Proof (by Strong Induction)

Basis: same as before

I.H.: given a fixed $k \in \mathbb{N}$, $k \geq 12$

$\forall\ 12 \leq r \leq k \rightarrow r = 5a + 4b$ for some $a, b \in \mathbb{N}$

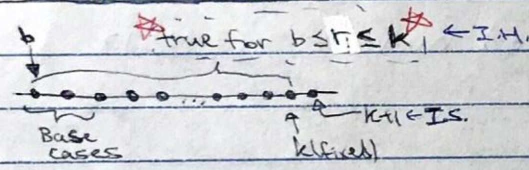
I.S.: Show that for $k+1$, it is true.

we need to show:

$k+1 = 5a' + 4b'$ for some $a', b' \in \mathbb{N}$

Very Important!

7/18 **Strong Induction**: use several base cases to show that everything up to k works/is true



I.S.: Show that the property is true for $k+1$

↳ Postage ≥ 12 with 4c & 5c

Idea: any postage k (property true)

I can make $k+1$ if I subtract 3 from k

ex) $k - 3 + 4c = k+1$

Base cases: Postage

	12	13	14	15
Stamps 4c	3	2	1	0
5c	0	1	2	3

I.H.: Assume that for a fixed $12 \leq k$, we can generate the $\forall\ 12 \leq r \leq k$ using 4c stamps and 5c stamps.

I.S.: We need to show that we can generate postage of $k+1$ using 4c and 5c.

2 cases

Cases: 1) $k+1 \leq 15$

$\rightarrow$ $k+1$ is one of base cases $\square$

2) $k+1 > 15$

$\rightarrow$ $k+1 \geq 16$

Subtract 4 on both sides

$k+1-4 \geq 16-4$

$\boxed{k-3 \geq 12}$ $\quad k-3 \leq k$ $\quad$ from I.H.

$\hookrightarrow$ $12 \leq k-3 \leq k$

$\rightarrow$ $k-3$ satisfies I.H.

$\rightarrow$ $k-3$ can be made with 4c and 5c stamps.

we can make $k+1$ just by adding a 4c stamp $\square$

ex) Polygons: (Simple and closed)

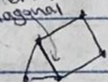
Simple

Polygon divides the plane

1) Inside

2) Outside

Not simple



we can always divide a polygon into triangles.

Property: any polygon with n sides can be divided into $n-2$ triangles.

Property: Any polygon with n sides can be divided into $n-2$ triangles. $(n \geq 3)$

Proof (by Strong induction on n)

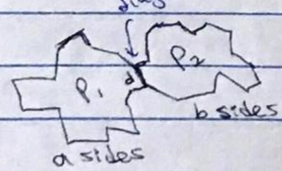
Basis: $n=3$

Δ number of triangles $= 1$

$= n-2$ $\square$

I.H.: given a fixed $k \geq 3$, every polygon with $3 \leq r \leq k$ sides can be divided into $r-2$ sides.

I.S.: we need to show that a polygon with $k+1$ sides can be divided into $k+1-2 = k-1$ triangles.



$k+1 \geq 4$ $\rightarrow$ has at least one interior diagonal

we know $\boxed{a+b = k+1}$

sides of P_1 : $a+1$

sides of P_2 : $b+1$

Since we found an interior diagonal $\rightarrow a \geq 2 \rightarrow a+1 \geq 3$

$b \geq 2 \rightarrow b+1 \geq 3$

clearly $\left.\begin{array}{l} a+1 \leq k \\ b+1 \leq k \end{array}\right\}$

Formal

$\rightarrow$ Let P be a polygon with $k+1$ sides

we know that $k+1 \geq 4$

$\rightarrow \exists$ at least one interior diagonal d

$\rightarrow P$ is divided into two polygons by d

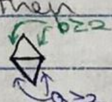
P_1 has $a+1$ sides

P_2 has $b+1$ sides

and $a+b = k+1$

Since d is an interior diagonal then

$a \geq 2$ and $b \geq 2$



*has at least 2 sides not including diagonal

$\rightarrow a+1 \geq 3$ and $b+1 \geq 3$

we can also see that

$a+1 \leq k$ and $b+1 \leq k$

Satisfy the conditions of the I.H.

Because of I.H.

$\left\{\begin{array}{l} P_1 \text{ can be triangulated with } a+1-2 = a-1 \text{ triangles} \\ P_2 \text{ " " " " } b+1-2 = b-1 \text{ "} \end{array}\right.$

$\rightarrow P$ can be triangulated with

$(a-1) + (b-1)$ triangles.

$a+b-2$ triangles

$\rightarrow k+1-2$ triangles. Since we know $a+b = k+1$

$\rightarrow k-1$ triangles $\square$

Sets (naive set theory):

What is a set? • Collection of elements (no restriction of elements)
↳ includes sets

Define a set:

Infinite / Finite

- Provide a list of all the elements
- Provide a property that all elements must satisfy $\{x \mid P(x)\}$ ← Domain
  ↳ sets can be infinite

ex) $X = \{1, 3, 9\}$ (list) $= \{n \in \mathbb{N} \mid n \text{ is odd} \wedge n \le 9 \wedge n \ne 7 \wedge n \ne 5\}$

$Y = \{n \in \mathbb{N} \mid n \text{ is even}\}$

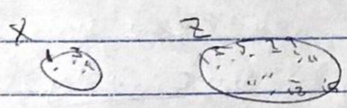
- Universal set U all the elements in the domain.
- $\emptyset = \{\}$ empty set
- a is an element of set A
  $a \in A$
- a is not an element of set A
  $\neg(a \in A) \equiv a \notin A$
- $A \subseteq B$ (A is a subset of B)
  $\leftrightarrow \forall x \in A \rightarrow x \in B$

ex) $3 \in X \rightarrow$ true

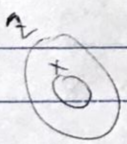
$3 \in Y \rightarrow$ false (3 does not satisfy the cond.)

Venn Diagrams

$Z = \{x \in \mathbb{N} \mid x \text{ is odd}\}$



Can we prove that $X \subseteq Z$?



Proof

$1 \in X$, 1 is odd $\rightarrow 1 \in Z$
$3 \in X$, 3 " " $\rightarrow 3 \in Z$
$9 \in X$, 9 " " $\rightarrow 9 \in Z$

$\therefore \forall a \in X \rightarrow a \in Z \rightarrow X \subseteq Z$

- $A \subset B$ A is a proper subset of B
  $\leftrightarrow \forall x \in A \rightarrow x \in B \wedge A \ne B$

Is it true $A \subseteq A$?
Yes: $\forall a \in A \rightarrow a \in A$

must prove that $A \neq B$!

ex) Is it true that $y \subset \mathbb{N}$?

True: Let $x \in y$

$\rightarrow x \in \mathbb{N} \wedge x$ is even

$\rightarrow x \in \mathbb{N}$

To show that $A \neq B$, we can either

a) show $a \in A$ but $a \notin B$

or b) show $a \in B$ but $a \notin A$

to show that $y \neq \mathbb{N}$, we find an element

$a \in \mathbb{N}$ s.t. $a \notin y$

$\rightarrow a = 3$ $a \in \mathbb{N}$

$a \notin y$ because 3 is not even.

$\boxed{\therefore y \neq \mathbb{N}}$

$\rightarrow y \subset \mathbb{N}$

must show both sides

Def.

- $A = B \longleftrightarrow \underline{A \subseteq B \wedge B \subseteq A}$
- $\nsubseteq$, $\neq$, $\not\subset$

ex) Show that $\emptyset \subseteq S$ ($\forall$ sets S)

(every element in $\emptyset$ is in S)

vacuous proof

we need to show that $\forall a \in \emptyset \rightarrow a \in S$.

To show that the property is false, we need an element exists, then the property is vacuously true.

$* \neg(\emptyset \subseteq S) = \emptyset \nsubseteq S$

ex) Show that $\emptyset \not\subset S$ is true

$\rightarrow$ we need an element in S s.t. it is not in $\emptyset$

$\rightarrow$ any element in S will do as long as $S \neq \emptyset$

ex) $\{2\} \in A$

Trick Questions

1) $A = \{x \in \mathbb{R} \mid x$ is an integer $> 1\}$

$\{2, 3, 4, \cdots\}$

$(2 \in A \neq \{2\} \in A)$ $\boxed{\therefore \text{False!}}$

2) $A = \{2, \{2\}\}$ True

3) $A = \{\{\{2\}\}\}$ False

$\{\{2\}\} \in A$

Tricky!

Study this!

4) $\emptyset \in \{\emptyset\}$? True

5) $\emptyset \in \{\{\}\}$ True $(\emptyset = \{\})$

6) $\emptyset \subseteq \{0\}$ True

7) $\{0\} \in \{0\}$ False

8) $\emptyset \in \{x\}$ False

9) $\emptyset \in \{\{\}, x\}$ True

Cardinality of a set: $|A|$

number of elements

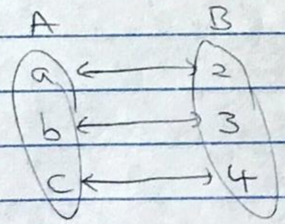
Theorem:

$A \subseteq B \rightarrow |A| \leq |B|$

$A \subset B \rightarrow |A| < |B|$ ... false

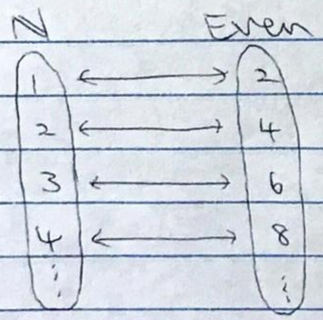
Two sets have the $\boxed{\text{same cardinality } |A| = |B|}$ if there is a one-to-one matching of A to B.

ex) $A = \{a, b, c\}$, $B = \{2, 3, 4\}$



$|A| = 3$

$|B| = 3$



Even $\subset \mathbb{N}$

$\longrightarrow |\mathbb{N}| = |\text{Even}|$

* we extend = to the case of ∞ sets.

ex) $S = \{S, 3, \{1, 8\}\}$

1) $S \in S$? Yes!

Classify Sets:

- Normal Sets: A is a normal set if $A \notin A$

(ex) S is not normal set)

ex) Set of all normal sets

$N = \{A \mid A \notin A\}$

we already said $S \notin N$ ← normal set

$\{a,b,c\} \in N$

$\mathbb{R} \in N$

know this example

Question: $N \in N$?

1) Assume true $N \in N \to N \notin N$ false

2) Assume false $N \notin N \to \neg(N \notin N) = N \in N$ true

$\Rightarrow$ Paradox (Proposition true & false at same time)

know this

07/20 Recap:

$A \subseteq B$	$a \in A$
$A \subset B$	$A = \{x \mid P(x)\}$
$A = B$	$a \in A,\ P(a)$

Normal sets:

$N = \{A \mid A \notin A\}$

N is normal

$C = \{C, 1, 2, 3, \{1,3\}\}$ C is not normal

$N \in N$?

Assume $N \in N \to N \notin N$

Assume $N \notin N \to \neg(N \notin N) \to N \in N$

Does not satisfy the condition

↳ Russell's Paradox

- Peano – formalize $\mathbb{N}$ (natural numbers)

know this