Exam 1 Proof

Show that $3x+2$ is even $\leftrightarrow$ x^2 is even

Proof into 2 parts

Proof: $3x+2$ is even $\rightarrow x^2$ is even
($\rightarrow$) (By contrapositon)

(we want) x^2 is odd $\rightarrow$ $3x+2$ is odd

x^2 is odd $\rightarrow$ x is odd (from class) $\rightarrow x = 2k+1, k \in \mathbb{N}$

In class, we were able to show
x is even $\leftrightarrow x^2$ is even
x is odd $\leftrightarrow x^2$ is odd

$\rightarrow 3(2k+1)+2 \rightarrow 6k+3+2 \rightarrow 2(3k+2)+1$

Let $r = 3k+2$, $r \in \mathbb{N}$
$\rightarrow 2r+1$
$\rightarrow 3x+2$ is odd $\square$

Proof: x^2 is even $\rightarrow 3x+2$ is even
($\leftarrow$) Let x^2 be even $\rightarrow$ x is even (shown in class)

$\rightarrow x = 2k, k \in \mathbb{N}$
$\rightarrow 3(2k)+2$
$\rightarrow 6k+2$
$\rightarrow 2(3k+1)$

Let r be $3k+1$, $r \in \mathbb{N}$ (closure of $\mathbb{N}$)
$\rightarrow 2r$
$\rightarrow 3x+2$ is even $\square$

Set of fractions

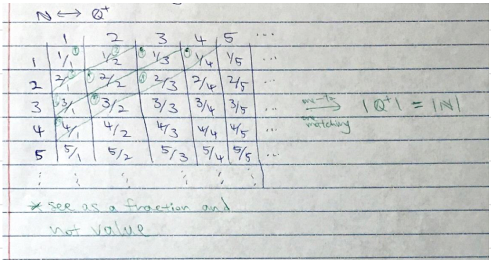
$F^+ = \{\frac{p}{q} \mid p, q \in \mathbb{N},\ q \neq 0\}$ ← seen as fractions

ex) $1/1,\ 1/2,\ 1/3\ \cdots\ 2/1,\ 2/2,\ 2/3, \cdots$

$\mathbb{Q}^+ = \{\frac{p}{q} \mid p, q \in \mathbb{N},\ q \neq 0\}$ ← seen as numbers (rational numbers)

$\mathbb{N} \subseteq \mathbb{Q}^+$ ... easy $n \leadsto \frac{n}{1}$

↳ $\underbrace{|\mathbb{N}| \leq |\mathbb{Q}^+|}_{\text{from the theorem}} \leq \underbrace{|F^+|}_{\text{notice } F^+ \text{ has duplicates with respect to } \mathbb{Q}^+}$



$\mathbb{N} \longleftrightarrow \mathbb{Q}^+$

	1	2	3	4	5	...
1	1/1	1/2	1/3	1/4	1/5	...
2	2/1	2/2	2/3	2/4	2/5	...
3	3/1	3/2	3/3	3/4	3/5	...
4	4/1	4/2	4/3	4/4	4/5	...
5	5/1	5/2	5/3	5/4	5/5	...
⋮	⋮	⋮	⋮	⋮	⋮	

$\xrightarrow[\text{matching}]{\text{one-to-one}}$ $|\mathbb{Q}^+| = |\mathbb{N}|$

* see as a fraction and not value

Since $\exists$ a one-to-one matching of $\mathbb{N}$ to F^+

$|F^+| \leq |\mathbb{N}|$

$|\mathbb{N}| \leq |\mathbb{Q}^+| \leq |F^+| \leq |\mathbb{N}|$

$\rightarrow |\mathbb{N}| = |\mathbb{Q}^+|$

<u>Diagonalization:</u>

Restriction: $[0, 1]$

we will try to find a one-to-one matching from $\mathbb{N}$ to $[0,1]$

→ Next pg.

$\mathbb{N} \subset \mathbb{Q}^+ \subset \mathbb{R}^+$ ← rational & irrational

we were able to show that $\sqrt{2} \in \mathbb{R}^+$ but $\sqrt{2} \notin \mathbb{Q}^+$

Real Numbers $[0, 1]$

$\{x \mid x \in [0,1]\}$

0 ——— 1

$0.d_1 d_2 d_3 d_4 \cdots$

$0 \leq d_i \leq 9$

Let $x = 0.\overline{9}$

$\rightarrow 10x = 9.\overline{9}$

$\rightarrow 10x - x = 9.\overline{9} - 0.\overline{9}$

$\rightarrow 9x = 9$

$\rightarrow x = 1$

$\therefore 0.9999\cdots = 1$ $\square$

Let $x = 0.\overline{3}$

$\rightarrow 10x = 3.\overline{3}$

$\rightarrow 10x - x = 3.\overline{3} - 0.\overline{3}$

$\rightarrow 9x = 3$

$\rightarrow x = \frac{9}{3} = \frac{1}{3}$

Let $x = 0.323232\cdots$

$= 0.\overline{32}$

$\rightarrow 100x = 32.\overline{32}$

$\rightarrow 100x - x = 32$

$x = \frac{32}{99}$

Assume that $\exists$ a one-to-one matching from $\mathbb{N} \rightarrow [0,1]$

$\mathbb{N}$	$[0,1]$
1.	$0.d_{11}d_{12}d_{13}d_{14}\cdots$
2.	$0.d_{21}d_{22}d_{23}d_{24}\cdots$
3.	$0.d_{31}d_{32}d_{33}\cdots$
4.	$0.d_{41}d_{42}d_{43}\cdots$
5.	$0.d_{51}d_{52}d_{53}\cdots$
$\vdots$	$\vdots$

from #1, digit #1

ex)

1 – $0.321584\cdots$

2 – $0.215428\cdots$

3 – $0.314579\cdots$

4 – $0.151403\cdots$

$\rightarrow$ all $[0,1]$ must be accounted for

We can find a number that is not in on the list

(in this ex:

0.2735

$\neq$ #3, $\neq$ #1, $\neq$ #4, $\neq$ #1

in general, select

$0.a_1a_2a_3a_4\cdots$

s.t. $a_1 \neq d_{11}$, $a_2 \neq d_{22}$ $\cdots$

$a_i \neq d_{ii}$

$\hookrightarrow$

Assume $\exists$ one to one mapping from $\mathbb{N}$ to $[0,1]$

$\rightarrow$ we get a contradiction

$a = 0.a_1 a_2 \cdots$

s.t. $a_i \neq d_{ii} \quad \forall i \in \mathbb{N}$

is <u>NOT</u> on the list

$\therefore \nexists$ one to one matching from $\mathbb{N}$ to $[0,1]$

$|A| = |B| \leftrightarrow \exists$ a one to one matching from A to B

since $\rightarrow |\mathbb{N}| \neq \underbrace{|[0,1]|}_{\text{larger infinity}}$

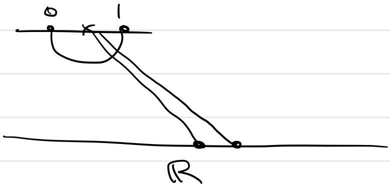
★ There is a <u>hierarchy of infinity.</u>

$\mathbb{N}$

$0 \rightsquigarrow 1 \rightsquigarrow 2 \rightsquigarrow 3$

*we can get to next inf. using "powerset"!

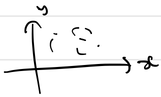
$[0,1]$ vs $\mathbb{R}$



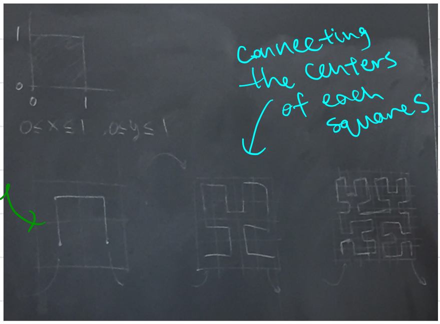
$(0,1)$ to $\mathbb{R} \rightarrow |[0,1]| = |\mathbb{R}|$

$\mathbb{R}$

$\mathbb{R}^2$



touch each point on the plane



connecting the centers of each squares

$\therefore \mathbb{R}$ one to one matching to $\mathbb{R}^2$

(Peano curves)

Countable Sets:

Cardinality: a) finite
or b) $|\mathbb{N}|$

$\mathbb{Q}^+$ is countable
$\mathbb{Q}$ is countable
$\mathbb{R}$ is not countable
$\{1,2,3\}$ is countable

Those that have cardinality $= |\mathbb{N}|$ are called countably infinite.

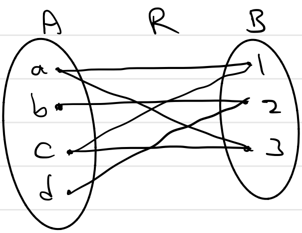
$\mathbb{Q}^+$ is countably infinite.
$\{1,2,3\}$ is not countably infinite.

Binary Relations:

Definition: A binary relation R on A and B is a subset of the cartesian product $A \times B$

$\boxed{R \subseteq A \times B} \rightsquigarrow A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

ex)



$R = \{(a,1), (a,3), (b,2), (c,1), (d,2)\}$

a is related to 1 → $(a,1) \in R$
→ aR_1

a is not related to 2 → $a \not R_2$
→ $(a,2) \notin R$

Binary Matrix

	1	2	3
A	1	0	1
B	0	1	0
C	1	0	0
D	0	1	0

Relation on A

$R \subseteq A \times A$

ex) $A = \{1,2,3,4\}$

$R_{div} = \{(a,b) \mid a \text{ divides } b\}$ (R_{div} is a relation on A)

$R_{div} = \{(1,1), (1,2), (1,3),$
$(1,4), (2,2), (2,4),$
$(3,3), (4,4)\}$

As matrix

	1	2	3	4
1	1	1	1	1
2	0	1	0	1
3	0	0	1	0
4	0	0	0	1

$R \subseteq A \times A$

$R \subseteq \{(a,b) \mid a \in A \wedge b \in B\}$

each R

$\hookrightarrow 2^{A \times A}$ ← all possible binary relations

Cardinality of the set of all binary relations on A

$\rightarrow \boxed{2^{|A|^2}}$

ex) List of all possible Binary Relations on $\{1,2\}$

$= \{\{\}, \{1,1\}, \{1,2\}, \{2,1\}, \{2,2\}, \{(1,1),(1,2)\}, \{(1,1),(2,1)\},$
$\{(1,1),(2,2)\}, \{(1,2),(2,1)\}, \{(1,2),(2,2)\}, \{(2,1),(2,2)\},$
$\{(1,1),(1,2),(2,1)\}, \{(1,1),(1,2),(2,2)\}, \{(1,2),(2,1),(2,2)\},$
$\{(1,1),(1,2),(2,1),(2,2)\}, \{(1,1),(2,1),(2,2)\}\}$

$2^{|A|^2} = 2^{2^2} = 2^4 = \boxed{16}$

Ex) $R_{\neq}$ on $A = \{1,2,3,4\}$

$R_{\neq} = \{(a,b) \mid a \neq b\}$

$R_{\neq} = \{(1,2), (1,3), (3,4) \cdots\}$

matrix

	1	2	3	4
1	0	1	1	1
2	1	0	1	1
3	1	1	0	1
4	1	1	1	0

Properties:

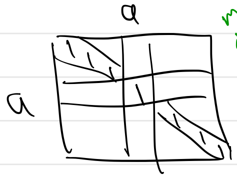
know all these properties

Reflexive

Def: A relation R on A is reflexive $\leftrightarrow \forall a \in A,\ aRa$

all the elements must be satisfied

matrix



main diagonal is 1.

Irreflexive

Def: $a \not R a \quad \forall a \in A$

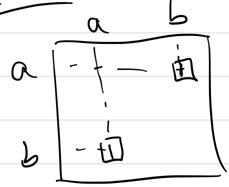
ex) Is $R_{ex} = \{(1,1), (2,1), (3,3)\}$ irreflexive?

R_{ex} is not irreflexive for ex) $(1,1) \in R_{ex}$

ex) Is R_{div} reflexive?
yes (look at main diagonal)

ex) Is $R_{\neq}$ irreflexive?
Yes, main diagonal is all 0s.

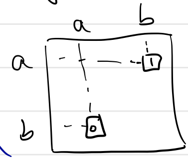
matrix



Symmetric

- Def: A binary relation R on A is symmetric $\leftrightarrow (a,b) \in R \rightarrow (b,a) \in R$

$m_{ab} = m_{ba}$
↳ symmetric



wherever $a \neq b$

AntiSymmetric:

- Def: A binary relation R on A is antisymmetric $\leftrightarrow (a,b) \in R \wedge (b,a) \in R \rightarrow a = b$ ← main diagonal

$m_{ab} = 1 \rightarrow m_{ba} = 0$
↳ antisymmetric

Ex) 1) Is R_{div} Symmetric? ... No!

2) $R_{\neq}$ Symmetric? ..... Yes!

3) R_{ex} is not Symmetric, because $(2,1) \in R$ but $(1,2) \notin R$

ex) Is R_{ex} antisymmetric?

Antisymmetric? Yes! ←

Not antisymmetric? No, we cannot find a counter example.

R_{ex}

	1	2	3
1	1	0	0
2	1	0	0
3	0	0	1

* Don't look at main diagonal.

* Find 1 and its symmetric point. If it's 0, then → antisymmetric.

Transitive:

- a binary relation on R on A is transitive $\Leftrightarrow$ $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

ex) Is R_{div} transitive? ... yes

Is $R_{\neq}$ transitive?

counter ex) $1 \neq 2 \wedge 2 \neq 1$

$\rightarrow (1,2) \in R_{\neq}$, $(2,1) \in R_{\neq}$ but $(1,1) \notin R_{\neq}$

$\boxed{\therefore \text{ not transitive}}$

Is R_{ex} transitive? ... Yes

$\rightarrow (1,1) \in R_{ex}$, $(2,1) \in R_{ex}$ and $(1,1) \in R_{ex}$