

EE5609 Assignment 5

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The python solution code is available at

https://github.com/Shantanu2508/Matrix_Theory/blob/master/Assignment_5/assignment5part2.py

From (2.0.1) and (2.0.5)

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \quad (3.0.4)$$

$$\Rightarrow 12 \begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2} \begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ -\frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2} \begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \quad (3.0.5)$$

$$|\mathbf{V}| = -\frac{529}{4} < 0 \quad (3.0.6)$$

The given equation therefore represents two intersecting lines. Since $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix \mathbf{P} such that

$$\mathbf{PVP}^T = \mathbf{D} = \text{diag}(\lambda_1 \ \lambda_2) \quad (3.0.7)$$

or equivalently

$$\mathbf{V} = \mathbf{PDP}^T \quad (3.0.8)$$

Eigen vectors of real symmetric matrix \mathbf{V} are orthogonal. The characteristic equation of \mathbf{V} is obtained by evaluating the determinant

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 12 & -\frac{7}{2} \\ -\frac{7}{2} & \lambda + 10 \end{vmatrix} = 0 \quad (3.0.9)$$

$$\Rightarrow \lambda^2 - 2\lambda - \frac{529}{4} = 0 \quad (3.0.10)$$

$$\Rightarrow \lambda_1 = \frac{527}{50}, \quad \lambda_2 = \frac{627}{50} \quad (3.0.11)$$

The eigen vector \mathbf{p} is defined as

$$\mathbf{Vp} = \lambda \mathbf{p} \quad (3.0.12)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (3.0.13)$$

1 PROBLEM

Prove that the equation

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$$

represents two straight lines and find the angle between the lines.

2 EXPLANATION

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (2.0.4)$$

(2.0.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.0.5)$$

The lines intersect if

$$|\mathbf{V}| < 0 \quad (2.0.6)$$

3 SOLUTION

From (2.0.3) and (2.0.4)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (3.0.2)$$

$$f = -35 \quad (3.0.3)$$

The orthogonal eigen-vector matrix

$$\mathbf{P} = \frac{1}{\|\mathbf{p}\|} (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} \frac{3}{20} & -\frac{49}{50} \\ -\frac{49}{50} & -\frac{3}{20} \end{pmatrix} \quad (3.0.14)$$

$$\mathbf{D} = \begin{pmatrix} -\frac{527}{50} & 0 \\ 0 & \frac{627}{50} \end{pmatrix} \quad (3.0.15)$$

Let $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$ with $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$. Substituting in (2.0.2)

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (3.0.16)$$

$$\Rightarrow \mathbf{y}^T \mathbf{D} \mathbf{y} = 0 \quad (3.0.17)$$

$$\Rightarrow \begin{pmatrix} \pm \sqrt{\frac{\lambda_1}{\lambda_2}} & 1 \end{pmatrix} \mathbf{y} = 0 \quad (3.0.18)$$

Substituting $\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c}$ in (3.0.18)

$$\Rightarrow \begin{pmatrix} \pm \sqrt{\frac{10}{12}} & 1 \end{pmatrix} (\mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c}) = 0 \quad (3.0.19)$$

The equations of lines are

$$\begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} = 5 \quad (3.0.20)$$

$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = -7 \quad (3.0.21)$$

The angle between the lines can be expressed in terms of normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (3.0.22)$$

as

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (3.0.23)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \quad (3.0.24)$$

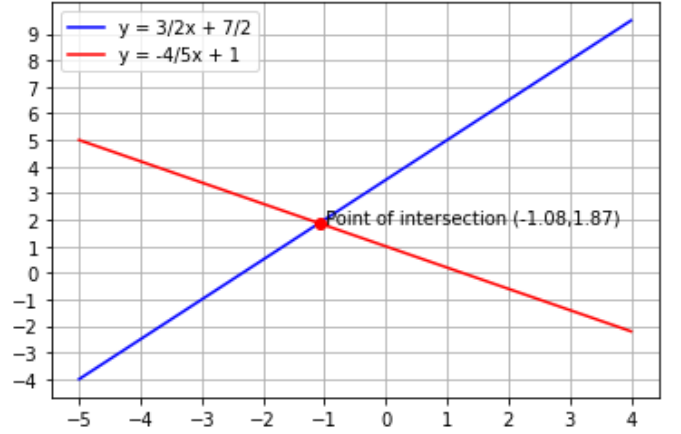


Fig. 0