EE5609 Assignment 8

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The python solution code is available at

https://github.com/Shantanu2508/Matrix_Theory/blob/master/Assignment_8/assignment8.py

1 Problem

Find the foot of perpendicular from point $B = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ to the plane $\begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \mathbf{x} = -5$.

2 Solution

Let us consider orthogonal vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ to the given normal vector $\mathbf{n}$. Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Then,

$$\mathbf{m}^T \mathbf{n} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 0 \tag{2.0.2}$$

$$\implies 2a + 3b - 4c = 0 \tag{2.0.3}$$

Let $a = 1,\ \ b = 0$, so that

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.4}$$

and $a = 0,\ \ b = 1$, so that

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \tag{2.0.5}$$

We, now, solve the equation

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{2.0.6}$$

which, upon substitution, becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{2.0.7}$$

Any $m \times n$ matrix $\mathbf{M}$ can be factorized in SVD form as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.8}$$

where $\mathbf{U}$ and $\mathbf{V}$ are matrices of eigen vectors which are orthogonal. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is the diagonal matrix of singular values of $\mathbf{M}$ of the eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \tag{2.0.9}$$

Putting (2.0.8) into (2.0.6), we get

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \tag{2.0.10}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \tag{2.0.11}$$

where $\mathbf{S}_+$ is the Moore-Penrose Pseudoinverse of $\mathbf{S}$.

The eigenvalues of $\mathbf{M}^T\mathbf{M}$:

$$\left| \mathbf{M}^T\mathbf{M} - \lambda \mathbf{I} \right| = 0 \tag{2.0.12}$$

$$\implies \begin{vmatrix} \frac{10}{8} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{vmatrix} = 0 \tag{2.0.13}$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{116}{64} = 0 \tag{2.0.14}$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \tag{2.0.15}$$

$$\lambda_2 = 1 \tag{2.0.16}$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector $\mathbf{v_1}$ can be calculated using row reduction as :

$$\begin{pmatrix} -\frac{9}{16} & \frac{3}{8} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -\frac{16}{9}R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \tag{2.0.17}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \tag{2.0.18}$$

Hence,

$$\mathbf{v_1} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{2.0.19}$$

Similarly, for $\lambda_2 = 1$,

$$\mathbf{v_2} = \begin{pmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{2.0.20}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{2.0.21}$$

Now,

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{2.0.22}$$

Now, calculating eigenvalues of $\mathbf{MM}^T$

$$\begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-\lambda \end{vmatrix} = 0 \tag{2.0.23}$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \tag{2.0.24}$$
$$\lambda_2 = 1 \tag{2.0.25}$$
$$\lambda_3 = 0 \tag{2.0.26}$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector can be computed as:

$$\begin{pmatrix} 1-\frac{29}{16} & 0 & \frac{1}{2} \\ 0 & 1-\frac{29}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-\frac{29}{16} \end{pmatrix} \tag{2.0.27}$$

$$\leftrightarrow \begin{pmatrix} -\frac{13}{16} & 0 & \frac{1}{2} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \tag{2.0.28}$$

$$\xleftrightarrow{R_1 \leftarrow -\frac{16}{13}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \tag{2.0.29}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \tag{2.0.30}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{16}{13}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \tag{2.0.31}$$

$$\xleftrightarrow{R_2 \leftarrow R_3 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.32}$$

Hence, the eigen vector $\mathbf{u}_1$:

$$\mathbf{u_1} = \begin{pmatrix} \frac{8}{\sqrt{377}} \\ \frac{12}{\sqrt{377}} \\ \frac{13}{\sqrt{377}} \end{pmatrix} \tag{2.0.33}$$

For $\lambda_2 = 1$, the eigen vector is:

$$\begin{pmatrix} 1-1 & 0 & \frac{1}{2} \\ 0 & 1-1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-1 \end{pmatrix} \tag{2.0.34}$$

$$\leftrightarrow \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -\frac{3}{16} \end{pmatrix} \tag{2.0.35}$$

Hence, the eigen vector $\mathbf{u}_2$:

$$\mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \tag{2.0.36}$$

Similarly, for $\lambda_3 = 0$, the eigen vector is:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{2.0.37}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1 - \frac{3}{4}R_2} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.38}$$

Hence, the eigen vector $\mathbf{u}_3$:

$$\mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{29}} \\ \frac{3}{\sqrt{29}} \\ -\frac{4}{\sqrt{29}} \end{pmatrix} \tag{2.0.39}$$

So, the orthonormal matrix $\mathbf{U}$ of eigen vectors is:

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & -\frac{4}{\sqrt{29}} \end{pmatrix} \tag{2.0.40}$$

The matrix of singular values of $\mathbf{M}$ is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.41}$$

The Moore-Penrose pseudoinverse of $\mathbf{S}$ is computed as

$$\mathbf{S}_+ = (\mathbf{SS}^T)^{-1}\mathbf{S}^T \tag{2.0.42}$$
$$= \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.43}$$

To solve for $\mathbf{x}$ in (2.0.11), noting that $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \\ 0 \end{pmatrix} \tag{2.0.44}$$

$$\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \tag{2.0.45}$$

Thus, the foot of perpendicular is:

$$\mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \tag{2.0.46}$$

$$\implies \quad \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{2.0.47}$$

The solution can be verified using

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.48}$$

The LHS gives

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{2.0.49}$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{2.0.50}$$

Now, finding $\mathbf{x}$ from

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{2.0.51}$$

Solving the augmented matrix, we get

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} & -3 \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -\frac{3}{10}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \tag{2.0.52}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & \frac{29}{20} & \frac{58}{20} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & 1 & 2 \end{pmatrix} \tag{2.0.53}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix} \tag{2.0.54}$$

Hence, the solution is given by

$$\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{2.0.55}$$

Comparing the results in Eq.(2.0.47) and (2.0.55), it is concluded that the solution is verified.