

## ~~Proofs:~~

- Mathematics is a "game"
  - rules (made up)
    - ↳ Axioms - proposition that we assume to be true
- \* Axiom System: Set of axioms
- Theorem - A proposition that can be proven to be true based on the given axiom system
  - i) Lemma ← theorem used as intermediate step.
  - ii) Corollary ← simple conclusion from a theorem
- Definitions - Proposition = name
  - \* matter of convenience

## ~~Axiom Systems: (Properties)~~

- Consistent - a property is either true or false with respect to the axiom system
- Independent - no axiom is a theorem
  - cannot prove an axiom based on other axioms
- Complete - there are no properties that cannot be proven (T or F)

## Geometry:

- A straight line segment can be drawn joining any two points
- A segment can be extended in a straight line
- Given a segment, we can draw a circle having one end point as the center and the segment as its radius.
- All right angles are congruent.
- If two lines intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines must intersect on that side

ex)  $\text{for less than } 180^\circ$

Concept

Concept

geometries  
Riemann  
Rochowski

## Equivalence of Propositions:

- Proposition  $p(x_1, x_2, \dots, x_n)$
- Proposition  $q(x_1, x_2, \dots, x_n)$
- $p \equiv q$

$p$  has the same value of  $q$  for every possible assignment of the variables  $x_1, \dots, x_n$

- ~~Method~~
- ① Use truth tables
  - OR
  - ② Use properties (logic)

Know how to do both ways

Ex 1. De Morgan (axb)

a)  $\neg(p \wedge q) \equiv \neg p \vee \neg q$

$p$	$q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Same. Proven True

b)  $\neg(p \vee q) \equiv \neg p \wedge \neg q$

$p$	$q$	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

c)  $\neg(\neg p) \equiv p$

$p$	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

# Given but know basics

## Useful Equivalences

### • Identity Laws

$$- p \wedge T \equiv p$$

$$- p \vee F \equiv p$$

### • Domination Laws

$$- p \vee T \equiv T$$

$$- p \wedge F \equiv F$$

### • Idempotent Laws

$$- p \vee p \equiv p$$

$$- p \wedge p \equiv p$$

### • Double Negation

$$- \neg(\neg p) \equiv p$$

### • Commutative

$$- p \vee q \equiv q \vee p$$

$$- p \wedge q \equiv q \wedge p$$

### • Associative

$$- (p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$- (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

### • Distributive

$$- p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$- p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

\*Prove equivalence without truth tables

$$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$$

DeMorgan  $\equiv \neg p \wedge (\neg(\neg p \wedge q))$

DeMorgan  $\equiv \neg p \wedge ((\neg(\neg p)) \vee \neg q)$

Double Neg.  $\equiv \neg p \wedge (p \vee \neg q)$

Distributive  $\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$

$\equiv F \vee (\neg p \wedge \neg q)$

Identity  $\equiv \neg p \wedge \neg q$

### • DeMorgan

$$- \neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$- \neg(p \vee q) \equiv \neg p \wedge \neg q$$

### • Absorption

$$- p \vee (p \wedge q) \equiv p$$

$$- p \wedge (p \vee q) \equiv p$$

$$\therefore (p \vee p) \wedge (p \vee q)$$

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

\* Definition of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

## LECTURE 4

Example (proving equivalences without using truth tables):

Show that  $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

- We start from one side:  $\neg(p \vee (\neg p \wedge q))$
- De Morgan:  $\equiv \neg p \wedge (\neg(\neg p \wedge q))$
- De Morgan again:  $\equiv \neg p \wedge (p \vee \neg q)$
- Distributive:  $\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$
- Negation:  $\equiv F \vee (\neg p \wedge \neg q)$
- Identity:  $\equiv (\neg p \wedge \neg q)$

A tautology is a compound proposition that is always true regardless of the truth assignment of its variables

An example of a tautology is:  $p \vee \neg p$

A contradiction is a compound proposition that is always false regardless of the truth assignment of its variables

An example of a contradiction is:  $p \wedge \neg p$

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### Examples:

Show that  $\neg(p \wedge q) \vee (p \vee q)$  is a tautology.

Proof:

- $\neg(p \wedge q) \vee (p \vee q)$
- De Morgan:  $\equiv (\neg p \vee \neg q) \vee (p \vee q)$
- Associative:  $\equiv (\neg p \vee p) \vee (\neg q \vee q)$
- Negation:  $\equiv T \vee T \equiv T$

### 3.1 The Satisfiability problem

Given a proposition, Is there an assignment of truth variables that will make the proposition true?

Notice that a proposition can be written as a function of several variables:

$$P(q_1, q_2, q_3, q_4, \dots, q_n)$$

So the question is whether there is an assignment to each one of the variables, such as:

$$\begin{array}{rcl} q_1 & = & T \\ q_2 & = & F \\ q_3 & = & F \\ \vdots & & \\ q_n & = & T \end{array}$$

that makes the proposition true?  
Example

In Conjunctive Normal Form (Conjunction of disjunctions):

Determine if the following formula is satisfiable:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

Start, for example, with  $p = T$ .

What do we need to show to prove that a proposition is NOT satisfiable?

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### 3.2 Quantifiers and proofs

#### ~~Propositional functions~~

Their value depends on the values of one or several variables.  
For example, let  $P(x)$  be the propositional function given by  $x > 5$ . The value of this proposition will depend on the value that we give to the variable  $x$ . So  $P(3)$  would be the proposition  $3 > 5$  which is false, therefore  $P(3) = F$ .  $P(7) = T$ , and so on.

It is possible to have propositions that depend on more than one variable, such as  $P(x,y)$  corresponding to the proposition  $x + y > 3$ .

For example:

$$\begin{array}{ll} P(x,y) & x + y > 3 \\ P(5,1) & 5 + 1 > 3 \quad T \\ P(1,1) & 1 + 1 > 3 \quad F \end{array}$$

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#### Quantifiers

- Universal quantifier:  $\forall$  (we read it as "For All")
- Existential quantifier:  $\exists$  (we read it as "Exists")

#### Using quantifiers and propositions

- $\forall x P(x)$  means that for every possible value of  $x$ ,  $P(x)$  is True.
- $\exists x Q(x)$  means that there is at least one value of  $x$  such that  $Q(x)$  is True.

#### Domain:

Notice that  $x$  represents values taken from some domain that might not be explicitly given since in many cases it is clear from the context of the problem.

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#### Example:

Let  $P(x)$  be the statement  $x + 5 > x$ . Is  $\forall x P(x)$  true?

#### Proof.

Let  $x$  be a real number, we know that  $5 > 0$ . Adding the same number to each side of a given inequality does not change the value of the inequality. Therefore we have that:

$$\begin{aligned} \rightarrow & 5 > 0 \\ \rightarrow & \forall z \quad 5 + z > 0 + z \\ \rightarrow & \forall z \quad 5 + z > z \\ \rightarrow & \forall z P(z) \end{aligned}$$

Notice that at each step we write a true statement that is true because the previous statement is true.

#### Example

Let  $P(x)$  be the statement  $x \geq 0$ . Is  $\forall x P(x)$  True?

We should make sure that we are clear with respect to the domain. The answer is not the same if the domain are all Natural numbers or if the domain are all integers, or even all real numbers.

In the case of Natural numbers, the statement is True.

In the case of Real numbers there is at least one counterexample,  $x = -1$  so that  $P(-1)$  is False. Therefore,  $\forall x P(x)$  is False.

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**Example:**

Let the domain of  $x$  consist of all positive integers less than or equal to 4.  
The domain is 1, 2, 3, 4.

Let  $P(x)$  be the statement  $x^2 < 10$ .

$\forall x P(x) = P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

$x$	$x^2$	$x^2 < 10$
1	1	T
2	4	T
3	9	T
4	16	F

So we have that  $\forall x P(x) = T \wedge T \wedge T \wedge F = F$

**Example:**

Let  $P(x)$  be the statement  $x^2 - 1 < 0$

Is  $\exists x P(x)$  True?

If the domain is the set of all Real numbers

We just show that if  $x = 0$  then  $x^2 - 1 < 0$ .

If the domain is the set of all positive integers

Since every positive integer satisfies that  $x \geq 1$ , then  $x^2 > 1$ , which means that  $\forall x (\neg P(x))$ . Therefore  $\exists x P(x)$  is false.

**Example**

Let  $Q(x)$  be the statement  $x > x + 3$ . If the domain of  $x$  is the set of all real numbers, is  $\exists x Q(x)$  True?

The argument is similar to the previous example. Notice that  $0 < 3$  then  $\forall x (x < x + 3)$ , then  $\forall x \neg Q(x)$ , therefore there does not exist any real number  $x$  that can make  $Q(x)$  true, therefore  $\exists x Q(x)$  is false.

**How to state the domain**

In the examples given before, the domain is given by context. It is possible to provide the domain directly in the statement:

$\forall x > 1 P(x)$

or

$\exists x < 0 Q(x)$

However, each one of these propositions can be written in such a way that the domain is included in the proposition inside of the quantifier:

In the case of  $\forall x > 1 P(x)$  we can write it as  $\forall (x > 1) \rightarrow P(x)$

In the case of  $\exists x < 0 Q(x)$  we can write it as  $\exists (x < 0) \rightarrow Q(x)$

HW

Scan and Submit

#### Scope

The scope of the quantifiers is given in a similar way as in the case of programming. We can see a proposition with quantifiers as nested loops each with its own local variables.

#### Example

The proposition  $\forall y > 0 \exists x (x + y = 1)$  can be seen as a program in pseudo code:

```
for every y in the domain of y  
  v=FALSE  
  for every x in the domain of x  
    if x+y=1 then v=True  
    if v=False then  
      return FALSE  
    HALT  
return TRUE
```

This is just an idea to think about scope, it is not a program that would actually work unless the domains were finite.

$$\forall x \exists y P(x, y)$$

$$\exists x \forall y P(x, y)$$

Do not reverse order!

$$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$$

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$$

\* A tautology is a proposition that is always true

(for every assignment of its variables)

ex)  $p \vee \neg p$  ← Always TRUE

\* A contradiction is always false.

ex)  $p \wedge \neg p$  ← Always FALSE

Example: Show that  $\neg(p \wedge q) \vee (p \vee q)$  is a tautology

$$\equiv (\neg p \vee \neg q) \vee (p \vee q) \quad \text{DeMorgan}$$

$$\equiv (\neg p \vee p) \vee (\neg q \vee q) \quad \text{Rearrange}$$

$$\equiv T \vee T$$

$$\equiv T$$

Show  
not contradiction

Ex) Determine if  $(p \vee \neg q) \wedge (\neg q \vee \neg r) \wedge (r \vee \neg p)$  is satisfiable.

if  $p = T$ ,

$(p \vee \neg q) = T \rightarrow (\neg q \vee \neg p)$  has to be T

so  $r = T$ ,

$(\neg q \vee \neg r)$  has to be T

so  $q = F$

All 3 has to be  
true

∴ Satisfiable

Propositional functions:

ex)  $p(x)$  is  $x > 5$ ,  $p(8) \leftarrow$  true

ex)  $p(x, y)$  is  $x < y$ ,  $p(3, 2) \leftarrow$  false  $p(2, 7) \leftarrow$  true

Quantifiers:

• Universal - for all  $\forall$

• Existential - exists  $\exists$

ex)  $\forall x P(x) \leftarrow$  for every value of  $x$  in domain,  $P(x)$  is true.

ex)  $\exists x P(x) \leftarrow$  there exists at least one value of  $x$  such that  $P(x)$  is true

Very  
Important!

Ex) Let  $P(x)$  be  $x+5 > x$ , is  $\forall x P(x)$  true?

$$\begin{array}{c} 5 > 0 \\ \rightarrow x+5 > x \end{array}$$

YES, IT IS TRUE!

Ex) Let  $P(x)$  be  $x \geq 0$ , is  $\forall x P(x)$  true?

$\rightarrow$  no, counterexample:  $x = -1$ ,  $-1 \not\geq 0$ ,  $P(-1)$  is False.

To show that  $\forall x P(x)$  is false  
we need to find an  $x$  such  
that  $P(x)$  is false

Ex) Domain all positive integers  $\leq 4$

Let  $P(x)$  be  $x^2 < 10$ , is  $\forall x P(x)$  true?

$\rightarrow$  False. Counter ex.:  $x=4$ ,  $x^2 = 16 \not< 10$

$P(4)$  is False.

Ex) Let  $P(x)$  be  $x^2 - 1 < 0$ , is  $\exists x P(x)$  true?

$\rightarrow$  True, ex:  $x=0$ ,  $0^2 - 1 < 0$ ,  $P(0)$  = True

↑ Domain: positive Int.  $x \geq 1$

$\rightarrow x^2 - 1 \geq 0 \Rightarrow P(x)$  is False

$\therefore \exists x P(x)$  is False

### Domain

$\forall x > 0 P(x)$

↳ Domain

$\forall (x > 0 \rightarrow P(x))$

$Q(x)$

$\forall x Q(x)$

Know this very well

\* Study Table 1 in pg 60

$\forall x [\exists y (x^2 = y)]$  ← for every  $x$

True

Let  $x \in \mathbb{R}$   
Let  $y = x^2$   
So  $x^2 = y$

True

for every  $y$   
if  $\dots$  is  
return true  
return false

True



**Examples:**

1.  $n$  is even  $\rightarrow n^2$  is even.
2. The product of two perfect squares is a perfect square.
3.  $3n + 2$  is odd  $\rightarrow n$  is odd.
4.  $n = ab \rightarrow a \leq \sqrt{n} \vee b \leq \sqrt{n}$ .
5. The sum of two rational numbers is a rational number.
6.  $n^2$  is even  $\rightarrow n$  is even.
7.  $\sqrt{2}$  is irrational.
8.  $n$  is odd  $\leftrightarrow n^2$  is odd.
9. Show that the following statements are equivalent:
  - (a)  $n$  is even
  - (b)  $n - 1$  is odd
  - (c)  $n^2$  is even
10.  $1 = 2?$
11. Rational + Irrational = Irrational