IT24632

Q. Prove that the set of rational numbers $\mathbb{Q}$, equipped with the two binary operations of addition & multiplication, forms a field.

⇒ We take the rational numbers $\mathbb{Q}$ to be the set of equivalence classes of ordered pairs (a,b) with $a, b \in \mathbb{Z}$ and $b \neq 0$, where $(a,b) \sim (a', b')$ iff $ab' = a'b$. We identify the class of (a,b) with the ~~usal~~ usual fraction $\frac{a}{b}$. Define addition & multiplication in the usual way:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{ac}{bd} = \frac{a}{b} \cdot \frac{c}{d},$$

for $b \neq 0$, $d \neq 0$. Below we show these operations make $\mathbb{Q}$ a field.

1. The operations are well defined

We must check that if $\frac{a}{b} = \frac{a'}{b'}$ & $\frac{c}{d} = \frac{c'}{d'}$ then

$$\frac{ad+bc}{bd} = \frac{a'd'+b'c'}{b'd'} \;\&\; \frac{ac}{bd} = \frac{a'c'}{b'd'}$$

From $\frac{a}{b} = \frac{a'}{b'}$ & $\frac{c}{d} = \frac{c'}{d'}$ we have $ab' = a'b$ & $c'd = cd'$.

Compute

$$(ad+bc)b'd' = (ab')(dd') + (bc)(b'd') = (a'b)(dd') + (bc)(b'd')$$

& similarly expand the right-hand numerator times $bdb'd'$. Rearranging and using $ab' = a'b$, $cd' = c'd$ shows both cross-products are equal therefore the sums represent the same equivalence class. So addition & multiplication are well-defined.

2. $(\mathbb{Q}, +)$ is an abelian group

Take any $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$.

Closure: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ is a rational number since $bd \neq$

Associativity: follow form associativity of integer addition

$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{ad+bc}{bd} + \frac{e}{f} = \frac{f(ad+bc) + e(bd)}{(bd)f}$ & a similar expansion for $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$: both give the same numerator by associativity of integer operation

Identity: $0 = \frac{0}{1}$ satisfy $\frac{a}{b} + 0 = \frac{a}{b}$

Inverse: additive inverse of $\frac{a}{b}$ is $-\frac{a}{b} = \frac{-a}{b}$ because

$$\frac{a}{b} + \frac{-a}{b} = \frac{0}{b} = 0$$

Commutativity: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} = \frac{bc+ad}{db} = \frac{c}{d} + \frac{a}{b}$

Thus $(\mathbb{Q}, +)$ is an abelian group.

3. Multiplication on $\mathbb{Q}\setminus\{0\}$ is an abelian group.

Closure: product $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ is rational since $bd\neq 0$

Associativity & commutativity: follow form associativity & commutativity of integer multiplication.

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f}=\frac{ac}{bd}\cdot\frac{e}{f}=\frac{(ac)e}{bd\cdot f}=\frac{ce\cdot a}{df\cdot b}=\frac{a}{b}\cdot\frac{ce}{df}=\frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right)$$

Multiplicative identity: $1=\frac{1}{1}$ satisfies $\frac{a}{b}\cdot 1=\frac{a}{b}$

Distributivity: For addition & multiplication

$$\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}\cdot\frac{cf+ed}{df}=\frac{a(cf+ed)}{b\cdot df}=\frac{acf+aed}{bdf}=\frac{ac}{bd}+\frac{ae}{bf}$$

$$=\frac{a}{b}\cdot\frac{c}{d}+\frac{a}{b}\cdot\frac{e}{f}$$

using integer distributivity.

So $\mathbb{Q}$ is a commutative ring with unity 1.

4. Multiplicative inverse exist for nonzero rationals.

Take a nonzero rational $\frac{a}{b}$. Its multiplicative inverse is $\frac{b}{a}$ because

$$\cancel{\frac{a}{b}}=\frac{a}{b}\cdot\frac{b}{a}=\frac{ab}{ba}=\frac{1}{1}=1.$$

We must check this inverse is well-defined: if $\frac{a}{b}=\frac{a'}{b'}$ & $a\neq 0$, then $ab'=a'b$. Multipling both sides by $1/(aa')$ is informal but the correct check is $\frac{b}{a}=\frac{b'}{a'}$ if & only if $ba'=b'a$ but from $ab' = b'a'$ we get exactly $ba' = b'a$, so inverse ~~agrees~~ agree for different representative.

5. Nontriviality : $0 \neq 1$

Clearly $\frac{0}{1} \neq \frac{1}{1}$ because if $0 \cdot 1 = 1 \cdot 1$ then $0 = 1$, contradicting integer's properties. So the field is not the zero ring.