TRIANGLES

9^{th}Math - Chapter 7

This is Problem-8 from Exercise 7.1

In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that DM = CM. Point D is joined to point B (see Figure 1). Show that:

(i) $\triangle AMC \cong \triangle BMD$

(ii) $\angle DBC$ is a right angle.

(iii) $\triangle DBC \cong \triangle ACB$

(iv) $CM = \frac{1}{2}AB$

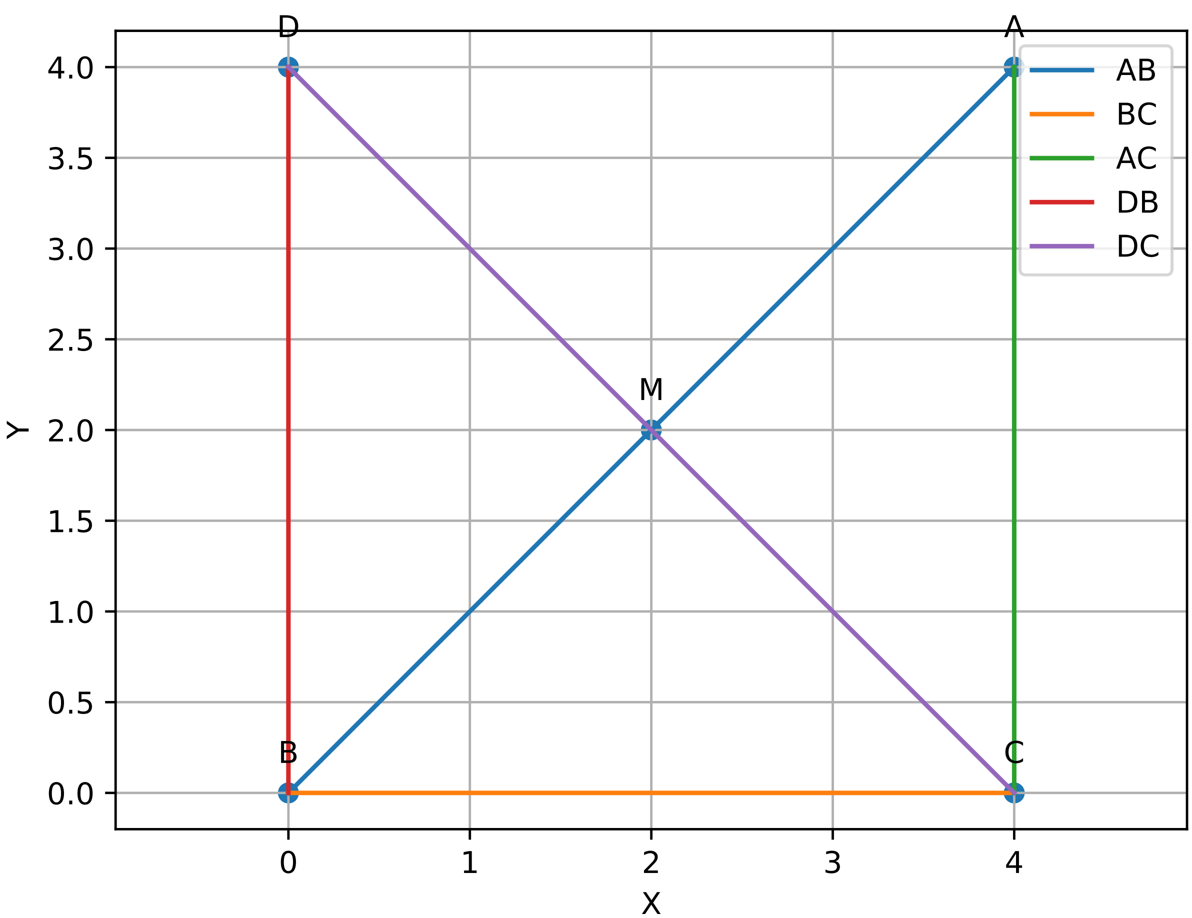

Figure 1

Construction:

Symbol	Values	Description
b	4	assumed constant
$\mathbf{e}_1$	$\begin{pmatrix}1\\0\end{pmatrix}$	eigen vector
$\mathbf{e}_2$	$\begin{pmatrix}0\\1\end{pmatrix}$	eigen vector
A	$\begin{pmatrix}4\\4\end{pmatrix}$	point A
B	$\begin{pmatrix}0\\0\end{pmatrix}$	point B
C	$\begin{pmatrix}4\\0\end{pmatrix}$	$b * e_1$
D	$\begin{pmatrix}0\\4\end{pmatrix}$	$b * e_2$
M	$\begin{pmatrix}2\\2\end{pmatrix}$	M=$\frac{A+B}{2}$

Solution: Given

$$\mathbf{M} = \frac{A+B}{2} \tag{1}$$

$$\mathbf{D} - \mathbf{M} = \mathbf{C} - \mathbf{M} \tag{2}$$

$$\angle ACB = 90^\circ \tag{3}$$

(i) $\triangle AMC \cong \triangle BMD$

$$\text{from (1) we can write as,} \tag{4}$$

$$\mathbf{A} - \mathbf{M} = \mathbf{B} - \mathbf{M} \tag{5}$$

$$\angle AMC = \angle DMB \tag{6}$$

$$\implies \cos^{-1}\left(\frac{(\mathbf{A}-\mathbf{M})^\top(\mathbf{C}-\mathbf{M})}{\|\mathbf{A}-\mathbf{M}\|\|\mathbf{C}-\mathbf{M}\|}\right) = \cos^{-1}\left(\frac{(\mathbf{D}-\mathbf{M})^\top(\mathbf{B}-\mathbf{M})}{\|\mathbf{D}-\mathbf{M}\|\|\mathbf{B}-\mathbf{M}\|}\right) \tag{7}$$

from (2), (5) and (7) taking norm of the respctive sides,

$$\triangle AMC \cong \triangle BMD \tag{8}$$

$$\text{from (8) we can say that,} \tag{9}$$

$$\mathbf{D} - \mathbf{B} = \mathbf{A} - \mathbf{C} \tag{10}$$

(ii) $\angle DBC$ is a right angle

$$\theta = \angle DBC \tag{11}$$

$$\cos\theta = \frac{(\mathbf{D}-\mathbf{B})^\top(\mathbf{C}-\mathbf{B})}{\|\mathbf{D}-\mathbf{B}\|\|\mathbf{C}-\mathbf{B}\|} \tag{12}$$

$$\text{if } \theta = 90^\circ, \tag{13}$$

$$\implies \cos\theta = 0 \tag{14}$$

$$\implies (\mathbf{D}-\mathbf{B})^\top(\mathbf{C}-\mathbf{B}) = 0 \tag{15}$$

$$\therefore \angle DBC = 90^\circ \tag{16}$$

(iii) $\triangle DBC \cong \triangle ACB$

$$\text{from (3) and (16)} \tag{17}$$

$$\angle ACB = \angle DBC \tag{18}$$

$$\mathbf{B} - \mathbf{C} = \mathbf{C} - \mathbf{B} \tag{19}$$

from (10), (18) and (19) taking norms of the respective sides,

$$\triangle DBC \cong \triangle ACB \tag{20}$$

(iv) $CM = \frac{1}{2}AB$

$$\begin{aligned}
\text{from (20) we can say that,} & \tag{21}\\
\mathbf{A} - \mathbf{B} &= \mathbf{D} - \mathbf{C} \tag{22}\\
&= (\mathbf{D} - \mathbf{M}) + (\mathbf{C} - \mathbf{M}) \tag{23}\\
\text{from (2) we can write as,} & \tag{24}\\
&= (\mathbf{C} - \mathbf{M}) + (\mathbf{C} - \mathbf{M}) \tag{25}\\
&= 2\left(\mathbf{C} - \mathbf{M}\right) \tag{26}\\
\implies \frac{\mathbf{A} - \mathbf{B}}{2} &= \mathbf{C} - \mathbf{M} \tag{27}
\end{aligned}$$