EE5609 Assignment 5

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The python solution code is available at

https://github.com/Shantanu2508/Matrix_Theory/blob/master/Assignment_5/assignment5.py

1 Problem

Prove that the equation

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$$

represents two straight lines and find the angle between the lines.

2 Explanation

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

(2.0.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.5}$$

The lines intercept if

$$\left|\mathbf{V}\right| < 0 \tag{2.0.6}$$

3 Solution

From (2.0.3) and (2.0.4)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \tag{3.0.2}$$

$$f = -35 \tag{3.0.3}$$

From (2.0.1) and (2.0.5)

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \tag{3.0.4}$$

$$\implies 12\begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2}\begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2}\begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \tag{3.0.5}$$

$$\left|\mathbf{V}\right| = -\frac{529}{4} < 0 \tag{3.0.6}$$

From (3.0.5) and (3.0.6) it can be concluded that the given equation represents a pair of intersecting lines. Let the equations of lines be

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{3.0.7}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{3.0.8}$$

Since (2.0.2) represents a pair of straight lines it must satisfy

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_1}^T\mathbf{x} - c_1) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.0.9}$$

where

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \tag{3.0.10}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} \tag{3.0.11}$$

$$c_1c_2 = f \tag{3.0.12}$$

Slopes of the lines can be obtained by solving

$$cm^2 + 2bm + a = 0 \tag{3.0.13}$$

$$-10m^2 + 7m + 12 = 0 \tag{3.0.14}$$

$$\implies m_1 = \frac{-4}{5}, m_2 = \frac{3}{2} \tag{3.0.15}$$

The normal vectors can be expressed in terms of corresponding slopes of lines as

$$\mathbf{n} = k\begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{3.0.16}$$

$$\implies \mathbf{n_1} = k_1\begin{pmatrix} \frac{4}{5} \\ 1 \end{pmatrix} \tag{3.0.17}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \tag{3.0.18}$$

Substituing (3.0.17) and (3.0.18) in (3.0.10) we get

$$k_1k_2 = -10 \tag{3.0.19}$$

Assuming $k_1 = 5$ and $k_2 = -2$

$$\mathbf{n_1} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{3.0.20}$$

Verification using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 4 & 0 \\ 5 & 4 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \tag{3.0.21}$$

From (3.0.11) we have

$$c_2\begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_1\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -13 \\ -45 \end{pmatrix} \tag{3.0.22}$$

Solving the augmented matrix

$$\begin{pmatrix} 4 & 3 & -13 \\ 5 & -2 & -45 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 4R_2 - 5R_1} \begin{pmatrix} 4 & 3 & -13 \\ 0 & -23 & -115 \end{pmatrix} \tag{3.0.23}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{R_2}{23}} \begin{pmatrix} 4 & 3 & -13 \\ 0 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 4 & 0 & -28 \\ 0 & 1 & 5 \end{pmatrix} \tag{3.0.24}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \end{pmatrix} \tag{3.0.25}$$

$$\implies \quad c_1 = -7, \ c_2 = 5 \tag{3.0.26}$$

Thus the equation of lines are

$$\begin{pmatrix} 4 & 5 \end{pmatrix}\mathbf{x} = 5 \tag{3.0.27}$$

$$\begin{pmatrix} 3 & -2 \end{pmatrix}\mathbf{x} = -7 \tag{3.0.28}$$

The angle between the lines can be expressed in-terms of normal vectors

$$\mathbf{n_1} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{3.0.29}$$

as

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{3.0.30}$$

$$\implies \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \tag{3.0.31}$$

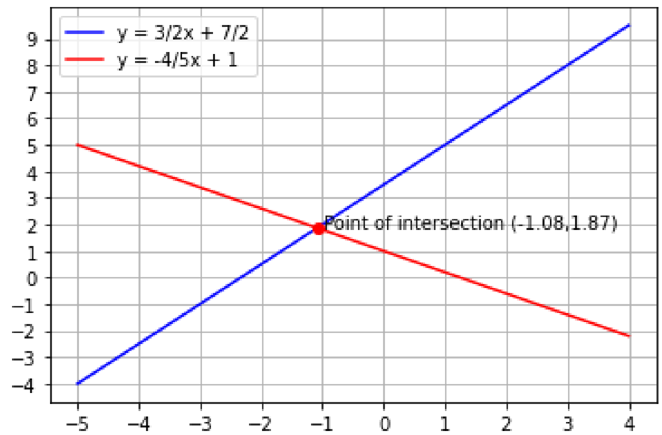

Fig. 0