EE5609 Assignment 12

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1 Problem

Let $\mathbf{W}$ be the subspace of $\mathbf{C}^3$ spanned by $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$.

a) Show that α_1 and α_2 form a basis for $\mathbf{W}$.

b) Show that the vectors $\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\beta_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$ are in $\mathbf{W}$ and form another basis for $\mathbf{W}$.

c) What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for $\mathbf{W}$.

2 Explanation

a) It is given that α_1 and α_2 span $\mathbf{W}$. For α_1 and α_2 to be the basis for $\mathbf{W}$ they must be linearly independent. Let

$$S_1 = \{\alpha_1, \alpha_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \\ -1 \end{pmatrix} \right\} \tag{2.0.1}$$

Using row reduction on matrix $\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ i & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - iR_1} \begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ 0 & -i \end{pmatrix} \tag{2.0.2}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + iR_2} \begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (i+1)R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.4}$$

Since $\mathbf{A}$ is a full-rank matrix the column vectors are linearly independent. Therefore $S_1 = \{\alpha_1, \alpha_2\}$ is a basis set for $\mathbf{W}$.

b)

$$\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.0.5}$$

$$\beta_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \tag{2.0.6}$$

Since column vectors of $\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}$ are basis for $\mathbf{W}$ and if β_1 and $\beta_2 \in \mathbf{W}$ there exist a unique solution $\mathbf{x}$ such that

$$\begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \mathbf{x} = \beta_1 \tag{2.0.7}$$

Using row reduction on augmented matrix

$$\begin{pmatrix} 1 & 1+i & | & 1 \\ 0 & 1 & | & 1 \\ i & -1 & | & 0 \end{pmatrix} \tag{2.0.8}$$

$$\xleftrightarrow{R3 \leftarrow R_3 - iR-1} \begin{pmatrix} 1 & 1+i & | & 1 \\ 0 & 1 & | & 1 \\ 0 & -i & | & -i \end{pmatrix} \tag{2.0.9}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1+i & | & 1 \\ 0 & 1 & | & 1 \\ 0 & 0 & | & 0 \end{pmatrix} \tag{2.0.10}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (i+1)R_2} \begin{pmatrix} 1 & 0 & | & -i \\ 0 & 1 & | & 1 \\ 0 & 0 & | & 0 \end{pmatrix} \tag{2.0.11}$$

$$\implies \mathbf{x} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{2.0.12}$$

Therefore $\beta_1 \in \mathbf{W}$.
Similarly for $\beta_2 \in \mathbf{W}$ there must exist a unique solution $\mathbf{x}$ such that

$$\begin{pmatrix} \alpha_2 & \alpha_2 \end{pmatrix} \mathbf{x} = \beta_2 \tag{2.0.13}$$

Using row reduction on augmented matrix

$$\begin{pmatrix} 1 & 1+i & | & 1 \\ 0 & 1 & | & i \\ i & -1 & | & 1+i \end{pmatrix} \tag{2.0.14}$$

$$\xleftrightarrow{R3 \leftarrow R_3 - iR-1} \begin{pmatrix} 1 & 1+i & | & 1 \\ 0 & 1 & | & i \\ 0 & -i & | & 1 \end{pmatrix} \tag{2.0.15}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + iR_2} \begin{pmatrix} 1 & 1+i & | & 1 \\ 0 & 1 & | & i \\ 0 & 0 & | & 0 \end{pmatrix} \tag{2.0.16}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (i+1)R_2} \begin{pmatrix} 1 & 0 & | & 2-i \\ 0 & 1 & | & i \\ 0 & 0 & | & 0 \end{pmatrix} \tag{2.0.17}$$

$$\implies \mathbf{x} = \begin{pmatrix} 2-i \\ i \end{pmatrix} \tag{2.0.18}$$

Therefore $\beta_2 \in \mathbf{W}$.
Consider

$$S_2 = \{\beta_1, \beta_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \right\} \tag{2.0.19}$$

and also let

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & i \\ 0 & 1+i \end{pmatrix} \tag{2.0.20}$$

Using row reduction on matrix $\mathbf{B}$

$$\begin{pmatrix} 1 & 1 \\ 1 & i \\ 0 & 1+i \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & i-1 \\ 0 & 1+i \end{pmatrix} \tag{2.0.21}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{i-1}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1+i \end{pmatrix} \tag{2.0.22}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - (i-1)R_2]{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.23}$$

Since $\mathbf{B}$ is a full rank matrix the column vectors are linearly independent.
Let α be any vector in the subspace $\mathbf{W}$, then it can be expressed as span $\{\alpha_1, \alpha_2\}$ i.e

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \mathbf{x_1} = \mathbf{A}\mathbf{x_1} \tag{2.0.24}$$

$S_2 = \{\beta_1, \beta_2\}$ spans $\mathbf{W}$ if any vector $\alpha \in \mathbf{W}$ can be expressed as

$$\alpha = \begin{pmatrix} \beta_1, \beta_2 \end{pmatrix} \mathbf{x_2} = \mathbf{B}\mathbf{x_2} \tag{2.0.25}$$

From (2.0.24) and (2.0.25) we conclude

$$\mathbf{B}\mathbf{x_2} = \mathbf{A}\mathbf{x_1} \tag{2.0.26}$$

$$\implies \mathbf{x_2} = \mathbf{B}^{-1}\mathbf{A}\mathbf{x_1} \tag{2.0.27}$$

Therefore from (2.0.27) $\mathbf{x_2}$ exists if $\mathbf{B}$ is invertible. From (2.0.23) we conclude $\mathbf{x_2}$ exists and hence any vector $\alpha \in \mathbf{W}$ can be expressed as span$\{\beta_1, \beta_2\}$. Therefore $\{\beta_1, \beta_2\}$ is basis for $\mathbf{W}$.

c) Since $\alpha_1, \alpha_2 \in \mathbf{W}$ and $\{\beta_1, \beta_2\}$ are ordered basis for $\mathbf{W}$ there must exist unique value of $\mathbf{x}$ such that

$$\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \mathbf{x} = \alpha_1 \tag{2.0.28}$$

$$\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \mathbf{x} = \alpha_2 \tag{2.0.29}$$

Using row reduction on (2.0.28) we get,

$$\begin{pmatrix} 1 & 1 & | & 1 \\ 1 & i & | & 0 \\ 0 & 1+i & | & i \end{pmatrix} \tag{2.0.30}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & | & 1 \\ 0 & i-1 & | & -1 \\ 0 & 1+i & | & i \end{pmatrix} \tag{2.0.31}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{i-1}} \begin{pmatrix} 1 & 1 & | & 1 \\ 0 & 1 & | & \frac{1+i}{2} \\ 0 & 1+i & | & i \end{pmatrix} \tag{2.0.32}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - (i+1)R_2} \begin{pmatrix} 1 & 1 & | & 1 \\ 0 & 1 & | & \frac{1+i}{2} \\ 0 & 0 & | & 0 \end{pmatrix} \tag{2.0.33}$$

$$\xleftrightarrow{R_1 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 0 & | & \frac{1-i}{2} \\ 0 & 1 & | & \frac{1+i}{2} \\ 0 & 0 & | & 0 \end{pmatrix} \tag{2.0.34}$$

$$\implies \mathbf{x} = \frac{1}{2} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} \tag{2.0.35}$$

and now applying row reduction on (2.0.29) we get,

$$\begin{pmatrix} 1 & 1 & | & 1+i \\ 1 & i & | & 1 \\ 0 & 1+i & | & -1 \end{pmatrix} \tag{2.0.36}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & | & 1+i \\ 0 & i-1 & | & -i \\ 0 & 1+i & | & -1 \end{pmatrix} \tag{2.0.37}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{i-1}} \begin{pmatrix} 1 & 1 & | & 1+i \\ 0 & 1 & | & \frac{-1+i}{2} \\ 0 & 1+i & | & -1 \end{pmatrix} \tag{2.0.38}$$

$$\xleftrightarrow{R_3 \leftarrow R_3-(i+1)R_2} \begin{pmatrix} 1 & 1 & | & 1+i \\ 0 & 1 & | & \frac{-1+i}{2} \\ 0 & 0 & | & 0 \end{pmatrix} \tag{2.0.39}$$

$$\xleftrightarrow{R_1 \leftarrow R_2-R_1} \begin{pmatrix} 1 & 0 & | & \frac{3+i}{2} \\ 0 & 1 & | & \frac{-1+i}{2} \\ 0 & 0 & | & 0 \end{pmatrix} \tag{2.0.40}$$

$$\implies \mathbf{x} = \frac{1}{2}\begin{pmatrix} 3+i \\ -1+i \end{pmatrix} \tag{2.0.41}$$