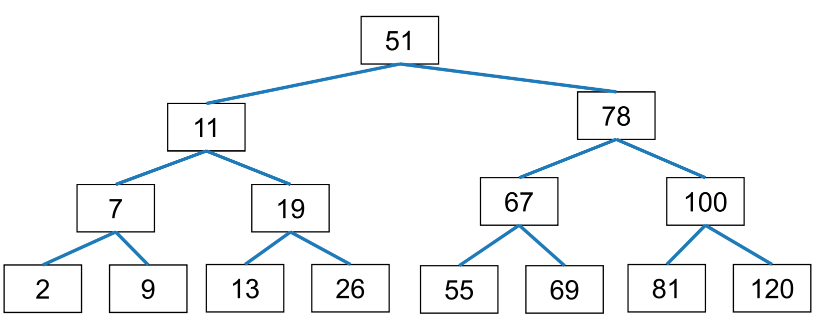
Trees

Yi-Shin Chen
Institute of Information Systems and Applications
Department of Computer Science
National Tsing Hua University
yishin@gmail.com

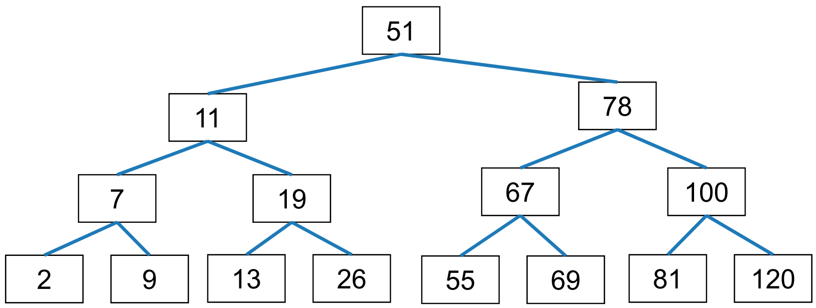
Tree Structure

- Data in a tree structure are organized in a **hierarchical** manner



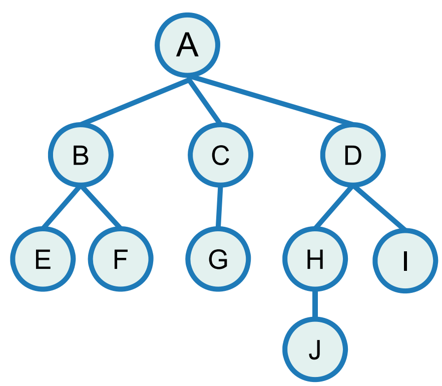
Tree Definition

- A **tree** is a finite set of one or more nodes
  - There is one ***root***
  - The remaining nodes can be partitioned into n disjointed sets $T_1, T_2, \ldots T_n$ ($n \geq 0$)
  - Each subset T_i is a tree (also called ***subtrees*** of the root)



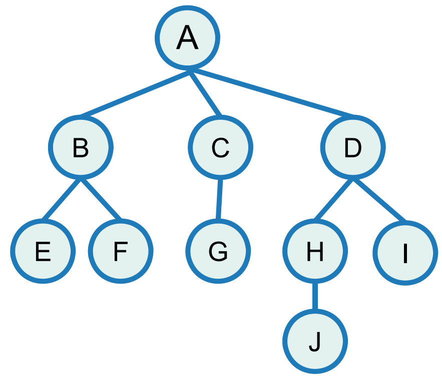
Terminology

- Degree of a node
  - The number of subtrees
  - E.g., deg(A) =3; deg(C) =1
- Leaf or Terminal nodes
  - The node whose degree is 0
  - E.g., E, F, G, J, I
- Non-terminals
- Degree of a tree
  - The maximum degree of the nodes in the tree
  - E.g., Deg. of the tree = 3



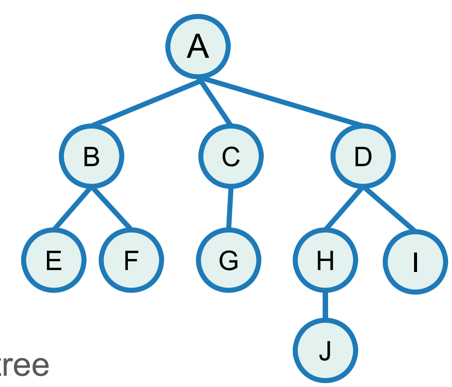
Terminology (Contd.)

- Parent / Children
- Sibling
  - Children of the same parent
  - E.g., B, C are siblings
- Ancestors
  - All nodes along the path from the root to that node
  - E.g., ancestor of J: H, D, A
- Descendants
  - All nodes in the subtrees



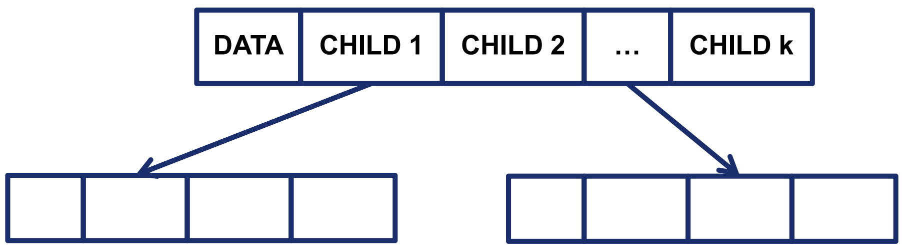
Terminology (Contd.)

- Level of a node
  - Level(root) = 1
  - Level(n) = ℓ + 1
    - if level of n's parent is ℓ
  - E.g., level(G) = 3
- Height or depth of a tree
  - Maximum level of any node in the tree
  - E.g., : Height of the tree = 4



List Representation

- Each tree node holds
  - A **data field**
  - Several **link fields** pointing to subtrees
    - Based on the degree of each node
    - E.g., For tree of **degree k**, allocate **k** link field for each node

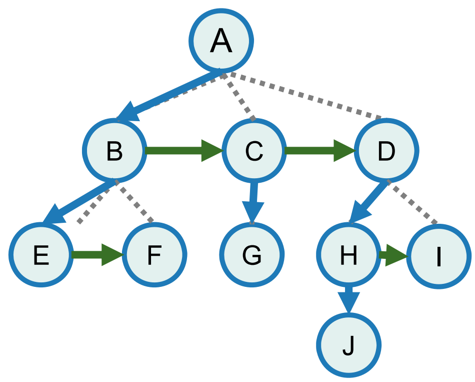


List Representation

- Disadvantage: Wasting the memory!
  - If T is a tree of degree k with n nodes.
  - The total # of link fields are n*k
  - Might have a lot of unused links

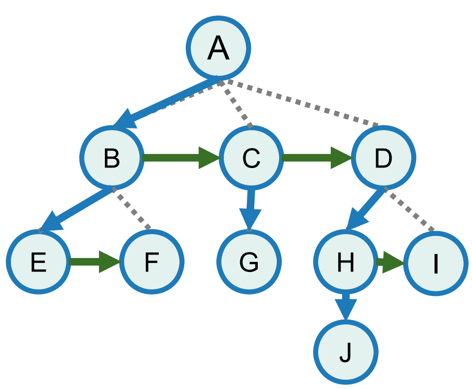
Left Child-Right Sibling Representation

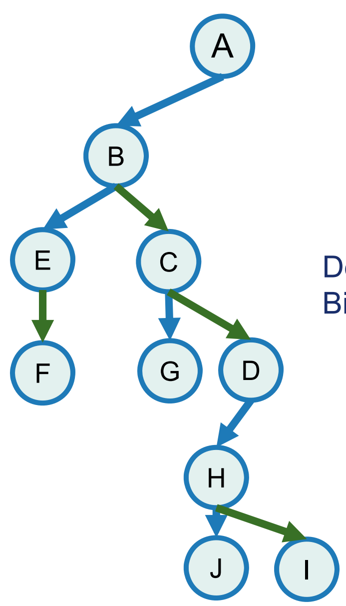
- Each node has exactly two link fields
  - Left link(child): points to leftmost child node
  - Right link(sibling): points to closest sibling node



Left Child-Right Sibling Representation

- Rotate clockwise 45^{o}

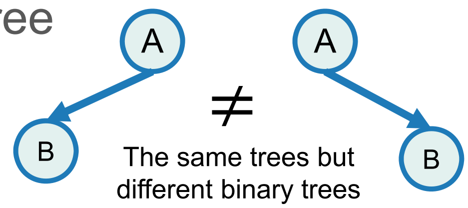




Degree-2 tree
Binary Tree

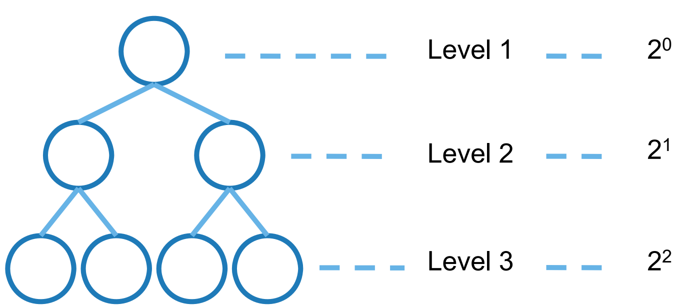
Overview of Binary Trees

- A binary tree is a finite set of nodes:
  - Either is empty
  - Or consists of
    - A root
    - Two disjoint binary trees
      - The left subtree
      - The right subtree
- Binary tree != Regular tree



Properties of Binary Tree

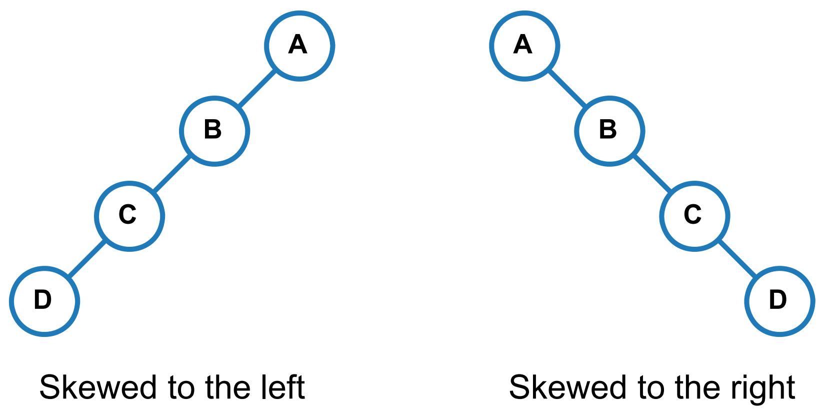
- [Maximum number of nodes]
  - The max. # of nodes on level i is $2^{(i-1)}$
  - The max. # of nodes in a binary tree with depth k is $2^k - 1$



Total # of node is $1 + 2 + 2^2 + 2^3 + \ldots + 2^{(k-1)} = 2^k - 1$

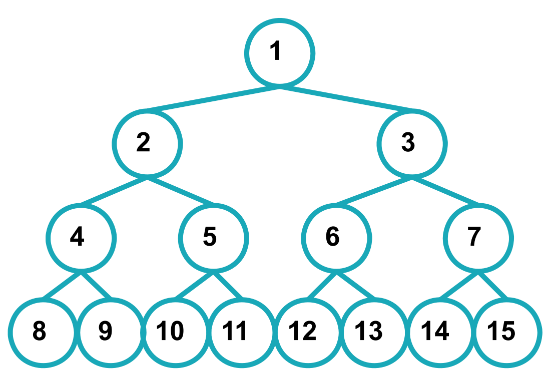
Special Binary Trees

- **Skewed** tree



Full Binary Tree

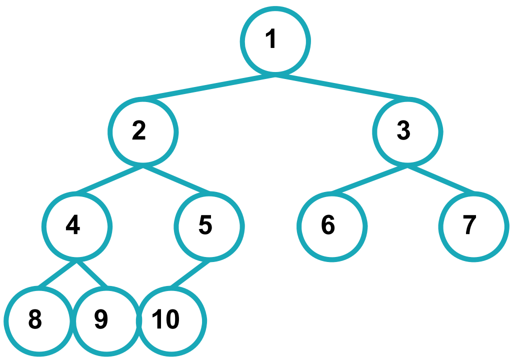
- A binary tree of depth k which has $2^k - 1$ nodes



A full binary tree of depth 4

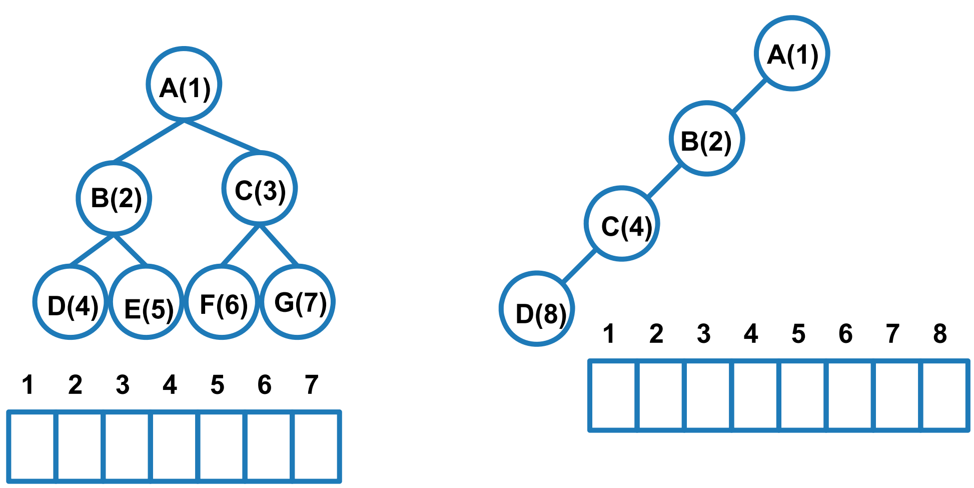
Complete Binary Tree

- A binary tree of depth k with n node is called complete
  - iff its nodes correspond to the nodes numbered from 1 to n in the full binary tree



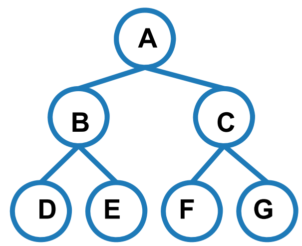
Array Representation

- The numbering scheme suggests to use a 1-D array



Array Representation

- Advantages: Easy to determine the locations of the parent, left child, and right child of any node.
- Let node i be in position i (array[0] is empty)
  - Parent(i) = i / 2   if i ≠ 1. If i=1, i is the root and has no parent
  - leftChild(i) = 2i if 2i ≤ n. If 2i > n, the i has no left child
  - rightChild(i) = 2i+1 if 2i+1 ≤ n, if 2i+1 > n, the i has no right child

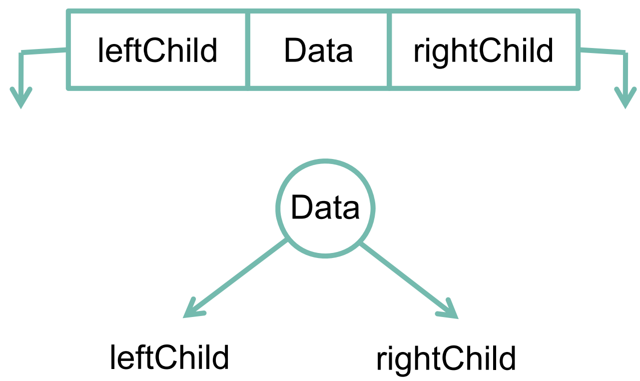


Array Representation (Contd.)

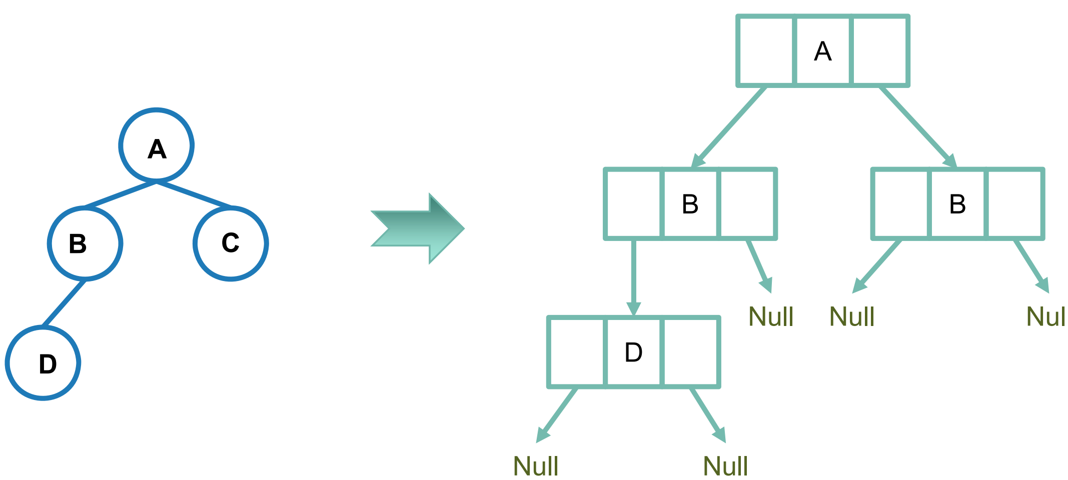
- Disadvantages:
  - Wasteful of space for a skewed tree
  - Insertion and deletion of nodes require move a large parts of existing nodes
    - To maintain sorted

Linked Representation

- Each tree node consists of three fields
  - Data, leftChild, and rightChild

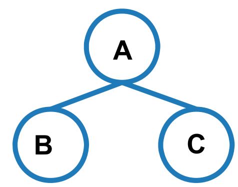


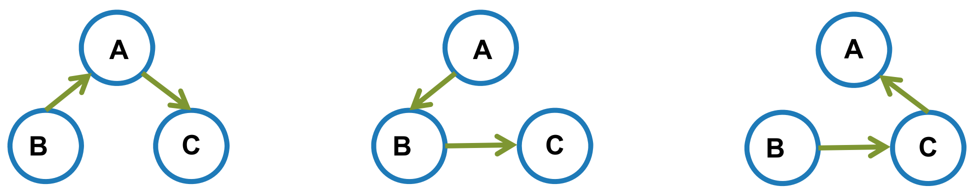
Linked Representation



Binary Tree Traversal

- Visit each node in a tree exactly once
- Treat each node and its subtrees in the same fashion
  - Inorder: left -> root -> right
  - Preorder: root -> left -> right
  - Postorder: left -> right -> root





Inorder Traversal

- Steps of traversal:
  - Step1: Moving down the tree toward the **left** until you can go no farther
  - Step2: **Visit** the node
  - Step3: Move one node to the **right** and continue
- Use recursion to describe this traversal

A / B * C * D + E

Inorder Traversal : Codes

```
template < class T >
void Tree<T>::Inorder()
{ // Start a recursive inorder traversal
  // This function is a public member function of Tree
  Inorder(root);
}

template <class T>
void Tree<T>::Inorder(TreeNode<T>* currentNode)
{ // Recursive inorder traversal function
  // This function is a private member function of Tree
  if(currentNode){
     Inorder(currentNode->leftChild);
     Visit(currentNode); // e.g., printout information
     Inorder(currentNode->RightChild);
  }
}
```


Preorder Traversal

- Steps of traversal:
  - Step1: Visit a node
  - Step2: Traverse left, and continue
  - Step3:When cannot continue, move right and begin again
- Use recursion to describe this traversal

+ * * / A B C D E

Preorder Traversal : Codes

```
template < class T >
void Tree<T>::Preorder()
{ // Start a recursive preorder traversal
  // This function is a public member function of Tree
  Preorder(root);
}

template <class T>
void Tree<T>::Preorder(TreeNode<T>* currentNode)
{ // Recursive preorder traversal function
  // This function is a private member function of Tree
  if(currentNode){
     Visit(currentNode); // e.g., printout information
     Preorder(currentNode->leftChild);
     Preorder(currentNode->RightChild);
  }
}
```


Postorder Traversal

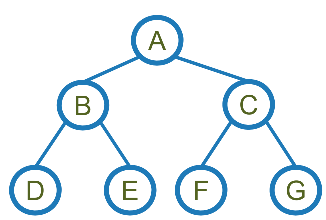
- Steps of traversal:
  - Step1: Moving down the tree toward the left until you can go no farther
  - Step2: Move one node to the right
  - Step3: Move back to visit the node and go right
- Use recursion to describe this traversal

A B / C * D * E +

Postorder Traversal : Codes

```
template < class T >
void Tree<T>::Postorder()
{ // Start a recursive postorder traversal
  // This function is a public member function of Tree
  Postorder(root);
}

template <class T>
void Tree<T>::Postorder(TreeNode<T>* currentNode)
{ // Recursive postorder traversal function
  // This function is a private member function of Tree
  if(currentNode){
      Postorder(currentNode->leftChild);
      Postorder(currentNode->RightChild);
      Visit(currentNode); // e.g., printout information
  }
}
```


Exercise



Traversal	Output ordered list
Inorder	
Preorder	
Postorder	

Tree Iterator

- We would like to visit nodes by using **iterator** visit elements in a container
  - Recursive traversal is not suitable
- An **iterative** version
  - Using stack to store non-visited nodes

Non-Recursive Inorder Traversal

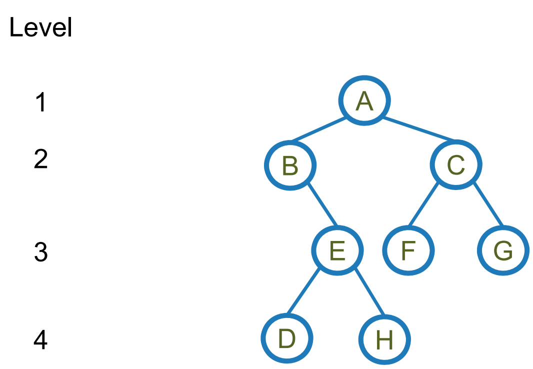
```
template < class T >
void Tree<T>::NonrecInorder()
{ // Non recursive inorder traversal using stack
  Stack<TreeNode<T>*> s;    // declare and init a stack
  TreeNode<T>* currentNode = root;
  while(1){
    while(currentNode){      // move down leftChild field
       s.Push(currentNode); // add to stack
       currentNode = currentNode->leftChild;
    }
    if(s.IsEmpty()) return; // all nodes are visited
    currentNode = s.Top();
    s.Pop();
    Visit(currentNode);      // e.g., printout information
    currentNode = currentNode->rightNode;
  }
}
```

Inorder Iterator

```
Class InorderIterator{ // A nested class within Tree
public:
  InorderIterator() { currentNode = root}
  T* Next();
private:
  Stack<TreeNode<T>*> s;
  TreeNode<T>* currentNode;
};

T* InorderIterator::Next()
{
   while(currentNode){      // Move down leftChild field
      s.Push(currentNode); // Add to stack
      currentNode = currentNode->leftChild;
   }
   if(s.IsEmpty()) return NULL; // All nodes are visited
   currentNode = s.Top();
   s.Pop();
   T& temp = currentNode->data;
   currentNode = currentNode->rightNode;
   return &temp;
}
```


Level-Order Traversal

- Visit nodes in a top to down, left to right manner



Preorder	Inorder	Postorder	

Level-Order Traversal : Codes

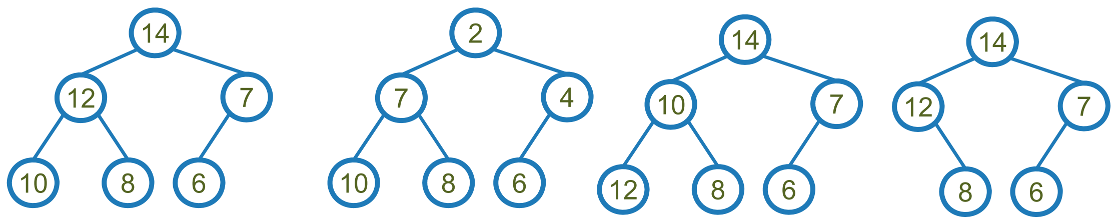
```
template <class T>
void Tree<T>::LevelOrder()
{ // Traverse the binary tree in level order
  Queue<TreeNode<T>*> q;
  TreeNode<T>* currentNode = root;
  while(currentNode){
    Visit(currentNode);
    if(currentNode->leftChild) q.Push(currentNode->leftChild);
    if(currentNode->rightChild) q.Push(currentNode->rightChild);
    if(q.IsEmpty()) return;
    currentNode = q.Front();
    q.Pop();
  }
}
```


Self-Study Topics

- Binary tree operations
  - Preorder traversal (Non-recursive & iterator)
  - Postorder traversal (Non-recursive & iterator)
  - Copying Binary Trees
  - Testing Equality

Heaps

- A specialized tree-based data structure
  - Satisfy the heap property
    - The value of parent node is either
      - Greater than or equal to the value of child (Max Heap)
      - Less than or equal to the value of child (Min Heap)
    - A complete binary tree

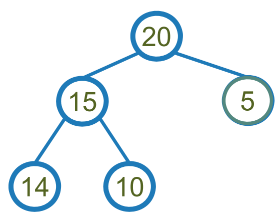


Max Heap : Representation

- Can adopt "**Array Representation**"
  - Since it is a complete binary tree
- Let node i be in position i (array[0] is empty)
  - Parent(i) = i / 2 if $i \neq 1$. If i=1, i is the root and has no parent
  - leftChild(i) = 2i if $2i \leq n$. If $2i > n$, the i has no left child.
  - rightChild(i) = 2i+1 if $2i+1 \leq n$, if $2i+1 > n$, the i has no right child

Max Heap : Insert

- Insert new node
- Make sure it is a complete binary tree
- Check if the new node is greater than its parent
  - If so, swap two nodes



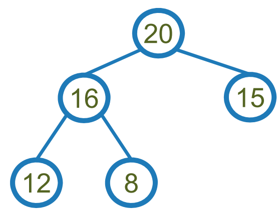
Max Heap : Insert Codes

```
template < class T >
void MaxPQ<T>::Push(const T& e)
{ // Insert e into max heap
  // Make sure the array has enough space here…
  // …
  int currentNode = ++heapSize;
  while(currentNode != 1 && heap[currentNode/2] < e)
  { // Swap with parent node
    heap[currentNode]=heap[currentNode/2];
    currentNode /= 2; // currentNode now points to parent
  }
  heap[currentNode]=e;
}
```

- Time Complexity:
  - Travel at most the height of a tree, therefore is O(logn)

Max Heap : Delete

- Priority Queues
  - The element to be deleted is the one with highest priority
- In priority queues
  1. Always delete the root
  2. Move the last element to the root (maintain a complete binary tree)
  3. Swap with larger and largest child (if any)
  4. Continue step 3 until the max heap is maintained (trickle down)

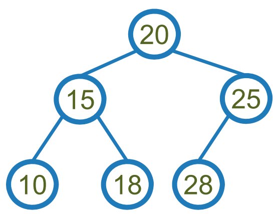


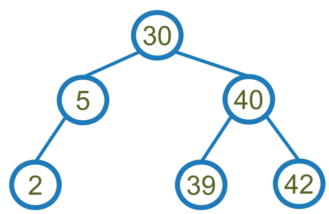
Max Heap : Delete Codes

```
template < class T >
void MaxPQ<T>::Pop()
{ //Delete max element
  if(IsEmpty()) throw "Heap is empty";
  heap[1].~T(); // delete max element (always the root!)
  // Remove the last element from heap
  T lastE = heap[heapSize--];

  // trickle down
  int currentNode = 1; // root
  int child = 2; // A child of currentNode
  while(child <= heapSize) {
    // Set child to larger child of currentNode
    if (child < heapSize && heap[child] < heap[child + 1]) child++;

    // Can we put lastE in currentNode?
    if (lastE >= heap[child]) break; // Yes!

    // No!
    heap[currentNode] = heap[child]; // Move child up
    currentNode = child; child *=2;  // Move down a level
  }
  heap[currentNode] = lastE;
}
```

Binary Search Tree (BST)

- A binary search tree (BST) is a binary tree which satisfies:
  - Every element has a key
    - No two elements have the same key
  - The keys in the left subtree are smaller than the key in the root
  - The keys in the right subtree are larger than the key in the root
  - The left and right subtrees are also BST





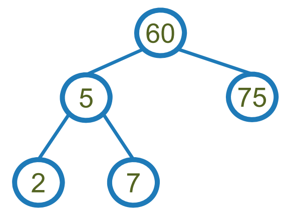
Inorder traversal of a BST will result in a sorted list

BST : Operations

- Search an element in a BST
- Search for the r^{th} smallest element in a BST
- Insert an element into a BST
- Delete max/min from a BST
- Delete an arbitrary element from a BST

BST : Search an Element

- Search for key 7
- Search process
  1. Start from root
  2. Compare the key with root
     - '<' search the left subtree
     - '>' search the right subtree
  3. Repeat step 3 until the key is found or a leaf is visited



BST : Recursive Search Codes

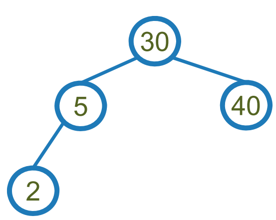
```
template < class K, class E >
pair<K,E>* BST<K,E>::Get(const K& k)
{ // Search the BST for a pair with key k
  // If the this pair is found, return a pointer to this
  // pair, otherwise return 0
  return Get(root, k);
}

template < class K, class E >
pair<K,E>* BST<K,E>::Get(TreeNode<pair<K,E>>* p, const K& k)
{
  if(!p) return 0;
  if(k < p->data.first) return Get(p->leftChild, k);
  if(k > p->data.first) return Get(p->rightChild, k);
  return &p->data;
}
```

BST : Iterative Search Codes

```
template < class K, class E >
pair<K,E>* BST<K,E>::Get(const K& k)
{
  TreeNode < pair<K, E> > *currentNode = root;
  while (currentNode) {
     if (k < currentNode->data.first)
        currentNode = currentNode->leftChild;
     else if (k > currentNode->data.first)
        currentNode = currentNode->rightChild;
     else return & currentNode->data;
  }
  return NULL; // no match found
}
```


BST : Search an Element by Rank

- Definition of **rank**:
  - A ***rank*** of a node is its position in inorder traversal



The r^{th} smallest element is the node with rank r

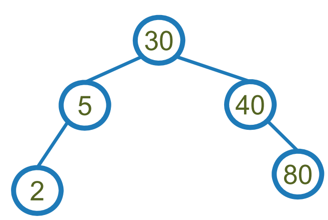
BST : Search by Rank Codes

- For each node, we store “leftSize”
  - which is 1 + (# of nodes in the left subtree)

```
template < class K, class E >
pair<K,E>* BST<K,E>::RankGet(int r)
{ // Search BST for the rth smallest pair
  TreeNode<pair<K,E>>* currentNode = root;
  while(currentNode){
    if(r < currentNode->leftSize)
      currentNode = currentNode->leftChild;
    else if(r > currentNode->leftSize) {
      r -= currentNode->leftSize;
      currentNode = currentNode->rigthChild;
    }
    else return &currentNode->data;
  }
  return 0;
}
```


BST : Insert

- To insert an element with key 80
- Search process
  1. Search for the existence of the element
  2. If the search is unsuccessful, then the element is inserted at the point the search terminates

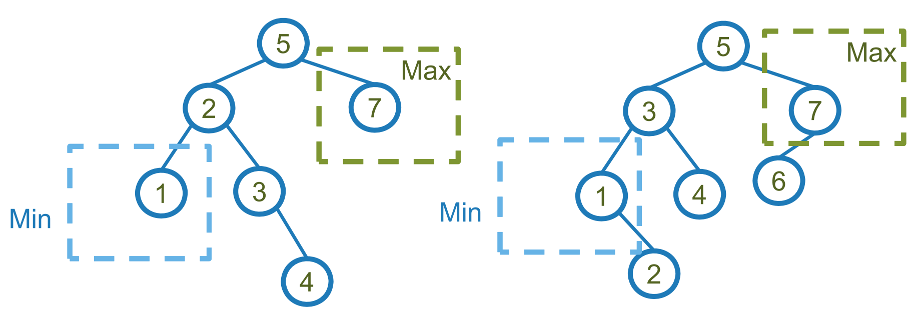


BST : Insert Codes

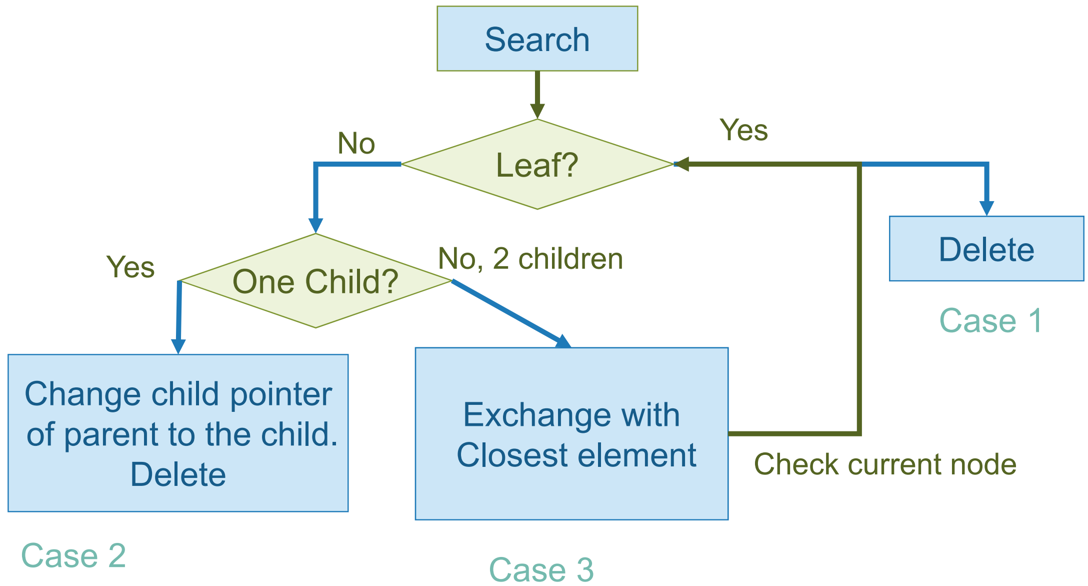
```
template < class K, class E >
void BST<K,E>::Insert(const pair<K,E>& thePair)
{ // Search for key "thePair.first", pp is the parent of p
  TreeNode<pair<K,E>>* p = root, *pp=0;
  while(p){
    pp = p;
    if(thePair.first < p->data.first)
      p = p->leftChild;
    else if(thePair.first > p->data.first)
      p = p->rightChild;
    else // Duplicate, update the value of element
    { p->data.second = thePair.second; return; }
  }
  // Perform the insertion
  p = new pair<K,E>(thePair);
  if(root) // tree is not empty
    if(thePair.first < pp->data.first) pp->leftChild = p;
    else pp->rightChild = p;
  else root = p;
}
```


Min (Max) Element in BST

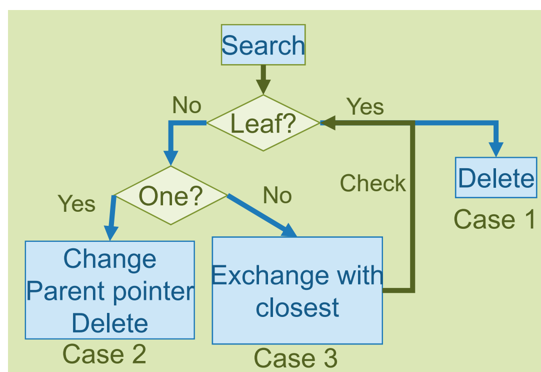
- Min (Max) element is at the leftmost (rightmost) one
- Min or max are not always terminal nodes
- Min or max has at most one child



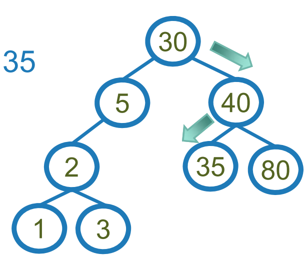
BST: Flow Chart of Deletion



BST: Delete (Case1)

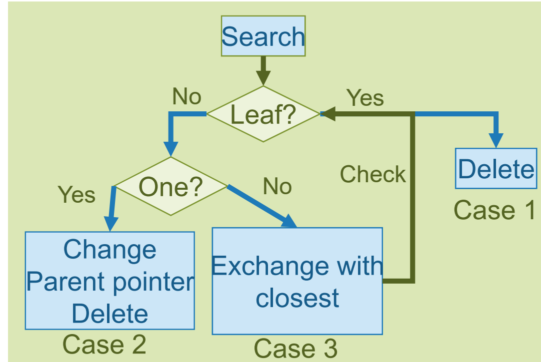


Delete 35

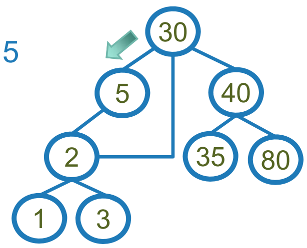


- Case 1 : The element is a leaf node
- The child field of parent node is set to NULL
- Dispose the node

BST: Delete (Case2)

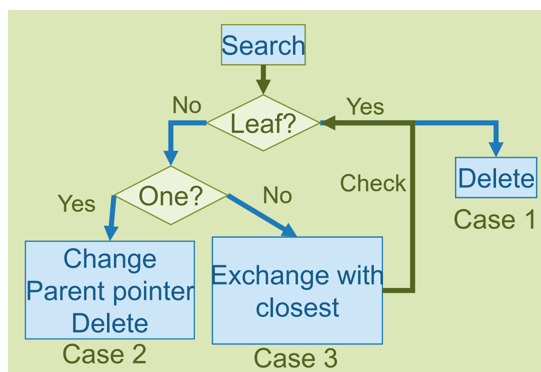


Delete 5

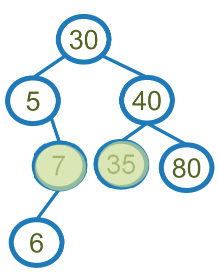


- Case 2 : The element is a non-leaf node with one child
- Change the pointer from the parent node to the single-child node
- Dispose the node

BST: Delete (Case3)

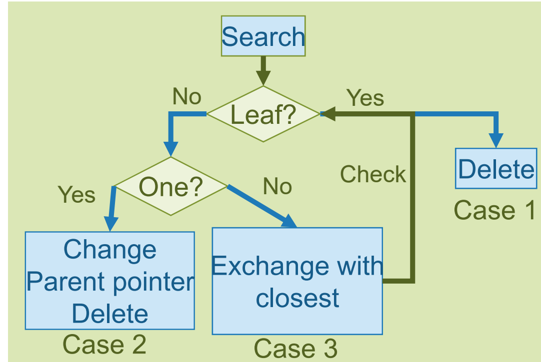


Delete 30

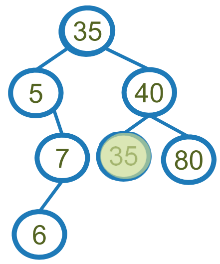


- Case 3 : The element is a non-leaf node with two children
- The deleted element is replaced by the closest one, either
  - The smallest element in right subtree
  - The largest element in left subtree

BST: Delete (Case3)

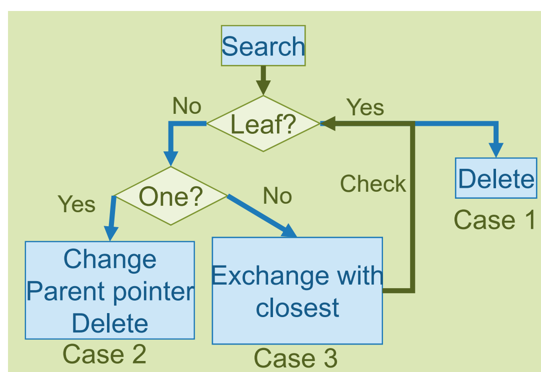


Delete 30

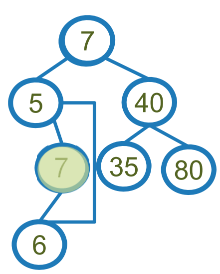


- Case 3 : The element is a non-leaf node with two children
- The deleted element is replaced by the closest one, either
  - The smallest element in right subtree
  - Delete this leaf node -> Case 1

BST: Delete (Case3)



Delete 30

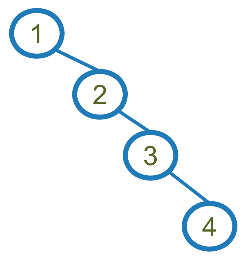


- Case 3 : The element is a non-leaf node with two children
- The deleted element is replaced by the closest one, either
  - The largest element in left subtree
  - Delete this node -> Case 2

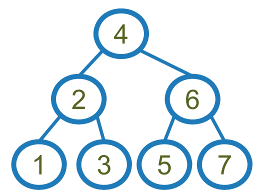
BST : Time Complexity

- Search, insertion, or deletion takes O(*h*)
- h = Height of a BST

- Worst case h=n
  - Insert keys: 1, 2, 3, 4, ..



- Best case h=$\log_2 n$
  - Insert keys : 4, 2, 6, 1, 3, 5, 7



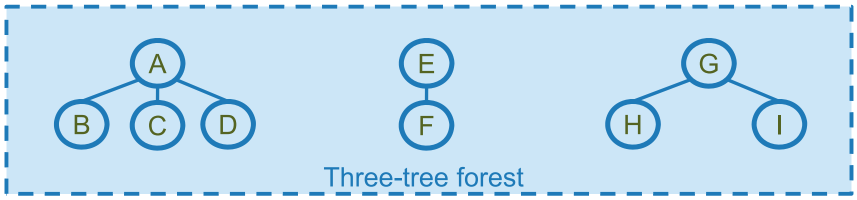
BST depends on how elements are inserted and deleted from the tree

Self-Study Topics

- Write pseudo codes of BST deletion
- Selection trees

Forests

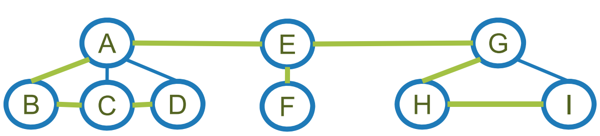
- Definition : A forest is a set of n ≥ 0 disjoint trees

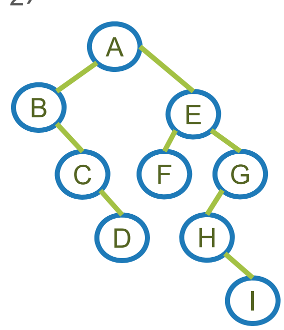


- Operations :
  - Transforming a forest to binary tree
  - Forest traversals

Transforming a Forest to Binary Tree

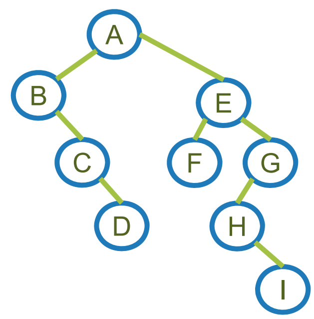
- Apply left child-right sibling approach
  - Convert each tree into binary tree
  - Connect two binary trees, T_1 and T_2, by setting the rightChild of root(T_1) to the root(T_2)





Forest Traversals

- Assume we have a forest **F** and binary tree **T**
- The following are equivalent
  - *Preorder* traversal of T
    - A B C D E F G H I
  - Visiting the nodes of F in *forest preorder*
    - Root: A
    - Left forest: B C D
    - Right forest: E F G H I



Disjoint Sets

- Assume a set S of n integers $\{0, 1, 2, \cdots, n-1\}$ is divided into several subsets $S_1, S_2, \ldots, S_k$
- $S_i \cap S_j = \emptyset\ for\ any\ i, j\ \in \{1, \cdots, k\}$ and $i \neq\ j$

- Operations:
  - Union disjoint sets: Union (S_i, S_j)
    - $S_i = S_i \cup S_j$ or $S_j = S_i \cup S_j$
  - Find the set containing element x : Find(x)

Disjoint Sets : Example

- Set
  - $S = \{ 0,1, 2, 3, 4, 5 \}$
- Disjoint subsets
  - $S_1 = \{ 0, 2, 3 \}$
  - $S_2 = \{ 1 \}$
  - $S_3 = \{ 4, 5 \}$
- $\text{Union}(S_1, S_2) = \{ 0, 1, 2, 3 \}$
- $\text{Find}(5) = 3$

DS: Array Representation

- $S = \{0, 1, 2, 3, 4, 5\}$ with subsets
  - $S_1 = \{0, 2, 3\}$, $S_2 = \{1\}$ and $S_3 = \{ 4, 5 \}$
- Using a sequential mapping array
  - Index represents set members
  - Array value indicates set name

DS Operation: Find(x)

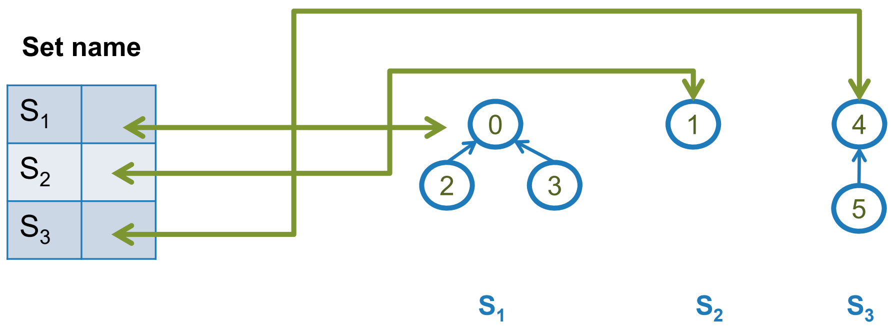
- Find the set which contains element x is easy
  - Find(5) = S[5] = set 3
    Find(3) = S[3] = set 1
  - Complexity = O(1)

DS Operation: Union(S_i, S_j)

- Assume we always merge the 2nd set to 1st set
  - $S_i = S_i \cup S_j$
- Scan the array and set S[k] to i if S[k]==j
  - S_2=Union(S_2, S_3)

DS: Tree Representation

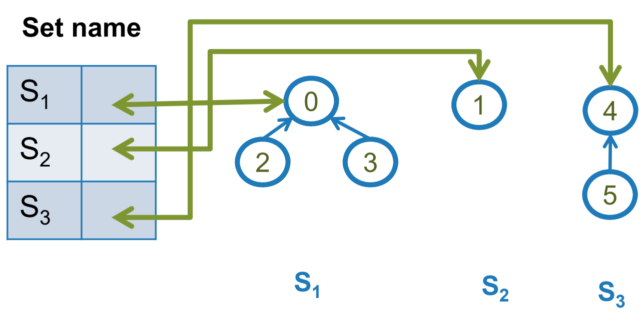
- Link elements of a subset to form a tree
  - Link children to root
  - Link root to set name



DS: Tree Representation

- Use an array to store the tree
- Identify the set by the root of the tree

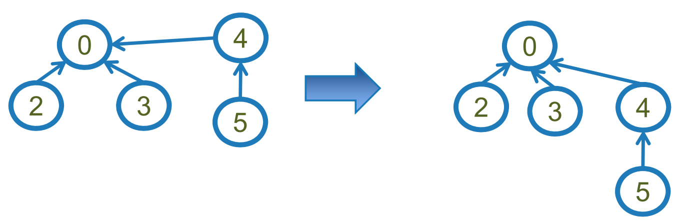
S_1 = {0, 2, 3}, S_2 = {1} and S_3 = { 4, 5 }



T[0]	**-1**
T[1]	**-1**
T[2]	**0**
T[3]	**0**
T[4]	**-1**
T[5]	**4**

DS Operation: Union(S_i, S_j)

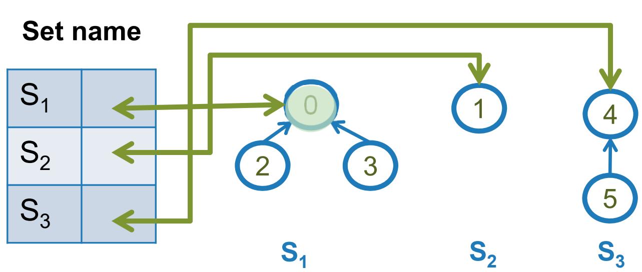
- Set the parent field of one of the root to the other root
  - S_1=Union(S_1, S_3)
  - Time complexity : O(1)



T[0]	-1
T[1]	-1
T[2]	0
T[3]	0
T[4]	-1
T[5]	4

DS Operation: Find(x)

- Following the index starting at x
- Tracing the tree structure
  - Until reaching a node with parent value = -1
- Use the root to identify the set name

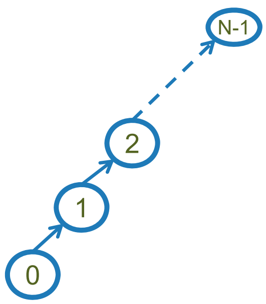


Find (3) = S_1

T[0]	-1
T[1]	-1
T[2]	0
T[3]	0
T[4]	-1
T[5]	4

DS Time Complexity

- S = { 0, 1, 2, … , n-1 }
  - S_1 = { 0 }, S_2 = { 1 }, S_3 = { 2 }, … , S_n = { n-1 }
- Perform a sequence Union
  - Union(S_2, S_1), Union(S_3, S_2), …, Union(S_n, S_{n-1})

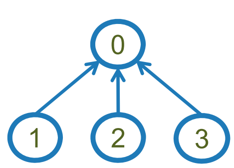


Followed by a sequence of Find
Find(0), Find(1), …, Find(n-1)

Total time complexity = $O(\sum_{i=1}^{n} i) = O(n^2)$

Improved Union(S_i, S_j)

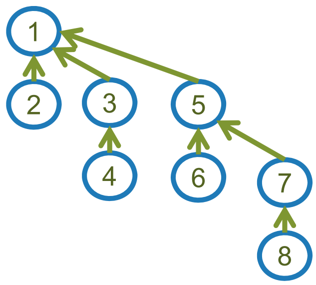
- Do not always merge two sets into the first set
- Adopt a Weighting rule to union operation
  - $S_i = S_i \cup S_j$, if $|S_i| >= |S_j|$
  - $S_j = S_i \cup S_j$, if $|S_i| < |S_j|$
- S = { 0, 1, 2, … , n }
  - S_1 = { 0 }, S_2 = { 1 }, S_3 = { 2 }, … , S_n = { n-1 }
  - Union (1, 2)->Union (1, 3)->Union (1, 4)



Time Complexity

- The following sequence produces the height of log n

- Union(1, 2)
- Union(3, 4)
- Union(5, 6)
- Union(7, 8)
- Union(1, 3)
- Union(5, 7)
- Union(1, 5)



Improved Find(x)

- Adopt a Collapsing rule for find(x)
  - If *j* is a node on the path from *i* to the root, set parent[j] to root(i)