EE5609 Assignment 14

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1 Problem

Let $\mathbf{T}$ be a linear operator on $\mathbf{R}^3$ defined by

$$\mathbf{T}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}3x_1\\x_1-x_2\\2x_1+x_2+x_3\end{pmatrix}$$

Is $\mathbf{T}$ invertible? If so, find a rule for $\mathbf{T}^{-1}$ like the one which defines T.

2 Explanation

The transformed vector can be re-written by expanding the columns as follows

$$\begin{pmatrix}3x_1\\x_1-x_2\\2x_1+x_2+x_3\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix}x_1 + \begin{pmatrix}0\\-1\\1\end{pmatrix}x_2 + \begin{pmatrix}0\\0\\1\end{pmatrix}x_3 \tag{2.0.1}$$

$$= \begin{pmatrix}3&0&0\\1&-1&0\\2&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \tag{2.0.2}$$

$$\implies \mathbf{T} = \begin{pmatrix}3&0&0\\1&-1&0\\2&1&1\end{pmatrix} \tag{2.0.3}$$

Using Gauss-Jordan Elimination to find the inverse of $\mathbf{T}$, if it exists

$$\left(\begin{array}{ccc|ccc}3&0&0&1&0&0\\1&-1&0&0&1&0\\2&1&1&0&0&1\end{array}\right) \tag{2.0.4}$$

$$\xleftrightarrow{R_1\leftarrow\frac{R_1}{3}} \left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\1&-1&0&0&1&0\\2&1&1&0&0&1\end{array}\right) \tag{2.0.5}$$

$$\xleftrightarrow[R_3\leftarrow R_3-2R_1]{R_2\leftarrow R_2-R_1} \left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&-1&0&-\frac{1}{3}&1&0\\0&1&1&-\frac{2}{3}&0&1\end{array}\right) \tag{2.0.6}$$

$$\xleftrightarrow{R_2\leftarrow\ -R_2} \left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&1&0&\frac{1}{3}&-1&0\\0&1&1&-\frac{2}{3}&0&1\end{array}\right) \tag{2.0.7}$$

$$\xleftrightarrow{R_3\leftarrow R_3-R_2} \left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&1&0&\frac{1}{3}&-1&0\\0&0&1&-1&1&1\end{array}\right) \tag{2.0.8}$$

Since $rank(\mathbf{T}) = 3$, $\mathbf{T}$ is invertible and the inverse is

$$\mathbf{T}^{-1} = \begin{pmatrix}\frac{1}{3}&0&0\\\frac{1}{3}&-1&0\\-1&1&1\end{pmatrix} \tag{2.0.9}$$

Now consider any vector $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbf{R}^3$, then

$$\mathbf{T}^{-1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}\frac{1}{3}&0&0\\\frac{1}{3}&-1&0\\-1&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \tag{2.0.10}$$

$$= \begin{pmatrix}\frac{x_1}{3}\\\frac{x_1}{3}-x_2\\-x_1+x_2+x_3\end{pmatrix} \tag{2.0.11}$$

Therefore the transformation $\mathbf{T}^{-1}$ is defined on $\mathbf{R}^3$ as

$$\mathbf{T}^{-1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}\frac{x_1}{3}\\\frac{x_1}{3}-x_2\\-x_1+x_2+x_3\end{pmatrix} \tag{2.0.12}$$