AI 1103 Assignment-2

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Download all latex-tikz codes from

`https://github.com/Shantanu467/AI1103/blob/main/Assignment_2/Assignment_2.tex`

Problem

GATE EC: Question-74
Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $\mathrm{P}\left(X_1 < X_2\right)$ is equal to.....

Solution

$$\mathrm{Exp}(\lambda) = \lambda e^{-x\lambda} \tag{1}$$
$$\text{Mean} = E\left(X\right) = 1/\lambda \tag{2}$$
$$\text{Given:} \quad E\left(X\right) = 1 \tag{3}$$
$$\text{So,} \quad \lambda = 1 \tag{4}$$
$$X_1 \sim \mathrm{Exp}(1) = e^{-x} \tag{5}$$

Now,

$$G(a,\lambda) = \frac{a^\lambda}{\Gamma\lambda} e^{ax} x^{\lambda-1} \tag{6}$$
$$\text{Mean} = \frac{\lambda}{a} = 2 \text{ (Given)} \implies \lambda = 2a \tag{7}$$
$$\text{Variance} = \frac{\lambda}{a^2} = \frac{2a}{a^2} = 2 \text{ (Given)} \tag{8}$$
$$\implies a = 1, \lambda = 2 \tag{9}$$
$$X_2 \sim G(1,2) = \frac{1}{\Gamma 2} e^x x \tag{10}$$
$$= e^x x \tag{11}$$

Now,

$$\mathrm{P}(X_1 < X_2) = \mathrm{P}\left(X_1 < X_2 \mid X_1 = X_2\right) \tag{12}$$
$$= \int_0^\infty f_{X_2}\left(x_2\right) \times \int_0^{x_2} f_{X_1}\left(x_1\right)\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{13}$$
$$= \int_0^\infty x_2 \mathrm{e}^{-x_2} \times \int_0^{x_2} \mathrm{e}^{-x_1}\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{14}$$
$$= \int_0^\infty x_2 \mathrm{e}^{-x_2} \times \left(1 - \mathrm{e}^{-x_2}\right)\, \mathrm{d}x_2 \tag{15}$$

Upon solving the definite integral, We get :

$$\mathrm{P}(X_1 < X_2) = \boxed{\frac{3}{4}} \tag{16}$$