ex) $\exists y \forall x (x^2 = y)$

↳ False

ex) $y = 5$

$\forall x (x^2 = 5) \leftarrow$ False

⚠ Order is Important!

07/06 ☆ Negation of quantifiers:

$p \to q \equiv \neg p \vee q$

ex) Negation of $(L = \lim_{x \to x_0} f(x))$

$$\neg(\forall \varepsilon > 0 \ \exists \delta > 0 \ (0 < |x - x_0| < \delta \to |f(x) - L| < \varepsilon))$$

$$\equiv \exists \varepsilon > 0 \ \forall \delta > 0 \ \neg (0 < |x - x_0| < \delta \to |f(x) - L| < \varepsilon)$$

$$\equiv \exists \varepsilon > 0 \ \forall \delta > 0 \ (0 < |x - x_0| < \delta \wedge \neg(|f(x) - L| < \varepsilon))$$

$$\equiv \exists \varepsilon > 0 \ \forall \delta > 0 \ (0 < |x - x_0| < \delta \wedge |f(x) - L| \geq \varepsilon)$$

$\neg(p \to q) \equiv p \wedge \neg q$ (can be solved logically)

ex) $\neg \exists x \forall y \exists z (x^2 + y^2 = z^2)$

$\forall x \exists y \forall z (x^2 + y^2 \neq z^2)$

Properties:

- $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

✗ $\forall x (P(x) \vee Q(x)) \not\equiv \forall x P(x) \vee \forall x Q(x)$

↖ This is not true! because

$\forall x (\underbrace{x \leq 4}_{P(x)} \vee \underbrace{x > 4}_{Q(x)})$ ... True

$\neq$

$\forall x \underbrace{(x \leq 4)}_{\text{False}} \vee \forall x \underbrace{(x > 4)}_{\text{False}}$ ... FALSE!

Proof methods:

ex) $p \to q$

p — assume p is True

$\therefore q$

modus ponens (prove)

ex) $p \to q$

$\neg q$

$\therefore \neg p$

modus tollens

such that

$\left\{ \begin{array}{l} 1 \in \mathbb{N} \\ \text{if } m \in \mathbb{N} \\ \rightarrow m+1 \in \mathbb{N} \end{array} \right\}$ natural numbers

1. Direct Proof:

Definition

n is even $\leftrightarrow$ 1) $n \in \mathbb{N}$
2) $\exists k \in \mathbb{N}$ s.t. $n = 2k$

ex) Show that if n is even then n^2 = even

n is even $\rightarrow n^2$ is even

Assume n is even $\rightarrow \exists k \in \mathbb{N}$ st $n = 2k$

$n^2 = (2k)^2 = 4k^2 = 2\underbrace{(2k^2)}$ → Axiom of integers (Natural numbers) closed under product

→ Product of two ints are int.

$2k^2$ is a natural number.

Let $r = 2k^2$, $r \in \mathbb{N}$

Use the definition other way around

$\rightarrow \boxed{n^2 = 2r}$

$\therefore n^2$ is even

ex) The product of two perfect squares is a perfect square.

Definition

n is a perfect square
$\leftrightarrow$ i) $n \in \mathbb{N}$
ii) $\exists k \in \mathbb{N}$ s.t. $n = k^2$

If r and s are perfect squares
$\rightarrow rs$ is a perfect square.

Let r and s be perfect squares

$\rightarrow \exists k_1 \in \mathbb{N}$ s.t. $r = k_1^2$ and $\exists k_2 \in \mathbb{N}$ s.t. $s = k_2^2$

$\rightarrow rs = (k_1)^2 (k_2)^2$ ← We know $x^2 \cdot y^2 = (xy)^2$

$\rightarrow rs = (k_1 k_2)^2$ ← closure of $\mathbb{N}$ under product, $k_1 k_2 \in \mathbb{N}$

Let $t = k_1 k_2 \in \mathbb{N}$

$\rightarrow rs = t^2$ ← Satisfies definition of a perfect square

$\therefore rs$ is a perfect square $\square$ ← end of proof

Hard to directly do it

2. Contraposition

$p \rightarrow q$

$\neg q \rightarrow \neg p$

ex) n^2 is even $\rightarrow n$ is even (Proof by Contraposition)

- We will show that n is not even $\rightarrow n^2$ is not even

n is not even $\rightarrow n^2$ is not even

n is odd $\rightarrow n^2$ is odd

$\rightarrow \exists k \in \mathbb{N} \cup \{0\}$ s.t. $n = 2k+1$

$\rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Let $r = 2k^2 + 2k \in \mathbb{N} \rightarrow n^2 = 2r + 1$

$\boxed{\therefore n^2 \text{ is odd} \square}$

odd

n is odd

can include 0 as odd int.

$\leftrightarrow$ i) $n \in \mathbb{N} \cup \{0\}$
ii) $\exists k \in \mathbb{N} \cup \{0\}$ s.t. $n = 2k+1$

any $n \in \mathbb{N}$ is either even or odd

3. Contradiction

ex) $p \to q$

assume $\neg q$ and p

$\to \dots \to$ contradiction to an axiom, a theorem

4. Vacuous Proof

· In proof ... $\forall x P(x)$

If the domain of x is empty,

then $P(x)$ is vacuously true.

(no counter example)

ex) $3n+2$ is odd $\to$ n is odd

Direct { ① $3n+2 = 2k+1$ ✗ Complicated

$3n = 2k-1$ }

② n is even $\to$ $3n+2$ is even

Assume n is even, $n = 2k$

$\to 3n+2 = 3(2k)+2$

$= 2\underbrace{(3k+1)}_{\in \mathbb{N}}$

Contra-position (much simpler)

Let $r = 3k+1 \in \mathbb{N}$

$\to 3n+2 = 2r$

$\to 3n+2$ is even $\square$

ex) $n = ab \to a \le \sqrt{n} \lor b \le \sqrt{n}$

Contraposition $a > \sqrt{n} \land b > \sqrt{n} \to n \ne ab$

Assume $a > \sqrt{n} \land b > \sqrt{n}$

$\to ab > \sqrt{n}\sqrt{n}$

$\to ab > n$

$\to n \ne ab$

assume $(n>0, a>0, b>0)$

$0 < x < y$

$0 < z < w$

$xz < yw$

ex) The sum of two rational numbers is rational.

Definition
x is rational $\nearrow x \in \mathbb{Q}$
$\leftrightarrow \exists p, q$ integers $\nearrow \mathbb{Z}$
s.t. $x = \frac{p}{q}$ $(q \neq 0)$

If x, y are rational numbers

$\rightarrow x+y$ is rational.

Let x, y be two rational numbers

$\rightarrow x = \frac{p_1}{q_1}, \; y = \frac{p_2}{q_2}, \quad p_1, p_2, q_1, q_2$ are int.

$\rightarrow x+y = \frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$ = from closure of $\mathbb{Z}$ under product

$\rightarrow p_3 = p_1 q_2 + p_2 q_1$ is an integer.

$q_3 = q_1 \cdot q_2$ is an integer

$\rightarrow x+y = \frac{p_3}{q_3}$ $\quad p_3, q_3$ are integers

$\therefore x+y$ is rational $\square$

ex) $\sqrt{2}$ is not rational (cannot be written as fraction)

Assume that $\sqrt{2}$ is rational

$\rightarrow \sqrt{2} = \frac{p}{q}$ ... p and q have no common factors

$\rightarrow (p)^2 = (\sqrt{2} q)^2$

$\rightarrow p^2 = 2q^2$ $\quad \begin{cases} p, q \text{ are int.} \\ q^2 \text{ is an int.} \end{cases}$

$\rightarrow p^2$ is even

$\rightarrow p$ is even

$\exists k \in \mathbb{N}$ s.t.

$\rightarrow p = 2k$

$\rightarrow p^2 = 2q^2$

$(2k)^2 = 2q^2$

$\rightarrow 4k^2 = 2q^2$

$\rightarrow 2k^2 = q^2$ $\quad \begin{cases} k \text{ is an int.} \\ k^2 \text{ is an int.} \end{cases}$

$\rightarrow q^2$ is even.

$\rightarrow q$ is even. ∇ contradiction

$\therefore \sqrt{2}$ is not rational.

Proof by cases:

[there are two non-rational numbers x, y such that x^y is rational.

Idea - $(\sqrt{2})^2 = (\sqrt{2})^{\sqrt{2}\cdot\sqrt{2}} = \left(\underbrace{(\sqrt{2})^{\sqrt{2}}}_{x}\right)^{\overbrace{\sqrt{2}}^{y}} = 2$

non-rational $x = (\sqrt{2})^{\sqrt{2}}$

non-rational $y = \sqrt{2}$

s.t. $x^y = 2$ rational

Cases:

1) $x = (\sqrt{2})^{\sqrt{2}}$, $y = \sqrt{2}$

   x is not rational

   $\rightarrow x^y = 2$ is rational

can be rational or irrational

2) $(\sqrt{2})^{\sqrt{2}}$ is rational

   $x = \sqrt{2}$, $y = \sqrt{2}$

   x^y = rational

exam 1

07/11 Proof Methods:

Existence Proof ($\exists x P(x)$) by construction

ex) Given any $n \in \mathbb{N}$

   $\exists\, m > n, n \in \mathbb{N}$ s.t. m is prime

Definition

$n \in \mathbb{N}$ is prime

$\leftrightarrow$ n is divisible only by n and 1

remainder $= 0$

Proof (by construction)

Let $n \in \mathbb{N}$

let $p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 5, p_5 = 7 \cdots p_r = n$

be the r prime numbers $\leq n$

(ex) $n = 7$; $p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 5, p_5 = 7$)

Let $P = p_1 p_2 p_3 p_4 \cdots p_5 + 1$

   if p is divided by p_k $\quad 1 \leq k \leq r$

(ex. $P = 1 \cdot 2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$ $\rightarrow$ $2\overline{)211}$ 1 $\quad 3\overline{)211}$ 1 $\quad 5\overline{)211}$ 1 $\quad 7\overline{)211}$ 1

$\rightarrow$ the remainder is 1

$\rightarrow$ P is not divisible by $\cancel{p_1}, p_2, p_3 \cdots p_r = n$