EE5609 Assignment 10

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1 Problem

If $\mathbf{F}$ is a field, verify that vector space of all ordered n-tuples $\mathbf{F}^n$ is a vector space over the field $\mathbf{F}$.

2 Solution

Let $\mathbf{F}^n$ be a set of all ordered n-tuples over $\mathbf{F}$ i.e

$$\mathbf{F}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \ldots, a_n \in \mathbf{F} \right\} \tag{2.0.1}$$

For $\mathbf{F}^n$ to be a vector space over $\mathbf{F}$ it must satisfy the closure property of vector addition and scalar multiplication.

Vector Addition in $\mathbf{F}^n$:
Let $\alpha = \begin{pmatrix} a_i \end{pmatrix}$ and $\beta = \begin{pmatrix} b_i \end{pmatrix}$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\alpha + \beta = \begin{pmatrix} a_i \end{pmatrix} + \begin{pmatrix} b_i \end{pmatrix} \tag{2.0.2}$$

$$= \begin{pmatrix} a_i + b_i \end{pmatrix} \tag{2.0.3}$$

Since

$$a_i + b_i \in \mathbf{F}\ \forall\ i = 1, 2, \cdots, n \tag{2.0.4}$$

$$\implies \alpha + \beta \in \mathbf{F}^n \tag{2.0.5}$$

Scalar multiplication in $\mathbf{F}^n$ over $\mathbf{F}$:
Let $\alpha = \begin{pmatrix} a_i \end{pmatrix}$ $\forall\ i = 1, 2, \cdots, n\ \in \mathbf{F}^n$ and $a \in \mathbf{F}$ then

$$a\alpha = \begin{pmatrix} aa_i \end{pmatrix} \tag{2.0.6}$$

Since

$$aa_i \in \mathbf{F}\ \forall\ i = 1, 2 \cdots, n \tag{2.0.7}$$

$$\implies a\alpha \in \mathbf{F}^n \tag{2.0.8}$$

Associativity of addition in $\mathbf{F}^n$:
Let $\alpha = \begin{pmatrix} a_i \end{pmatrix},\ \beta = \begin{pmatrix} b_i \end{pmatrix},\ \gamma = \begin{pmatrix} g_i \end{pmatrix}$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\alpha + (\beta + \gamma) = \begin{pmatrix} a_i \end{pmatrix} + \begin{pmatrix} b_i + g_i \end{pmatrix} \tag{2.0.9}$$

$$= \begin{pmatrix} a_i + b_i + g_i \end{pmatrix} \tag{2.0.10}$$

$$= \begin{pmatrix} a_i + b_i \end{pmatrix} + \begin{pmatrix} g_i \end{pmatrix} \tag{2.0.11}$$

$$= (\alpha + \beta) + \gamma \tag{2.0.12}$$

Existence of additive identity in $\mathbf{F}^n$:
We have $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{F}^n$ and $\alpha = \begin{pmatrix} a_i \end{pmatrix}$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\begin{pmatrix} a_i \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} a_i + 0 \end{pmatrix} \tag{2.0.13}$$

$$= \begin{pmatrix} a_i \end{pmatrix} \tag{2.0.14}$$

Therefore $\mathbf{0}$ is the additive identity in $\mathbf{F}^n$.

Existence of additive inverse of each element of $\mathbf{F}^n$:
If $\alpha = \begin{pmatrix} a_i \end{pmatrix}$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then $\begin{pmatrix} -a_i \end{pmatrix} \in \mathbf{F}^n$. Also we have

$$\begin{pmatrix} -a_i \end{pmatrix} + \begin{pmatrix} a_i \end{pmatrix} = \mathbf{0} \tag{2.0.15}$$

Therefore $-\alpha = \begin{pmatrix} -a_i \end{pmatrix}$ is the additive inverse of α. Thus $\mathbf{F}^n$ is an abelian group with respect to addition.

Futher we observe that

1) If $a \in \mathbf{F}$ and $\alpha = \begin{pmatrix} a_i \end{pmatrix},\ \beta = \begin{pmatrix} b_i \end{pmatrix}$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$a(\alpha + \beta) = a\begin{pmatrix} a_i + b_i \end{pmatrix} \tag{2.0.16}$$

$$= \begin{pmatrix} a[a_i + b_i] \end{pmatrix} \tag{2.0.17}$$

$$= \begin{pmatrix} aa_i + ab_i \end{pmatrix} \tag{2.0.18}$$

$$\begin{pmatrix} aa_i \end{pmatrix} + \begin{pmatrix} ab_i \end{pmatrix} \tag{2.0.19}$$

$$= a\begin{pmatrix} a_i \end{pmatrix} + a\begin{pmatrix} b_i \end{pmatrix} \tag{2.0.20}$$

$$= a\alpha + a\beta \tag{2.0.21}$$

2) If $a,b \in \mathbf{F}$ and $\alpha = \left(a_i\right) \ \forall \ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$(a+b)\alpha = \left([a+b]a_i\right) \tag{2.0.22}$$

$$= \left(aa_i + ba_i\right) \tag{2.0.23}$$

$$= \left(aa_i\right) + \left(ba_i\right) \tag{2.0.24}$$

$$= a\left(a_i\right) + b\left(a_i\right) \tag{2.0.25}$$

$$= a\alpha + b\alpha \tag{2.0.26}$$

3) If $a,b \in \mathbf{F}$ and $\alpha = \left(a_i\right) \ \forall \ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$(ab)\alpha = \left([ab]a_i\right) \tag{2.0.27}$$

$$= \left(a[ba_i]\right) \tag{2.0.28}$$

$$= a\left(ba_i\right) \tag{2.0.29}$$

$$= a(b\alpha) \tag{2.0.30}$$

4) If 1 is the unity element of $\mathbf{F}$ and $\alpha = \left(a_i\right) \ \forall \ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$1\alpha = \left(1a_i\right) \tag{2.0.31}$$

$$= \left(a_i\right) \tag{2.0.32}$$

$$= \alpha \tag{2.0.33}$$

Hence $\mathbf{F}^n$ is a vector space over $\mathbf{F}$.