EE5609 Assignment 13

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1 Problem

a) Let $\mathbf{F}$ be a field and let $\mathbf{V}$ be the space of polynomial functions f from $\mathbf{F}$ into $\mathbf{F}$, given by

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n$$

Let $\mathbf{D}$ be a linear differentiation transformation defined as

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx}$$

Then find the range and null space of $\mathbf{D}$.

b) Let $\mathbf{R}$ be the field of real numbers and let $\mathbf{V}$ be the space of all functions from $\mathbf{R}$ into $\mathbf{R}$ which are continous. Let $\mathbf{T}$ be linear transformation defined by

$$(\mathbf{T}f)(x) = \int_0^x f(t)\,dt$$

Find the range and null space of $\mathbf{T}$.

2 Explanation

Let the vector space of n-dimension be deined as

$$\mathbf{V} = \left\{ f : \mathbf{F} \to \mathbf{F} : f(x) = \sum_{k=0}^{n} c_k x^k,\ c_k \in \mathbf{F} \right\} \tag{2.0.1}$$

The corresponding standard basis for $\mathbf{V}$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x^{n-1} \end{pmatrix} \right\} \tag{2.0.2}$$

a) Let f and $g \in \mathbf{V}$ and let α and $\beta \in \mathbf{F}$ then

$$\mathbf{D}(\alpha f + \beta g) = \frac{d(\alpha f(x) + \beta g(x))}{dx} \tag{2.0.3}$$
$$= \alpha \frac{df(x)}{dx} + \beta \frac{dg(x)}{dx} \tag{2.0.4}$$
$$= \alpha(\mathbf{D}f) + \beta(\mathbf{D}g) \tag{2.0.5}$$

Therefore $\mathbf{D}$ is a linear transformation.
The $\mathbf{D}$ transformation maps the k^{th} basis vector as follows

$$\mathbf{D} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^k \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ kx^{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.6}$$

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ kx^{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \tag{2.0.7}$$

the range of $\mathbf{D}$ is the vector space $\mathbf{V}$. Thus the transformation is defined as $\mathbf{D} : \mathbf{V} \to \mathbf{V}$. Therefore the $\mathbf{D}$ Transformation on the basis vector set is

$$\mathbf{D} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{2.0.8}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.0.9}$$

Thus the **D** transformation coefficient matrix is

$$D = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.0.10}$$

Since D contains a zero row hence $\left|D\right| = 0$. Therefore **D** transformation matrix is singular. The nullspace for differentiation transformation is defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{D}f = 0\} \tag{2.0.11}$$

Let the coefficient matrix of $f \in \mathbf{V}$ be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{2.0.12}$$

then

$$\mathbf{D}f = 0 \tag{2.0.13}$$

$$\implies \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \tag{2.0.14}$$

Since D is in row reduced echolon form and $rank(D) = n - 1$ the solution of (2.0.13) is

$$\mathbf{f} = \begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.15}$$

where $k \in \mathbf{R}$. Therefore the nullspace for $\mathbf{D} : \mathbf{V} \to \mathbf{V}$ is

$$\mathbf{N} = \left\{ \begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R} \right\} \tag{2.0.16}$$

b) Let f and $g \in \mathbf{V}$ and let α and $\beta \in \mathbf{F}$ then

$$\mathbf{T}(\alpha f + \beta g) = \int_0^x (\alpha f(t) + \beta g(t))\, dt \tag{2.0.17}$$

$$= \alpha \int_0^x f(t)\, dt + \beta \int_0^x g(t)\, dt \tag{2.0.18}$$

$$= \alpha(\mathbf{T}f) + \beta(\mathbf{T}g) \tag{2.0.19}$$

Therefore **T** is a linear transformation.
The **T** transformation maps the k^{th} basis vector as follows

$$\mathbf{T}\begin{pmatrix} 0 \\ \vdots \\ x^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.20}$$

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \tag{2.0.21}$$

the range of **T** is the vector space **V**. Thus the transformation is defined as $\mathbf{T} : \mathbf{V} \to \mathbf{V}$. Therefore the **T** Transformation on the basis vector set is

$$\mathbf{T}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{2.0.22}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \tag{2.0.23}$$

Thus the **T** transformation coefficient matrix is

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \tag{2.0.24}$$

Since T contains a zero row hence $\left|T\right| = 0$. Therefore **T** transformation matrix is singular. The nullspace for integration transformation is defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{T}f = 0\} \tag{2.0.25}$$

Let the coefficient matrix of $f \in \mathbf{V}$ be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{2.0.26}$$

then

$$\mathbf{T}f = 0 \tag{2.0.27}$$

$$\implies \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \tag{2.0.28}$$

Since T is in row reduced echolon form and $rank(T) = n$ the solution of (2.0.27) is

$$\mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.0.29}$$

where $k \in \mathbf{R}$. Therefore the nullspace for $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ is

$$\mathbf{N} = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R} \right\} \tag{2.0.30}$$