08/01

Relations: (Recap)

- $R \subseteq A \times B \rightarrow$ matrix $(0,1)$
- $R \subseteq A \times A$

Properties: (Recap)

- Reflexive
- Irreflexive
- Symmetric
- Antisymmetric
- Transitive

↳ Since Relations are sets

- $R_1 \cup R_2$ $\quad \underbrace{M_{R_1}}_{p} \quad \underbrace{M_{R_2}}_{q}$

$\rightarrow \underbrace{M_{R_1 \cup R_2}}_{m}$

$\rightarrow m_{ij} = p_{ij} \vee q_{ij}$

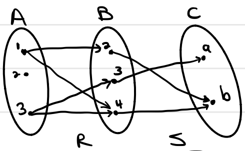
Ex)

$$M_{R_1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad M_{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow M_{R_1 \cup R_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

- $R_1 \cap R_2$

matrix $m_{ij} = p_{ij} \wedge q_{ij}$

Composition of Relations:



$\underline{R}$ $\quad \underline{S}$

matrix $\quad r_{ij} \quad s_{ij}$

$R \subseteq A \times B$

$R = \{(1,2), (1,4), (3,3), (3,4)\}$

$S \subseteq B \times C$

$S = \{(2,b), (3,a), (4,b)\}$

↳ $S \circ R = \{(x,y) \mid \exists z \text{ s.t. } (x,z) \in R \wedge (z,y) \in S\}$

In this ex)

$S \circ R = \{(1,b), (3,a), (3,b)\}$

Matrix:

$$\begin{array}{c|ccc} & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} & a & b \\ \hline 2 & 0 & 1 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{array} \qquad = \qquad \begin{array}{c|cc} & a & b \\ \hline 1 & 0 & 1 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{array}$$

$\quad R \qquad\qquad S$

* Matrix multiplication with OR operator

↳ $m_{ij} = \bigvee_{k=1}^{|B|} (r_{ik} \wedge s_{kj})$

$\rightarrow M_{S \circ R} = M_R \odot M_S$

Composition: (Relation on a set)

$R \subseteq A \times A$

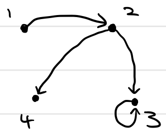
$R^2 = R \circ R$

ex) $A = \{1,2,3,4\}$, $R = \{(1,2),(2,3),(2,4),(3,3)\}$

$R^2 = ?$

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \odot \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

↳ **Diagram** ($R \subseteq A \times A$)



Graph $= (V, E)$

$V = A$

$E = \{(u,v) \mid (u,v) \in R\}$

→ Directed Graph (Diagraph)

$R^2 = ?$

↳ Path → $v_0 v_1 v_2 v_3 \cdots v_n$

$(v_0, v_1) \in R$, $(v_1, v_2) \in R$, $\cdots$, $(v_{n-1}, v_n) \in R$

Path $v_0 v_1 \cdots v_n$ s.t. $(v_{i-1}, v_i) \in R$ $\forall i = 1 \ldots n$ (Path from v_0 to v_n)

* Vertex Repetition is allowed

In this ex)

from 1 to 3; 1,2,3 (length 2)

from 1 to 4; 1,2,4 (length 2)

from 2 to 3; 2,3,3 (length 2)

* Length of the path is equal to the number of edges traversed.

R^2 represents paths of length 2.

Paths of length 3

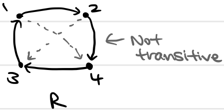
↳ $\boxed{R^3 = R^2 \circ R}$

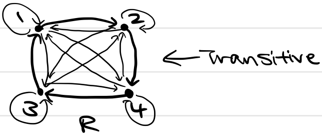
$R^4 = R^3 \circ R$ $\Rightarrow$ $\boxed{R^{n+1} = R^n \circ R}$

$R^5 = R^4 \circ R$

Theorem:

R on A is transitive $\longleftrightarrow R^n \subseteq R$ (any path of any length should be in R)





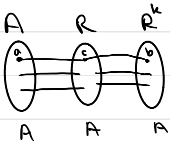
i) $(\rightarrow)$ R is transitive $\rightarrow R^n \subseteq R$

Assume that R is transitive

We will show that $R^n \subseteq R$ using induction (on n)

Basis: $n=1$

$R^1 \subseteq R$ $\square$



I.H.:

$R^k \subseteq R$ for a fixed $k \geq 1$

I.S.: We need to show that

$R^{k+1} \subseteq R$

Definition of R^{n+1}

Let $(a,b) \in R^{k+1} = R^k \circ R$

Def. of $R \circ R$ $\rightarrow \exists\, c \in A$ s.t. $(a,c) \in R \wedge (c,b) \in R^k$ $(\subseteq R)$

I.H. $\rightarrow (a,c) \in R \wedge (c,b) \in R$ because $R^k \subseteq R$

$\rightarrow (a,b) \in R$

$\therefore R^{k+1} \subseteq R$ $\square$

ii) $(\leftarrow)$ $R^n \subseteq R \rightarrow R$ is transitive

Assume $R^n \subseteq R$ $\forall n \geq 1$ in particular if $n=2$,

$R^2 \subseteq R$

We will show that if $R^2 \subseteq R \rightarrow R$ is transitive.

$R^2 = R \circ R$

Let $\underline{(a,b)} \in R \wedge \underline{(b,c)} \in R$

Notice that since $R^2 \subseteq R$

$\rightarrow (a,c) \in R^2$ $\xrightarrow{\text{Def. of } R \circ S} (a,c) \in R$

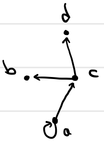
$\therefore R$ is transitive. $\square$

☆Very Important!

Relation on a Set

☆+ Reflexive

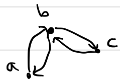
$(a,a) \in R$
$\forall a \in A$



*Not reflexive must have [loop] a $\forall a \in A$

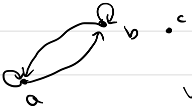
☆+ Symmetric

$(a,b) \in R \rightarrow (b,a) \in R$



*Yes, Symmetric

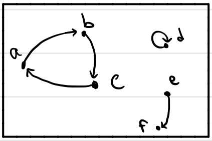
☆+ Transitive (All paths are there)



Yes, transitive

$\Rightarrow$

ex)



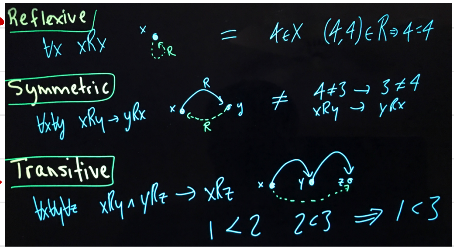
Transitive?
$(a,b) \in R$, $(b,c) \in R$
but $(a,c) \notin R$

$\rightarrow$ Not transitive!

$$\underbrace{(a,b) \in R \wedge (b,c) \in R}_{p} \rightarrow \underbrace{(a,c) \in R}_{q}$$

$$p \rightarrow q$$

Summary



Closures:

☆Reflexive Closure

- Start with R (might not be reflexive)
- minimal set R s.t. $\underbrace{R \cup R_1}_{\text{Reflexive Closure}}$ is reflexive.

Ex) over $\mathbb{Z}$

$R = \{(a,b) \mid a < b\}$ Not reflexive

$R_1 = \{(a,b) \mid a = b\}$

can add $\{(3,3), \cancel{(5,7)}\}$ but would not be minimal

$R \cup R_1 = \{(a,b) \mid a < b\} \cup \{(a,b) \mid a = b\}$

$\rightarrow$ Reflexive closure of R $= \{(a,b) \mid a \leq b\}$

☆Symmetric Closure of R

- Find minimal set R_1
- s.t. $\underbrace{R \cup R_1}_{\text{Symmetric Closure}}$ is Symmetric.

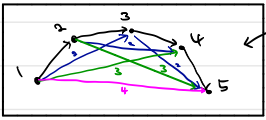
Ex) Find the symmetric closure of $R = \{(a,b) \mid a > b\}$

$R_1 = \{(a,b) \mid a < b\}$

$R \cup R_1 = \{(a,b) \mid a > b\} \cup \{(a,b) \mid a < b\}$

→ Symmetric closure of R $= \{(a,b) \mid a \neq b\}$

Transitive Closure



← Transitive closure

In general,

$$R^* = \bigcup_{i=1}^{\infty} R^i$$

R^* is the transitive closure of R.

$(a,b) \in R^* \wedge (b,c) \in R^*$
$\to (a,c) \in R^*$

Algorithm to compute the transitive closure

m is a matrix of R

$m^* = m$

for $i = 2$ to n

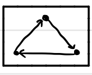
$m^* = m^* \odot m \cup m^*$

*More efficient Algorithm:
-Warshall's Algorithm

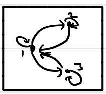
Equivalence Relations:

Def.: a relation R is an equivalence relation if R is
- Reflexive
- Symmetric
- Transitive

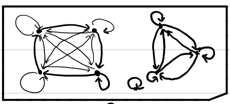
Ex)



R_1



R_2



R_3

- R_1 is not eq. relation
- R_2
  - Reflexive? Yes
  - Symmetric? Yes
  - Transitive? $(3,1) \in R \wedge (1,2) \in R$ but $(3,2) \notin R$

  ⇒ Not eq. relation
- R_3
  - Reflexive? Yes
  - Symmetric? Yes
  - Transitive? Yes

  ⇒ eq. relation

✪ Study examples

Example definition

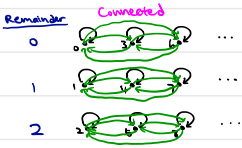
$a \equiv b \mod m \leftrightarrow a \mod m = b \mod m$

Congruent

$\mathbb{N} \cup \{0\}$

$R = \{(a,b) \mid \underbrace{a \equiv b \mod 3}_{a \mod 3 = b \mod 3}\}$

↳ $A = \{0,1,2,3,4,5,6,7,8\}$



→ Looks as if Equivalence Relations partition a set.

Ex) Show that $R = \{(x,y) \mid x,y \in \mathbb{R} \wedge |x-y| < 1\}$ is not an equivalence relation.

- Reflexive: $(x,x) \in R$? $|x-x| = 0 < 1$ ✓
- Symmetric: $(x,y) \in R \rightarrow |x-y| < 1$

  $\rightarrow |y-x| < 1$

  $\rightarrow (y,x) \in R$

  Yes, it is Symmetric. ✓
- Transitive?

Let $x = 0$, $y = 0.8$, $z = 1.6$

$|x-y| = 0.8 < 1 \rightarrow (x,y) \in R$

$|y-z| = 0.8 < 1 \rightarrow (y,z) \in R$

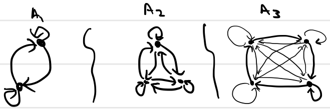
However, $|x-z| = 1.6$ not $< 1 \rightarrow (x,z) \notin R$ ✗

∴ Not eq. relation.

08/03 #Recap

Equivalence Relations

- Reflexive
- Symmetric
- Transitive



$R \subseteq A \times A$

✪ 3 prop. of equivalence Relations