf{uu}^*) \quad (2.0.6)$$ and $$rank\,(\mathbf{M}\,(\mathbf{I} - \mathbf{uu}^*)) \leq min\,(rank\,(\mathbf{M})\,, rank\,(\mathbf{I} - \mathbf{uu}^*)) \quad (2.0.7)$$ $$\implies rank\,(\mathbf{M}\,(\mathbf{I} - \mathbf{uu}^*)) \leq k \quad (2.0.8)$$ Thus we have from (2.0.5) and (2.0.8) that $$rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) = k - 1 \text{ or } k \quad (2.0.9)$$ Consider a matrix $$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.10)$$

Objective	Explanation
	such that $rank\,(M) = 1$. The eigenvalue of $\mathbf{M}$ is $\lambda = 1$ and the corresponding eigenvector is $$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.11)$$ Thus we have, $$\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} \quad (2.0.12)$$ $$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.13)$$ $$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.14)$$ $$\implies rank\,(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = 0 \quad (2.0.15)$$ Hence if $rank\,(\mathbf{M}) = k$ then $rank\,(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k - 1$.
$(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n\mathbf{u}\mathbf{u}^*$	Let the given statement be P(n):$(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n\mathbf{u}\mathbf{u}^*$. It can be seen that P(1) is true. Assume P(n) is true for some $k \in \mathbf{N}$ such that $$(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^k = \mathbf{M}^k - \lambda^k\mathbf{u}\mathbf{u}^* \quad (2.0.16)$$ Now to prove that P(k+1) is true we have $$(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^{k+1} = (\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^k \quad (2.0.17)$$ $$= (\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)\left(\mathbf{M}^k - \lambda^k\mathbf{u}\mathbf{u}^*\right) \quad (2.0.18)$$ $$= \mathbf{M}^{k+1} - \lambda^k\mathbf{M}\mathbf{u}\mathbf{u}^* - \lambda\mathbf{M}^k\mathbf{u}\mathbf{u}^* + \lambda^{k+1}\mathbf{u}\mathbf{u}^*\mathbf{u}\mathbf{u}^* \quad (2.0.19)$$ $$= \mathbf{M}^{k+1} - \lambda^{k+1}\mathbf{u}\mathbf{u}^* - \lambda^{k+1}\mathbf{u}\mathbf{u}^* + \lambda^{k+1}\mathbf{u}\,\|\mathbf{u}\|^2\,\mathbf{u}^* \quad (2.0.20)$$ $$= \mathbf{M}^{k+1} - 2\lambda^{k+1}\mathbf{u}\mathbf{u}^* + \lambda^{k+1}\mathbf{u}\mathbf{u}^* \quad (2.0.21)$$ $$= \mathbf{M}^{k+1} - \lambda^{k+1}\mathbf{u}\mathbf{u}^* \quad (2.0.22)$$ Hence, by the Principle of Mathematical Induction P(n) is true for all n.
Answer	(1) and (4)

TABLE I