EE5609 Assignment 17

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1 Problem

Let $\mathbf{A}$ be a $n \times n$ real matrix with $\mathbf{A}^2 = \mathbf{A}$. Then

1) the eigenvalues of $\mathbf{A}$ are either 0 or 1
2) $\mathbf{A}$ is a diagonal matrix with diagonal entries 0 or 1
3) $rank(\mathbf{A}) = trace(\mathbf{A})$
4) if $rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A})$

2 Explanation

<table>
<tr><th>Objective</th><th>Explanation</th></tr>
<tr><td>Eigenvalues of $\mathbf{A}$</td><td>Since

$$\mathbf{A}^2 = \mathbf{A} \tag{2.0.1}$$
$$\implies \mathbf{A}^2 - \mathbf{A} = \mathbf{O} \tag{2.0.2}$$

From Cayley-Hamilton Theorem we have,

$$\lambda^2 - \lambda = 0 \tag{2.0.3}$$
$$\implies \lambda(\lambda - 1) = 0 \tag{2.0.4}$$
$$\implies \lambda = 0, 1 \tag{2.0.5}$$
</td></tr>
<tr><td>Relation between rank and trace of $\mathbf{A}$</td><td>A matrix $\mathbf{A}$ satisfying $\mathbf{A}^2 = \mathbf{A}$ is an idempotent matrix with eigen values equal to 0 or 1. Rank of matrix is defined as the number of non-zero eigenvalues. Since number of non-zero matrix is 1,

$$rank(\mathbf{A}) = 1 \tag{2.0.6}$$
$$trace(\mathbf{A}) = \sum_i \lambda_i = 0 + 1 = 1 \tag{2.0.7}$$
$$\implies rank(\mathbf{A}) = trace(\mathbf{A}) \tag{2.0.8}$$
</td></tr>
<tr><td>Relation between rank and trace of $\mathbf{I} - \mathbf{A}$</td><td>Now for the matrix $\mathbf{I} - \mathbf{A}$ we have,

$$(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) \tag{2.0.9}$$
$$= \mathbf{I}^2 - \mathbf{IA} - \mathbf{AI} + \mathbf{A}^2 \tag{2.0.10}$$
$$= \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} \tag{2.0.11}$$
$$= \mathbf{I} - \mathbf{A} \tag{2.0.12}$$

Hence $\mathbf{I} - \mathbf{A}$ is an idempotent matrix. Therefore we conclude,

$$rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A}) \tag{2.0.13}$$
</td></tr>
<tr><td>Diagonalizibility of $\mathbf{A}$</td><td>Since the characteristic equation of $\mathbf{A}$ is

$$f(\lambda) = \lambda(\lambda - 1) = 0 \tag{2.0.14}$$

and since the matrix $\mathbf{A}$ satisfies its charateristic equation such that

$$f(\mathbf{A}) = \mathbf{0} \tag{2.0.15}$$

so f annhiliates $\mathbf{A}$ and the minimal polynomial of the charateristic equation is

$$m(\lambda) = \lambda(\lambda - 1) \tag{2.0.16}$$

Since the roots of $m(\lambda)$ are distinct we can conclude that $\mathbf{A}$ is diagonalizable.</td></tr>
<tr><td>Answer</td><td>(1),(3) and (4)</td></tr>
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TABLE I