08/08

Functions:

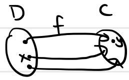
A function is a relation $f \subseteq D \times C$ s.t.

Domain, Codomain

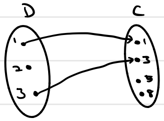
$f: D \to C$

[$\forall x \in D$ $\exists$ a unique $y \in C$ s.t. $y = f(x)$

$[x f y]$

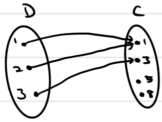


ex 1)



Not a function because $2 \in D$ and not related to elements in codomain.

ex 2)



Yes, it is a function

What do we need to properly define a function?

$f: D \to C$

- We need to give D and C
- We need the mapping

ex) $f(x) = \sqrt{x}$ D and C missing (Not properly defined)

ex) $f(x) = \sqrt{x}$ $f: \mathbb{R} \to \mathbb{R}$ Properly defined but not a function b/c $-1 \in \mathbb{R}$ (Domain) but $f(-1)$ does not exist (in C)

ex) $f(x) = \sqrt{x}$ $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a function

ex) $f(x) = \sqrt{x}$ $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ Tricky
Not a function b/c $0 \in$ Domain but $f(0) = 0 \notin$ Codomain

Operations: $f, g : D \to C$

Domain stay same

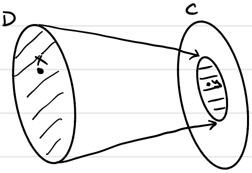
Codomain may change

$f \cdot g\,(x) = f(x) \cdot g(x)$ $f \cdot g : D \to \boxed{}$

$f + g\,(x) = f(x) + g(x)$ $f + g : D \to$

?

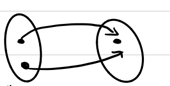
Range of a Function



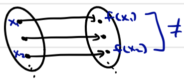
Range $= \{ y \in C \mid y = f(x) \text{ for some } x \in D \}$

$\ast$ Range $\subseteq C$

One-to-One Functions (Injective):



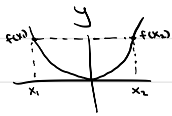
this cannot happen



A function $f: D \to C$ is one-to-one $\longleftrightarrow$ $\forall\, x_1, x_2 \in D$ s.t. $x_1 \neq x_2 \to f(x_1) \neq f(x_2)$

ex1) Is $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ one to one?

Counter example



$\left.\begin{array}{l} x_1 = -1 \\ x_2 = 1 \end{array}\right\} x_1 \neq x_2$

However $f(x_1) = (-1)^2 = 1$

$f(x_2) = (1)^2 = 1$

$\to f(x_1) = f(x_2)$ //

ex2) Is $f: \mathbb{R}^+ \to \mathbb{R}$ $f(x) = x^2$ one to one?

Yes, it is one to one.

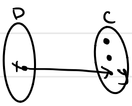
Let $x_1 \neq x_2$ s.t. $x_1, x_2 \in \mathbb{R}^+$ $(x_1 > 0 \wedge x_2 > 0)$

$\to x_1^2 \neq x_2^2$

Onto functions (Surjective):

A function $f: D \to C$ is onto $\longleftrightarrow$ $\forall y \in C \; \exists x \in D$ s.t. $y = f(x)$

for every y there exists x.



$\ast$ If f is Surjective

$\to$ Range $= C$

$\hookrightarrow$ Proof next pg

We can see that (f surjective)

$C \subseteq$ Range

Let $y \in C \rightarrow \exists x \in D$ s.t. $y = f(x)$

$\rightarrow y \in$ Range

$\therefore$ Range $= C$ $\square$

Bijection:

A function $f: C \rightarrow D$ is bijective (or a bijection)

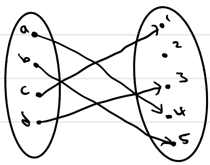
$\Leftrightarrow$ i) f is one-to-one $\wedge$ ii) f is onto

ex 1) $f: \{a,b,c,d\} \rightarrow \{1,2,3,4,5\}$

$f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 3$

i) one to one? — Yes

ii) onto? — No, $2 \in C$ and $\nexists x \in D$ s.t. $f(x) = 2$



$*$Range $= \{1,3,4,5\}$

Study this ex

ex 2) $f(x) = x+1$ $\quad f: \mathbb{R} \rightarrow \mathbb{R}$

i) one to one? Yes

ii) onto? Yes

iii) Bijection? Yes

iv) Range? Yes

one to one

$\rightarrow$ $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$

$\rightarrow$ $x_1 = x_2 \leftarrow f(x_1) = f(x_2)$

one to one: Assume that $f(x_1) = f(x_2)$

$\rightarrow x_1 + 1 = x_2 + 1$

$\rightarrow x_1 = x_2$ //

onto? $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}$ s.t. $y = f(x)$

Let $y \in \mathbb{R}$

$y = f(x) = x+1$

$\rightarrow y = x+1$

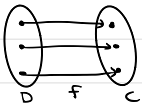
$\rightarrow x = y-1 \in \mathbb{R}$ $\square$

Range? $x \in \mathbb{R}$, $x+1 \in \mathbb{R}$ Range $\in \mathbb{R}$

Bijective? f is onto and one to one. Therefore, f is bijective.

Inverse of a function:

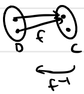
$f: D \rightarrow C$



Create new function

$f^{-1}: C \rightarrow D$

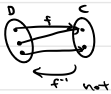
cases : 1)



reverse

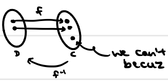
we cannot compute f^{-1}

2)



not a function

3) one to one



$\rightarrow$ Only bijective functions can be inverted.

ex) 1) $f: \mathbb{R} \rightarrow \mathbb{R}$ $\quad f(x) = x+1$

find f^{-1}

1) Set $y = f(x)$

2) Solve for x

3) Rewrite $(x \leftrightarrow y)$

1) $y = x+1$

2) $x = y-1$

3) $y = x-1 \rightarrow f^{-1}(x) = x-1$

$f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$

2) $f(x) = \sqrt{x}$ $\quad f: \mathbb{R}^+ \rightarrow \mathbb{R}$

Does it have f^{-1}, if so, what is it?

- one to one $\quad f(x_1) = f(x_2)$

$\rightarrow \sqrt{x_1} = \sqrt{x_2}$

$\rightarrow (\sqrt{x_1})^2 = (\sqrt{x_2})^2$

$\rightarrow x_1 = x_2$

- onto: $\forall y \in C$ s.t. $y = f(x) \longrightarrow \exists x \in D$ s.t. $f(x) = y$

Let $y = f(x)$

$y = \sqrt{x}$

$\rightarrow \underbrace{x = y^2}$ satisfies $y = \sqrt{y^2} = y$

$y = \pm\sqrt{x}$

$y = 4 = f(x)$

Let $x = 4^2 = 16$

$y = f(x) = f(16) = \pm\sqrt{16} = \pm 4$

Does not work!

☆ Study this ex

$\forall y \in \mathbb{R} \quad \exists x \in D$ s.t. $f(x) = y$

↳ Not onto

$y = -1 \in \mathbb{R}$ but $\nexists x \in \mathbb{R}$ s.t. $-1 = \sqrt{x}$

$\therefore$ No inverse.

3) $f(x) = \sqrt{x}$ $\quad f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$

- one to one: Yes (see ②)

- onto: $\forall y \in \mathbb{R}^+ \quad \exists x \in \mathbb{R}^+$ s.t. $y = f(x)$

Let $y \in \mathbb{R}^+$, we find x

$y = f(x)$

$y = \sqrt{x}$

$\therefore x = y^2 \rightsquigarrow y = \sqrt{x}$ works since $y \in \mathbb{R}^+$

$\rightarrow$ it is onto.

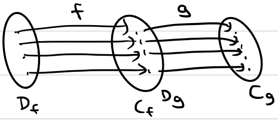
Inverse: 1) $y = \sqrt{x}$

2) $x = y^2$

3) $y = x^2$

$\boxed{f^{-1}(x) = x^2}$

Composition of functions



$f: D_f \to C_f$	$g(f(x))$
$g: D_g \to C_g$	$g \circ f(x) = g(f(x))$

$D_g \subseteq \text{Range}_f$

ex) $f(x) = 2x+3$ | $f: \mathbb{R} \to \mathbb{R}$

$g(x) = 3x+2$ | $g: \mathbb{R} \to \mathbb{R}$

$f \circ g(x) = f(g(x)) = f(3x+2)$

$= 2(3x+2)+3$

$= 6x+4+3$

$= 6x+7$

$g \circ f(x) = g(f(x))$

$= g(2x+3)$

$= 3(2x+3)+2$

$= 6x+11$

$\rightarrow f \circ g \neq g \circ f$

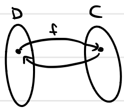
$f \circ f^{-1} = f^{-1} \circ f = i$ Identity function

$i: D \to D$

$i(x) = x$

$f(f^{-1}(x)) = x$

$f \circ f^{-1} = i$



Machines:

Given a String

abbababb → $\boxed{M}$ → { Yes (accepts) / No

ex) Accept those strings with two consequtive bs

aaab ababa → $\boxed{M}$ → No

Machine: yes if prime

no if otherwise

311 → $\boxed{M_p}$ → { Yes / No

Build M that determines if a number has 10 consequtive 9's

π? π → $\boxed{M}$ →

*Accept or reject String

Vending machine

products

Coke	(C)	30¢
Diet Coke	(D)	30¢
Sprite	(S)	30¢

D ← Coins

- n (nickles)
- d (dime)
- q (quarter)

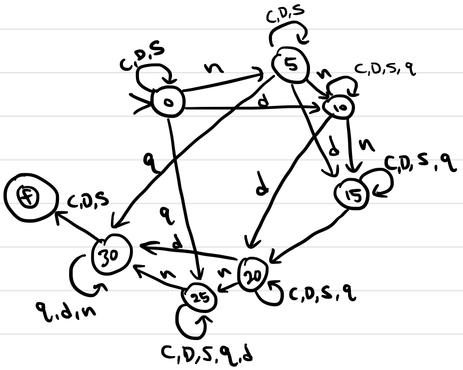
dnD
- Insert dime
- Insert nickel
- Press Diet coke

no soda → NO

qnC Yes

Use States: (total amount of money inserted)

Start state, Final State

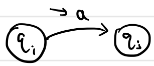


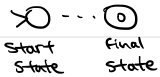
← Finite Automaton

☆ know how to do this.

08/10

Deterministic Finite Automata (DFA)





Symbols: $\{n, d, q, C, D, S\}$

Alphabet $\Sigma = \{n, d, q, C, D, S\}$

We need:

☆

- A set of states Q
- Alphabet Σ
- Transition function $\delta : Q \times \Sigma \to Q$ delta
- A Start State q_0 (only one)
- A set of Final States F

$M = (Q, \Sigma, \delta, q_0, F)$

machine