AI 1103 Assignment-2

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Download all latex-tikz codes from

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https://github.com/Shantanu467/AI1103/
blob/main/Assignment_2/Assignment_2.tex
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Problem

GATE EC: Question-74
Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $\mathrm{P}\left(X_1 < X_2\right)$ is equal to.....

Solution

Definition 1 *The probability density function (pdf) of an* ***exponential distribution*** *is*

$$f(x;\lambda) = \begin{cases} \lambda \mathrm{e}^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{1}$$

Here $\lambda > 0$ is the parameter of the distribution, often called the rate parameter.

Lemma 0.1 *The* ***mean*** *of an exponentially distributed random variable X with rate parameter λ is given by*

$$\mathrm{E}(X) = \frac{1}{\lambda} \tag{2}$$

Proof 0.1

$$\mathrm{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\, \mathrm{d}x \tag{3}$$

$$\mu = \int_{-\infty}^{0} 0 \times x\, \mathrm{d}x + \int_{0}^{+\infty} \lambda \mathrm{e}^{-\lambda x} \times x\, \mathrm{d}x \tag{4}$$

$$= 0 + \left(-\frac{(\lambda x + 1)\, \mathbf{e}^{-\lambda x}}{\lambda} \right) \Big|_0^{+\infty} \tag{5}$$

$$= 0 - \left(-\frac{(0+1)\, \mathrm{e}^{0}}{\lambda} \right) \tag{6}$$

$$\mu = \frac{1}{\lambda} \tag{7}$$

Now, using (7) and $\mu = 1$ (*Given*), we get;

$$\mu = \frac{1}{\lambda} = 1 \tag{8}$$

$$\implies \lambda = 1 \tag{9}$$

So,

$$X_1 \sim f(x;1) = \mathrm{e}^{-x} \tag{10}$$

Definition 2 *The probability density function (pdf) of an* ***gamma distribution*** *is*

$$f(x;\lambda) = \begin{cases} \frac{a^\lambda}{\Gamma\lambda} e^{ax} x^{\lambda-1}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{11}$$

Here, $\lambda > 0$, $a > 0$ and

$$\Gamma n = a^n \int_0^\infty \mathrm{e}^{ax} x^{n-1}\, \mathrm{d}x \tag{12}$$

$$\Gamma n = (n-1)! \tag{13}$$

Lemma 0.2 *The* ***mean*** *and* ***variance*** *of an exponentially distributed random variable X with λ and a is given by*

$$\mu = \frac{\lambda}{a} \tag{14}$$

$$\sigma^2 = \frac{\lambda}{a^2} \tag{15}$$

Proof 0.2

$$\mathrm{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\, \mathrm{d}x \tag{16}$$

$$\mu = 0 + \int_0^\infty x \times \frac{a^\lambda}{\Gamma\lambda} e^{ax} x^{\lambda-1}\, \mathrm{d}x \tag{17}$$

$$= 0 + \frac{a^\lambda}{\Gamma\lambda} \int_0^\infty e^{ax} x^{\lambda}\, \mathrm{d}x \tag{18}$$

Now by using, $n = \lambda + 1$ *in* (12) *we get;*

$$\Gamma(\lambda+1) = a^{\lambda+1} \int_0^\infty e^{ax} x^\lambda \, \mathrm{d}x \tag{19}$$

Using (19) *in* (18), *we get;*

$$\mu = \frac{a^\lambda}{\Gamma\lambda} \times \frac{\Gamma(\lambda+1)}{a^{\lambda+1}} \tag{20}$$
$$= \frac{1}{a} \times \frac{\Gamma(\lambda+1)}{\Gamma\lambda} \tag{21}$$

Using (13) *to find* $\Gamma\lambda$

$$\mu = \frac{1}{a} \times \frac{\lambda!}{(\lambda-1)!} \tag{22}$$
$$\mu = \frac{\lambda}{a} \tag{23}$$

Similarly we can find $E(X^2)$.

$$\mathrm{E}(X) = \frac{\lambda(\lambda+1)}{a^2} \tag{24}$$
$$\sigma^2 = (\mathrm{E}(X))^2 - \mathrm{E}\left(X^2\right) \tag{25}$$
$$= \frac{\lambda}{a^2} \tag{26}$$

Now by using (23) (26) And given info;

$$\mu = \frac{\lambda}{a} = 2 \implies \lambda = 2a \tag{27}$$
$$\sigma^2 = \frac{\lambda}{a^2} = \frac{2a}{a^2} = 2 \tag{28}$$
$$\implies a = 1, \lambda = 2 \tag{29}$$
$$X_2 \sim G(1,2) = \frac{1}{\Gamma 2} e^x x \tag{30}$$
$$= e^x x \tag{31}$$

Now by the help of (10) and (31), we can solve for $\mathrm{P}(X_1 < X_2)$.

$$\mathrm{P}(X_1 < X_2) = \mathrm{P}(X_1 < X_2 \mid X_1 = X_2) \tag{32}$$
$$= \int_0^\infty f X_2(x_2) \times \int_0^{x_2} f X_1(x_1) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \tag{33}$$
$$= \int_0^\infty x_2 \mathrm{e}^{-x_2} \times \int_0^{x_2} \mathrm{e}^{-x_1} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \tag{34}$$
$$= \int_0^\infty x_2 \mathrm{e}^{-x_2} \times \left(1 - \mathrm{e}^{-x_2}\right) \, \mathrm{d}x_2 \tag{35}$$

Upon solving the definite integral, We get :

$$\mathrm{P}(X_1 < X_2) = \mathbf{\frac{3}{4}} \tag{36}$$