✩ Study examples

Example definition

$a \equiv b \bmod m \leftrightarrow a \bmod m = b \bmod m$

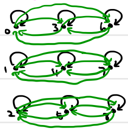
Congruent

$\mathbb{N} \cup \{0\}$

$R = \{(a,b) \mid \underbrace{a \equiv b \bmod 3}_{a \bmod 3 = b \bmod 3}\}$

$\hookrightarrow A = \{0,1,2,3,4,5,6,7,8\}$

Remainder | Connected
0 ...
1 ...
2 ...



→ Looks as if Equivalence Relations partition a set.

Ex) Show that $R = \{(x,y) \mid x,y \in \mathbb{R} \wedge |x-y| < 1\}$ is not an equivalence relation.

- Reflexive: $(x,x) \in R$? $|x-x| = 0 < 1$ ✓
- Symmetric: $(x,y) \in R \rightarrow |x-y| < 1$

$\rightarrow |y-x| < 1$

$\rightarrow (y,x) \in R$

Yes, it is Symmetric. ✓

- Transitive?

Let $x = 0$, $y = 0.8$, $z = 1.6$

$|x-y| = 0.8 < 1 \rightarrow (x,y) \in R$

$|y-z| = 0.8 < 1 \rightarrow (y,z) \in R$

However, $|x-z| = 1.6$ not $< 1 \rightarrow (x,z) \notin R$ ✗

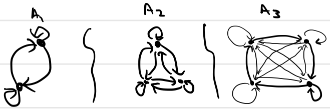
∴ Not eq. relation.

08/03 #Recap

Equivalence Relations

- Reflexive
- Symmetric
- Transitive

✩ 3 prop. of equivalence Relations



$R \subseteq A \times A$

Partition of a Set A

$= A_1, A_2, A_3, \cdots A_n$

s.t. 1) $\bigcup_{i=1}^{n} A_i = A$

2) If $A_i \neq A_j \rightarrow A_i \cap A_j = \emptyset$

$A_1 \cup A_2 \cup A_3 \cdots A_n = A$

A

$\leftarrow$ No overlap

ex) $A = \{1,2,3,4,5\}$

$A_1 = \{1,2\}$, $A_2 = \{3,4\}$, $A_3 = \{5\}$

$\rightarrow$ This is a partition of A

Equivalence classes of $R \subseteq A \times A$

Let $a \in A$

$[a]_R$ is the equivalence class of a with respect to R

$\rightarrow [a]_R = \{x \in A \mid (a,x) \in R\}$

ex) $A = \{0,1,2,3,4,5\}$

$R = \{(a,b) \mid a \equiv b \bmod 3\}$

Find the equivalence classes.

$[0] = \{0,3\}$

$(0,x) \in R \rightarrow \frac{0}{3}$ remainder $= 0$

$[1] = \{1,4\}$

$(1,x) \in R \rightarrow \frac{1}{3}$ remainder $= 1$

$[2] = \{2,5\}$

$$\left\{ \begin{array}{l} [0] = [3] \\ [1] = [4] \\ [2] = [5] \end{array} \right.$$

Equivalence Classes

$A_1 = \{0,3\}$, $A_2 = \{1,4\}$, $A_3 = \{2,5\}$

↳ Is this a partition?

$A_1 \cup A_2 \cup A_3 = A$

$A_i \neq A_j \rightarrow A_i \cap A_j = \emptyset$

$\therefore$ Yes, it is a partition.

Goal: to prove that Equivalence Classes are a partition

Prove that the following are $\equiv$

$(a,b) \in R \leftrightarrow [a]=[b] \leftrightarrow [a] \cap [b] \neq \emptyset$

Proof

R is an Equivalence Relation.

- $P_1 \leftrightarrow P_2 \leftrightarrow P_3$

4 proofs

- $P_1 \to P_2$, $P_2 \to P_3$, $P_3 \to P_1$ — 3 proofs ✓

1) $(a,b) \in R \to [a]=[b]$

Assume $(a,b) \in R$

We need to show i) $[a] \subseteq [b]$

$\wedge$ ii) $[b] \subseteq [a]$

i) Let $x \in [a] \to (a,x) \in R \wedge (a,b) \in R$

Symmetric $\to (b,a) \in R \wedge (a,x) \in R$

Transitive $\to (b,x) \in R$

Def. $\to x \in [b]$ //

ii) Let $x \in [b] \to (b,x) \in R \wedge (a,b) \in R$

$\to (a,b) \in R \wedge (b,x) \in R$

Transitive $\to (a,x) \in R$

Def. $\to x \in [a]$ //

*Need $x \in [a]$, $(a,x) \in R$

2) $[a]=[b] \to [a] \cap [b] \neq \emptyset$

$\forall a \in A,\ a \in [A]$ (Reflexive) $\therefore [a] \neq \emptyset$

Let $[a]=[b]$,

Notice that $[a] \cap [b] = [a] \cap [a] = [a] \neq \emptyset$

3) $[a] \cap [b] \neq 0 \to (a,b) \in R$

Assume that $[a] \cap [b] \neq \emptyset \to \exists\, c \in A$ s.t. $c \in [a] \wedge c \in [b]$

$\to (a,c) \in R \wedge (b,c) \in R$

Symmetric $\to (a,c) \in R \wedge (c,b) \in R$

Transitive $\to (a,b) \in R$ //

To show that equivalence classes Partition A

i) $\bigcup_{a \in A} [a] = A$

ii) $[a] \neq [b] \rightarrow [a] \cap [b] = \emptyset$

i) Since $\forall a \in A$ $a \in [a]$

$\rightarrow \bigcup_{a \in A} [A] = A$

ii) $[a] \neq [b] \rightarrow [a] \cap [b] = \emptyset$

We were able to show $\underbrace{[a] = [b] \leftrightarrow [a] \cap [b] \neq \emptyset}$

Using contrapositive of

$\neg([a] = [b]) \rightarrow \neg([a] \cap [b] \neq \emptyset)$

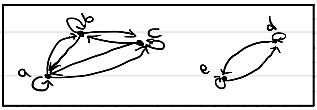
$[a] \neq [b] \rightarrow [a] \cap [b] = \emptyset$

☆ If R is an equivalence relation on a set A then R partitions A into equivalence classes.

☆ Study this ex

ex) a) Find the <u>smallest</u> Equivalence Relation on $A = \{a, b, c, d, e, f\}$ containing $R = \{(a,b), (a,c), (d,e)\}$

b) What are its equivalence classes?

a) 

← smallest R_1 such that $R \subseteq R_1$ and R is transitive

b) Equivalence Classes

$A_1 = \{a, b, c\}$

$A_2 = \{d, e\}$

✡ Very Important!

Partial order:

$(A, \le)$

$R_\le \leftarrow \begin{cases} \text{Reflexive} \\ \text{Antisymmetric} \\ \text{Transitive} \end{cases}$

$(a,b) \in R \wedge (b,a) \in R$
$\rightarrow a = b$

Poset (A is partial ordered set)

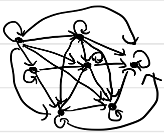
ex)

✓ Reflexive
✓ Antisymmetric
✗ Transitive

Cannot be

* The problem is the cycle.

↳ Partial Order Relations have no cycles



is a partial order.

ex) Show that $(\mathbb{Z}, \ge)$ is a poset

i) Reflexive : $\forall a \in \mathbb{Z},\ a \ge a$ ✓

ii) Antisymmetric : $a \ge b \wedge b \ge a \rightarrow a = b$

$a \ge b \ge a$

iii) Transitive : $a \ge b,\ b \ge c \rightarrow a \ge c$? Yes ✓

✡

ex) $A = \{1, 2, 3\}$

$(2^A, \subseteq)$

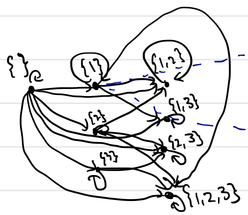
i) Reflexive : $a \in 2^A \rightarrow a \subseteq a$

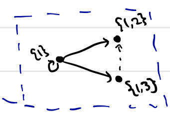
ii) Antisymmetric : $a, b \in 2^A$

$a \subseteq b \wedge b \subseteq a \rightarrow a = b$

iii) Transitive : $a \subseteq b \wedge b \subseteq c \rightarrow a \subseteq c$

→ Is a poset.





{1,2}, {1,3} are not comparable. ✡

* Two elements $a, b \in S$ are ✡ comparable $\longleftrightarrow a \le b \wedge b \le a$

Total Order:

$(A, \leq)$, $\forall a,b \in A$ * a & b are comparable.

poset

* Total Order $\subseteq$ Partial Ordered

$(\mathbb{Z}, \geq)$ is a total order

$a,b \in \mathbb{Z}$, either $a \geq b$ or $b \geq a$

We say that a is less than or equal to b if $a \leq b$

The least element of A is $s \in A$ s.t. $\forall a \in A \rightarrow s \leq a$

What is the least element of $\mathbb{Z}$ in $(\mathbb{Z}, \leq)$

Least $s \in \mathbb{Z}$ s.t. $\forall a \in \mathbb{Z}$, $s \leq a$?

-10000
-10001

Not all sets have a least number.

$\Rightarrow$ Not a Well-Ordered Set

Well Order:

$(A, \leq)$

① $(A, \leq)$ is a total order.

② $\forall S \subseteq A$ s.t. $S \neq \emptyset$

S has a least element. → ex) $\mathbb{Z}^+$ (not $\mathbb{Z}$, $\mathbb{R}$)

smallest value

no smallest element

→ Induction works because

- Basis – (first element)

ex) Is $(\mathbb{N}, \leq)$ a well ordered set?

1) Show that it is a total order

i) Poset (Ref, antisym., transitive)

a) $a \in \mathbb{N} \rightarrow a \leq a$ ✓

b) $a \leq b \wedge b \leq a \rightarrow a = b$ ✓

c) $a \leq b \wedge b \leq c \rightarrow a \leq c$ ✓

ii) Total Order

$\forall a, b,\ a \leq b$ or $b \leq a$ ✓

2) $\forall S \subseteq \mathbb{N}$, S has a least element

Let $x = \min(S)$

$x \in S$ and $x \leq a$

$\forall a \in S$

Induction:

$P(x)$ is true $\forall x \in A \longleftrightarrow P(x)$ is true $\forall \underbrace{x < y}$ $\rightarrow P(y)$ is true

$x \leq y$ but $x \neq y$

* Notice A has a least element (Since A is well ordered)

↳ $x_0 \in A$ is this element

Lexicographical Order (on String) [restrict to Strings of length 2]

$x = x_1 x_2$ | $x = ab$ | ex)

$y = y_1 y_2$ | $y = aX$ |

$\underline{a}\ \underline{d}$

$x\ z$) $\leq$

$\underline{b}\ \underline{s}$

$d\ x$

$x_1 x_2 \leq y_1 y_2$

If $x_1 < y_1$ or ($x_1 = y_1$ and $x_2 \leq y_2$)

Strings of length 2

$x_1 x_2 \leq y_1 y_2 \longleftrightarrow (x_1 < y_1) \vee (x_1 = y_1 \wedge x_2 \leq y_2)$

Show that
- Reflexive
- Antisymmetric
- Transitive

☆ Try this ex

Reflexive:

We need to show $x_1x_2 \leq x_1x_2$

$$\underbrace{\underbrace{(x_1 < x_1)}_{F} \vee (\underbrace{\underbrace{x_1 = x_1}_{T} \wedge \underbrace{x_2 \leq x_2}_{T}}_{T})}$$

$= T$ //

Antisymmetric:

we need to show that

if $\underbrace{x_1x_2 \leq y_1y_2}$ $\wedge$ $\underbrace{y_1y_2 \leq x_1x_2}$ $\rightarrow x_1x_2 = y_1y_2$

Proof

$((x_1 < y_1) \vee (x_1 = y_1 \wedge x_2 \leq y_2)) \wedge ((y_1 < x_1) \vee (y_1 = x_1 \wedge y_2 \leq x_2))$

$\equiv \underbrace{((x_1 < y_1) \wedge (y_1 < x_1))}_{F} \vee \underbrace{((x_1 < y_1) \wedge (y_1 = x_1) \wedge (y_2 \leq x_2))}_{F} \vee$

$\underbrace{((x_1 = y_1) \wedge (x_2 \leq y_2) \wedge (y_1 < x_1))}_{F} \vee \underbrace{((x_1 = y_1) \wedge (x_2 \leq y_2) \wedge (y_1 = x_1 \wedge y_2 \leq x_2))}_{T}$

$\rightarrow (x_1 = y_1) \wedge (x_2 = y_2)$

$\rightarrow x_1x_2 = y_1y_2$ //

Transitive

DIY