EE5609 Challenge 1

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The python solution code is available at

https://github.com/Shantanu2508/Matrix_Theory/blob/master/Challenge1/skew_shortest_distance_2.py

1 Problem Statement

Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \tag{1.0.1}$$

2 Explanation

Let the two lines be L_1 and L_2

$$L_1 : \mathbf{x} = \mathbf{A} + \lambda_1 \mathbf{m_1} \tag{2.0.1}$$

$$L_2 : \mathbf{x} = \mathbf{B} + \lambda_2 \mathbf{m_2} \tag{2.0.2}$$

Let $\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix}$

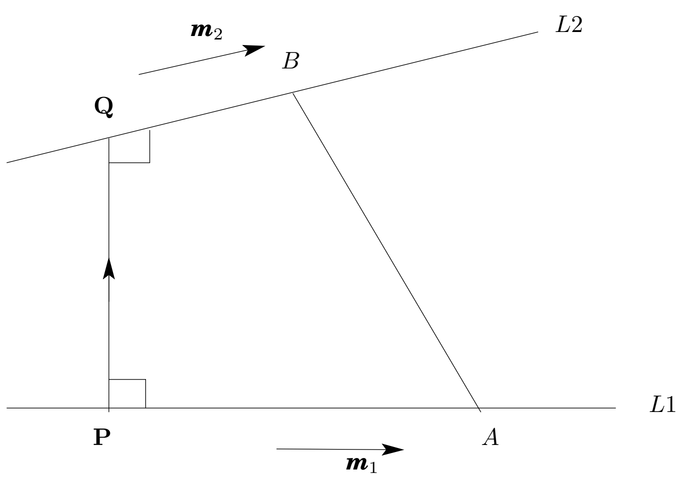

Fig. 0

Since P lies on L_1 and Q lies on L_2, the points should satisfy equations (2.0.1) and (2.0.2), respectively.

$$\mathbf{P} = \mathbf{A} + \lambda_1 \mathbf{m_1} \tag{2.0.3}$$

$$\mathbf{Q} = \mathbf{B} + \lambda_2 \mathbf{m_2} \tag{2.0.4}$$

$$\implies \mathbf{PQ} = \mathbf{Q} - \mathbf{P} = \mathbf{B} - \mathbf{A} + \mathbf{M} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \lambda \tag{2.0.5}$$

Since $\mathbf{PQ}$ is parallel to $\mathbf{m_1} \times \mathbf{m_2}$:

$$\mathbf{PQ^T M} = (\mathbf{B} - \mathbf{A})^{\mathbf{T}} \mathbf{M^T} + \lambda^{\mathbf{T}} \begin{pmatrix} -1 & 1 \end{pmatrix} \|\mathbf{M}\| = 0 \tag{2.0.6}$$

$$\implies \lambda^{\mathbf{T}} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{M^T} = (\mathbf{B} - \mathbf{A})^{\mathbf{T}} \tag{2.0.7}$$

$$\implies \mathbf{M} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \lambda = \mathbf{B} - \mathbf{A} \tag{2.0.8}$$

The value of λ can be obtained from (2.0.8) and the points $\mathbf{P}$ and $\mathbf{Q}$ can be evaluated.

3 Solution :

The lines intersect if :

$$\begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \tag{3.0.1}$$

$$\implies \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} - \lambda_2 \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} \tag{3.0.2}$$

$$\implies \begin{pmatrix} 1 & -3 \\ -2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -10 \\ -2 \\ -3 \end{pmatrix} \tag{3.0.3}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 1 & -3 & -10 \\ -2 & 2 & -2 \\ 2 & 2 & -3 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 2R_1]{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & -3 & -10 \\ 0 & -4 & -22 \\ 0 & 8 & 17 \end{pmatrix} \tag{3.0.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 2R_2} \begin{pmatrix} 1 & -3 & -10 \\ 0 & -4 & -22 \\ 0 & 0 & -23 \end{pmatrix} \tag{3.0.5}$$

Since the above matrix has rank =3 the lines do not intersect. The direction vector of the lines are also not same and hence not parallel. Therfore the lines lie in a different plane. Such lines are known as skew lines.

From equation (2.0.5) we have

$$\mathbf{PQ} = \mathbf{Q} - \mathbf{P} = \begin{pmatrix} -10 \\ -2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -2 & 2 \\ -2 & -2 \end{pmatrix} \lambda \tag{3.0.6}$$

From equation (4):

$$\begin{pmatrix} 3 & -9 \\ 17 & -3 \end{pmatrix} \lambda = \begin{pmatrix} 12 \\ 20 \end{pmatrix} \implies \lambda = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.0.7}$$

Substituting the values in equation to obtain:

$$\mathbf{P} = \begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix} \tag{3.0.8}$$

and

$$\mathbf{Q} = \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} \tag{3.0.9}$$

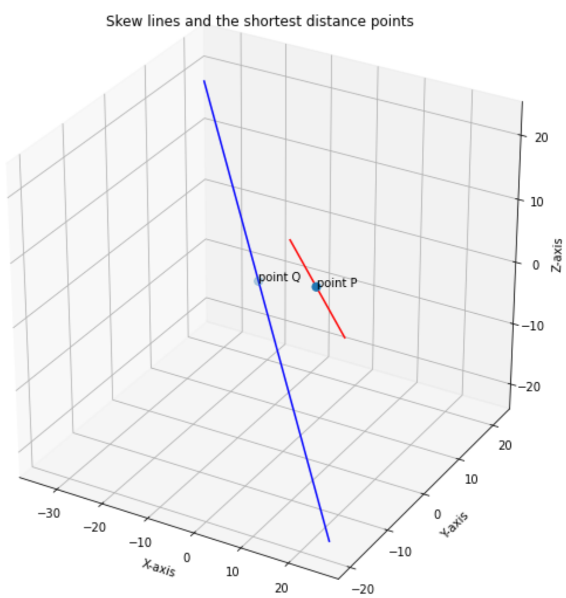

Fig. 0