IT 24132 MD. Amirul Hasan Shanto

Q.1. Is the set of odd numbers with the binary addition i.e. $\langle O, + \rangle$ an abelian group?

Answer: False.

Explanation: The set of odd number (O) is not under addition. For example, $a=3$, $b=1$ both odd numbers. Then $a+b=4$ which is even. Since closure fails $(O, +)$ is not even a group. So it can't be an abelian group.

Q.2 Let G be a group of order pq, where p & q are distinct primes. Prove that G is abelian.

Answer: False

Explanation: The symmetric group S_3, which has order $6 = 2 \times 3$. S_3 is non-abelian as it contains non-commut elements (12) & (123) permutations.

Q.3 Prove that if G is a group of order p^2, where p is prime, then G is abelian if and only if it has $p+1$ subgroups of order p.

Answer: False

Explanation: Every group of order p^2 is abelian. However, the number of subgroups of order p depends on the structure: if G is cyclic, it has exactly one subgroup of order p; if G is elementary abelian, it has $p+1$ subgroups of order p. "If & only if" condition fails because a cyclic group of order p^2 is abelian but doesn't have $p+1$ subgroup of order p.

Q.4 Let G be a finite group & H be a proper subgroup of G. Prove that the union of all conjugates of H can't be equal to G.

Answer: True

Explanation: This is a standard result in group theory. The union of all conjugates of a proper subgroup H is ~~a~~ proper subset of G. This can be shown using the formula for the ~~nu~~ number of conjugates & the fact that the intersection of conjugates has index at least 2, leading to a size contradiction if the union were equal to G.

Q.5. Let G be a group & N be a normal subgroup of G. If G/N is cyclic & N is cyclic, prove that G is abelian.

Answer: False.

Explanation: Let N be the alternating subgroup A_3 which is cyclic of order 3. Then G/N is cyclic order 2. However, S_3 is non-abelian, showing that the conditions do not gurantee that G is abelian.

Q.6. Prove that in any group G, the set of elements of finite order forms a subgroup of G.

Answer: False

Explanation: In the infinite dihedral group $D\alpha$, the elements of order are the reflections, but the product of two distinct reflections is a translation which has infinite order. Thus, the set of elements of finite order is not closed under multiplication & is not a subgroup.

Q.7. Let G be a finite group & p be the smallest prime diving $|G|$. Prove that any subgroup of index p in G is normal.

Answer: True.

Explanation: If H is a subgroup of index p in G & p is the smallest prime diving $|G|$ then H is normal. This can be proven using the action of G on the cosets of H and considering the homo-morphism into the symmetric group S_p.

Q.8. Let G be a group & $a,b \in G$. Prove that $a^4 = b^2$ & $ab = ba$ then $(ab)^6 = e$

Answer: False

Explanation: Let $G = \langle a \rangle$ where a has order 4. Set $b = e$ (the identy). Then $a^4 = e = b^2$ & $ab = ba$. But $(ab)^6 = a^6 . a^2 = e . a^2 = a^2 \neq e$. Thus the statement fails.