EE5609 Assignment 5

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The python solution code is available at

https://github.com/Shantanu2508/Matrix_Theory/blob/master/Assignment_5/assignment5.py

1 Problem

Find the value of k so that the following equation may represent pairs of straight lines.

$$12x^2 + xy - 6y^2 - 29x + 8y + k = 0$$

Also, find the equations of the lines.

2 Explanation

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

(2.0.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.5}$$

The lines intercept if

$$\left|\mathbf{V}\right| < 0 \tag{2.0.6}$$

3 Solution

From (2.0.3) and (2.0.4)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{u} = \begin{pmatrix} \frac{29}{2} \\ 4 \end{pmatrix} \tag{3.0.2}$$

From (2.0.1) and (2.0.5)

$$\begin{vmatrix} 12 & \frac{1}{2} & -\frac{29}{2} \\ \frac{1}{2} & -6 & 4 \\ -\frac{29}{2} & 4 & k \end{vmatrix} = 0 \tag{3.0.3}$$

$$\implies \quad 12\begin{vmatrix} -6 & 4 \\ 4 & k \end{vmatrix} - \frac{1}{2}\begin{vmatrix} \frac{1}{2} & 4 \\ -\frac{29}{2} & k \end{vmatrix} - \frac{29}{2}\begin{vmatrix} \frac{1}{2} & -6 \\ -\frac{29}{2} & 4 \end{vmatrix} = 0 \tag{3.0.4}$$

$$\implies \quad k = 14 \tag{3.0.5}$$

Since $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}\mathbf{V}\mathbf{P}^T = \mathbf{D} = diag\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \tag{3.0.6}$$

or equivalently

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{3.0.7}$$

Eigen vectors of real symmetric matrix $\mathbf{V}$ are orthogonal. The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 12 & -\frac{1}{2} \\ -\frac{1}{2} & \lambda + 6 \end{vmatrix} = 0 \tag{3.0.8}$$

$$\implies \quad \lambda^2 - 6\lambda - \frac{289}{4} = 0 \tag{3.0.9}$$

$$\implies \quad \lambda_1 = -\frac{483}{72}, \lambda_2 = \frac{865}{72} \tag{3.0.10}$$

The eigen vector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{3.0.11}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{3.0.12}$$

The orthogonal eigen-vector matrix

$$\mathbf{P} = \frac{1}{||\mathbf{p}||}\begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} \frac{24}{865} & \frac{-2596}{2597} \\ -\frac{2596}{2597} & \frac{24}{865} \end{pmatrix} \tag{3.0.13}$$

$$\mathbf{D} = \begin{pmatrix} -\frac{433}{72} & 0 \\ 0 & \frac{865}{72} \end{pmatrix} \tag{3.0.14}$$

Let $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$ with $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$. Substituting in (2.0.2)

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{3.0.15}$$

$$\implies \mathbf{y}^T\mathbf{D}\mathbf{y} = 0 \tag{3.0.16}$$

$$\implies \begin{pmatrix} \pm\sqrt{\frac{\lambda_1}{\lambda_2}} & 1 \end{pmatrix}\mathbf{y} = 0 \tag{3.0.17}$$

Substituting $\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c}$ in (3.0.17)

$$\implies \begin{pmatrix} \pm\frac{1}{\sqrt{2}} & 1 \end{pmatrix}(\mathbf{P^T}\mathbf{x} - \mathbf{P^T}\mathbf{c}) = 0 \tag{3.0.18}$$

The equations of lines are

$$\begin{pmatrix} \frac{49}{50} & \frac{37}{50} \end{pmatrix}\mathbf{x} = \frac{343}{200} \tag{3.0.19}$$

$$\begin{pmatrix} \frac{51}{50} & -\frac{34}{50} \end{pmatrix}\mathbf{x} = \frac{34}{50} \tag{3.0.20}$$

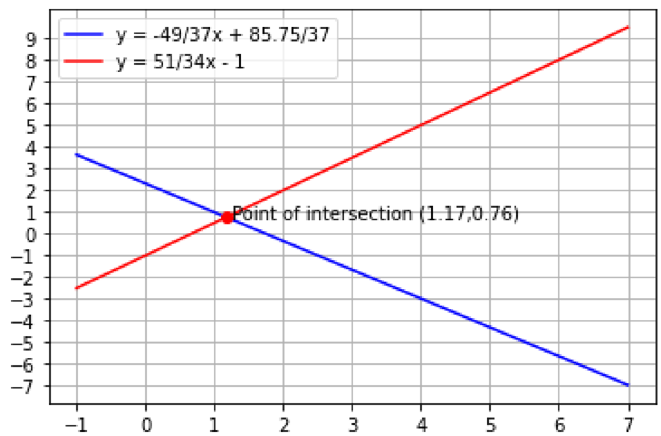

Fig. 0