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CHALLENGE PROBLEM 1

Lines and Planes

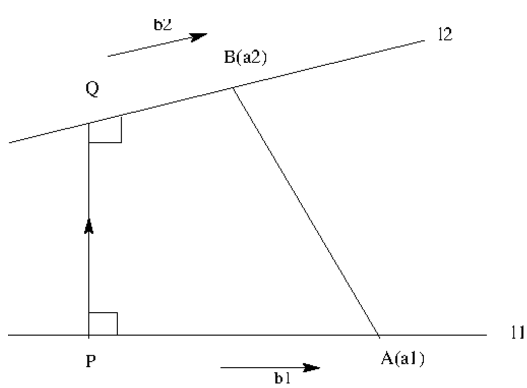
Problem Statement

Shortest distance between two skew lines

Let the two lines are l_1 and l_2. The equations of these lines are

$$\overrightarrow{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1} \quad \text{and} \quad \overrightarrow{r} = \overrightarrow{a_2} + \mu \overrightarrow{b_2} \quad \text{respectively.}$$

These lines pass through the points A and B whose position vectors are $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$. Let us assume that the lines are parallel to the vectors $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ respectively.



Let $\overrightarrow{PQ}$ be the shortest distance vector between l_1 and l_2. Then $\overrightarrow{PQ}$ is perpendicular to both l_1 and l_2 which are parallel to $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ respectively. Therefore, $\overrightarrow{PQ}$ is perpendicular to both $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$. But $\overrightarrow{b_1} \times \overrightarrow{b_2}$ is perpendicular to both $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$. Therefore, $\overrightarrow{PQ}$ is parallel to $\overrightarrow{b_1} \times \overrightarrow{b_2}$.

Let $\hat{n}$ is a unit vector along $\overrightarrow{PQ}$. Then, we have

$$\hat{n} = \pm \frac{\overrightarrow{b_1} \times \overrightarrow{b_2}}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|}$$

therefore, $$PQ = \text{projection of } \overrightarrow{AB} \text{ on } \overrightarrow{PQ}$$

$\implies$ $$PQ = \overrightarrow{AB} \cdot \hat{n}$$

$$\begin{aligned} PQ &= \pm(\overrightarrow{a_2} - \overrightarrow{a_1}) \cdot \left\{ \frac{\overrightarrow{b_1} \times \overrightarrow{b_2}}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|} \right\} \qquad (1) \\ &= \pm \frac{(\overrightarrow{b_1} \times \overrightarrow{b_2}) \cdot (\overrightarrow{a_2} - \overrightarrow{a_1})}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|} \end{aligned}$$

Since the distance PQ is to be taken as positive, hence

$$PQ = \left| \frac{(\overrightarrow{b_1} \times \overrightarrow{b_2}) \cdot (\overrightarrow{a_2} - \overrightarrow{a_1})}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|} \right| \qquad (2)$$