EE5609 Assignment 17

SHANTANU YADAV, EE20MTECH12001

1 Problem

Let $\mathbf{A}$ be a $n \times n$ real matrix with $\mathbf{A}^2 = \mathbf{A}$. Then

1) the eigenvalues of $\mathbf{A}$ are either 0 or 1
2) $\mathbf{A}$ is a diagonal matrix with diagonal entries 0 or 1
3) $rank(\mathbf{A}) = trace(\mathbf{A})$
4) if $rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A})$

2 Explanation

See next page.

<table>
<tr><th>Objective</th><th>Explanation</th></tr>
<tr><td>Eigenvalues of $\mathbf{A}$</td><td>Since

$$\mathbf{A}^2 = \mathbf{A} \tag{2.0.1}$$
$$\implies \mathbf{A}^2 - \mathbf{A} = \mathbf{O} \tag{2.0.2}$$

From Cayley-Hamilton Theorem we have,

$$\lambda^2 - \lambda = 0 \tag{2.0.3}$$
$$\implies \lambda(\lambda - 1) = 0 \tag{2.0.4}$$
$$\implies \lambda = 0, 1 \tag{2.0.5}$$

A matrix $\mathbf{A}$ satisfying $\mathbf{A}^2 = \mathbf{A}$ is an idempotent matrix with eigen values equal to 0 or 1.</td></tr>
<tr><td>Check if $\mathbf{A}$ is necessary diagonal</td><td>Consider

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.6}$$
$$\tag{2.0.7}$$

Then,

$$\mathbf{A}^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.8}$$
$$= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.9}$$
$$= \mathbf{A} \tag{2.0.10}$$

Hence $\mathbf{A}$ is idempotent but not diagonal.</td></tr>
<tr><td>Relation between rank and trace of $\mathbf{A}$</td><td>Rank of matrix is defined as the number of non-zero eigenvalues. Since number of non-zero eigenvalues is 1,

$$rank(\mathbf{A}) = 1 \tag{2.0.11}$$
$$trace(\mathbf{A}) = \sum_i \lambda_i = 0 + 1 = 1 \tag{2.0.12}$$
$$\implies rank(\mathbf{A}) = trace(\mathbf{A}) \tag{2.0.13}$$</td></tr>
<tr><td>Relation between rank and trace of $\mathbf{I} - \mathbf{A}$</td><td>Now for the matrix $\mathbf{I} - \mathbf{A}$ we have,

$$(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) \tag{2.0.14}$$
$$= \mathbf{I}^2 - \mathbf{IA} - \mathbf{AI} + \mathbf{A}^2 \tag{2.0.15}$$
$$= \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} \tag{2.0.16}$$
$$= \mathbf{I} - \mathbf{A} \tag{2.0.17}$$

Hence $\mathbf{I} - \mathbf{A}$ is an idempotent matrix. Therefore we conclude,

$$rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A}) \tag{2.0.18}$$</td></tr>
<tr><td>Answer</td><td>(1),(3) and (4)</td></tr>
</table>

TABLE I