Classify Sets:

- Normal Sets: A is a <u>normal set</u> if $A \notin A$

(ex) S is not normal set)

ex) Set of all normal sets

$N = \{A \mid A \notin A\}$

we already said $S \notin N$ ← normal set

$\{a,b,c\} \in N$

$\mathbb{R} \in N$

know this example

Question: $N \in N$?

1) Assume true $N \in N \rightarrow N \notin N$ false

2) Assume false $N \notin N \rightarrow \neg(N \notin N) = N \in N$ true

$\Rightarrow$ <u>Paradox</u> (Proposition true & false at same time)

know this

07/20 Recap:

$A \subseteq B$	$a \in A$
$A \subset B$	$A = \{x \mid P(x)\}$
$A = B$	$a \in A,\ P(a)$

Normal sets:

$N = \{A \mid A \notin A\}$

N is normal

$C = \{C, 1, 2, 3, \{1,3\}\}$ C is not normal

<u>$N \in N$?</u>

Assume $N \in N \rightarrow N \notin N$

Assume $N \notin N \rightarrow \neg(N \notin N) \rightarrow N \in N$

Does not satisfy the condition

↳ <u>Russell's Paradox</u>

- Peano – formalize $\mathbb{N}$ (natural numbers)

know this

Solutions: different set theories

Define a set as a collection of elements where it is always possible to determine if $x \in A$ or $x \notin A$. $(\forall x)$

- for every True proposition (p) within an axiom system, there should be a proof of P (p must be a theorem)

Godel

$\exists P$ that is true but cannot be proven. (Number Theory)

* math is not perfect

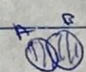
Any axiom system is incomplete

↳ Turing machine { Problem, Algorithms

\# of problem > # of algorithms

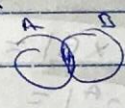
Important! ☆ Set operations:

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$



- Intersection: $A \cap B$

$A \cap B = \{x \mid x \in A \wedge x \in B\}$

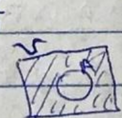


- Set Difference: $A - B$

$A - B = \{x \mid x \in A \wedge x \notin B\}$

- Complementation: $\overline{A}$

$\overline{A} = \{x \in V \mid x \notin A\}$



- Power Set: 2^A or $P(A)$

$2^A = \{B \mid B \subseteq A\}$

- Cartesian Product: $A \times B$

$A \times B = \{\underbrace{(x,y)}_{\text{List}} \mid x \in A \wedge y \in B\}$

ex. 1) $A = \{1,2,5\}$, $B = \{2,5,7\}$

a) $A \cup B = \{1,2,5,7\}$

☆ * Property of sets: <u>no duplicates</u>

$A = \{1,3\}$	$A = B$	$A \subseteq B$ ✓
$B = \{3,1\}$		$B \subseteq A$ ✓

* <u>Order does not matter</u>

* Lists: $(1,3,5,2)$

$\neq (3,1,2,5)$ ← order is important & duplicates allowed

* Bags: order is not important & duplicates allowed

$[1,2,2,9,7,3] = [2,1,2,7,3,9]$

example cont.

b) $A \cap B = \{2,5\}$

c) $A - B = \{1\}$

d) $\overline{A} = \{3,4,6,7,8,9,\ldots\} = \{x \in \mathbb{N} \mid x \notin \{1,2,5\}\}$

(Assuming $U = \{1,2,3,\ldots\}$

$|A \times B| = |A||B|$

e) $A \times B = \{(1,2),(1,5),(1,7),(2,2),(2,5),(2,7),(5,2),(5,5),(5,7)\}$

$\emptyset$ is a subset of any element

f) $2^A = \{\{1\},\{2\},\{5\},\{1,2\},\{1,5\},\{2,5\},\{1,2,5\},\{\}\}$

# of elements $= 2^3 = \boxed{8}$

$\emptyset \subseteq 2^A$ also $\emptyset \subseteq A$

$\rightarrow \emptyset \in 2^A$

Cardinalities:

Important!

- $|A \times B| = |A||B|$
- $|2^A| = 2^{|A|}$
- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cap B| = |A| + |B| - |A \cup B|$
- $A \subseteq B \rightarrow |A| \leq |B|$

ex.2) $A = \{x \mid x \text{ is odd}\}$

$B = \{x \mid x \text{ is prime}\}$

a) $A \cup B = A \cup \{2\}$

checking

1) $A \cup B \subseteq A \cup \{2\}$

Let $x \in A \rightarrow x \in A \cup \{2\}$

Let $x \in B \rightarrow x$ is prime

cases: 1) $x = 2$; $\rightarrow x \in A \cup \{2\}$

2) $x \neq 2$

x is prime and $x \neq 2 \rightarrow \begin{cases} x = 1 \\ x > 2 \end{cases}$

If $x > 2$, is x divisible by 2?

if $x > 2$ was divisible by 2

$\rightarrow x$ is not prime $\square$

$\rightarrow x = 1$ or x is odd

$\rightarrow x \in A \rightarrow x \in A \cup \{2\}$

ii) $A \cup \{2\} \subseteq A \cup B$

i) $x \in A \to x \in A \cup B$

ii) $x \in \{2\} \to x = 2 \to x$ is prime $\to x \in B$

$\to x \in A \cup B$ ▫

b) $A \cap B = B - \{2\}$

c) $|A| = \infty$

d) $|B| = \infty$

ex) $\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R} \wedge y \in \mathbb{R}\}$

$= \mathbb{R}^2$

can only use on propositions

ex) $x \in A \vee x \in B$

$\to x \in A \cup B$

Identities:

Identity: • $A \cap U = A$

• $A \cup \emptyset = A$

Domination: • $A \cup U = U$

• $A \cap \emptyset = \emptyset$

Idempotant: $A \cup A = A$

$A \cap A = A$

Complementation: $\overline{\overline{A}} = A$

Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Associative: $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

DeMorgan: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Complement: $A \cup \overline{A} = U$

$A \cap \overline{A} = \emptyset$

Absorption: $A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

know how to use this!

*All these identities are theorems & we can prove them using logic and definition.

☆ Very Important !!!

ex)

1) Show $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cap B} = \{x \mid x \notin \underbrace{(A \cap B)}_{\{x \mid x \in A \wedge x \in B\}}\}$

↳ seems difficult ...

Proof

1) $X = \{x \mid \_\_\_\} = \{x \mid \_\_\_\} = \{x \mid \_\_\_\} = Y$

2) • Let $x \in X \rightarrow x \in Y$ $[X \subseteq Y]$
   • Let $x \in Y \rightarrow x \in X$ $[Y \subseteq X]$ $\Rightarrow X = Y$

2') Let $x \in X \leftrightarrow \leftrightarrow \leftrightarrow x \in Y$

Proof method #2

i) $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

Let $x \in \overline{A \cap B} \rightarrow x \notin (A \cap B) \rightarrow \neg(x \in (A \cap B))$

$\rightarrow \neg(x \in A \wedge x \in B) \rightarrow \neg x \in A \vee \neg x \in B$

$\rightarrow x \notin A \vee x \notin B \rightarrow x \in \bar{A} \vee x \in \bar{B}$

$\rightarrow \bar{A} \cup \bar{B}$

ii) $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

Let $x \in \bar{A} \cup \bar{B} \rightarrow x \in \bar{A} \vee x \in \bar{B}$

$\rightarrow x \notin A \vee x \notin B$

$\rightarrow \neg(x \in A) \vee \neg(x \in B)$

$\rightarrow \neg(x \in A \wedge x \in B)$

$\rightarrow \neg(x \in (A \cap B))$

$\rightarrow x \notin A \cap B$

$\rightarrow x \in \overline{A \cap B}$

Proof method #2

☆ $\overline{A \cap B} = \{x \mid x \notin (A \cap B)\}$

$= \{x \mid \neg(x \in A \wedge x \in B)\}$ — ☆ can only operate on propositions

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$

$= \{x \mid x \notin A \vee x \notin B\}$

$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$

$= \{x \mid x \in \bar{A} \cup \bar{B}\}$

$= \bar{A} \cup \bar{B}$

☆ Infinite Sets:

$|\mathbb{N}| =$ infinite

$\frac{\text{even}}{\text{odd}} = \mathbb{N}$

$\mathbb{Q}^+ = \{\frac{p}{q}\}$ $p, q \in \mathbb{N}$, $q \neq 0$

ex) Show that $|\mathbb{N}| = |\mathbb{Q}^+|$

need a matching (one-to-one)

$\mathbb{N} \longleftrightarrow \mathbb{Q}^+$

	1	2	3	4	5	...
1	1/1	1/2	1/3	1/4	1/5	...
2	2/1	2/2	2/3	2/4	2/5	...
3	3/1	3/2	3/3	3/4	3/5	...
4	4/1	4/2	4/3	4/4	4/5	...
5	5/1	5/2	5/3	5/4	5/5	...
⋮	⋮	⋮	⋮	⋮	⋮	

$\xrightarrow[\text{matching}]{\text{one-to-one}} |\mathbb{Q}^+| = |\mathbb{N}|$

* see as a fraction and not value