EE5609 Assignment 16

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1 Problem

Let $\mathbf{W}$ be the space of $n \times n$ matrices over the field $\mathbf{F}$, and let $\mathbf{W}_0$ be the subspace spanned by the matrices C of the form $C = AB - BA$. Prove that $\mathbf{W}_0$ is exactly the subspace of matrices which have trace zero.

2 Explanation

Let there be two subspaces defined as

$$\mathbf{W}_0 = \{A \in \mathbf{W} : trace(A) = 0\} \tag{2.0.1}$$

and

$$\mathbf{W}_1 = \{C \in \mathbf{W} : C = AB - BA\} \tag{2.0.2}$$

Consider $C \in \mathbf{W}$ such that $C = AB - BA$ where $A, B \in \mathbf{R}^{N \times N}$ then

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \tag{2.0.3}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{NN} \end{pmatrix} \tag{2.0.4}$$

Then

$$\begin{aligned} trace(AB) &= \sum_{i=1}^{N} (AB)_{ii} && (2.0.5) \\ &= \sum_{i=1}^{N} \sum_{k=1}^{N} A_{ik} B_{ki} && (2.0.6) \\ &= \sum_{i=1}^{N} \sum_{k=1}^{N} B_{ki} A_{ik} && (2.0.7) \\ &= \sum_{k=1}^{N} \sum_{i=1}^{N} B_{ki} A_{ik} && (2.0.8) \\ &= \sum_{k=1}^{N} (BA)_{kk} && (2.0.9) \\ &= trace(BA) && (2.0.10) \\ &\implies trace(AB) - trace(BA) = 0 && (2.0.11) \\ &\implies trace(C) = 0 && (2.0.12) \end{aligned}$$

Thus any linear combination of matrices $C = AB - BA$ represents the subspace of all $N \times N$ matrices with trace equal to 0. Hence, the subspace $\mathbf{W}_0$ and $\mathbf{W}_1$ are equivalent.