VECTORS

12^{th}Math - Chapter 10

This is Problem-16 from Exercise 10.5

1. If θ is the angle between two vectors $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$,then$\overrightarrow{\mathbf{a}} \cdot \overrightarrow{\mathbf{b}} \geq 0$ only when
   (a) $0< \theta < \frac{\pi}{2}$
   (b) $0\leq \theta \leq \frac{\pi}{2}$
   (c) $0< \theta < \pi$
   (d) $0\leq \theta \leq \pi$

   Solution: Given $\mathbf{a}$, $\mathbf{b}$ are two vectors
   We know that

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right) \tag{1}$$

$$\implies \mathbf{a}^\top \mathbf{b} = \cos(\theta)\|\mathbf{a}\|\|\mathbf{b}\| \tag{2}$$

   (a) $0 < \theta < \frac{\pi}{2}$
   Assume, $\theta = \frac{\pi}{3}$

$$\mathbf{a}^\top \mathbf{b} = \cos(\frac{\pi}{6})\|\mathbf{a}\|\|\mathbf{b}\| \tag{3}$$

$$= \frac{1}{2}\|\mathbf{a}\|\|\mathbf{b}\| \tag{4}$$

$$\implies \mathbf{a}^\top \mathbf{b} > 0 \tag{5}$$

$$\implies \text{ but } \mathbf{a}^\top \mathbf{b} \neq 0 \tag{6}$$

(b) $0 \leq \theta \leq \frac{\pi}{2}$

$$\text{for } \theta = 0 \tag{7}$$
$$\mathbf{a}^\top \mathbf{b} = \cos(0)\|\mathbf{a}\|\|\mathbf{b}\| \tag{8}$$
$$= \|\mathbf{a}\|\|\mathbf{b}\| \tag{9}$$
$$\implies \mathbf{a}^\top \mathbf{b} > 0 \tag{10}$$
$$\text{for } \theta = \frac{\pi}{2} \tag{11}$$
$$\mathbf{a}^\top \mathbf{b} = \cos(\frac{\pi}{2})\|\mathbf{a}\|\|\mathbf{b}\| \tag{12}$$
$$= 0 \tag{13}$$
$$\implies \mathbf{a}^\top \mathbf{b} = 0 \tag{14}$$

(c) $0 < \theta < \pi$

Assume, $\theta = \dfrac{2\pi}{3}$

$$\mathbf{a}^\top \mathbf{b} = \cos(\frac{2\pi}{6})\|\mathbf{a}\|\|\mathbf{b}\| \tag{15}$$
$$= -\frac{1}{2}\|\mathbf{a}\|\|\mathbf{b}\| \tag{16}$$
$$\implies \mathbf{a}^\top \mathbf{b} < 0 \tag{17}$$

(d) $0 \leq \theta \leq \pi$

$$\text{for } \theta = 0 \tag{18}$$
$$\mathbf{a}^\top \mathbf{b} = \cos(0)\|\mathbf{a}\|\|\mathbf{b}\| \tag{19}$$
$$= \|\mathbf{a}\|\|\mathbf{b}\| \tag{20}$$
$$\implies \mathbf{a}^\top \mathbf{b} > 0 \tag{21}$$
$$\text{for } \theta = \pi \tag{22}$$
$$\mathbf{a}^\top \mathbf{b} = \cos(\pi)\|\mathbf{a}\|\|\mathbf{b}\| \tag{23}$$
$$= -\|\mathbf{a}\|\|\mathbf{b}\| \tag{24}$$
$$\implies \mathbf{a}^\top \mathbf{b} < 0 \tag{25}$$

$\therefore \theta$ is $0 \leq \theta \leq \frac{\pi}{2}$. So, Option (b) is correct.