07/13 **Proofs by Induction:**

$\forall n \in \mathbb{N},\ n \geq b,\ P(n)$

- Basis: show $P(b)$
- Inductive hypothesis: for a fixed $k \in \mathbb{N}$, $k \geq b$
  Assume $P(k)$
- Inductive step: show $P(k+1)$

Important! *read the note ch8

- ALWAYS LABEL EACH STEP!
- ALWAYS tell the variable used
  Proof (By induction on n)
- Always say "fixed" on I.H.

Ex) The sum of the first n odd integers is a perfect square.

$1 = 1 = (1)^2$ $\qquad n \geq 1$

$1+3 = 4 = (2)^2$

$1+3+5 = 9 = (3)^2$

$1+3+5+7 = 16 = (4)^2$

$\vdots$

first odd number $\rightarrow 2i-1$ $(i=1)$

Second odd number $\rightarrow 2i-1$ $(i=2)$

$\vdots$

ith oddnumber $\rightarrow \boxed{2i-1}$

$n \in \mathbb{N}$
$\forall n \geq 1 \left(\sum_{i=1}^{n} (2i-1) = n^2 \right) \leftarrow$ what we have to prove

Proof (by induction on n):

Basis: $(n=1)$

$\underbrace{\sum_{i=1}^{1} (2i-1) = 2(1)-1}_{LHS} = 1 = \underbrace{1^2}_{RHS}$ $\square$

Inductive Hypothesis:

Let $k \in \mathbb{N}$, $k \geq 1$ be a fixed number.

Assume that $\sum_{i=1}^{k} (2i-1) = k^2$

Inductive step:

We need to show that

$$\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$$

from LHS

$$\sum_{i=1}^{k+1} (2i-1) = \underbrace{\sum_{i=1}^{k} (2i-1)} + 2(k+1)-1$$

using I.H. $= k^2 + 2k+2-1$

$= k^2+2k+1$

$= (k+1)^2$

ex.2) $\forall n \in \mathbb{N}$, $n \geq 0$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

if $n=3$ then

$2^0 + 2^1 + 2^2 + 2^3 = 2^{3+1} - 1$

$1+2+4+8 = 16-1$

$15 = 15$

Proof (by induction on n)

Basis: $n=0$

LHS $2^0 = 1$

RHS $2^1 - 1 = 2-1 = 1$ $\square$

Inductive Hypothesis:

Let k be fixed s.t. $k \in \mathbb{N}$ and $k \geq 0$

Assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

Inductive Step:

We want to show $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$

$= 2^{(k+2)} - 1$

LHS: $\sum_{i=0}^{k+1} 2^i = \underbrace{\sum_{i=0}^{k} 2^i}_{} + 2^{k+1}$

$\overset{I.H.}{=} 2^{k+1} - 1 + 2^{k+1}$

$= 2 \cdot (2^{k+1}) - 1$

$= 2^{k+2} - 1$ □

Ex.3) $\sum_{i=0}^{n} ar^i = \frac{ar^{n+1} - a}{r-1}$ $r \neq 1,\ n \geq 0,\ n \in \mathbb{N}$

Proof (by induction on n)

Basis: $n=0$

LHS: $ar^0 = a$

RHS: $\frac{ar^1 - a}{r-1} = \frac{a\cancel{(r-1)}}{\cancel{r-1}} = a$ □

Inductive Hypothesis

Let $k \in \mathbb{N}$ be a fixed number s.t. $k \geq 0$

Assume

$$\sum_{i=0}^{k} ar^i = \frac{ar^{k+1} - a}{r-1}$$

Inductive step:

We need to show that

$$\sum_{i=1}^{k+1} ar^i = \frac{ar^{(k+1)+1} - a}{r-1} = \boxed{\frac{ar^{k+2} - a}{r-1}}$$

LHS: $\sum_{i=1}^{k+1} ar^i = \underbrace{\sum_{i=1}^{k} ar^i}_{} + ar^{k+1}$

$\overset{I.H.}{=} \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$

$= \frac{ar^{k+1} - a + (ar^{k+1})(r-1)}{r-1}$

$= \frac{\cancel{ar^{k+1}} - a + ar^{k+2} - \cancel{ar^{k+1}}}{r-1}$

$= \boxed{\frac{ar^{k+2} - a}{r-1}}$ □

Ex.4) $2^n < n!$ $\forall n \in \mathbb{N}$ s.t. $n \geq 4$

Proof (by induction on n)

Basis ($n=4$)

LHS $2^4 = 16$

RHS $4! = 24$

$\rangle$ ~~$16 < 24$~~ $\square$

Inductive Hypothesis:

Let k be fixed int s.t. $k \in \mathbb{N}$ and $k \geq 4$

Assume $2^k < k!$

Inductive Step:

We have to show that

$2^{k+1} < (k+1)!$

from LHS $\rightarrow$ works fine with summation

$2^{k+1} \ldots ?$

We know $2^k < k!$

$2 < k+1$

$\rangle$ = ~~$2^{k+1} < (k+1)!$~~

$2(2^k) < k!(k+1)$

this is true,

b/c $k \geq 4$

$\rightarrow 2 \cdot 2^k < (k+1)k!$

$\rightarrow \boxed{2^{k+1} < (k+1)!}$ $\square$

Ex.5) $n^3 - n$ is divisible by 3 ($n \in \mathbb{N}$, $n \geq 2$)

$\exists s \in \mathbb{N}$ s.t. $n^3 - n = 3s$

Basis ($n=2$)

LHS $2^3 - 2 = 6$ divisible by 3

$= 3(2)$

Inductive Hypothesis

For a fixed $k \in \mathbb{N}$, $k \geq 2$

$\exists s \in \mathbb{N}$ s.t. $k^3 - k = 3s$

Inductive Step:

We need to show that

$(k+1)^3 + (k+1) = 3r$ for some $r \in \mathbb{N}$

LHS $(k+1)^3 - (k+1)$

$= k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$

$= \underbrace{k^3 - k} + 3k^2 + 3k$

I.H. $= 3s + 3k^2 + 3k$

$= 3(\underbrace{s + k^2 + k}_{r})$

$= 3r$ $\square$ where $r = s + k^2 + k \in \mathbb{N}$

Ex. 6) $n+1 \leq 2n \quad \forall n \geq 1,\ n \in \mathbb{N}$

Proof (by induction on n)

Basis: $(n=1)$

$2 \leq 2$

I.H.: Let k be an arbitrary fixed number, $k \in \mathbb{N}, k \geq 1$

Assume $k+1 \leq 2k$ & $1 < 2$

I.S.: We want to show

$(k+1)+1 \leq 2(k+1)$

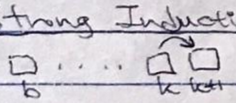
$\boxed{k+2 \leq 2k+2}$

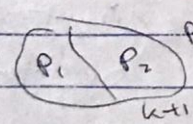
I.H. $\left\{ \begin{array}{l} k+1 \leq 2k \\ 1 < 2 \end{array} \right.$

$k+1+1 \leq 2k+2$

$\boxed{k+2 \leq 2k+2}$ $\square$

Strong Induction



I.S. 

Problem: property true for every possible value $\leq k$

$\underbrace{P_1 \leq k}$ and $\underbrace{P_2 \leq k}$

Property P_1 is true + Property P_2 is true. $\xrightarrow{\text{Combine}}$ Property P is true

Inductive Hypothesis:
$\rightarrow$ given a fixed $k \in \mathbb{N}$, $k \geq b$
Assum the property is true
$\forall\, b \leq r \leq k$

Inductive Step:
Show that the property is true for $k+1$

* Basis: we might need to show for several values

ex1) $\forall n \in \mathbb{N}$, $n > 1 \rightarrow n$ can be written as a product of primes

Proof (by (Strong) induction on n)
Basis: $n=2$ $\quad n=2 \leftarrow$ prime

I.H.: Let k be a fixed number $k \in \mathbb{N}$ and $k > 1$
Assume $\forall\, 1 < r \leq k$ (r can be written as a product of primes)

I.S.: We want to show that
$k+1$ can be written as a product of primes

$k+1 > 2$
$k > 1$
$1 < r \leq k$
$\rightarrow 1 < k \leq k$

case A) $k+1$ is prime
$\rightarrow k+1$ is a product of primes
case B) $k+1$ is not prime (composite)
$\rightarrow k+1 = ab \begin{cases} a \geq 2 \\ b \geq 2 \rightarrow \frac{1}{2} \leq \frac{1}{b} \end{cases}$

$[n+1 \leq 2n] \rightarrow \frac{1}{2}(n+1) \leq n$

$k+1 = ab$
$a = (k+1)\frac{1}{b} \leq (k+1)\frac{1}{2} \leq k$
$\boxed{\therefore a \leq k}$
Same in the case of $b \leq k$
$\rightarrow$ I.H. is true for a and b
$\rightarrow$ a can be written as a product of primes & b can be written as a product of primes

$\rightarrow \therefore k+1 = ab$ can be written as a product of primes.

☆ Study this example + exam 2 #5

Ex. 2) Every postage of ≥ 12 cents can be formed using only 4 cent stamps and 5 cent stamps.

$\boxed{\text{formally, } \forall n \in \mathbb{N},\ n \geq 12 \quad \exists a, b \in \mathbb{N} \text{ s.t. } n = 5a + 4b}$

$\begin{cases} a \text{ is number of 5 cent stamps} \\ b \text{ " " " 4 " "} \end{cases}$

Proof (by induction on n)

Basis: $n = 12$

$n = 4(3)$ $\begin{cases} b = 3 \\ a = 0 \end{cases}$

I.H.: Let $k \in \mathbb{N}$ be a fixed number s.t. $k \geq 12$

Assume: $\exists a, b \in \mathbb{N}$ s.t. $k = 5a + 4b$

I.S.: Need to show that $\boxed{k+1 = 5a' + 4b'}$ for some $a', b' \in \mathbb{N}$

From I.H. we know $\exists a, b$ s.t. $k = 5a + 4b$

$a \to \underbrace{a+1}_{a'}$, $b \to \underbrace{b-1}_{b'}$

case A) $b \geq 1$

$5(a+1) + 4(b-1)$

$= 5a + 4b + 5 - 4$

$= \underbrace{5a + 4b}_{} + 1$

$\text{I.H.} = \boxed{k+1 \quad = 5a' + 4b'}$

case B) $b = 0 \rightarrow$ all stamps must be 5 cent stamps

$k \geq 15$ $\quad [k = 5a + 4b$

$b = 0 \quad a \geq 3]$

↳ Then, replace 3 5 cent stamps with 4 4 cent stamps.

$k + 1 = 5\underset{a-3}{a'} + 4\underset{b+4}{b'}$

$k + 1 = 5(a-3) + 4(b+4)$

$= \underbrace{5a + 4b}_{\text{I.H.}} - 15 + 16 = \boxed{k+1 \rightarrow = 5a' + 4b'}$

Proof (by Strong Induction)

Basis: same as before

I.H.: given a fixed $k \in \mathbb{N}$, $k \geq 12$

$\forall\, 12 \leq r \leq k \rightarrow r = 5a + 4b$ for some $a, b \in \mathbb{N}$

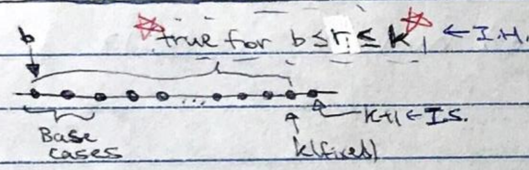
I.S.: Show that for $k+1$, it is true.

we need to show:

$k+1 = 5a' + 4b'$ for some $a', b' \in \mathbb{N}$

Very Important!

7/18 **Strong Induction**: use several base cases to show that everything up to k works/is true



I.S.: Show that the property is true for $k+1$

↳ Postage ≥ 12 with 4c & 5c

Idea: any postage k (property true)

I can make $k+1$ if I subtract 3 from k

ex) $k - 3 + 4c = k+1$

Base cases: Postage

		12	13	14	15
Stamps	4c	3	2	1	0
	5c	0	1	2	3

I.H.: Assume that for a fixed $12 \leq k$, we can generate the $\forall\, 12 \leq r \leq k$ using 4c stamps and 5c stamps.

I.S.: We need to show that we can generate postage of $k+1$ using 4c and 5c.