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Linear Algebra I

Unit 4: Vector Spaces

Vector Spaces:

Before we define what a vector space is, it is important to note once again that the 10 properties of vectors, matrices, and linear mappings also apply here.

A vector space is just a set of vectors that can support two basic operations, addition and scalar multiplication. The sum of two vector spaces is just the sum of two sets, and scalar multiplication is a scalar multiplied by everything inside of the set. However, since this is a non-standard operation, we can define a custom operation for addition or scalar multiplication, but the definition of this operation must comply with the 10 properties.

A vector space must at least contain one vector, the zero vector. To prove that the vector space is valid or not, we either must prove that it violates one of the 10 properties(counter example) or find a counterexample where this property doesn't hold.

Ex. Is the following set a vector space? $\mathbb{S} = \{(x, y) \in \mathbb{R} | x, y \in \mathbb{R}\}$.

> We define addition as $(x_1, y_1) \oplus (x_2, y_2) = (2x_1 + x_2, y_1 + 2y_2)$and scalar multiplication as:
> $t \odot (x_1, y_1) = (tx_1, ty_1)$. We can see that the scalar multiplication operation is just standard. But, the addition operation seems a little strange. Let's test if A + B = B + A.
>
> $(1, 2) \oplus (2, 3) = (2, 3) \oplus (1, 2)$
> $(4, 8) = (5, 7)$
> Thus, we have found a counter example and this set isn't a vector set.

Subspaces:

Subspaces: In previous units, we defined a subspace. A subspace is a non-empty subset of a *vector space*(Now a vector space) that is closed under addition and scalar multiplication. Exactly the same as previously, to check if a subset is a subset of the vector space, use the definitions of addition and multiplication of the vector space and check for the same 3 conditions.

3 Conditions:
1) Is the subspace not empty?(check for the zero vector)
2) Is it closed under addition? I.e if $A, B \in \mathbb{S}$, is $A + B \in \mathbb{S}$?
3) Is it closed under scalar multiplication? Is $cA \in \mathbb{S}$?

Example: Let $\mathbb{S} = \{ax^2 + bx + c \in P_2 | a^2 - b^2 = 0\}$, is this a subspace?

We take a look at the constraints and immediately find that this is almost guaranteed to not be, as the constraint isn't linear. We can prove this with a counter example

Let $x = 3x^2 - 3x + c$ and $y = 3x^2 + 3x + c$, the c doesn't matter in this case so I omitted it.

Both polynomials satisfy the constraints, but if we add them, the polynomial isn't closed under addition. Thus, it isn't a subspace.

Spanning, Subspace, and Linear Independence:

Spanning applies in the exact same way as before with vector sets, and the span of a vector space B is just a linear combination of all vectors in the vector space.

Theorem 4.1.3: if we have a given vector space V, then Span{V} is a subspace of V
Theorem 4.1.4: if v_k is part of the set Span{V}, then $Span\{v_1, ..., v_k\} = Span\{v_1, ..., v_{k-1}\}$
Linear Independence of Vector Spaces: If $(c_1 \otimes v_1) \oplus ... \oplus (c_k \otimes v_k) = 0$, where c is not the trivial solution, then the set is linearly independent.

Please note that the three definitions stated above are the same as in unit 1, but now they apply specifically to the definitions of addition and scalar multiplication of the vector space.

Bases:

Back in unit 1, basis sets existed for vectors in euclidean space, to act sort of like the coordinate space for a given set of vectors. More specifically, they were a linearly independent set that spans some given vectors. Bases also exist for vector spaces, and they are defined in the exact same way.

If you haven't noticed, vector spaces are very very *similar* to vectors in euclidean space.

Unique Representation Theorem: if B is the basis for a given vector space V, then for each vector inside of V, there is a ***unique*** linear combination in B to form this vector.

Standard Bases: Just like the standard basis vectors of $\mathbb{R}^n$, the standard basis for polynomial and matrix space just has a unique entry for each power of x or each spot in the matrix.

For instance, the standard matrix for a 2x2 matrix is as follows:

$$\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

Algorithm to Find a Basis:

1) Write x in the most general form
2) Write x as a linear combination of vectors/matrices given constraints
3) Repeat:
    a) Is this set a linear combination?
    b) If not, reduce it down.

Ex. Find a basis for $\mathbb{S} = \{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in M_{2x2} | \, x_1 + x_2 + x_3 = 0\}$

We first write the matrix in it's most general form, which is trivial. $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$
We then use the constraints of the problem, mainly being how $x_1 + x_2 + x_3 = 0$, and we find that $x_1 = -x_2 - x_3$. We substitute this value into the matrix.

$$\begin{bmatrix} -x_2 - x_3 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

Extracting the basis matrices of this set, we get:

$$\{x_2 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, x_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, x_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

We check if the set is linearly independent using row reduction and find that it is, thus the basis has been found.

Dimension of a Vector Space:

Theorem 4.2.1: if B is a set of vectors that is a basis in a vector space V, and C is a linearly independent set of vectors inside of the same set, then dimension of C is less than or equal to the dimension of B.

Proof: This theorem is true because since B is a basis for the vector space, it spans the entire space and is linearly independent. It's impossible that another linearly independent set inside of the same vector space can have more vectors than this basis.

What this theorem also tells us is how another interpretation of the basis is a maximal linearly independent set of a vector space. This also proves the following:

Theorem 4.2.2: If v and w are both a basis for a vector space V, then the dimension of v and w has to be the same.

Definitions of Dimension in Vector Spaces:
Vector: $\mathbb{R}^n$ has dimension n
Matrix: M_{mxn} matrices has dimension mn
Polynomial: P^n has dimension n + 1(all n exponents and 1)

If a given vector space has a dimension n, then the following properties apply:
1) No more than n vectors in the set can be linearly independent
2) N vectors are needed to span the set

3) A set B with n elements spans a vector space V if it is linearly independent.

Converting a linearly independent set to a basis:

Sometimes, you will be given a linearly independent set and you need to find a basis for the entire space. You can do this by finding enough vectors that don't exist inside the current set until you have all n vectors that are all linearly independent.

How can we do this without guess and checking? We can do this by adding the basis vectors of the space more generally, and then reduce down the set(trying to preserve the original vectors). The only basis vectors that will remain are the ones needed to extend the set to the basis vector.
Theorem 4.2.4: This means that if there is a k dimensional subspace of a vector space, then there exists n - k other basis vectors to fully span the vector space.

Theorem 4.2.5: If S is a subspace of V, then dim S <= dim V.

Coordinates:

Since bases are just a space where each vector can be mapped to a unique coordinate, it would be useful to have a specific formulation of a coordinate vector under a basis. We can express a vector as a coordinate under a basis by writing it as a linear combination of the basis vectors. You can see that this is even true in $\mathbb{R}^n$

Ex. $\vec{x} = c_1\vec{x_1} + \cdots + c_n\vec{x_n}$

The notation for a vector with respect to a basis is: $[v]_B$, and it is read as a vector with respect to a basis B. This vector is defined as the vector made up of the constants needed to form x. Note that this means that we need an ordered basis, since the order of the constants will change otherwise. This also means that if you know what the ordered basis is and you know $[v]_B$, you can simply do a linear combination of vectors using the constants in $[v]_B$to get the original vector v.

The inverse is also true. If you have vector v and ordered basis B, and want to find $[v]_B$, you simply write v as a linear combination of B and use the constants.

Theorem 4.3.2: $[sv + tw]_B = s[v]_B + t[w]_B$. In other words, the conversion of coordinates is a linear operation and thus is distributable

Change of Coordinates:

Sometimes, we need to change coordinate vectors from one vector to another. We may be given a vector in B-coordinates, not knowing the original vectors, and you want to change it to C-coordinates. We can do this by leveraging Theorem 4.3.2.

To do this by finding a transition matrix that will map from B-Coordinate basis vectors to the C-coordinate equivalents, this will allow for a transition matrix that can map any vector to the

C-coordinates you need, by definition of basis and basis vectors. This transition matrix is denoted as CP_B and represents the matrix needed from B to C coords.

Sample Problem: Find CP_B if $C = \{1, x, x^2\}$ *and* $B = \{1, x + 1, (x + 1)^2\}$.
We first need to form the augmented matrix to solve for a linear combination of the vectors in B to form the C vectors. We get the following systems of equations:

$$1 = a_1(1) + a_2(x + 1) + a_3(x + 1)^2$$
$$1 = a_1 + a_2x + a_2 + a_3(x^2 + 2x + 1)$$
$$1 = a_1 + a_2x + a_2 + a_3x^2 + 2a_3x + a_3$$
$$1 = (a_1 + a_2 + a_3) + (a_2 + 2a_3)x + a_3(x^2)$$

Similarly,

$$x = (b_1 + b_2 + b_3) + (b_2 + 2b_3)x + b_3(x^2)$$
$$x^2 = (c_1 + c_2 + c_3) + (c_2 + 2c_3)x + c_3(x^2)$$

We then form the augmented matrix and row reduce to get:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Thus, to get any C-coordinate version of a B-coordinate based vector, we simply matrix multiply that matrix with the vector. A sanity check would be to simply apply the change of basis vector to the B-coordinate bases and necessitate that this yield the C-coordinate basis vectors.

Note that multiplying $CP_B[BP_C]$ will yield the identity matrix, as it will convert the vectors to C-coordinate and then back to B-coordinate, doing nothing.

Next Unit: Inverses.