EE5609 Assignment 6

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The python solution code is available at

https://github.com/Shantanu2508/Matrix_Theory/blob/master/Assignment_6/assignment6.py

1 Problem

What conic does the following equation represent? Find its equation and centre.

$$3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$$

2 Solution

The general equation of second degree can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.3}$$

From (2.0.2) and (2.0.3)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -\frac{13}{2} \end{pmatrix} \tag{2.0.5}$$

$$\left|\mathbf{V}\right| = \begin{vmatrix} 3 & -4 \\ -4 & 3 \end{vmatrix} = -25 \tag{2.0.6}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{2.0.7}$$

Since $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}\mathbf{V}\mathbf{P}^T = \mathbf{D} = diag\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \tag{2.0.8}$$

or equivalently

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.9}$$

Eigen vectors of real symmetric matrix $\mathbf{V}$ are orthogonal. The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 3 & 4 \\ 4 & \lambda + 3 \end{vmatrix} = 0 \tag{2.0.10}$$

$$\implies \lambda^2 - 25 = 0 \tag{2.0.11}$$

$$\implies \lambda_1 = -5, \lambda_2 = 5 \tag{2.0.12}$$

From (2.0.7) and (2.0.12) the equation represents a hyperbola. The eigen vector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.13}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{2.0.14}$$

For $\lambda_1 = -5$:

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}]{R_1 \leftarrow -\frac{R_1}{4}} \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \tag{2.0.15}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.16}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.17}$$

Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$\mathbf{p_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.0.18}$$

The orthogonal eigen-vector matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \tag{2.0.19}$$

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} \tag{2.0.20}$$

Let $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$ with $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$. Substituting in (2.0.1)

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.0.21}$$

with centre

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} -\frac{41}{25} \\ \frac{1}{50} \end{pmatrix} \tag{2.0.22}$$

and minor and major axes parameters as

$$\sqrt{\frac{\lambda_1}{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}} = \sqrt{\frac{500}{33}}, \quad \sqrt{\frac{\lambda_2}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}} = \sqrt{\frac{500}{33}} \tag{2.0.23}$$

The equation of hyperbola is

$$\frac{y_2^2}{\frac{33}{500}} - \frac{y_1^2}{\frac{33}{500}} = 1 \tag{2.0.24}$$

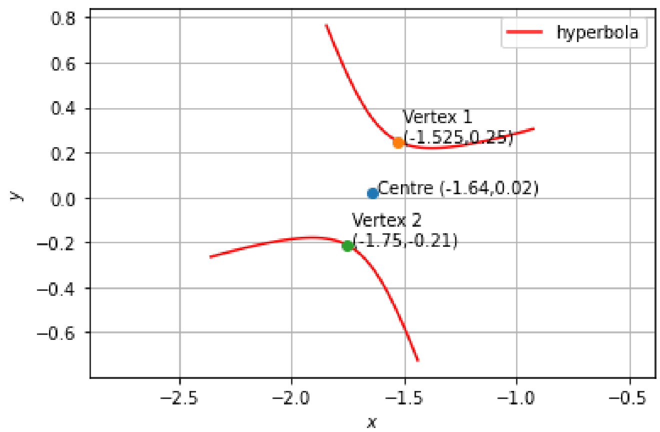

Fig. 0