EE5609 Assignment 9

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The python solution code is available at

https://github.com/Shantanu2508/Matrix_Theory/blob/master/Assignment_9/assignment9.py

1 Problem

Discover whether

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{1.0.1}$$

is invertible, and find $\mathbf{A^{-1}}$ if it exists.

2 Solution

The matrix $\mathbf{A}$ is in row reduced echolon form with four pivot elements. Therefore the rank($\mathbf{A}$) is 4. Hence the rows of matrix $\mathbf{A}$ constitute of 4 linearly independent vectors. Thus it can be concluded that matrix $\mathbf{A}$ is invertible. Using Gauss-Jordan Elimination, if there exists an elimentary matrix $\mathbf{E}$ such that $\mathbf{E}[\mathbf{A}\ \mathbf{I}] = [\mathbf{I}\ \mathbf{E}]$ then $\mathbf{E}$ is the inverse of A i.e $\mathbf{E} = \mathbf{A^{-1}}$.

$$[\mathbf{A}\ \mathbf{I}] = \begin{pmatrix} 1 & 2 & 3 & 4 & | & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & | & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.1}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 4 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & | & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & | & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 4 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & | & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.3}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_4} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & | & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & | & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & | & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.4}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}\ R_3 \leftarrow \frac{R_3}{3}]{R_4 \leftarrow \frac{R_4}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} = [\mathbf{I}\ \mathbf{E}] \tag{2.0.5}$$

Therefore, for the given problem,

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{2.0.6}$$

3 Generalization of above result to a matrix of any arbitrary size:

Let

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{3.0.1}$$

Then

$$\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{3.0.2}$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{3.0.3}$$

$$\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix}\begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{3.0.4}$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{3.0.5}$$

Proceeding in similar manner, we get

$$\mathbf{E}_N\mathbf{E}_{N-1}\dots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{3.0.6}$$

$$= \text{diag}\begin{pmatrix} a_1 & a_2 & \dots & a_N \end{pmatrix} \tag{3.0.7}$$

$$\implies \quad \mathbf{A} = \mathbf{LU} \tag{3.0.8}$$

where $\quad \mathbf{L} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\dots\mathbf{E}_N^{-1}$

$$\implies \quad \mathbf{A}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1} \tag{3.0.9}$$

$$\implies \quad \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & \frac{1}{a_N} \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \tag{3.0.10}$$

Therefore

$$\mathbf{A^{-1}} = \begin{pmatrix} \frac{1}{a_1} & -\frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & -\frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & -\frac{1}{a_3} & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{a_N} \end{pmatrix} \tag{3.0.11}$$

From (3.0.11) for the above problem

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{3.0.12}$$