Advanced Functions

Unit 4: Rational Functions

What is a Rational Function:

Rational Functions: A function in the form of *G(x) / F(x)*, where *G(x)* and *F(x)* are polynomials and *F(x) IS NOT* 0. Ex. $f(x) = 1 / (2x + 3)$

Note that rational functions have to be valid polynomials, and thus absolute values and square roots are not allowed.

How does the reciprocal function map to the base function?

Note that the reciprocal function is defined as $f(x) = 1 / g(x)$. Let's discuss some patterns found between *f(x) and g(x)*.

Asymptotes and zeros: The Asymptotes of *f(x)* are the zeros of *g(x).*
Positives and Negatives: Wherever *g(x)* is positive, *f(x)* is also positive
Domain: The domain of *f(x)* is $\{x \in \mathbb{R} | f(x) \neq 0\}$
Increasing and Decreasing: When *g(x)* is decreasing, *f(x)* is increasing.
Local Minima and Maxima: When *g(x)* reaches a local minimum, *f(x)* is at a local maximum.
Even and Odd Functions: g(x) and f(x) are of the same function type(Even = Even, Odd = Odd, not Even nor Odd = not Even nor Odd)

Using this information, we can easily sketch a reciprocal using the following algorithm:

1) Graph the original function
2) Determine the vertical asymptotes by finding the zeros
3) When drawing the reciprocal, use local minima and maxima as reference points, make sure that the signs match and the direction of movement are the same
4) If any other transformations are applied after the reciprocal, apply them now as usual.

This algorithm can be used to roughly sketch the graph of any rational function, cubic or quartic, etc.

Vertical Asymptotes:

All rational functions in the form of $f(x) = \frac{g(x)}{h(x)}$ will have a vertical asymptote at a, if $h(a) = 0$. An asymptote is a line that the graph approaches, but never touches. But, we need to communicate what direction the graph is "asymptoting" at. We can do this with the language of *limits*.

We can communicate that a graph approaches the value of(but never reaches), the value of 5 using arrow notation: $x \rightarrow 5$. We can specifically indicate the direction that the graph approaches with + and - signs, ex. $x \rightarrow 5^{-}$ to indicate that x approaches from the left and $x \rightarrow 5^{+}$, to indicate from the right.

Then, we can use limit notation to express where the function heads towards as it moves towards 5, using the following:

$\lim_{x \to 5-} f(x) = -\infty$. This denotes that as x approaches 5(closest possible value to 5), f(x) approaches - infinity.

Now, using all of this, we can express the asymptote of any given graphs with ease, using limit notation.

Sample Problem: represent the asymptotes present in the graph of $f(x) = \frac{-2x+4}{x^2-x-2}$

We first factor everything out, getting:

$f(x) = \frac{-2(x-2)}{(x+1)(x-2)}$

Conveniently, we can knock out the numerator, yielding:

$f(x) = \frac{-2}{x+1}$

Now, we know this will be invalid at x = -1, meaning the domain will be $\{x \in \mathbb{R} | x \neq -1\}$.

Taking a look at the asymptote to see it's behaviour, we test values around -1, yielding the following table:

x	f(x)
-1.5	4
-1.1	20
-0.9	-20
-0.5	4

We can evidently see that $\lim_{x \to -1^-} f(x) = \infty$and $\lim_{x \to -1^+} f(x) = -\infty$.

Note something very important, there is a *point of discontinuity* in this example, at x = 2. Remember how we factored out (x-2) from the numerator and denominator. That is possible, but remember that 0/0 is still undefined, so x cannot also be 2. Therefore the domain is $\{x \in \mathbb{R} | x \neq -1, 2\}$

We can also use the knowledge of limits and asymptotes to solve for rational functions given their properties.

Sample Problem: Find a rational equation where a point of discontinuity exists at (4, -2) and a vertical asymptote exists at x = 0.

We know that for this to occur, x - 4 needs to be in both the numerator and denominator. Furthermore, for a vertical asymptote to exist at x = 0, x by itself must be in the denominator. Thus, we form the function: $f(x) = \frac{k(x-4)}{x(x-4)}$. We solve for k, knowing that (4, -2) should exist on the graph, ignoring the restriction on the domain for now.

$f(x) = k/x$
$-2 = k/4$
$-8 = k$

Thus, a valid function would be: $f(x) = \frac{-8(x-4)}{x(x-4)}$ or fully expanded: $f(x) = \frac{-8x+32}{x^2-4x}$

Horizontal Asymptotes:

Previously, we just worked with vertical asymptotes and described them using limits. How can we do the same for horizontal asymptotes? To do so, we need to figure out the end behaviours of a function, i.e $\lim_{x \to \infty} f(x) = ?$.

Similar to vertical asymptotes, we want to keep testing values as they get closer and closer to infinity and see where the graph converges. Combining this with testing with Vertical Asymptotes, we can model the behaviour of a graph very accurately.

Technically, this testing for horizontal asymptotes isn't actually "mandatory". We can use a basic limit principle if the numerator and denominator have the same degree.

Ex. What is the horizontal asymptote of: $f(x) = \frac{2x-3}{x+1}$.

> When we look at 2x-3 and x+1, as x approaches infinity, this -3 and 1 becomes completely insignificant, and thus only 2x / x remains. If we divide this, we find that the horizontal asymptote will be found at y = 2.

By using this method, we can quickly find the horizontal asymptote rather than testing every single value. Note the general rule, if the numerator is greater than the denominator(In degree), there is no horizontal asymptote. If they are equal, then the horizontal asymptote is the ratio of the leading coefficients. If the numerator is less than the denominator, then the asymptote is at y = 0.

Oblique Asymptotes:

However, oblique(or linear) asymptotes can exist where the numerator is one degree higher than the denominator.

You can compute this *slant/oblique* asymptote by finding the quotient of the rational function. The remainder will fade away as x approaches infinity.

Rational Functions in the form of $\frac{ax+b}{cx+d}$

So far, we have only looked at simple rational functions. How can we graph rational functions in the numerator and denominator?

Rational Functions in the form of $\frac{ax+b}{cx+d}$ behave really similarly to a reciprocal counterpart in the form of $f(x) = \frac{a}{b(x-h)} + k$. Take the example of $f(x) = \frac{-2}{x-3}$+1, we can rewrite this into a rational function:
$f(x) = \frac{-2}{x-3} + \frac{x-3}{x-3} = \frac{x-5}{x-3}$. From this form, we can see that the zero in the function exists at x = 5.

Through switching between these two forms, we can find points of discontinuity in the graph, asymptotes, x and y intercepts, etc., which will be very useful in graphing the function. Note how the y-intercept is found at b/d and the x intercept at a/b.

Solving Rational Equations

Solving rational equations is the exact same as solving regular algebraic equations. However, sometimes, you may have to cross multiply before solving. Also, please recall the principle of points of discontinuity. These can exist in the graph and can lead to invalid roots.

Sample Problem: Find the POI of $f(x) = \frac{2}{x^2-1}$, $g(x) = \frac{x}{x+1}$.

$f(x) = g(x)$

$\frac{2}{x^2-1} = \frac{x}{x+1}$

$\frac{2}{(x+1)(x-1)} = \frac{x}{x+1}$

We remove x+1, taking in note that this is an invalid root)(POD)

Furthermore, x-1 is also a POD in the graph. This means -1 and 1 cannot be a root.

$\frac{2}{x-1} = x$

$2 = x(x - 1)$

$2 = x^2 - x$

$0 = x^2 - x - 2$

$0 = x^2 - x - 2$

$0 = (x - 2)(x + 1)$

Thus, only x = 2 is a valid root.

Substituting in for x, f(2) = 2/3

Thus, the POI is (2, ⅔)

Tip: When solving rational equations and one of the sides is in the form of $Q(X) + R(x) / D(X)$, simply rearrange it first to: $(Q(X)D(X) + R(X))/D(X)$, which allows you to cross multiply.

A similar principle can be applied to rational inequalities.