EE5609 Assignment 15

SHANTANU YADAV, EE20MTECH12001

1 Problem

The linear operator $\mathbf{T}$ on $\mathbf{R}^2$ defined by

$$\mathbf{T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Prove that if $\mathbf{S}$ is a linear operator on $\mathbf{R}^2$ such that $\mathbf{S}^2 = \mathbf{S}$, then $\mathbf{S} = \mathbf{0}$, or $\mathbf{S} = \mathbf{I}$, or there is an ordered basis $\mathbf{B}$ for $\mathbf{R}^2$ such that $[\mathbf{S}]_B = \mathbf{A}$.

2 Explanation

If a linear operator $\mathbf{S}$ is defined on $\mathbf{R}^2$ such that $\mathbf{S}^2 = \mathbf{S}$, then

$$\mathbf{S}^2 - \mathbf{S} = \mathbf{0} \tag{2.0.1}$$
$$\mathbf{S}(\mathbf{S} - \mathbf{I}) = \mathbf{0} \tag{2.0.2}$$
$$\implies \mathbf{S} = \mathbf{0}, \mathbf{S} = \mathbf{I} \tag{2.0.3}$$

The transformation of a vector $\mathbf{x} \in \mathbf{R}^2$ can be represented as

$$\mathbf{S}\mathbf{x} = \mathbf{y} \tag{2.0.4}$$
$$\implies \mathbf{S}(\mathbf{S}\mathbf{x}) = \mathbf{S}\mathbf{y} \tag{2.0.5}$$
$$\implies \mathbf{S}^2\mathbf{x} = \mathbf{S}\mathbf{y} \tag{2.0.6}$$
$$\implies \mathbf{S}\mathbf{x} = \mathbf{S}\mathbf{y} \tag{2.0.7}$$
$$\implies \mathbf{x} = \mathbf{y} \tag{2.0.8}$$

Therefore the transformation of a vector $\mathbf{x} \in \mathbf{R}^2$ can be given as

$$\mathbf{S}\mathbf{x} = \mathbf{x} \ \forall \ \mathbf{x} \in \mathbf{R}^2 \tag{2.0.9}$$

Consider the ordered basis set

$$B = \{\epsilon_1, \epsilon_2\} \in \mathbf{R}^2 \tag{2.0.10}$$

and if

$$[\mathbf{S}]_{\mathbf{B}} = \mathbf{A} \tag{2.0.11}$$
$$\implies [\mathbf{S}]_{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

Thus we can re-write the column vectors of $[\mathbf{S}]_{\mathbf{B}}$ using (2.0.9) as

$$\mathbf{S}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.13}$$
$$\mathbf{S}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.14}$$

Therefore, any vector $\mathbf{x}$ in column space of $[\mathbf{S}]_{\mathbf{B}}$ can be uniquely expressed by $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, hence it forms the basis for column space of $[\mathbf{S}]_{\mathbf{B}}$. Therefore one of the basis vector of B is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The other basis vector can be any vector which is linearly independent to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. One of the ordered basis set can be

$$B = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \tag{2.0.15}$$