EE5609 Assignment 13

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1 Problem

Let $\mathbf{F}$ be a field and let $\mathbf{V}$ be the space of polynomial functions f from $\mathbf{F}$ into $\mathbf{F}$, given by

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n$$

Let $\mathbf{D}$ be a linear differentiation transformation defined as

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx}$$

Then find the range and null space of $\mathbf{D}$.
Let $\mathbf{R}$ be the field of real numbers and let $\mathbf{V}$ be the space of all functions from $\mathbf{R}$ into $\mathbf{R}$ which are continous. Let $\mathbf{T}$ be linear transformation defined by

$$(\mathbf{T}f)(x) = \int_0^x f(t)\,dt$$

Find the range and null space of $\mathbf{T}$.

2 Explanation

Let the vector space $\mathbf{V}$ be defined as

$$\mathbf{V} = \left\{ f : \mathbf{F} \to \mathbf{F} : f(x) = \sum_{k=0}^{n} c_k x^k,\ c_k \in \mathbf{F} \right\} \tag{2.0.1}$$

Differentiation transformation is defined as a function which maps the vectors in $\mathbf{F}$ into $\mathbf{F}$ such that

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx} \tag{2.0.2}$$

$$\implies \mathbf{D}f = \sum_{k=0}^{n} k c_k x^{k-1} = g(x) \tag{2.0.3}$$

Since $g(x) \in \mathbf{V}$ therefore the transformation $\mathbf{D}$ is defined from $\mathbf{V}$ into $\mathbf{V}$. Thus the range of $\mathbf{D}$ is the entire vector space $\mathbf{V}$. Now consider the nullspace for differentiation transformation defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{D}f = 0\} \tag{2.0.4}$$

$$\mathbf{D}f = 0 \implies f = c \tag{2.0.5}$$

where c is a constant. Such a polynomial is known as constant polynomial. Therefore

$$\mathbf{N} = \{f = c : f \in \mathbf{V}, c \in \mathbf{F} \text{ where c is a constant}\} \tag{2.0.6}$$

Now consider the vector space defined as

$$\mathbf{V} = \{f : \mathbf{R} \to \mathbf{R} : \text{f is continous}\} \tag{2.0.7}$$

Integration transformation is defined as

$$(\mathbf{T}f)(x) = \int_0^x f(t)\,dt \tag{2.0.8}$$

Let

$$F(x) = \int_0^x f(t)\,dt \tag{2.0.9}$$

Since f is continous function we have $|f(t)| \leq M$ $\forall\ t \in [0, x]$ and $|M| \geq 0$, it follows that

$$|F(x+h) - F(x)| = \left| \int_0^h f(t)\,dt \right| \leq M\,|h| \tag{2.0.10}$$

which shows that F(x) is also continous and thus

$$F(x) \in \mathbf{V} \tag{2.0.11}$$

Therefore the transformation $\mathbf{T}$ is defined from $\mathbf{V}$ into $\mathbf{V}$. Thus the range of $\mathbf{T}$ is the entire vector space $\mathbf{V}$. Now consider the nullspace for intergration transformation defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{T}f = 0\} \tag{2.0.12}$$

$$\mathbf{T}f = 0 \implies \int_0^x f(t)\,dt = 0 \tag{2.0.13}$$

$$\implies f(t) = 0 \tag{2.0.14}$$

Therefore nullspace for integration transformation is

$$\mathbf{N} = \{0\} \tag{2.0.15}$$