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Advanced Functions

Unit 3: Higher Polynomial Functions

Division of Polynomials:

In the past, we have only ever worked with the addition, subtraction, and multiplication of polynomials. However, division can also occur with polynomials, just in a slightly weirder way. Recall back to grade 4, when you learned long division. It is unlikely that you have ever used it again past grade 7, but we can now apply this principle of long division to divide polynomials.

As an example, divide $2x^2 + 5x - 8$ by $x + 3$. Since long division symbols aren't allowed in LaTeX, I will use the square root to replace it, don't mind it.

> $x + 3\sqrt{2x^2 + 5x - 8}$. We first divide the largest polynomial term(x) by the largest polynomial term in $2x^2 + 5x - 8$. We get our first divisor, 2x. We then subtract 2x(x+3), and this eliminates 2x^2 from the original equation. It's really hard to show long division in LaTeX but I hope this is understandable.
>
> $2x^2 + 5x - 8 - (2x(x + 3))$
> $2x^2 + 5x - 8 - (2x^2 + 6x)$
> $-x - 8$
>
> We repeat this process again:
> $-x - 8 - (-1(x + 3))$
> $-x - 8 + x + 3$
> $-5.$

We now have our quotient: $2x - 1$and our remainder: -5.
You can see this works by rewriting the original terms as a combination of the quotient and remainder, since it is evident that $P(x) = Q(x)D(x) + R(x)$, where Q = Quotient, D = Divisor, R = remainder

> $2x^2 + 5x - 8 = (2x - 1)(x + 3) - 5$
> $2x^2 + 5x - 8 = 2x^2 + 5x - 3 - 5$
> $2x^2 + 5x - 8 = 2x^2 + 5x - 8.$

Tip: When performing polynomial division, make sure to restrict the domain of the function to assure that the divisor is not 0, else you run the risk of dividing by 0.

Tip 2: When there are polynomial terms that are missing in the dividend or divisor, make sure to add placeholders(Ex. 0x^2), to keep all terms aligned properly.

Synthetic Division: Dividing Polynomials quicker and easier.

Synthetic division can allow you to skip dealing with the variables, *as long as* the divisor is a binomial of form (x - n). It is extremely difficult to show in LaTeX, so I will be simply showing images of how to do it.

Sample Problem: Divide $2x^2 + 5x - 8$by $x + 3$.

We first form x + 3 as a binomial of (x - n), so (x - (-3)), we extract n and form a frame, using the coefficients of the dividend(Make sure to add placeholders if needed).

$$\begin{array}{r|rrr} -3 & 2 & 5 & -8 \\ & \downarrow & & \\ \hline & 2 & & \end{array}$$

You can see the n values goes on the outside and the first coefficient is just brought down. Then, multiply the brought down value by n and add it to the second coefficient to get the next "brought-down" value.

$$\begin{array}{r|rrr} -3 & 2 & 5 & -8 \\ & \downarrow & -6 & \\ \hline & 2 & -1 & \end{array}$$

Repeat until all is finished. Note how much faster this is. To interpret the solution, the first numbers are the coefficients of the quotient polynomial and the last digit is the remainder polynomial's coefficient. The reasoning behind why synthetic division doesn't work for anything other than a binomial is due to how the remainder can't be a polynomial if it's just one digit.

Tip: Sometimes, when dividing using synthetic division, the leading coefficient of the divisor isn't a 1. Thus, to get it into the form of (x - b), you need to divide the binomial by the leading coefficient. This is okay, but make sure to multiply the final result at the end by the same amount to keep the equation balanced.

For non-linear binomials, the problem with synthetic division becomes even more difficult, and it becomes no longer worth it to use synthetic division.

The Remainder and Factor Theorem:

Now, we can start moving onto really interesting ideas, factoring higher order polynomials. But first, we need to know the *remainder theorem.*

The remainder theorem shows that we can actually skip all of this long division, simply due to how factors work. We know already that division can be expressed as multiplication + remainder.

$P(x) = d(x)q(x) + r(x)$.

If we want to divide the polynomial by some binomial x - c, like in synthetic division, our equation becomes:

> $P(x) = (x - c)q(x) + r(x)$.
> But, instead of computing the division between P(x) and d(x), we can substitute in c as x:
> $P(x) = (c - c)q(x) + r(x)$
> $P(x) = r(x)$

Thus, the remainder theorem states that rather than computing the long division between p(x) and d(x), we just substitute c into P(x) to get the remainder. Similarly to synthetic division, the divisor has to be a binomial in the form of x - c to work, and thus if it has another constant, you will need to factor out the GCF of the binomial first.

Factor Theorem: The factor theorem comes pretty straight forward after, stating that if the remainder theorem gets a result of 0, then this c value is a factor of the polynomial. This is obvious, as if there is no remainder, this number is a factor. This allows us to quickly brute-force search for divisors in a polynomial.

Factoring Polynomials:

Since there is no “quadratic formula” for this, we identify a polynomial as factorable iff their roots are rational. At the current point, we only know how to see if a root is valid, and how to graph the polynomial function with the roots. But, we don’t know how to find roots better than random guessing. We can actually narrow down which roots are possible using the rational root theorem.

Rational Root Theorem: Integer roots can only come from the constant term, in the form of q / r, where q is the constant term and r is the factor. This means that given a polynomial, we only ever need to test the roots that cleanly divide into the constant term and can ignore the other terms. To factor the polynomial, we need to test every option, and when we find one, we can reduce the polynomial down immediately to reduce the work needed.

Sample Problem: $factor\ x^3 - 4x^2 - 3x + 18$

> We first look at the constant term 18, and can form a set of possible factors: $\{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18\}$. Testing all factors, the first one I found was (x + 2)(Using the factor theorem and using -2 as the input value).
>
> The next thing we do is divide the polynomial by x + 2, reducing the problem to a quadratic expression and then we can simply factor it as usual, getting $(x + 2)(x - 3)^2$as the factored form of the equation.

An alternative to performing the long division is the idea of *Have and Need* Factoring, where based on the term we already have(In the case of the previous example, x + 2), we simply perform the following:

We know $(x + 2)(ax^2 + bx + c) = x^3 - 4x^2 - 3x + 18$, and since the $x(ax^2)$term during FOIL will be the only cubic term, we know a = 1, and similarly, 2c will be the only constant term, so we know c = 9. Thus, we can skip ahead and simply plug in:
$(x + 2)(x^2 + 9) = x^3 - 4x^2 - 3x + 18$, and solve for the middle terms we need.
$(x + 2)(x^2 + 9) = x^3 + 2x^2 + 9x + 18$.

We subtract $x^3 - 4x^2 - 3x + 18$ by $x^3 + 2x^2 + 9x + 18$ to find what we need, which is: $- 6x^2 + 12x$, and then we factor out the GCF to get: $- 6x(x + 2)$. This polynomial term must be the same as the original x+2, or else there is an error. We add the -6x term in, to get the quadratic term we need: $x^2 - 6x + 9$.

Some people find have and need factoring easier, but I find it rather convoluting rather than beneficial. However, it is a valid option.

For factoring even higher degree polynomials, you just have to repeat the process over and over again.

Methods to Factor Higher Order Polynomials:

At this point, we have a few strategies to factor high order polynomials: including GCF factoring, Factor Theorem, and Rational Root Theorem.

When tackling a factoring/solving problem, we immediately try to factor out any GCF's. After doing this, if the degree is still higher than 2, we have to leverage the rational root theorem and factor theorem to find all factors.

One final strategy that you might be able to use is factoring by grouping, grouping together the largest terms possible in the polynomial. This is similar to factoring by grouping in quadratics, but now groups can be larger and there may be more of them.

Special Factors:
Just like the sum of squares and difference of squares factoring methods, there are also special factoring rules for sums and differences of cubes. Here they are:
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

We can use these rules to quickly factor equations and avoid the factor theorem entirely.
Factoring when there is a skip of factors:
Sometimes, you have a "seemingly" large equation to factor, but in reality it is a quadratic, or lower degree in disguise. For instance:

$f(x) = ax^4 + bx^2 + c$

It looks annoying to deal with, but you should consider how this is really just a quadratic, with the input x^2.

$f(x^2) = a(x^2)^2 + b(x^2)^1 + c$

Note how it now looks more like a quadratic. When such special functions appear, it makes factoring significantly easier.

Solving Higher Degree Equations:

Now that we know how to factor high degree polynomials, solving them becomes trivial. We simply need all terms on one side of the equation and factor to solve for the root.

Sample Problem: Solve $2x^3 + 8x^2 - 3x - 12 = 0$.

Luckily, all terms are already on the left side, and thus it simply becomes a problem of factoring. We notice that this problem can be solved using factoring by grouping, where it becomes:

$2x^2(x + 4) - 3(x + 4) = 0$

$(2x^2 - 3)(x + 4) = 0$

Thus, we can simply solve for the two sides:

$2x^2 = 3$

$x^2 = 3/2$

$x = \pm\sqrt{\frac{3}{2}}$

And:

$x + 4 = 0$

$x = -4$

Higher Order Inequalities:

Solving inequalities is relatively simple, just solve it like an equation but make sure to flip the direction when multiplying or dividing by a negative number. When presenting the solution for an inequality, you can show it in either the form of an interval or a set.

Ex. in set notation or: $(-\infty, -3]$ in interval notation.

In terms of solving the set of the inequality, you need to solve for the roots, and then you have 2 approaches: Algebra, or Graphing. Given the roots, you can graph the inequality and visually extract the intervals of when the inequality is true, or you can algebraically reason when a certain amount of products is positive or negative and use this to solve the inequality(The other side has to be 0 for this to work).

Ex. *Find the interval of when* $2x^2 - x^3 \geq 2 - x$.

We first factor this equation, moving all terms to one side, yielding the inequality:

$x^3 - 2x^2 - 2x + 2 \leq 0$. After factoring by grouping, we get: $(x - 2)(x + 1)(x - 1) \leq 0$.

Now, based on these factors, we need to find the intervals where this inequality is true. The easiest and least-error prone way to do this is using a table. We identify the intervals where anything changes, being $x \leq -1$, $-1 \leq x \leq 1$, $1 \leq x \leq 2$, and $x \geq 2$. Then, using a table, you see if the binomial is positive or negative and multiplying the signs together, you know if it's positive or negative.

Ex.

	x+1	x-1	x-2	=
x≤-1	-	-	-	-
-1≤x≤1	-	+	-	+
1≤x≤2	-	+	+	-
x≥ 2	+	+	+	+

Thus, the intervals are: $(-\infty, -1] \cup [1, 2]$

Next Unit: Rational Functions