EE5609 Assignment 15

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1 Problem

The linear operator $\mathbf{T}$ on $\mathbf{R}^2$ defined by

$$\mathbf{T}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\0\end{pmatrix}$$

is represented by the matrix

$$\mathbf{A} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

Prove that if $\mathbf{S}$ is a linear operator on $\mathbf{R}^2$ such that $\mathbf{S}^2 = \mathbf{S}$, then $\mathbf{S} = \mathbf{0}$, or $\mathbf{S} = \mathbf{I}$, or there is an ordered basis $\mathbf{B}$ for $\mathbf{R}^2$ such that $[\mathbf{S}]_B = \mathbf{A}$.

2 Explanation

Let us consider a matrix $\mathbf{S}$ formed from the linear combination of columns of standard ordred basis matrix $\mathbf{I}$ of $\mathbf{R}^2$ where

$$\mathbf{I} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{2.0.1}$$

and therefore,

$$\mathbf{S} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.2}$$

$$= \begin{pmatrix}\alpha & 0\\0 & \beta\end{pmatrix} \tag{2.0.3}$$

Then,

$$\mathbf{S}^2 = \begin{pmatrix}\alpha & 0\\0 & \beta\end{pmatrix}\begin{pmatrix}\alpha & 0\\0 & \beta\end{pmatrix} \tag{2.0.4}$$

$$= \begin{pmatrix}\alpha^2 & 0\\0 & \beta^2\end{pmatrix} \tag{2.0.5}$$

From the question if $\mathbf{S}^2 = \mathbf{S}$ then,

$$\implies \mathbf{S}^2 - \mathbf{S} = \mathbf{0} \tag{2.0.6}$$

$$\implies \begin{pmatrix}\alpha^2 & 0\\0 & \beta^2\end{pmatrix} - \begin{pmatrix}\alpha & 0\\0 & \beta\end{pmatrix} = \mathbf{0} \tag{2.0.7}$$

$$\implies \begin{pmatrix}\alpha(\alpha-1) & 0\\0 & \beta(\beta-1)\end{pmatrix} = \mathbf{0} \tag{2.0.8}$$

Thus the solution set is

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\} \tag{2.0.9}$$

For $\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ we get,

$$\mathbf{S} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} = \mathbf{0} \tag{2.0.10}$$

For $\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ we get,

$$\mathbf{S} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \tag{2.0.11}$$

For $\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ we get,

$$\mathbf{S} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \mathbf{A} \tag{2.0.12}$$