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Linear Algebra I

Unit 1: Vectors

Notation of Vectors:

Euclidean Space: the set of all points where all dimensions are real numbers
Ex: $R^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$

While points just represent a spot in Euclidean space, vectors represent a displacement from the origin.

Ex. in R^2, a vector is represented as: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and graphically, this would represent an arrow starting from the origin and ending at the point (x_1, x_2)

A vector can both contain variables and numbers, where if it is filled with numbers, it simply represents a displacement. If it contains variables, it can represent lines, planes, or hyperplanels.

Operations on Vectors:

Equality: if $\vec{x} = \vec{y}$, then: $x_i = y_i \, for \; 1 \leq i \leq n$

Addition: $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \dots \\ x_i + y_i \end{bmatrix}$

This addition has the geometric effect of concatenating the two vectors, head to tail.

Scalar Multiplication: $c\vec{x} = \begin{bmatrix} cx_1 \\ \dots \\ cx_i \end{bmatrix}$

This has the geometric effect of scaling the vector, or flipping it's direction if it's negative.

Note: Subtraction of two vectors is really just: $\vec{x} - \vec{y} = \vec{x} + (-1)\vec{y}$

Linear Combination of vectors: $\vec{L} = c_1\vec{x_1} + \cdots + c_i\vec{x_i}$

Properties of Vectors:

The 10 Fundamental Properties of Vectors in R^n (Theorem 1.1.1)
If $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$, then:
V1: $\vec{x} + \vec{y} \in \mathbb{R}^n$, closure under addition
V2: $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$, associative property

V3: $\vec{x}+\vec{y}=\vec{y}=\vec{x}$, commutative property
V4: $\vec{0}$ is called the 0 vector, and $\vec{x}+\vec{0}=\vec{x}$, additive identity
V5: $\vec{x}+(-\vec{x})=\vec{0}$, additive inverse
V6: $c\vec{x}\in\mathbb{R}^2$, closure under multiplication
V7: $c(d\vec{x})=(cd)\vec{x}$, associative property
V8: $(c+d)\vec{x}=c\vec{x}+d\vec{x}$, scalar distributive property
V9: $c(\vec{x}+\vec{y})=c\vec{x}+c\vec{y}$, vector distributive property
V10: $1\vec{x}=\vec{x}$, multiplicative identity

While many of these properties are obvious, these ten properties will come up often.

Spans:

Given some set of vectors B, the span of this set is the span of all linear combinations of each of these vectors.

$S = SpanB = \{c_1v_1 + c_2v_2 + \cdots + c_iv_i | c_1, \ldots, c_k \in \mathbb{R}^n\}$

We can say that B spans S, as the linear combinations of the vectors in B can make up exactly S. Geometrically, we can see that a span represents all the points that can be constructed with linear combinations of the vectors. For instance, if only one vector is given in $\mathbb{R}^2$, then the span can only create points in a line.

Converting a Spanning Set to a Vector Equation:

To reduce the number of operations needed on a spanning set, it is often useful to simplify the spanning set as much as possible. To "simplify", basically means to remove any vectors that can be created with the other vectors in the spanning set.

In order to work with an equation to do so, we can simplify a spanning set as so:
$\vec{x} = t_1\vec{v_1} + \cdots + t_i\vec{v_i}$

This basically means that given some spanning set, we can construct a new vector with a linear combination of the others.
Theorem 1.1.2: if vector k+1 can be constructed by the other vectors in the set, then:
$span(\vec{v_1}, \ldots, \vec{v_k}) = span(\vec{v_1}, v_k \vec{+} 1)$

This is the first important proof of Linear Algebra I:
To create a Proof, this is the general format:

1) *State your assumptions.* In this case, we assume that vector k + 1 can actually be constructed by the other vectors in the set
2) *Identify Specifically what you want to prove.* In this case, we want to show that the two spans are the same
3) *How are you going to prove it?* In this case, if the vector is constructed by the others, just rewrite it as a linear combination, and then we can remove duplicate vectors.

Proof:

$v_k = c_1 v_1 + ... + c_i v_i$

$x = d_1 v_1 + ... + d_k(c_1 v_1 + ... + c_i v_i)$

We distribute the d_k and group terms

$x = (d_1 + d_k c_1)v_1 + ... + (d_{k-1} + d_k c_{k-1})v_{k-1}$

$d_1 + d_k c_1$ is just some arbitrary constant, and thus we can just replace it with e

$x = e_1 v_1 + ... + e_{k-1} v_{k-1}$

Thus, we have just rewritten it and simplified the spanning set.

To be able to do this process without inspection(Just picking out vectors that are linear combinations by hand), we need to figure out a method to algebraically reduce spanning sets

We "decompose" this problem by splitting it into two parts: Checking if the equation is *Linearly Independent* and finding the vector that can be replaced

We can do this with simple algebraic manipulation

To prove that vector v_i can be created with a linear combination of the others:

$c_1 v_1 + ... + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + c_k v_k = c_i v_i$

Simply move $c_i v_i$ to the other side and we find that we need to find a set of constants that the linear combination is 0

$c_1 v_1 + ... + c_k v_k = 0$

Now, there is always the *trivial solution,* which occurs when all constants are 0. For the set to be linearly independent, there needs to exist another solution.

For instance, with the set of vectors:

$$B = \{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\}$$

Is it linearly independent? If so, what can it be replaced by?

We first distribute the constants

$$\vec{0} = \begin{bmatrix}c_1\\c_1\end{bmatrix} + \begin{bmatrix}c_2\\2c_2\end{bmatrix} + \begin{bmatrix}c_3\\3c_3\end{bmatrix}$$

We then add the vectors together

$$\vec{0} = \begin{bmatrix}c_1 + c_2 + c_3\\c_1 + 2c_2 + 3c_3\end{bmatrix}$$

Now, we have two equations to solve.

$c_1 + c_2 + c_3 = 0$

$c_1 + 2c_2 + 3c_3 = 0$

By Elimination, we get
$-c_2 - 2c_3 = 0$
or,
$c_2 = -2c_3$
We substitute this in
$c_1 - 2c_3 + c_3 = 0$
$c_1 - c_3 = 0$
$c_1 = c_3$
We replace once again into the original matrices
$$c_3 \begin{bmatrix}1\\1\end{bmatrix} - 2c_3 \begin{bmatrix}1\\2\end{bmatrix} + c_3 \begin{bmatrix}1\\3\end{bmatrix}$$
We can then simply solve for any of the vectors and find how to replace one of the vectors with the other two.

Note: Should the coefficient of the vector be 0, that vector is not replaceable, and thus another vector to be replaced is needed.
Also Note: If the zero vector is inside the set, then the set is linearly dependent

Bases:

Bases: A set of vectors that spans a space and is also linearly independent.

- When working with potential bases, the checking for linear independence can be done as above
- To see if a set spans a space, assure that every vector that you need(if it's an entire dimension, then use variables) can be represented as a linear combination of the set

Standard Bases: The axes of a given dimension n, can be represented as: $\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}\}$
Where each "e_i" vector has it's ith value as 1, and the rest as 0

Ex. For $\mathbb{R}^2$, the standard bases would be: $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$, or the x and y axis.

Surfaces in Different Dimensions

You can apply points in any dimension that is larger than 0-D, and this is in the form(x, y, … n).

Lines are applicable in larger than 1D dimensions, and are represented as $c_0\vec{v_0} + b$

Given that the set $\{v_0,\ v_1\}$ is linearly independent, a plane can be represented as $c_0v_0 + c_1v_1 + b$

Higher dimensions are called hyperplanes, and follow a similar form.

Subspaces:

Subspaces are non-empty sets of vectors, that satisfy all of the vector properties(detailed in Theorem 1.1.1)

The subspace test is a way to identify if a given set of vectors is a subspace, and instead of proving all 10 properties detailed in Theorem 1.1.1, we only realistically need to check 2:

- ❖ Closure under addition: $\vec{x} + \vec{y} \in \mathbb{S}$
- ❖ Closure under scalar multiplication: $c\vec{x} \in \mathbb{S}$

We additionally need to check that the set is non-empty, and we do this by assuring that the 0 vector exists in the subset.

Sample Problem: if $\mathbb{S} = \{\begin{bmatrix} \vec{x_1} \\ \vec{x_2} \end{bmatrix} \in \mathbb{R}^2 | \vec{x_1} - \vec{x_2} = 0\}$

The 0 vector is inside this subset, as 0 - 0 is 0, and thus the set is not empty

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

We know x_1 - x_2 = 0, and y_1 - y_2 = 0

Closure under addition: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$

(x_1 + y_1) - (x_2 + y_2) = 0

(x_1 - x_2) + (y_1 - y_2) = 0

0 + 0 = 0

Therefore, it is closed under addition.

The set is also closed under multiplication as cx_1 - cx_2 = c(x_1 - x_2) = c(0) = 0

Therefore, this set is a subspace of $\mathbb{R}^2$

This process also exposes a new theorem.

Theorem 1.2.2: if a set of vectors is inside of $\mathbb{R}^n$, then the span of that set is a subspace for $\mathbb{R}^n$

Proof:

We know that the zero vector is present in the span, as we can simply use the trivial solution of setting all coefficients to 0.

Let $\vec{x} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$, simply a linear combination of the set

Let $\vec{y} = t_1\vec{v_1} + \cdots + t_n\vec{v_n}$, another combination

$$x + y = c_1 v_1 + ... + c_n v_n + t_1 v_1 + ... + t_n v_n$$
$$x + y = (c_1 + t_1) v_1 + ... + (c_n + t_n) v_n$$

Thus, it is closed under addition

$$cy = c(t_1 v_1 + ... + t_n v_n)$$
$$cy = ct_1 v_1 + ... + ct_n v_n$$

Thus, it is also closed under scalar multiplication.

Finding a Basis of a Subspace:

3 Step Algorithm to find a basis of a subspace:

1) First, replace as many variables as possible, using information about the subspace
2) Make the general form of each vector
3) Simplify the spanning set as much as possible

For instance, find the basis of: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} | x_1 - x_2 + x_4 = 0$

We first use the information that $x_1 - x_2 + x_4 = 0$, and we know: $x_1 = x_2 - x_4$

We replace x_1 in the vector definition:

$$\begin{bmatrix} x_2 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Then separate out each variable for the general form of the vectors

$$x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Given this form, we can construct the basis vectors, being:

$$\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}$$

This cannot be simplified further, and thus this is the basis of this set.

Dot Product and Cross Product:

In $\mathbb{R}^2$, the dot product is simply defined as $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2$

This definition is equivalent to: $\vec{x} \cdot \vec{y} = ||x||||y||cos(\theta)$

Since the dot product is so useful in $\mathbb{R}^2$, we extend the definition to $\mathbb{R}^n$ with the following:

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \dots \\ y_n \end{bmatrix} = x_1y_1 + \cdots + x_ny_n$$

Alternatively, you can show this in summation notation:

$$x \; dot \; y = \sum_{i=1}^{n} x_i y_i$$

The dot product is also called the *standard inner product*.

Theorem 1.3.2:

1) $\vec{x} \cdot \vec{x} \geq 0$, and if $\vec{x} \cdot \vec{x} = 0$, $\vec{x} = 0$
2) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
3) $\vec{x} \cdot (c_1\vec{y} + c_2\vec{z}) = c_1(\vec{x} \cdot \vec{y}) + c_2(\vec{x} \cdot \vec{z})$

The first property of theorem 1.3.2 is due to how the magnitude of a vector is defined as: $||x|| = \sqrt{x_1 * x_1 + ... + x_n * x_n}$, or just the square root of the dot product with itself. Thus, the length cannot be negative.

Theorem 1.3.3: Properties of Length:

1) $||\vec{x}|| \geq 0$ and $||\vec{x}|| = 0$ iff $\vec{x} = \vec{0}$
2) $||c\vec{x}|| = |c|||\vec{x}||$
3) $||\vec{x} \cdot \vec{y}|| \leq ||\vec{x}||||\vec{y}||$
4) $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$

Also note: Two vectors are *Orthogonal(90 degrees apart)* if their dot products are 0, since this means that their cosine is 0.

Also, since the dot product can be defined as: $\vec{x} \cdot \vec{y} = ||x||||y||cos(\theta)$, we can isolate for theta and find that the angle between two vectors is:

$\theta = arccos((\vec{x} \cdot \vec{y})/(||x||||y||))$

Cross Product:

While the inner product provides a scalar value for every two vectors, the cross product(or vector product), returns a vector in $\mathbb{R}^3$. Note that the cross product only works with two vectors in $\mathbb{R}^3$, and returns a vector orthogonal to both vectors.

This means that the cross product will return a vector that has a dot product of *0* with both vectors, and it is defined as so:

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y - 1 \end{bmatrix}$$

Theorem 1.3.4: Properties of the Cross Product

1) if $\vec{n} = \vec{x} \times \vec{y}$, then for any $z \in span(x, y)$, $\vec{z} \cdot \vec{n} = 0$
2) $\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$
3) $\vec{x} \times \vec{x} = \vec{0}$
4) $\vec{x} \times \vec{y} = 0$ iff $\vec{x} = \vec{0}$ or $\vec{y} = vec0$, or $\vec{x} = c\vec{y}$
5) $\vec{n} \times (\vec{x} + \vec{y}) = \vec{n} \times \vec{x} + \vec{n} \times \vec{y}$
6) $c\vec{x} \times \vec{y} = c(\vec{x} \times \vec{y})$
7) $||\vec{x} \times \vec{y}|| = ||\vec{y}||||\vec{w}|||sin(\theta)|$

Note: The cross product is not associative. $\vec{x} \times (\vec{y} \times \vec{z}) \neq (\vec{x} \times \vec{y}) \times \vec{z}$

Scalar Equations and Projections:

Given vectors $\vec{v}, \vec{w}, \vec{b} \in \mathbb{R}^3$, and we represent x as:
$\vec{x} = s\vec{v} + t\vec{w} + \vec{b}$, and find the normal vector between vector v and vector w($\vec{n} = \vec{v} \times \vec{w}$).

If we subtract vector b from x, we end up with a plane (x - b) where all points are perpendicular to this normal vector. Thus,

$(\vec{x} - \vec{b}) \cdot \vec{n} = 0$

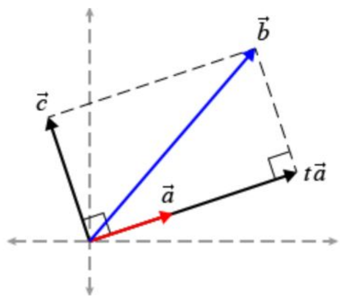
We distribute vector n

$\vec{x} \cdot \vec{n} - \vec{x} \cdot \vec{b} = 0$

$\vec{x} \cdot \vec{n} = \vec{x} \cdot \vec{b}$

Thus, $n_1x_1 + n_2x_2 + n_3x_3 \;=\; b_1x_1 + b_2x_2 + b_3x_3$.

This is known as the scalar equation of a plane and there are infinitely many of them. Each plane has a different normal vector. You can extend this definition to different dimensions with the general form: $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{b}$



Projections:

To solve for the projection of vector b onto vector a, we need to solve for t * vector a.

We know that if vector c is perpendicular to vector a:

$$\vec{b} = t\vec{a} + \vec{c}$$
$$\vec{b} \cdot \vec{a} = (t\vec{a} + \vec{c}) \cdot \vec{a}$$
$$\vec{b} \cdot \vec{a} = t||\vec{a}||^2$$
$$(\vec{b} \cdot \vec{a})/||\vec{a}||^2 = t$$

Thus, we get that:

$$proj(\vec{a})(\vec{b}) = ((\vec{b} \cdot \vec{a})/||\vec{a}||^2) \cdot \vec{a}$$

Similarly, $perp(\vec{a})(\vec{b}) = b - proj(\vec{a})(\vec{b})$.

To project a vector onto a plane, you instead find the perpendicular vector to the plane's normal vector.

In other words, $proj(P)(\vec{a}) = perp(\vec{n})(\vec{a})$

End of Unit 1, Next Unit: Matrices