

EE5609 Assignment 8

SHANTANU YADAV, EE20MTECH12001

The python solution code is available at

https://github.com/Shantanu2508/Matrix_Theory/blob/master/Assignment_8/assignment8.py

1 PROBLEM

Find the foot of perpendicular from point $B = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ to the plane $(2 \ 3 \ -4)\mathbf{x} = -5$.

2 SOLUTION

Let us consider orthogonal vectors \mathbf{m}_1 and \mathbf{m}_2 to the given normal vector \mathbf{n} . Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Then,

$$\mathbf{m}^T \mathbf{n} = 0 \quad (2.0.1)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 0 \quad (2.0.2)$$

$$\Rightarrow 2a + 3b - 4c = 0 \quad (2.0.3)$$

Let $a = 1$, $b = 0$, so that

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (2.0.4)$$

and $a = 0$, $b = 1$, so that

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \quad (2.0.5)$$

We, now, solve the equation

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.6)$$

which, upon substitution, becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad (2.0.7)$$

Any $m \times n$ matrix \mathbf{M} can be factorized in SVD form as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (2.0.8)$$

where \mathbf{U} and \mathbf{V} are matrices of eigen vectors which are orthogonal. Columns of \mathbf{V} are the eigen vectors of $\mathbf{M}^T\mathbf{M}$, columns of \mathbf{U} are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and \mathbf{S} is the diagonal matrix of singular values of \mathbf{M} of the eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \quad (2.0.9)$$

Putting (2.0.8) into (2.0.6), we get

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (2.0.10)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \quad (2.0.11)$$

where \mathbf{S}_+ is the Moore-Penrose Pseudoinverse of \mathbf{S} .

The eigenvalues of $\mathbf{M}^T\mathbf{M}$:

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (2.0.12)$$

$$\Rightarrow \begin{vmatrix} \frac{10}{8} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{vmatrix} = 0 \quad (2.0.13)$$

$$\Rightarrow \lambda^2 - \frac{45}{16}\lambda + \frac{116}{64} = 0 \quad (2.0.14)$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \quad (2.0.15)$$

$$\lambda_2 = 1 \quad (2.0.16)$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector \mathbf{v}_1 can be calculated using row reduction as :

$$\begin{pmatrix} -\frac{9}{16} & \frac{3}{8} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \xrightarrow{R_1 \leftarrow -\frac{16}{9}R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \quad (2.0.17)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad (2.0.18)$$

Hence,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (2.0.19)$$

Similarly, for $\lambda_2 = 1$,

$$\mathbf{v}_2 = \begin{pmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad (2.0.20)$$

Thus,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \quad (2.0.21)$$

Now,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \quad (2.0.22)$$

Now, calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$

$$\begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-\lambda \end{vmatrix} = 0 \quad (2.0.23)$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \quad (2.0.24)$$

$$\lambda_2 = 1 \quad (2.0.25)$$

$$\lambda_3 = 0 \quad (2.0.26)$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector can be computed as:

$$\begin{pmatrix} 1 - \frac{29}{16} & 0 & \frac{1}{2} \\ 0 & 1 - \frac{29}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \frac{29}{16} \end{pmatrix} \quad (2.0.27)$$

$$\leftrightarrow \begin{pmatrix} -\frac{13}{16} & 0 & \frac{1}{2} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \quad (2.0.28)$$

$$\xleftrightarrow{R_1 \leftarrow -\frac{16}{13}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{13} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \quad (2.0.29)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{13} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ 0 & -\frac{1}{16} & -\frac{9}{13} \end{pmatrix} \quad (2.0.30)$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{16}{13}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{13} \\ 0 & 1 & -\frac{12}{13} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \quad (2.0.31)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{13} \\ 0 & 1 & -\frac{12}{13} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.32)$$

Hence, the eigen vector \mathbf{u}_1 :

$$\mathbf{u}_1 = \begin{pmatrix} \frac{8}{\sqrt{377}} \\ \frac{\sqrt{377}}{12} \\ \frac{\sqrt{377}}{13} \end{pmatrix} \quad (2.0.33)$$

For $\lambda_2 = 1$, the eigen vector is:

$$\begin{pmatrix} 1-1 & 0 & \frac{1}{2} \\ 0 & 1-1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-1 \end{pmatrix} \quad (2.0.34)$$

$$\leftrightarrow \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -\frac{3}{16} \end{pmatrix} \quad (2.0.35)$$

Hence, the eigen vector \mathbf{u}_2 :

$$\mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (2.0.36)$$

Similarly, for $\lambda_3 = 0$, the eigen vector is:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \quad (2.0.37)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1 - \frac{3}{4}R_2} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.38)$$

Hence, the eigen vector \mathbf{u}_3 :

$$\mathbf{u}_3 = \begin{pmatrix} \frac{2}{\sqrt{29}} \\ \frac{3}{\sqrt{29}} \\ -\frac{4}{\sqrt{29}} \end{pmatrix} \quad (2.0.39)$$

So, the orthonormal matrix \mathbf{U} of eigen vectors is:

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{29}} \\ \frac{\sqrt{377}}{12} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{29}} \\ \frac{\sqrt{377}}{13} & 0 & -\frac{4}{\sqrt{29}} \end{pmatrix} \quad (2.0.40)$$

The matrix of singular values of \mathbf{M} is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.41)$$

The Moore-Penrose pseudoinverse of \mathbf{S} is computed as

$$\mathbf{S}_+ = (\mathbf{S}\mathbf{S}^T)^{-1}\mathbf{S}^T \quad (2.0.42)$$

$$= \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.43)$$

To solve for \mathbf{x} in (2.0.11), noting that $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \\ 0 \end{pmatrix} \quad (2.0.44)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \quad (2.0.45)$$

Thus, the foot of perpendicular is:

$$\mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \quad (2.0.46)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (2.0.47)$$

The solution can be verified using

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (2.0.48)$$

The LHS gives

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (2.0.49)$$

$$\Rightarrow \mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (2.0.50)$$

Now, finding \mathbf{x} from

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (2.0.51)$$

Solving the augmented matrix, we get

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} & -3 \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \xrightarrow{R_1 \leftarrow -\frac{3}{10} R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \quad (2.0.52)$$

$$\xrightarrow{R_2 \leftarrow R_2 - \frac{3}{8} R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & \frac{29}{20} & \frac{58}{20} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & 1 & 2 \end{pmatrix} \quad (2.0.53)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{3}{10} R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix} \quad (2.0.54)$$

Hence, the solution is given by

$$\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (2.0.55)$$

Comparing the results in Eq.(2.0.47) and (3.0.8), it is concluded that the solution is verified.