AI 1103 Assignment-2

Shantanu Pandey
CS20BTECH11046

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Download all latex-tikz codes from

```
https://github.com/Shantanu467/AI1103/
blob/main/Assignment_2/Assignment_2.tex
```

Problem

GATE EC: Question-74
Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $\mathrm{P}(X_1 < X_2)$ is equal to.....

Solution

Definition 1 *The probability density function (pdf) of an* ***exponential distribution*** *is*

$$f(x;\lambda) = \begin{cases} \lambda \mathrm{e}^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{1}$$

Here $\lambda > 0$ is the parameter of the distribution, often called the rate parameter.

Lemma 0.1 *The* ***mean*** *of an exponentially distributed random variable X with rate parameter λ is given by*

$$\mathrm{E}(X) = \frac{1}{\lambda} \tag{2}$$

Proof 0.1

$$\mathrm{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\,\mathrm{d}x \tag{3}$$

$$\mu = \int_{-\infty}^{0} 0 \times x\,\mathrm{d}x + \int_{0}^{+\infty} \lambda \mathrm{e}^{-\lambda x} \times x\,\mathrm{d}x \tag{4}$$

$$= 0 + \left(-\frac{(\lambda x + 1)\,\mathbf{e}^{-\lambda x}}{\lambda} \right)\Bigg|_0^{+\infty} \tag{5}$$

$$= 0 - \left(-\frac{(0+1)\,\mathrm{e}^{0}}{\lambda} \right) \tag{6}$$

$$\mu = \frac{1}{\lambda} \tag{7}$$

Now, using (7) and $\mu = 1$ (*Given*), we get;

$$\mu = \frac{1}{\lambda} = 1 \tag{8}$$

$$\implies \lambda = 1 \tag{9}$$

So,

$$X_1 \sim f(x;1) = \mathrm{e}^{-x} \tag{10}$$

Definition 2 *The probability density function (pdf) of an* ***gamma distribution*** *is*

$$f(x;\lambda) = \begin{cases} \dfrac{a^{\lambda}}{\Gamma\lambda} e^{ax} x^{\lambda-1}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{11}$$

Here, $\lambda > 0$, $a > 0$ and

$$\Gamma n = a^n \int_0^{\infty} \mathrm{e}^{ax} x^{n-1}\,\mathrm{d}x \tag{12}$$

$$\Gamma n = (n-1)! \tag{13}$$

Lemma 0.2 *The* ***mean*** *and* ***variance*** *of an exponentially distributed random variable X with λ and a is given by*

$$\mu = \frac{\lambda}{a} \tag{14}$$

$$\sigma^2 = \frac{\lambda}{a^2} \tag{15}$$

Proof 0.2

$$\mathrm{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\,\mathrm{d}x \tag{16}$$

$$\mu = 0 + \int_0^{\infty} x \times \frac{a^\lambda}{\Gamma\lambda} e^{ax} x^{\lambda-1}\,\mathrm{d}x \tag{17}$$

$$= 0 + \frac{a^\lambda}{\Gamma\lambda} \int_0^{\infty} e^{ax} x^\lambda\,\mathrm{d}x \tag{18}$$

$$\textit{Now, Using } n = \lambda + 1 \textit{ in } (12) \tag{19}$$

$$\textit{We get: } \Gamma(\lambda+1) = a^{\lambda+1} \int_0^{\infty} e^{ax} x^\lambda\,\mathrm{d}x \tag{20}$$

$$\textit{Now, } \mu = \frac{a^\lambda}{\Gamma\lambda} \times \frac{\Gamma(\lambda+1)}{a^{\lambda+1}} \tag{21}$$

$$= \frac{1}{a} \times \frac{\Gamma(\lambda+1)}{\Gamma\lambda} \tag{22}$$

$$\textit{Using } (13) \tag{23}$$

$$= \frac{1}{a} \times \frac{\lambda!}{(\lambda-1)!} \tag{24}$$

$$\mu = \frac{\lambda}{a} \tag{25}$$

Similarly we can find $E\big(X^2\big)$.

$$\mathrm{E}(X) = \frac{\lambda(\lambda+1)}{a^2} \tag{26}$$

$$\sigma^2 = (\mathrm{E}(X))^2 - \mathrm{E}\left(X^2\right) \tag{27}$$

$$= \frac{\lambda}{a^2} \tag{28}$$

Now by using (25) (28) And given info;

$$\text{Mean} = \frac{\lambda}{a} = 2 \text{ (Given)} \implies \lambda = 2a \tag{29}$$

$$\text{Variance} = \frac{\lambda}{a^2} = \frac{2a}{a^2} = 2 \text{ (Given)} \tag{30}$$

$$\implies a = 1, \lambda = 2 \tag{31}$$

$$X_2 \sim G(1,2) = \frac{1}{\Gamma 2} e^x x \tag{32}$$

$$= e^x x \tag{33}$$

Now by the help of (10) and (33), we can solve for $\mathrm{P}(X_1 < X_2)$.

$$\mathrm{P}(X_1 < X_2) = \mathrm{P}(X_1 < X_2 \mid X_1 = X_2) \tag{34}$$

$$= \int_0^{\infty} fX_2(x_2) \times \int_0^{x_2} fX_1(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{35}$$

$$= \int_0^{\infty} x_2 \mathrm{e}^{-x_2} \times \int_0^{x_2} \mathrm{e}^{-x_1}\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{36}$$

$$= \int_0^{\infty} x_2 \mathrm{e}^{-x_2} \times \left(1 - \mathrm{e}^{-x_2}\right)\mathrm{d}x_2 \tag{37}$$

Upon solving the definite integral, We get :

$$\mathrm{P}(X_1 < X_2) = \mathbf{\frac{3}{4}} \tag{38}$$