EE5609 Assignment 13

SHANTANU YADAV, EE20MTECH12001

1 Problem

a) Let $\mathbf{F}$ be a field and let $\mathbf{V}$ be the space of polynomial functions f from $\mathbf{F}$ into $\mathbf{F}$, given by

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n$$

Let $\mathbf{D}$ be a linear differentiation transformation defined as

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx}$$

Then find the range and null space of $\mathbf{D}$.

b) Let $\mathbf{R}$ be the field of real numbers and let $\mathbf{V}$ be the space of all functions from $\mathbf{R}$ into $\mathbf{R}$ which are continous. Let $\mathbf{T}$ be linear transformation defined by

$$(\mathbf{T}f)(x) = \int_0^x f(t)\,dt$$

Find the range and null space of $\mathbf{T}$.

2 Explanation

a) Let the vector space $\mathbf{V}$ be defined as

$$\mathbf{V} = \left\{ f : \mathbf{F} \to \mathbf{F} : f(x) = \sum_{k=0}^{n} c_k x^k,\ c_k \in \mathbf{F} \right\} \tag{2.0.1}$$

Differentiation transformation is defined as a function which maps the vectors in $\mathbf{F}$ into $\mathbf{F}$ such that

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx} \tag{2.0.2}$$

$$\implies \mathbf{D}f = \sum_{k=0}^{n} k c_k x^{k-1} = g(x) \tag{2.0.3}$$

Since $g(x) \in \mathbf{V}$ therefore the transformation $\mathbf{D}$ is defined from $\mathbf{V}$ into $\mathbf{V}$. Thus the range of $\mathbf{D}$ is the entire vector space $\mathbf{V}$. Now consider the nullspace for differentiation transformation defined as

$$\mathbf{N} = \{ f \in \mathbf{V} : \mathbf{D}f = 0 \} \tag{2.0.4}$$

$$\mathbf{D}f = 0 \implies f = c \tag{2.0.5}$$

where c is a constant. Such a polynomial is known as constant polynomial. Therefore

$$\mathbf{N} = \{ f = c : f \in \mathbf{V}, c \in \mathbf{F} \text{ where c is a constant} \} \tag{2.0.6}$$

b) Now consider the vector space defined as

$$\mathbf{V} = \{ f : \mathbf{R} \to \mathbf{R} : \text{f is continous} \} \tag{2.0.7}$$

Integration transformation is defined as

$$(\mathbf{T}f)(x) = \int_0^x f(t)\,dt \tag{2.0.8}$$

Let

$$F(x) = \int_0^x f(t)\,dt \tag{2.0.9}$$

Since f is continous function we have $|f(t)| \leq M\ \forall\ t \in [0, x]$ and $|M| \geq 0$, it follows that

$$|F(x+h) - F(x)| = \left| \int_0^h f(t)\,dt \right| \leq M\,|h| \tag{2.0.10}$$

which shows that F(x) is also continous and thus

$$F(x) \in \mathbf{V} \tag{2.0.11}$$

Therefore the transformation $\mathbf{T}$ is defined from $\mathbf{V}$ into $\mathbf{V}$. Thus the range of $\mathbf{T}$ is the entire vector space $\mathbf{V}$. Now consider the nullspace for intergration transformation defined as

$$\mathbf{N} = \{ f \in \mathbf{V} : \mathbf{T}f = 0 \} \tag{2.0.12}$$

$$\mathbf{T}f = 0 \implies \int_0^x f(t)\,dt = 0 \tag{2.0.13}$$

$$\implies f(t) = 0 \tag{2.0.14}$$

Therefore nullspace for integration transformation is

$$\mathbf{N} = \{0\} \tag{2.0.15}$$

3 D and T Transformation Matrices for the Polynomial Vector Space

Let the vector space of n-dimension be deined as

$$\mathbf{V} = \left\{ f : \mathbf{F} \to \mathbf{F} : f(x) = \sum_{k=0}^{n} c_k x^k,\ c_k \in \mathbf{F} \right\} \tag{3.0.1}$$

The corresponding standard basis for $\mathbf{V}$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x^{n-1} \end{pmatrix} \right\} \tag{3.0.2}$$

a) Let f and $g \in \mathbf{V}$ and let α and $\beta \in \mathbf{F}$ then

$$\mathbf{D}(\alpha f + \beta g) = \frac{d(\alpha f(x) + \beta g(x))}{dx} \tag{3.0.3}$$

$$= \alpha \frac{df(x)}{dx} + \beta \frac{dg(x)}{dx} \tag{3.0.4}$$

$$= \alpha(\mathbf{D}f) + \beta(\mathbf{D}g) \tag{3.0.5}$$

Therefore $\mathbf{D}$ is a linear transformation.
The $\mathbf{D}$ transformation maps the k^{th} basis vector as follows

$$\mathbf{D} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^k \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ kx^{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.0.6}$$

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ kx^{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \tag{3.0.7}$$

the range of $\mathbf{D}$ is the vector space $\mathbf{V}$. Thus the transformation is defined as $\mathbf{D} : \mathbf{V} \to \mathbf{V}$. Therefore the $\mathbf{D}$ Transformation on the basis vector set is

$$\mathbf{D} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.0.8}$$

Thus the $\mathbf{D}$ transformation coefficient matrix is

$$D = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.0.9}$$

The nullspace for differentiation transformation is defined as

$$\mathbf{N} = \{ f \in \mathbf{V} : \mathbf{D}f = 0 \} \tag{3.0.10}$$

Let the coefficient matrix of $f \in \mathbf{V}$ be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{3.0.11}$$

then

$$\mathbf{D}f = 0 \tag{3.0.12}$$

$$\implies \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \tag{3.0.13}$$

Since D is in row reduced echolon form and $rank(D) = n - 1$ the solution of (**??**) is

$$\mathbf{f} = \begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.0.14}$$

where $k \in \mathbf{R}$. Therefore the nullspace for

$\mathbf{D} : \mathbf{V} \to \mathbf{V}$ is

$$\mathbf{N} = \left\{ \begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R} \right\} \tag{3.0.15}$$

b) Let f and $g \in \mathbf{V}$ and let α and $\beta \in \mathbf{F}$ then

$$\mathbf{T}(\alpha f + \beta g) = \int_0^x (\alpha f(t) + \beta g(t))\, dt \tag{3.0.16}$$

$$= \alpha \int_0^x f(t)\, dt + \beta \int_0^x g(t)\, dt \tag{3.0.17}$$

$$= \alpha(\mathbf{T}f) + \beta(\mathbf{T}g) \tag{3.0.18}$$

Therefore $\mathbf{T}$ is a linear transformation.
The $\mathbf{T}$ transformation maps the k^{th} basis vector as follows

$$\mathbf{T} \begin{pmatrix} 0 \\ \vdots \\ x^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \tag{3.0.19}$$

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \tag{3.0.20}$$

the range of $\mathbf{T}$ is the vector space $\mathbf{V}$. Thus the transformation is defined as $\mathbf{T} : \mathbf{V} \to \mathbf{V}$. Therefore the $\mathbf{T}$ Transformation on the basis vector set is

$$\mathbf{T} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \tag{3.0.21}$$

Thus the $\mathbf{T}$ transformation coefficient matrix is

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \tag{3.0.22}$$

The nullspace for integration transformation is defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{T}f = 0\} \tag{3.0.23}$$

Let the coefficient matrix of $f \in \mathbf{V}$ be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{3.0.24}$$

then

$$\mathbf{T}f = 0 \tag{3.0.25}$$

$$\implies \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \tag{3.0.26}$$

Since T is in row reduced echolon form and $rank(T) = n$ the solution of (**??**) is

$$\mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.0.27}$$

where $k \in \mathbf{R}$. Therefore the nullspace for $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ is

$$\mathbf{N} = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R} \right\} \tag{3.0.28}$$