

EE5609 Assignment 13

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1 PROBLEM

- a) Let \mathbf{F} be a field and let \mathbf{V} be the space of polynomial functions f from \mathbf{F} into \mathbf{F} , given by

$$f(x) = c_0 + c_1x + \cdots + c_nx^n$$

Let \mathbf{D} be a linear differentiation transformation defined as

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx}$$

Then find the range and null space of \mathbf{D} .

- b) Let \mathbf{R} be the field of real numbers and let \mathbf{V} be the space of all functions from \mathbf{R} into \mathbf{R} which are continuous. Let \mathbf{T} be linear transformation defined by

$$(\mathbf{T}f)(x) = \int_0^x f(t) dt$$

Find the range and null space of \mathbf{T} .

2 EXPLANATION

- a) Let the vector space \mathbf{V} be defined as

$$\mathbf{V} = \left\{ f : \mathbf{F} \rightarrow \mathbf{F} : f(x) = \sum_{k=0}^n c_k x^k, c_k \in \mathbf{F} \right\} \quad (2.0.1)$$

Differentiation transformation is defined as a function which maps the vectors in \mathbf{F} into \mathbf{F} such that

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx} \quad (2.0.2)$$

$$\Rightarrow \mathbf{D}f = \sum_{k=0}^n k c_k x^{k-1} = g(x) \quad (2.0.3)$$

Since $g(x) \in \mathbf{V}$ therefore the transformation \mathbf{D} is defined from \mathbf{V} into \mathbf{V} . Thus the range of \mathbf{D} is the entire vector space \mathbf{V} . Now consider

the nullspace for differentiation transformation defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{D}f = 0\} \quad (2.0.4)$$

$$\mathbf{D}f = 0 \Rightarrow f = c \quad (2.0.5)$$

where c is a constant. Such a polynomial is known as constant polynomial. Therefore

$$\mathbf{N} = \{f = c : f \in \mathbf{V}, c \in \mathbf{F} \text{ where } c \text{ is a constant}\} \quad (2.0.6)$$

- b) Now consider the vector space defined as

$$\mathbf{V} = \{f : \mathbf{R} \rightarrow \mathbf{R} : f \text{ is continuous}\} \quad (2.0.7)$$

Integration transformation is defined as

$$(\mathbf{T}f)(x) = \int_0^x f(t) dt \quad (2.0.8)$$

Let

$$F(x) = \int_0^x f(t) dt \quad (2.0.9)$$

Since f is continuous function we have $|f(t)| \leq M \forall t \in [0, x]$ and $|M| \geq 0$, it follows that

$$|F(x+h) - F(x)| = \left| \int_0^h f(t) dt \right| \leq M|h| \quad (2.0.10)$$

which shows that $F(x)$ is also continuous and thus

$$F(x) \in \mathbf{V} \quad (2.0.11)$$

Therefore the transformation \mathbf{T} is defined from \mathbf{V} into \mathbf{V} . Thus the range of \mathbf{T} is the entire vector space \mathbf{V} . Now consider the nullspace for integration transformation defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{T}f = 0\} \quad (2.0.12)$$

$$\mathbf{T}f = 0 \Rightarrow \int_0^x f(t) dt = 0 \quad (2.0.13)$$

$$\Rightarrow f(t) = 0 \quad (2.0.14)$$

Therefore nullspace for integration transformation is

$$\mathbf{N} = \{0\} \quad (2.0.15)$$