

EE5609 Assignment 15

SHANTANU YADAV, EE20MTECH12001

1 PROBLEM

The linear operator \mathbf{T} on \mathbf{R}^2 defined by

$$\mathbf{T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Prove that if \mathbf{S} is a linear operator on \mathbf{R}^2 such that $\mathbf{S}^2 = \mathbf{S}$, then $\mathbf{S} = \mathbf{0}$, or $\mathbf{S} = \mathbf{I}$, or there is an ordered basis \mathbf{B} for \mathbf{R}^2 such that $[\mathbf{S}]_{\mathbf{B}} = \mathbf{A}$.

2 EXPLANATION

If a linear operator \mathbf{S} is defined on \mathbf{R}^2 such that $\mathbf{S}^2 = \mathbf{S}$, then

$$\mathbf{S}^2 - \mathbf{S} = \mathbf{0} \quad (2.0.1)$$

$$\mathbf{S}(\mathbf{S} - \mathbf{I}) = \mathbf{0} \quad (2.0.2)$$

$$\implies \mathbf{S} = \mathbf{0}, \mathbf{S} = \mathbf{I} \quad (2.0.3)$$

The transformation of a vector $\mathbf{x} \in \mathbf{R}^2$ can be represented as

$$\mathbf{S}\mathbf{x} = \mathbf{y} \quad (2.0.4)$$

$$\implies \mathbf{S}(\mathbf{S}\mathbf{x}) = \mathbf{S}\mathbf{y} \quad (2.0.5)$$

$$\implies \mathbf{S}^2\mathbf{x} = \mathbf{S}\mathbf{y} \quad (2.0.6)$$

$$\implies \mathbf{S}\mathbf{x} = \mathbf{S}\mathbf{y} \quad (2.0.7)$$

$$\implies \mathbf{x} = \mathbf{y} \quad (2.0.8)$$

Therefore the transformation of a vector $\mathbf{x} \in \mathbf{R}^2$ can be given as

$$\mathbf{S}\mathbf{x} = \mathbf{x} \quad \forall \mathbf{x} \in \mathbf{R}^2 \quad (2.0.9)$$

Consider the ordered basis set

$$\mathbf{B} = \{\epsilon_1, \epsilon_2\} \in \mathbf{R}^2 \quad (2.0.10)$$

and if

$$[\mathbf{S}]_{\mathbf{B}} = \mathbf{A} \quad (2.0.11)$$

$$\implies [\mathbf{S}]_{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.12)$$

Thus we can re-write the column vectors of $[\mathbf{S}]_{\mathbf{B}}$ using (2.0.12) as

$$\mathbf{S} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.13)$$

$$\mathbf{S} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.14)$$

Therefore, any vector \mathbf{x} in column space of $[\mathbf{S}]_{\mathbf{B}}$ can be uniquely expressed by $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, hence it forms the basis for column space of $[\mathbf{S}]_{\mathbf{B}}$. Therefore one of the basis vector of \mathbf{B} is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The other basis vector can be any vector which is linearly independent to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. One of the ordered basis set can be

$$\mathbf{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad (2.0.15)$$