EE5609 Assignment 10

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1 Problem

If $\mathbf{F}$ is a field, verify that vector space of all ordered n-tuples $\mathbf{F}^n$ is a vector space over the field $\mathbf{F}$.

2 Solution

Let $\mathbf{F}^n$ be a set of all ordered n-tuples over $\mathbf{F}$ i.e

$$\mathbf{F}^n = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} : \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbf{F} \right\} \tag{2.0.1}$$

For $\mathbf{F}^n$ to be a vector space over $\mathbf{F}$ it must satisfy the closure property of vector addition and scalar multiplication.

Vector Addition in $\mathbf{F}^n$:

Let $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ and $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \in \mathbf{F}^n$ then

$$\alpha + \beta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \tag{2.0.2}$$

$$= \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \vdots \\ \alpha_n + \beta_n \end{pmatrix} \tag{2.0.3}$$

Since

$$\alpha_i + \beta_i \in \mathbf{F} \ \forall \ i = 1, 2, \cdots, n \tag{2.0.4}$$

$$\implies \alpha + \beta \in \mathbf{F}^n \tag{2.0.5}$$

Scalar multiplication in $\mathbf{F}^n$ over $\mathbf{F}$:

Let $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbf{F}^n$ and $a \in \mathbf{F}$ then

$$a\alpha = \begin{pmatrix} a\alpha_1 \\ a\alpha_2 \\ \vdots \\ a\alpha_n \end{pmatrix} \tag{2.0.6}$$

Since

$$a\alpha_i \in \mathbf{F} \ \forall \ i = 1, 2 \cdots, n \tag{2.0.7}$$

$$\implies a\alpha \in \mathbf{F}^n \tag{2.0.8}$$

Associativity of addition in $\mathbf{F}^n$:

Let $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$, $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$, $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} \in \mathbf{F}^n$ then

$$\alpha + (\beta + \gamma) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} + \begin{pmatrix} \beta_1 + \gamma_1 \\ \beta_2 + \gamma_2 \\ \vdots \\ \beta_n + \gamma_n \end{pmatrix} \tag{2.0.9}$$

$$= \begin{pmatrix} \alpha_1 + \beta_1 + \gamma_1 \\ \alpha_2 + \beta_2 + \gamma_2 \\ \vdots \\ \alpha_n + \beta_n + \gamma_n \end{pmatrix} \tag{2.0.10}$$

$$= \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \vdots \\ \alpha_n + \beta_n \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} \tag{2.0.11}$$

$$= (\alpha + \beta) + \gamma \tag{2.0.12}$$

Existence of additive identity in $\mathbf{F}^n$:

We have $\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix} \in \mathbf{F}^n$ and $\alpha = \begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix} \in \mathbf{F}^n$ then

$$\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix} + \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}\alpha_1+0\\\alpha_2+0\\\vdots\\\alpha_n+0\end{pmatrix} \tag{2.0.13}$$

$$= \begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix} \tag{2.0.14}$$

Therefore $\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$ is the additive identity in $\mathbf{F}^n$.

Existence of additive inverse of each element of $\mathbf{F}^n$:

If $\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix} \in \mathbf{F}^n$ then $\begin{pmatrix}-\alpha_1\\-\alpha_2\\\vdots\\-\alpha_n\end{pmatrix} \in \mathbf{F}^n$. Also we have

$$\begin{pmatrix}-\alpha_1\\-\alpha_2\\\cdots\\-\alpha_n\end{pmatrix} + \begin{pmatrix}\alpha_1\\\alpha_2\\\cdots\\\alpha_n\end{pmatrix} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix} \tag{2.0.15}$$

Therefore $\begin{pmatrix}-\alpha_1\\-\alpha_2\\\vdots\\-\alpha_n\end{pmatrix}$ is the additive inverse of $\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix}$.
Thus $\mathbf{F}^n$ is an abelian group with respect to addition.

Futher we observe that

1) If $a \in \mathbf{F}$ and $\alpha = \begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix}$, $\beta = \begin{pmatrix}\beta_1\\\beta_2\\\vdots\\\beta_n\end{pmatrix} \in \mathbf{F}^n$ then

$$a(\alpha+\beta) = a\begin{pmatrix}\alpha_1+\beta_1\\\alpha_2+\beta_2\\\vdots\\\alpha_n+\beta_n\end{pmatrix} \tag{2.0.16}$$

$$= \begin{pmatrix}a[\alpha_1+\beta_1]\\a[\alpha_2+\beta_2]\\\vdots\\a[\alpha_n+\beta_n]\end{pmatrix} \tag{2.0.17}$$

$$= \begin{pmatrix}a\alpha_1+a\beta_1\\a\alpha_2+a\beta_2\\\vdots\\a\alpha_n+a\beta_n\end{pmatrix} \tag{2.0.18}$$

$$\begin{pmatrix}a\alpha_1\\a\alpha_2\\\vdots\\a\alpha_n\end{pmatrix} + \begin{pmatrix}a\beta_1\\a\beta_2\\\vdots\\a\beta_n\end{pmatrix} \tag{2.0.19}$$

$$= a\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix} + a\begin{pmatrix}\beta_1\\\beta_2\\\vdots\\\beta_n\end{pmatrix} \tag{2.0.20}$$

$$= a\alpha + a\beta \tag{2.0.21}$$

2) If $a,b \in \mathbf{F}$ and $\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix} \in \mathbf{F}^n$ then

$$(a+b)\alpha = \begin{pmatrix}[a+b]\alpha_1\\[a+b]\alpha_2\\\vdots\\[a+b]\alpha_n\end{pmatrix} \tag{2.0.22}$$

$$= \begin{pmatrix}a\alpha_1+b\alpha_1\\a\alpha_2+b\alpha_2\\\vdots\\a\alpha_n+b\alpha_n\end{pmatrix} \tag{2.0.23}$$

$$= \begin{pmatrix}a\alpha_1\\a\alpha_2\\\vdots\\a\alpha_n\end{pmatrix} + \begin{pmatrix}b\alpha_1\\b\alpha_2\\\cdots, b\alpha_n\end{pmatrix} \tag{2.0.24}$$

$$= a\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix} + b\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix} \tag{2.0.25}$$

$$= a\alpha + b\alpha \tag{2.0.26}$$

3) If $a,b \in \mathbf{F}$ and $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbf{F}^n$ then

$$(ab)\alpha = \begin{pmatrix} [ab]\alpha_1 \\ [ab]\alpha_2 \\ \vdots \\ [ab]\alpha_n \end{pmatrix} \tag{2.0.27}$$

$$= \begin{pmatrix} a[b\alpha_1] \\ a[b\alpha_2] \\ \vdots \\ a[b\alpha_n] \end{pmatrix} \tag{2.0.28}$$

$$= a\begin{pmatrix} b\alpha_1 \\ b\alpha_2 \\ \vdots \\ b\alpha_n \end{pmatrix} \tag{2.0.29}$$

$$= a(b\alpha) \tag{2.0.30}$$

4) If 1 is the unity element of $\mathbf{F}$ and $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_n \end{pmatrix} \in \mathbf{F}^n$

then

$$1\alpha = \begin{pmatrix} 1\alpha_1 \\ 1\alpha_2 \\ \vdots \\ 1\alpha_n \end{pmatrix} \tag{2.0.31}$$

$$= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{2.0.32}$$

$$= \alpha \tag{2.0.33}$$

Hence $\mathbf{F}^n$ is a vector space over $\mathbf{F}$.