Andrew Shao

Advanced Functions

Unit 1: Exponential and Sinusoidal Functions

Basic Form of an Exponential Function:

The basic form of an exponential function is as follows: $f(x) = c^x$, where c > 0 and c != 1. When c > 1, then the function is classified as *exponential growth*, which can be modelled using: $f(x) = a * c^x$, where a is the initial state, c is the rate of growth, and x is the time step.

Ex. If we are given the information that a population of rabbits quadruples every year, and there are 30 rabbits initially, how many rabbits will exist in 8 years?

We can model the function as $f(x) = 30 * 4^x$. And thus: $f(8) = 30 * 4^8 = 1966080$

Exponential Decay: an exponential function where 0 < c < 1, and the function decreases by a decaying amount at every time step.

Ex. A radioactive isotope with a half-life of 1 year has 60g starting mass. How much remains after 5 years?

We can model the function as: $f(x) = 60 * (1/2)^x$, and thus: $f(5) = 60 * (1/2)^5 = 1.875g$

Properties of Exponential Functions:

Domain of an exponential function: All real numbers, since negative exponents represent moving back in time instead of forward in time. Fractional exponents represent not a full time step(ex. ½ = half an hour).

Y Intercept of an Exponential Function: The y-intercept doesn't depend on the base c, as $c^0 = 1$. Thus, the y-intercept is *a*.

X-intercept: The x intercept doesn't exist for an exponential function(Unless you add some constant factor d). The function will get closer and closer to 0, as x becomes more negative, but it will never strictly reach 0, meaning that the function has an asymptote at 0.

Range of an Exponential Function: Either 0 < y or y > 0, depending on the factor a, since exponential functions can grow infinitely, but asymptote at 0. The growth of the function depends on the factor a, for smaller a, the function is stretched out more.

Identifying and Writing the Equation for an Exponential Function:

When solving for exponential functions, we need to first identify if a function is exponential and not just quadratic or cubic. After this, we solve for the initial value a, and c.
We can actually do this given two Input-Output pairs only.

Ex. If we know that at the 2 hour mark, a drug has weight 50, and at the 3 hour mark, the drug has decayed to 25. What is the function denoting the exponential decay?

We know that f(2) = 50, f(3) = 25.

Thus, $f(2) = a * c^2 = 50,\ f(3) = a * c^3 = 25$

From linear algebra, we can divide the two equations and we get the following ratio:

$\frac{a*c^3}{a*c^2} = \frac{25}{50}$
$c = 1/2$

Then, we solve for a, using one point:

$f(2) = a * (1/2)^2 = 1/4\,a = 50$
Thus:
$a = 200$
Therefore the equation is: $f(x) = 200 * (1/2)^x$

From the graph representation of an exponential function, you can determine the exponential function from two points on the graph. But, it is easiest to pick the y-intercept, which can immediately tell you what the constant a is.

Finite Differences to Solve for an exponential function:

The finite differences method will be ineffective in terms of directly computing the base, since the finite difference will never be constant. However, when computing finite differences for an exponential function, the first, second, third, …, differences will all increase by a constant multiple. This multiple is the base of the exponential function.

Transformation of Exponential Equations:

Understanding how exponential equations get transformed is very useful to sketch and visualize their graphs. Recall that the equation is structured in the form of $f(x) = ac^{bx}$.

What is the effect of changing a?
- Changing the coefficient changes 3 factors: the direction of opening, the initial value, and vertical stretch and compression
- If $|x| > 1$, then the graph will be vertically stretched and vice-versa.

- If a < 0, then the graph is vertically reflected on the x-axis

What is the effect of changing b:

- if $|b| > 1$, then the graph will be horizontally compressed(vertically stretched) and vice-versa
- If b < 0, then the graph is horizontally reflected on the y-axis

Note: C cannot be changed, as this is a part of the original graph.

Translation of Exponential Curves:

Horizontal and Vertical Translations are the exact same as regular functions, mainly being the form of:

$f(x) = ac^{b(x-h)} + k$

When h is positive, the curve is shifted h units to the right, and when k is positive, the graph is shifted k units up.

Domain and Range of Exponential Functions:

In the normal exponential curve of form: $f(x) = c^x$, the domain is the $\{x \in \mathbb{R}\}$, and the range is $\{y0\}$. Please note that it is strictly greater, since there is a horizontal asymptote at 0.

However, transformations can drastically change the range of the function(domain isn't really affected). For a reflection on the x-axis, the $\{y0\}$ becomes $\{y0\}$. For a vertical translation, the range $\{y0\}$ becomes $\{yk\}$

When using this to sketch the graph, transformations should be applied in the following order:

1) Apply stretches, compressions, reflections
2) Apply Translations

Identical Exponential Curves:

A variety of curves can be represented using the exact same function. For instance, in representing an exponential decay of a radioactive material weighing one gram and a half life of 5730 years, we can represent this curve using two identical functions:

$f(x) = (0.999879)^t$ and $f(x) = (1/2)^{t/5730}$.

Note that in secret, these two functions are identical, due to the exponent of an exponent property. However, I would prefer to use the second function as it is much more clear to what it does.

Editing Exponential Curves:

We have seen previously that various exponential functions can be identical and some bases make more sense than others. Thus, we can change the base of the exponent using a multiplicative bias factor.

Ex. $f(x) = ac^x = ac'^{bx}$, where $bc' = c$

Modelling Exponential Decay:
You can also model exponential decay in a similar way. However, different models can be constructed due to the situation.

Ex. Newton's Law of Cooling: The temperature of an object cools at an exponentially decaying rate down to the temperatures of the surrounding.

Ex. $T(x) = a(initialTemp)^{t} + k$
Where k is the temperature of the surrounding temperature.

Example Problem: A cup of coffee is 60C, and in a room whose temperature is 20C. It's temperature halves every 10 minutes.

> We can model this equation using the function: $T(t) = 40 * (1/2)^{t/10} + 20$. We say the initial temp is 40, because at timestep 0, the temperature would be 40 + 20 = 60. Then, it decays at a rate of half every 10 minutes.
>
> Note that if the liquid is colder than the room temperature, you will find it will exponentially decay, but it has been reflected across the x-axis(Since it slowly grows in temperature)

Next Up: Sinusoidal Functions

Periodic Functions: Functions that repeat their outputs on a regular interval. Periodic functions don't have to be sinusoidal, they just have to repeat their outputs at regular intervals.

The horizontal gap between when the cycle repeats determines how frequently it repeats. This gap is called the *period* of the function.

Maximum and Minimum: The minimum and maximum points on the periodic function.

Axis: The midway line between the maximum and minimum points, the *amplitude* of the periodic function is the distance between the axis and the minimum/maximum point.

A function with period p can be written as $f(x) = f(x + np)$, where all points that are p intervals away have the same value.

While periodic functions can have arbitrary intervals repeating, it is often the sinusoidal periodic functions that are the most useful. These functions repeat in waves.

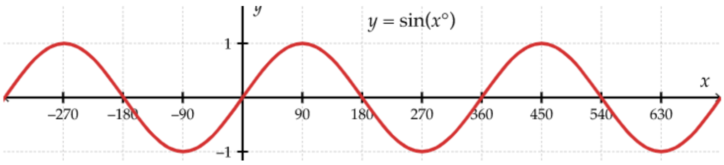
Sine Function:

Quick reminder of the sine function. The sine and cosine properties are used mostly in trigonometry, to solve for angles or sides of various triangles(Sine, Cosine, Tangent for Right angled triangles, Sine Law, Cosine Law for acute triangles).

The sine function represents the ratio between the opposite side and hypotenuse side of a right angled triangle. However, it has an interesting wave-like cycle, repeating every 360 values of x(360 degrees, or 2pi radians(But radians are harder to visualize on a graph)). This is due to the nature of the unit circle. If you imagine spinning a stick from the center of the unit circle, and plotting all y-values on the graph, you would find the sine function.

We use the y-value due to how the sine function is opposite / hypotenuse, where hypotenuse is always 1, and thus it is just the height of the triangle formed inside the unit circle.

Properties of the Sine Function:
Period: 360 degrees(360)
Maximum: 1
Minimum: -1
Axis: y = 0
Amplitude: 1

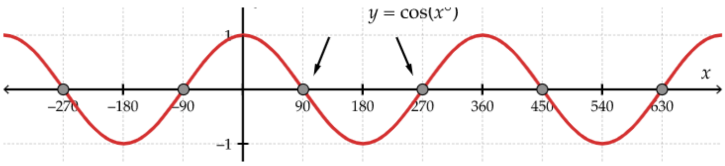


The sine function has a domain of $\{x \in \mathbb{R}\}$ and range of $\{y \in \mathbb{R} | -1 \geq y \geq 1\}$. It increases on the range of $-90 < x < 90$ and decreases on the range of $90 < x < 270$. The x-intercepts of the function includes all multiples of 180n.

The Cosine Function:

Recall from trigonometry that the cosine ratio is the ratio between adjacent and hypotenuse sides of a right triangle. It has a wave-like structure just like the sine function, but the wave looks slightly different.

Properties:
Period: 360
Minimum: -1
Maximum: 1
Amplitude: 1
Axis: y = 0
Domain: $\{x \in \mathbb{R}\}$. Range: $\{y \in \mathbb{R} | -1 \geq y \geq 1\}$.



Waves decrease from intervals of $0 < x < 180$ and increase at intervals $180 < x < 360$
X-Intercepts occur every 90 + 180n.

Transformations of the Sinusoidal Functions:

Knowing the fundamental sinusoidal functions is cool, but where they become actually useful is when you can change the properties of the functions while keeping the wave-like structure.

Vertical stretches and compressions work exactly the same as normal, in the form of $f(x) = af(x)$. Once again, for |a| > 1, the graph is vertically stretched, and vice-versa. A negative "a" value will result in the graph being reflected over the x-axis. Fundamentally, this changes the minimum and maximum by a, and also affects the amplitude of the curve in the same amount. This has no effect on the period of the function.

The maximum(starting from the base sinusoidal curve) goes from 1 to |a|, and the minimum becomes -|a|. The amplitude becomes |a|.

Horizontal stretches and compressions work the same as normal too, using the function $f(x) = f(bx)$. Values of |b| that are greater than 1 causes a horizontal compression, causing the period of the function to be smaller. Values of |b| less than 1 cause horizontal stretches, leading to larger periods. Values of b that are negative cause a reflection on the y-axis.

Note: When the sine function is reflected across the x-axis, it looks the same as being reflected on the y-axis. Functions like this are called *odd functions.* When the cosine function is reflected across the y-axis, it looks the exact same as being reflected on the x-axis. Functions like these are called *even functions*.

Graphing Sinusoidal Functions:
While this sounds like a daunting task, there are 5 key points that are evenly spaced on the sinusoidal curve, which can allow us to draw curves with high speed and accuracy. First point is the starting one, sine curves have 0 as this point, and cosine curves have 1 as this point. Second point and fourth points are the x-intercepts. Third point is the peak of the middle of the curve(either a max or a min). Fifth is when the cycle resets again.

Using these points, if we know the distance between each point, we can easily graph sinusoidal curves with little to no problem.

Using our knowledge of horizontal stretches and compressions, the period can be estimated as 360 / |b|, since 0<|b|<1 will stretch the graph and |b| > 1 will compress it.

Horizontal and Vertical Shifts:

Sometimes, we want the amplitude of the curve to remain the same, but we need to change the maximum and minimum. Or, we want the starting point of the curve to change. This is where vertical and horizontal shifts can come in handy.

Horizontal Shifts are called *phase shifts,* as they literally shift when the phases occur inside of the function. If two sinusoidal functions reach their maximum and minimum at different times but have the same period, they are called *out of phase* and *vice-versa.*

As usual, the shifts occur using the formula $f(x) = f(x - h) + k$, where h is the horizontal shift, and k is the vertical shift. Note that h has to be relatively large for this to work, since we are working with degrees here.

The value of the vertical shift affects the axis of the equation and the range of the equation. The value of h doesn't change anything noticeable from just properties, but visualizing the graph shows that it majorly changes the phases. To differentiate these equations compared to the others, we will need to specify the point of the origin, so one can realize what the vertical and horizontal shift is.

Graphing Sinusoidal Functions:

Strategy For Graphing Sinusoidal Functions:

1) Graph the base sinusoidal function(Sin(x) or Cos(x))

2) Apply reflections, stretches or compressions
    a) Stretches increase the amplitude, compressions decrease it
    b) Compute the amplitude and period and use this to compute the 5 key points of the transformed graph.
3) Apply Translations, simply shift the origin, and change the graph in the same amount.

Tip: Instead of doing horizontal flips, you might find converting the flip to a vertical flip to be easier(for sine, horizontal flip = vertical flip; for cosine, horizontal flip = no flip)

Keypoint Method of Graphing: An alternative to the transformation based approach would be the keypoint graphing based approach, where you directly compute the 5 keypoints using the transformed graph equation and plot them.

Parameters Vs Variables:

Parameter: Values *a, h, k, b*, they don't affect the type of function but transform it in some way

Variable: *x and y*, affects the type of function.

Modelling Equations from Graphs:

One thing that is absolutely necessary with sinusoidal functions is the ability to model a graph using a sinusoidal function. So how is the a, h, k, and b values related to the graph itself?

To immediately figure out the "a" value of the graph, simply compute the amplitude of the sinusoidal curve.

Then, compute the horizontal stretch using the period of the function, which nets you the b value of the sinusoidal curve

Compute the k value of the curve using the axis of the curve.

The h value is a little different in the sense that you need to find the peak of the curve and try to shift it to overlap with the actual curve given. This is slightly harder to do than the other parameters.

Note that it doesn't matter if you choose sine or cosine curves. This is because they can be transformed to each other.

The base information you need to accurately model a sinusoidal curve is:

- Period
- Minimum and Maximum

With only this, you can determine the axis, amplitude, horizontal shift, …, to model it. It may be given through the form of a graph or simply a list of information(word problems)

Next Unit: Higher Degree Polynomial Functions