EE5609 Assignment 10

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1 Problem

If $\mathbb{F}$ is a field, verify that vector space of all ordered n-tuples $\mathbb{F}^n$ is a vector space over the field $\mathbb{F}$.

2 Solution

Let $\mathbb{F}^n$ be a set of all ordered n-tuples over $\mathbb{F}$ i.e

$$\mathbb{F}^n = \{(a_1, a_2, \cdots, a_n) : a_1, a_2, \cdots, a_n \in \mathbb{F}\} \tag{2.0.1}$$

For $\mathbb{F}^n$ to be a vector space over $\mathbb{F}$ it must satisfy the closure property of vector addition and scalar multiplication.

Vector Addition in $\mathbb{F}^n$:
Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{F}^n$ then

$$\alpha + \beta = (\alpha_1, \alpha_2, \cdots, \alpha_n) + (\beta_1, \beta_2, \cdots, \beta_n) \tag{2.0.2}$$
$$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n) \tag{2.0.3}$$

Since

$$\alpha_i + \beta_i \in \mathbb{F} \ \forall \ i = 1, 2, \cdots, n \tag{2.0.4}$$
$$\implies \alpha + \beta \in \mathbb{F}^n \tag{2.0.5}$$

Scalar multiplication in $\mathbb{F}^n$ over $\mathbb{F}$:
Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{F}^n$ and a $\in \mathbb{F}$ then

$$a\alpha = (a\alpha_1, a\alpha_2, \cdots, a\alpha_n) \tag{2.0.6}$$

Since

$$a\alpha_i \in \mathbb{F} \ \forall \ i = 1, 2 \cdots, n \tag{2.0.7}$$
$$\implies a\alpha \in \mathbb{F}^n \tag{2.0.8}$$

Associativity of addition in $\mathbb{F}^n$:
Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$, $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{F}^n$ then

$$\alpha + (\beta + \gamma) = (\alpha_1, \alpha_2, \cdots, \alpha_n) + (\beta_1 + \gamma_1, \beta_2 + \gamma_2, \cdots, \beta_n + \gamma_n) \tag{2.0.9}$$
$$= (\alpha_1 + \beta_1 + \gamma_1, \alpha_2 + \beta_2 + \gamma_2, \cdots, \alpha_n + \beta_n + \gamma_n) \tag{2.0.10}$$
$$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n) + (\gamma_1, \gamma_2, \cdots, \gamma_n) \tag{2.0.11}$$
$$= (\alpha + \beta) + \gamma \tag{2.0.12}$$

Existence of additive identity in $\mathbb{F}^n$:
We have $(0, 0, \cdots, 0) \in \mathbb{F}^n$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{F}^n$ then

$$(\alpha_1, \alpha_2, \cdots, \alpha_n) + (0, 0, \cdots, 0) = (\alpha_1 + 0, \alpha_2 + 0, \cdots, \alpha_n + 0) \tag{2.0.13}$$
$$= (\alpha_1, \alpha_2, \cdots, \alpha_n) \tag{2.0.14}$$

Therefore $(0, 0, \cdots, 0)$ is the additive identity in $\mathbb{F}^n$.

Existence of additive inverse of each element of $\mathbb{F}^n$:
If $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{F}^n$ then $(-\alpha_1, -\alpha_2, \cdots, -\alpha_n) \in \mathbb{F}^n$. Also we have

$$(-\alpha_1, -\alpha_2, \cdots, -\alpha_n) + (\alpha_1, \alpha_2, \cdots, \alpha_n) = (0, 0, \cdots, 0) \tag{2.0.15}$$

Therefore $(-\alpha_1, -\alpha_2, \cdots, -\alpha_n)$ is the additive inverse of $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. Thus $\mathbb{F}^n$ is an abelian group with respect to addition.

Futher we observe that

1) If a $\in \mathbb{F}$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{F}^n$ then

$$a(\alpha + \beta) = a(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n) \tag{2.0.16}$$
$$= (a[\alpha_1 + \beta_1], a[\alpha_2 + \beta_2], \cdots, a[\alpha_n + \beta_n]) \tag{2.0.17}$$
$$= (a\alpha_1 + a\beta_1, a\alpha_2 + a\beta_2, \cdots, a\alpha_n + a\beta_n) \tag{2.0.18}$$
$$= (a\alpha_1, a\alpha_2, \cdots, a\alpha_n) + (a\beta_1, a\beta_2, \cdots, a\beta_n) \tag{2.0.19}$$
$$= a(\alpha_1, \alpha_2, \cdots, \alpha_n) + a(\beta_1, \beta_2, \cdots, \beta_n) \tag{2.0.20}$$
$$= a\alpha + a\beta \tag{2.0.21}$$

2) If a,b $\in \mathbb{F}$ and $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{F}^n$ then

$$(a + b)\alpha = ([a + b]\alpha_1, [a + b]\alpha_2, \cdots, [a + b]\alpha_n) \tag{2.0.22}$$

$$= (a\alpha_1 + b\alpha_1, a\alpha_2 + b\alpha_2, \cdots, a\alpha_n + b\alpha_n) \tag{2.0.23}$$

$$= (a\alpha_1, a\alpha_2, \cdots, a\alpha_n) + (b\alpha_1, b\alpha_2, \cdots, b\alpha_n) \tag{2.0.24}$$

$$= a(\alpha_1, \alpha_2, \cdots, \alpha_n) + b(\alpha_1, \alpha_2, \cdots, \alpha_n) \tag{2.0.25}$$

$$= a\alpha + b\alpha \tag{2.0.26}$$

3) If a,b $\in \mathbb{F}$ and $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{F}^n$ then

$$(ab)\alpha = ([ab]\alpha_1, [ab]\alpha_2, \cdots, [ab]\alpha_n) \tag{2.0.27}$$

$$= (a[b\alpha_1], a[b\alpha_2], \cdots, a[b\alpha_n]) \tag{2.0.28}$$

$$= a(b\alpha_1, b\alpha_2, \cdots, b\alpha_n) \tag{2.0.29}$$

$$= a(b\alpha) \tag{2.0.30}$$

4) If 1 is the unity element of $\mathbb{F}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_n) \in \mathbb{F}^n$ then

$$1\alpha = (1\alpha_1, 1\alpha_2, \cdots, 1\alpha_n) \tag{2.0.31}$$

$$= (\alpha_1, \alpha_2, \cdots, \alpha_n) \tag{2.0.32}$$

$$= \alpha \tag{2.0.33}$$

Hence $\mathbb{F}^n$ is a vector space over $\mathbb{F}$.