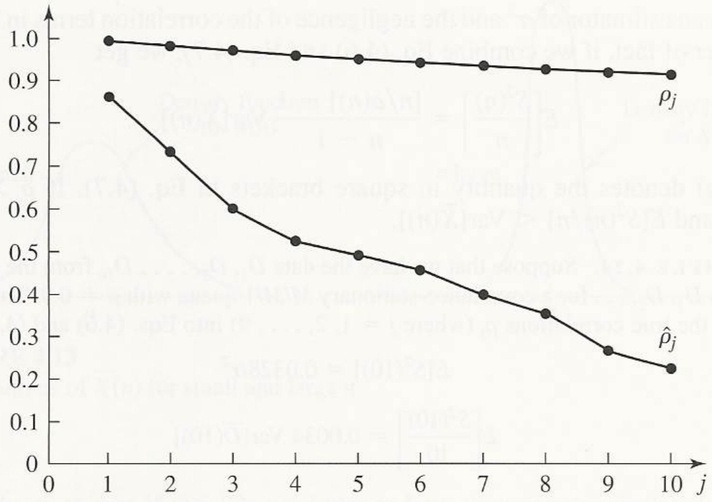

FIGURE 4.14
ρ_j and $\hat{\rho}_j$ of the process $D_1, D_2, \ldots$ for the *M*/*M*/1 queue with $\rho = 0.9$.

estimating variances. However, we shall see in Chap. 9 that it is often possible to group simulation output data into new "observations" to which the formulas based on IID observations *can* be applied. Thus, the formulas in this and the next two sections based on IID observations are *indirectly* applicable to analyzing simulation output data.

4.5
CONFIDENCE INTERVALS AND HYPOTHESIS TESTS FOR THE MEAN

Let $X_1, X_2, \ldots, X_n$ be IID random variables with finite mean μ and finite variance σ^2. (Also assume that $\sigma^2 > 0$, so that the X_i's are not degenerate random variables.) In this section we discuss how to construct a confidence interval for μ and also the complementary problem of testing the hypothesis that $\mu = \mu_0$.

We begin with a statement of the most important result in probability theory, the classical central limit theorem. Let Z_n be the random variable $[\bar{X}(n) - \mu]/\sqrt{\sigma^2/n}$, and let $F_n(z)$ be the distribution function of Z_n for a sample size of n; that is, $F_n(z) = P(Z_n \le z)$. [Note that μ and σ^2/n are the mean and variance of $\bar{X}(n)$, respectively.] Then the *central limit theorem* is as follows [see Chung (1974, p. 169) for a proof].

THEOREM 4.1. $F_n(z) \to \Phi(z)$ as $n \to \infty$, where $\Phi(z)$, the distribution function of a normal random variable with $\mu = 0$ and $\sigma^2 = 1$ (henceforth called a *standard normal random variable*; see Sec. 6.2.2), is given by

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy \qquad \text{for } -\infty < z < \infty$$

The theorem says, in effect, tha
will be approximately distribut
of the underlying distribution
sample mean $\bar{X}(n)$ is approxim
mean μ and variance σ^2/n.

The difficulty with using th
generally unknown. However,
gets large, it can be shown that
in the expression for Z_n. With
large, the random variable t_n =
as a standard normal random v

$$P\left(-z_{1-\alpha/2} \le \frac{\bar{X}(n) - \mu}{\sqrt{S^2(n)/n}} \le z_{1-\ldots\right.$$

$$= P\left[\bar{X}(n) - z_{1-\alpha/2}\ldots\right.$$

$$\approx 1 - \alpha$$

where the symbol $\approx$ means "
the upper $1 - \alpha/2$ critical p
Fig. 4.15 and the last line of
Therefore, if n is sufficiently la
interval for μ is given by

For a given set of data X_1,
$l(n, \alpha) = \bar{X}(n) - z_{1-\alpha/2}\sqrt{S^2(\ldots}$
$u(n, \alpha) = \bar{X}(n) + z_{1-\alpha/2}\sqrt{S^2(\ldots}$
of random variables) and the c

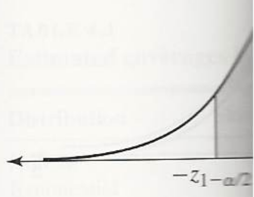

FIGURE 4.15
Density function for the

The theorem says, in effect, that if n is "sufficiently large," the random variable Z_n will be approximately distributed as a standard normal random variable, regardless of the underlying distribution of the X_i's. It can also be shown for large n that the sample mean $\overline{X}(n)$ is approximately distributed as a normal random variable with mean μ and variance σ^2/n.

The difficulty with using the above results in practice is that the variance σ^2 is generally unknown. However, since the sample variance $S^2(n)$ converges to σ^2 as n gets large, it can be shown that Theorem 4.1 remains true if we replace σ^2 by $S^2(n)$ in the expression for Z_n. With this change the theorem says that if n is sufficiently large, the random variable $t_n = [\overline{X}(n) - \mu]/\sqrt{S^2(n)/n}$ is approximately distributed as a standard normal random variable. It follows for large n that

$$\begin{aligned}P\left(-z_{1-\alpha/2} \le \frac{\overline{X}(n) - \mu}{\sqrt{S^2(n)/n}} \le z_{1-\alpha/2}\right) &= P\left[\overline{X}(n) - z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}} \le \mu \le \overline{X}(n) + z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}}\right] \\ &\approx 1 - \alpha \end{aligned} \tag{4.10}$$

where the symbol $\approx$ means "approximately equal" and $z_{1-\alpha/2}$ (for $0 < \alpha < 1$) is the upper $1 - \alpha/2$ critical point for a standard normal random variable (see Fig. 4.15 and the last line of Table T.1 of the Appendix at the back of the book). Therefore, if n is sufficiently large, an approximate $100(1 - \alpha)$ percent confidence interval for μ is given by

$$\overline{X}(n) \pm z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}} \tag{4.11}$$

For a given set of data $X_1, X_2, \ldots, X_n$, the lower confidence-interval endpoint $l(n, \alpha) = \overline{X}(n) - z_{1-\alpha/2}\sqrt{S^2(n)/n}$ and the upper confidence-interval endpoint $u(n, \alpha) = \overline{X}(n) + z_{1-\alpha/2}\sqrt{S^2(n)/n}$ are just numbers (actually, specific realizations of random variables) and the confidence interval $[l(n, \alpha), u(n, \alpha)]$ either contains μ

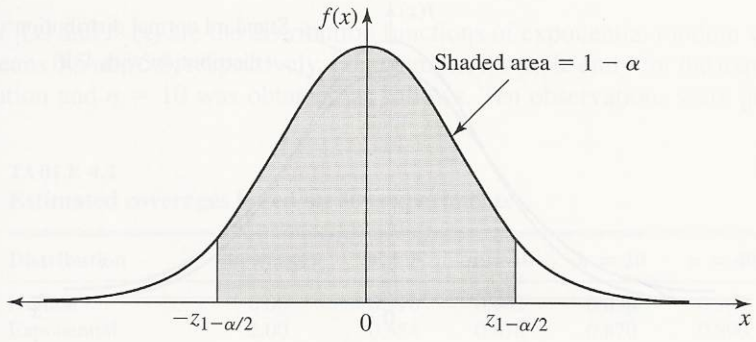

FIGURE 4.15
Density function for the standard normal distribution.

or does not contain μ. Thus, there is nothing probabilistic about the single confidence interval $[l(n, \alpha), u(n, \alpha)]$ *after* the data have been obtained and the interval's endpoints have been given numerical values. The correct interpretation to give to the confidence interval (4.11) is as follows [see (4.10)]: If one constructs a very large number of independent $100(1 - \alpha)$ percent confidence intervals, each based on n observations, where n is sufficiently large, the proportion of these confidence intervals that contain (cover) μ should be $1 - \alpha$. We call this proportion the *coverage* for the confidence interval.

The difficulty in using (4.11) to construct a confidence interval for μ is in knowing what "n sufficiently large" means. It turns out that the more skewed (i.e., nonsymmetric) the underlying distribution of the X_i's, the larger the value of n needed for the distribution of t_n to be closely approximated by $\Phi(z)$. (See the discussion later in this section.) If n is chosen too small, the actual coverage of a desired $100(1 - \alpha)$ percent confidence interval will generally be less than $1 - \alpha$. This is why the confidence interval given by (4.11) is stated to be only approximate.

In light of the above discussion, we now develop an alternative confidence-interval expression. If the X_i's are *normal* random variables, the random variable $t_n = [\bar{X}(n) - \mu]/\sqrt{S^2(n)/n}$ has a t distribution with $n - 1$ degrees of freedom (df) [see, for example, Hogg and Craig (1995, pp. 181–182)], and an *exact* (for any $n \geq 2$) $100(1 - \alpha)$ percent confidence interval for μ is given by

$$\bar{X}(n) \pm t_{n-1,1-\alpha/2}\sqrt{\frac{S^2(n)}{n}} \tag{4.12}$$

where $t_{n-1,1-\alpha/2}$ is the upper $1 - \alpha/2$ critical point for the t distribution with $n - 1$ df. These critical points are given in Table T.1 of the Appendix at the back of the book. Plots of the density functions for the t distribution with 4 df and for the standard normal distribution are given in Fig. 4.16. Note that the t distribution is less peaked and has longer tails than the normal distribution, so, for any finite n, $t_{n-1,1-\alpha/2} > z_{1-\alpha/2}$. We call (4.12) the *t confidence interval*.

The quantity that we add to and subtract from $\bar{X}(n)$ in (4.12) to construct the confidence interval is called the *half-length* of the confidence interval. It is a measure of how precisely we know μ. It can be shown that if we increase the sample

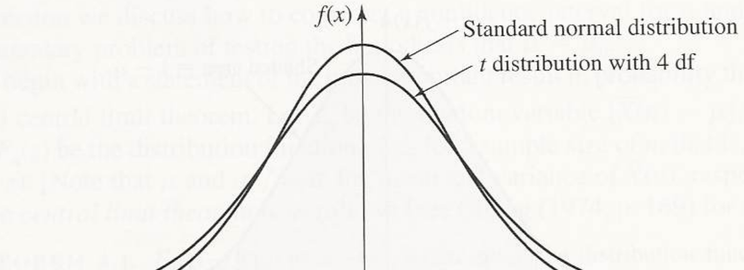

FIGURE 4.16
Density functions for the t distribution with 4 df and for the standard normal distribution.

size from n to $4n$ in (4.12), then the half-length is decreased by a factor of approximately 2 (see Prob. 4.20).

In practice, the distribution of the X_i's will rarely be normal, and the confidence interval given by (4.12) will also be approximate in terms of coverage. Since $t_{n-1,1-\alpha/2} > z_{1-\alpha/2}$, the confidence interval given by (4.12) will be larger than the one given by (4.11) and will generally have coverage closer to the desired level $1 - \alpha$. For this reason, we recommend using (4.12) to construct a confidence interval for μ. Note that $t_{n-1,1-\alpha/2} \to z_{1-\alpha/2}$ as $n \to \infty$; in particular, $t_{40,0.95}$ differs from $z_{0.95}$ by less than 3 percent. However, in most of our applications of (4.12) in Chaps. 9, 10, and 12, n will be small enough for the difference between (4.11) and (4.12) to be appreciable.

EXAMPLE 4.26. Suppose that the 10 observations 1.20, 1.50, 1.68, 1.89, 0.95, 1.49, 1.58, 1.55, 0.50, and 1.09 are from a normal distribution with unknown mean μ and that our objective is to construct a 90 percent confidence interval for μ. From these data we get

$$\bar{X}(10) = 1.34 \quad \text{and} \quad S^2(10) = 0.17$$

which results in the following confidence interval for μ:

$$\bar{X}(10) \pm t_{9,0.95}\sqrt{\frac{S^2(10)}{10}} = 1.34 \pm 1.83\sqrt{\frac{0.17}{10}} = 1.34 \pm 0.24$$

Note that (4.12) was used to construct the confidence interval and that $t_{9,0.95}$ was taken from Table T.1. Therefore, subject to the interpretation stated above, we claim with 90 percent confidence that μ is in the interval [1.10, 1.58].

We now discuss how the coverage of the confidence interval given by (4.12) is affected by the distribution of the X_i's. In Table 4.1 we give estimated coverages for 90 percent confidence intervals based on 500 independent experiments for each of the sample sizes $n = 5$, 10, 20, and 40 and each of the distributions normal, exponential, chi square with 1 df (a standard normal random variable squared; see the discussion of the gamma distribution in Sec. 6.2.2), lognormal (e^Y, where Y is a standard normal random variable; see Sec. 6.2.2), and hyperexponential whose distribution function is given by

$$F(x) = 0.9F_1(x) + 0.1\,F_2(x)$$

where $F_1(x)$ and $F_2(x)$ are the distribution functions of exponential random variables with means 0.5 and 5.5, respectively. For example, the table entry for the exponential distribution and $n = 10$ was obtained as follows. Ten observations were generated

TABLE 4.1
Estimated coverages based on 500 experiments

Distribution	Skewness ν	$n = 5$	$n = 10$	$n = 20$	$n = 40$
Normal	0.00	0.910	0.902	0.898	0.900
Exponential	2.00	0.854	0.878	0.870	0.890
Chi square	2.83	0.810	0.830	0.848	0.890
Lognormal	6.18	0.758	0.768	0.842	0.852
Hyperexponential	6.43	0.584	0.586	0.682	0.774

TABLE T.1

Critical points $t_{\nu,\gamma}$ for the t distribution with ν df, and z_γ for the standard normal distribution

$\gamma = P(T_\nu \le t_{\nu,\gamma})$, where T_ν is a random variable having the t distribution with ν df; the last row, where $\nu = \infty$, gives the normal critical points satisfying $\gamma = P(Z \le z_\gamma)$, where Z is a standard normal random variable

	γ															
ν	0.6000	0.7000	0.8000	0.9000	0.9333	0.9500	0.9600	0.9667	0.9750	0.9800	0.9833	0.9875	0.9900	0.9917	0.9938	0.9950
1	0.325	0.727	1.376	3.078	4.702	6.314	7.916	9.524	12.706	15.895	19.043	25.452	31.821	38.342	51.334	63.657
2	0.289	0.617	1.061	1.886	2.456	2.920	3.320	3.679	4.303	4.849	5.334	6.205	6.965	7.665	8.897	9.925
3	0.277	0.584	0.978	1.638	2.045	2.353	2.605	2.823	3.182	3.482	3.738	4.177	4.541	4.864	5.408	5.841
4	0.271	0.569	0.941	1.533	1.879	2.132	2.333	2.502	2.776	2.999	3.184	3.495	3.747	3.966	4.325	4.604
5	0.267	0.559	0.920	1.476	1.790	2.015	2.191	2.337	2.571	2.757	2.910	3.163	3.365	3.538	3.818	4.032
6	0.265	0.553	0.906	1.440	1.735	1.943	2.104	2.237	2.447	2.612	2.748	2.969	3.143	3.291	3.528	3.707
7	0.263	0.549	0.896	1.415	1.698	1.895	2.046	2.170	2.365	2.517	2.640	2.841	2.998	3.130	3.341	3.499
8	0.262	0.546	0.889	1.397	1.670	1.860	2.004	2.122	2.306	2.449	2.565	2.752	2.896	3.018	3.211	3.355
9	0.261	0.543	0.883	1.383	1.650	1.833	1.973	2.086	2.262	2.398	2.508	2.685	2.821	2.936	3.116	3.250
10	0.260	0.542	0.879	1.372	1.634	1.812	1.948	2.058	2.228	2.359	2.465	2.634	2.764	2.872	3.043	3.169
11	0.260	0.540	0.876	1.363	1.621	1.796	1.928	2.036	2.201	2.328	2.430	2.593	2.718	2.822	2.985	3.106
12	0.259	0.539	0.873	1.356	1.610	1.782	1.912	2.017	2.179	2.303	2.402	2.560	2.681	2.782	2.939	3.055
13	0.259	0.538	0.870	1.350	1.601	1.771	1.899	2.002	2.160	2.282	2.379	2.533	2.650	2.748	2.900	3.012
14	0.258	0.537	0.868	1.345	1.593	1.761	1.887	1.989	2.145	2.264	2.359	2.510	2.624	2.720	2.868	2.977
15	0.258	0.536	0.866	1.341	1.587	1.753	1.878	1.978	2.131	2.249	2.342	2.490	2.602	2.696	2.841	2.947
16	0.258	0.535	0.865	1.337	1.581	1.746	1.869	1.968	2.120	2.235	2.327	2.473	2.583	2.675	2.817	2.921
17	0.257	0.534	0.863	1.333	1.576	1.740	1.862	1.960	2.110	2.224	2.315	2.458	2.567	2.657	2.796	2.898
18	0.257	0.534	0.862	1.330	1.572	1.734	1.855	1.953	2.101	2.214	2.303	2.445	2.552	2.641	2.778	2.878
19	0.257	0.533	0.861	1.328	1.568	1.729	1.850	1.946	2.093	2.205	2.293	2.433	2.539	2.627	2.762	2.861
20	0.257	0.533	0.860	1.325	1.564	1.725	1.844	1.940	2.086	2.197	2.285	2.423	2.528	2.614	2.748	2.845
21	0.257	0.532	0.859	1.323	1.561	1.721	1.840	1.935	2.080	2.189	2.277	2.414	2.518	2.603	2.735	2.831
22	0.256	0.532	0.858	1.321	1.558	1.717	1.835	1.930	2.074	2.183	2.269	2.405	2.508	2.593	2.724	2.819
23	0.256	0.532	0.858	1.319	1.556	1.714	1.832	1.926	2.069	2.177	2.263	2.398	2.500	2.584	2.713	2.807
24	0.256	0.531	0.857	1.318	1.553	1.711	1.828	1.922	2.064	2.172	2.257	2.391	2.492	2.575	2.704	2.797
25	0.256	0.531	0.856	1.316	1.551	1.708	1.825	1.918	2.060	2.167	2.251	2.385	2.485	2.568	2.695	2.787
26	0.256	0.531	0.856	1.315	1.549	1.706	1.822	1.915	2.056	2.162	2.246	2.379	2.479	2.561	2.687	2.779
27	0.256	0.531	0.855	1.314	1.547	1.703	1.819	1.912	2.052	2.158	2.242	2.373	2.473	2.554	2.680	2.771
28	0.256	0.530	0.855	1.313	1.546	1.701	1.817	1.909	2.048	2.154	2.237	2.368	2.467	2.548	2.673	2.763
29	0.256	0.530	0.854	1.311	1.544	1.699	1.814	1.906	2.045	2.150	2.233	2.364	2.462	2.543	2.667	2.756
30	0.256	0.530	0.854	1.310	1.543	1.697	1.812	1.904	2.042	2.147	2.230	2.360	2.457	2.537	2.661	2.750
40	0.255	0.529	0.851	1.303	1.532	1.684	1.796	1.886	2.021	2.123	2.203	2.329	2.423	2.501	2.619	2.704
50	0.255	0.528	0.849	1.299	1.526	1.676	1.787	1.875	2.009	2.109	2.188	2.311	2.403	2.479	2.594	2.678
75	0.254	0.527	0.846	1.293	1.517	1.665	1.775	1.861	1.992	2.090	2.167	2.287	2.377	2.450	2.562	2.643
100	0.254	0.526	0.845	1.290	1.513	1.660	1.769	1.855	1.984	2.081	2.157	2.276	2.364	2.436	2.547	2.626
∞	0.253	0.524	0.842	1.282	1.501	1.645	1.751	1.834	1.960	2.054	2.127	2.241	2.326	2.395	2.501	2.576