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Advanced Functions

Unit 2: Higher Polynomial Functions

Note: Technically there was a previous unit covering basic functions again, but they discussed pretty much the same material as unit 1: Exponential and Sinusoidal functions. Thus, I'm skipping it.

Exploring Higher Order Polynomial Functions:

At this point, we have only ever seriously studied linear and quadratic functions, only rarely moving to higher degrees of equations. Let's first give a formal definition for a polynomial:

Polynomial: The sum of algebraic terms where: all coefficients are real numbers AND exponents of the algebraic terms must be non-negative *integers*. This means: No Absolute Values, no square roots, no negative exponents, no complex numbers and no sinusoidal functions.

Polynomial Function: A function with a polynomial in one variable.

Leading Coefficient: The coefficient of the polynomial term with the highest degree, the algebraic term that defines the degree of the polynomial(Leading Coefficient is the coefficient of this term)

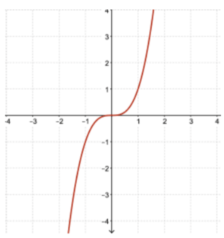
We just call functions as nth degree polynomial functions, but the names for degree 0-5 are: constant, linear, quadratic, cubic, quartic, quintic. After this, there is no "special name" for higher degrees.

Different Forms of Polynomial Functions:
You may have seen these in quadratic functions, but they are useful too in higher degree functions:

Standard Form: $ax^2 + bx + c$, Factored Form: $(x - h)(x - k)$

We may need to switch between the two to solve for higher degree equations, by factoring, or expanding to solve for the leading coefficient. Techniques for factoring higher degree polynomials will be discussed later.

Power Functions: Functions in the form of $f(x) = ax^n$
The Cubic Function, in the form of $f(x) = x^3$, has some interesting properties. Firstly, it has a domain and range of all real numbers. But not only this, it has what is called an *inflection point*, where the *concavity* of the function switches.



Inflection Point: The exact point where the *concavity* of a function changes.

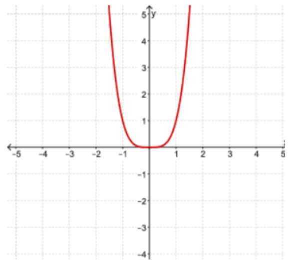
Concavity: the curve of the graph, concave up is when the line looks more like a U shaped graph, concave down is when it looks more like an N shaped graph.

Concavity is more a topic studied in Calculus I, but it can be seen on this curve of x^3. We will learn how to identify these points in that class.

Note that the curve of $f(x) = x^3$ is completely horizontal at x = 0 and is also symmetric around the origin.

The Quartic Function:

The quartic function resembles the quadratic function, which is why we can group odd and even polynomial functions. However, notice that it lingers around the origin a little longer than the quadratic function and then explodes upwards at a rate much quicker than the standard parabola.



Higher degree polynomial functions follow the pattern of cubic and quartic ones, but become even more extreme around the origin.

We do not fully discuss the graphs of fractional degree functions. This is due to how they are not polynomial. However, as the fraction gets larger, it matches closer to the high degree polynomial function. Even denominators will only have a positive domain(due to how there is a square root).

Transformations of Higher Degree Polynomial Functions:

Note that all transformations almost always have the same parameters, *a, b, h, and k*. However, how we present them varies greatly based on the type of function. For instance, in Grade 10, you learned about quadratics and how they can be presented in $f(x) = ax^2 + bx + c$. But, this form wasn't too easy to figure out what the transformations were. Thus, vertex form exists and it is: $f(x) = a(b(x - h))^2 + k$.

Thus, we create transformational form, which allows us to generally transform any given function. It is simply: $f'(x) = af(b(x - h)) + k$. Then, you can expand the higher order polynomial knowing the transformations are encoded inside already. The reasoning behind each of these parameters are the exact same as normal functions.

Note what transformations actually do for the function. They take a pair of (x, y) and converts it to: $(\frac{1}{b}x + h, ay + k)$.

Knowing this knowledge, we can now identify and draw higher degree polynomials with transformations as usual(I won't dive into the details as they are pretty much the same as quadratic, sinusoidal, or linear).

Characteristics of Polynomial Functions of degree 0-6:

Constant Functions: $f(x) = a$, Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R} | y = a\}$

Linear Functions: $f(x) = mx + b$, Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$

Quadratic Functions: $f(x) = ax^2 + bx + c$, Domain: $\{x \in \mathbb{R}\}$. The range varies a bit depending on the a value, with a > 0 leading to open up parabolas and a < 0 leading to open down parabolas.

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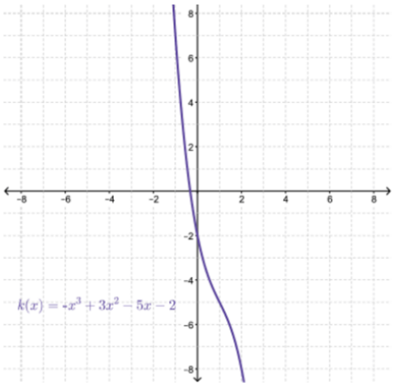
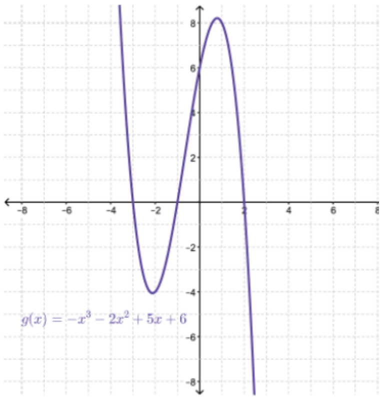
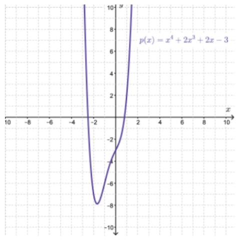
Cubic Functions: $f(x) = ax^3 + bx^2 + cx + d$.

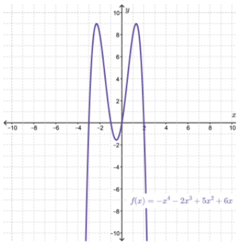
Cubic Functions have a few special properties, they can have 1, 2, or 3 solutions, and can have 0 or 2 turning points(1 or 3 inflection points). For instance, compare the graph of
$f(x) = -x^3 + 3x^2 - 5x - 2$ and $g(x) = -x^3 - 2x^2 + 5x + 6$.





Quartic Functions: $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. Quartic functions can either have 1 or 3 turning points. They can have 0, 1, 2, 3, or 4 solutions. Ex:





Quartic functions have a similar property to quadratic ones, being open up if a>0 else open down.

Odd Degree Functions: Have end behaviours like linear functions, where if the leading coefficient is positive, it moves from the bottom-left to the top-right. If it is negative, then the graph moves from the top-left to the bottom-right.

- Odd Degree Functions have an even number of turning points, up to n-1 points.
- They also have minimally 1 x-intercept, up to n x-intercepts

Even Degree: End behaviours like parabolas, they will have an odd number of turning points, up to n - 1 turning points. There will also be 0 up to n x-intercepts, and will have an absolute minimum and maximum point, similar to parabolas.

We can now apply this knowledge to graph the functions.

Graphs of Functions from Factored Form:

Given a higher order polynomial function in it's factored form, how can we draw the graph? We need a k value, which represents how much the function changes(like a slope) and the zeros, which don't change no matter what the k value is. Note that even if there can be 0, 1, 2, 3, …, zeros in the equation, in reality, there are a few zeros with a larger *multiplicity*, which means that the factored zeros are repeated.

Zeros with multiplicity 1 pass right through the x-axis, similar to a linear function. Zeros with multiplicity 2 have a turning point. For zeros with multiplicity 3, a point of inflection occurs. Any even multiplicity that is larger will be another turning point that lingers around the zero and any odd multiplicity that is larger will have a flatter point of inflection.

Combining all of this knowledge, we can now sketch any given graph given its factored form. We first need to determine the degree of the function, and then find the multiplicity and sketch each zero of the function.

Solving for Polynomial Functions:

Family of Polynomial Functions: “Families” of n-degree polynomial functions are functions that all have the same x-intercepts but different k-values.

If we are given the roots of a family and one point, we could solve for the equation of the graph. Note that this point cannot be a zero of the graph.

Sample Problem: Solve for the polynomial function with 2 real roots at $x = -5 \pm \sqrt{2}$and $x = 1 \pm i\sqrt{3}$, with y-intercept of -46.

> This seems daunting at first, but let’s work through the problem. Given the family of equations, we can set up a base equation
> $f(x) = k(x - (-5 + sqrt(2)))(x - (-5 - sqrt(2)))(x - (1 + isqrt(3)))(x - (1 - isqr$
> Expanding and Simplifying with Tedious FOIL:
> $f(x) = k(x^2 + 10x + 23)(x^2 - 2x + 4)$
> We then plug in the fact we know the y-intercept is -46.
> $-46 = k(23)(4)$
> $-1/2 = k$
> Thus,
> $f(x) = -1/2(x^2 + 10x + 23)(x^2 - 2x + 4)$

Now, we have the ability to solve for the equation of a graph given the graph itself, or the roots and a point.

Finite Differences:

You should already know how finite differences work to classify linear and quadratic functions. As you probably guessed, you can continue the pattern of finite differences to continue classifying higher degree polynomial functions. But, did you know about the connection of the finite differences to the leading coefficient of the higher degree function?

$\Delta^n y / n! / \Delta x = a$, this means that to compute the leading coefficient, we divide the finite differences with the factorial of the degree and the change in x.

Knowing this, we can now easily compute the equation of any polynomial function given a table of values.

Symmetry of Odd Functions and Even Functions:

Odd Functions: Functions that are symmetric around the origin, meaning $f(-x) = -f(x)$.
Even Functions: Functions that are symmetric around the y-axis, meaning $f(x) = f(-x)$.

Odd Terms: Polynomial Terms with odd degree
Even Terms: Polynomial Terms with even degree

Odd Functions can only be created from odd terms, we can prove this using algebra on a cubic example:

Let's start with the general form of a cubic function: $f(x) = ax^3 + bx^2 + cx + d$
$-f(x) = -(ax^3 + bx^2 + cx + d)$
$-f(x) = -ax^3 - bx^2 - cx - d$
$f(-x) = a(-x)^3 + b(-x)^2 + c(-x) + d$
$f(-x) = -ax^3 + bx^2 - cx + d$
You can see that it's impossible for the bx^2 and d terms from being equal, and this shows how only the odd terms can remain. Furthermore, we can do a similar proof of why cubic functions can never be even.

$f(x) = f(-x)$
$ax^3 + bx^2 + cx + d = a(-x)^3 + b(-x)^2 + c(-x) + d$
$ax^3 + bx^2 + cx + d = -ax^3 + bx - cx + d.$

Since a!= 0, then it's impossible for it to be even. This also shows that even functions can only be created from even terms.

Next Unit: Polynomial Equations and Inequalities