KerrQED Theory

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Abstract

This note clarifies all the notation and unit and equation issues. For direct usage, go to the summary.

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1 Settings

$$S = \int \mathrm{d}^D x \sqrt{-g} \mathcal{L}$$

$$\mathcal{L} = \frac{c^4}{16\pi \mathrm{G}} R + \mathcal{L}^{\mathrm{em}}$$

1.1 Novello's Notation

Define

$$\mathcal{F} = F^{\mu\nu} F_{\mu\nu} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2)$$

$$\mathcal{G} = F^{\mu\nu} (^*F)_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} = 4\boldsymbol{B} \cdot \boldsymbol{E}$$

$$\mu = \frac{2}{45} \alpha^2 \left(\frac{\hbar}{m_e c} \right)^3 \frac{1}{m_e c^2}$$

In Heaviside–Lorentz Units,

$$\mathcal{L}^{\text{em}} = \frac{1}{c}\left[-\frac{1}{4}\mathcal{F} + \frac{\mu}{4}\left(\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2\right)\right]$$

In Gaussian Units,

$$\mathcal{L}^{\text{em}} = \frac{1}{4\pi c}\left[-\frac{1}{4}\mathcal{F} + \frac{\mu}{16\pi}\left(\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2\right)\right]$$

1.2 Breton's Notation

$$X = \frac{1}{4}\mathcal{F},\, Y = \frac{1}{4}\mathcal{G},\, A = \frac{2}{\pi}\mu,\, B = \frac{7}{2\pi}\mu$$

In Heaviside–Lorentz Units

$$\mathcal{L}^{\text{em}} = \frac{1}{c}[-X + 2\pi A X^2 + 2\pi B Y^2]$$

In Gaussian Units,

$$\mathcal{L}^{\text{em}} = \frac{1}{4\pi c}\left[-X + \frac{A}{2}X^2 + \frac{B}{2}Y^2\right]$$

1.3 Hu and Zhong's Notation

$$
\begin{aligned}
\mathcal{G}^2 &= (F^{\mu\nu}\tilde{F}_{\mu\nu})^2 \\
&= \frac{1}{4}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\mu\nu\sigma\rho}F^{\alpha\beta}F^{\gamma\delta}F_{\mu\nu}F_{\sigma\rho} \\
&= -\frac{1}{4}\begin{vmatrix} \delta_\alpha^\mu & \delta_\alpha^\nu & \delta_\alpha^\sigma & \delta_\alpha^\rho \\ \delta_\beta^\mu & \delta_\beta^\nu & \delta_\beta^\sigma & \delta_\beta^\rho \\ \delta_\gamma^\mu & \delta_\gamma^\nu & \delta_\gamma^\sigma & \delta_\gamma^\rho \\ \delta_\delta^\mu & \delta_\delta^\nu & \delta_\delta^\sigma & \delta_\delta^\rho \end{vmatrix} F^{\alpha\beta}F^{\gamma\delta}F_{\mu\nu}F_{\sigma\rho} \\
&= -2(F_{\mu\nu}F^{\mu\nu})^2 + 4F_{\mu\nu}F_{\sigma\rho}F^{\mu\sigma}F^{\nu\rho}
\end{aligned}
$$

$$\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2 = -\frac{5}{2}(F_{\mu\nu}F^{\mu\nu})^2 + 7F_{\mu\nu}F_{\sigma\rho}F^{\mu\sigma}F^{\nu\rho}$$

In Heaviside–Lorentz Units,

$$
\begin{aligned}
\mathcal{L}^{\text{em}} &= \frac{1}{c}\left[-\frac{1}{4}\mathcal{F} + \frac{\mu}{4}\left(\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2\right)\right] \\
&= -\frac{1}{4c}F^{\mu\nu}F_{\mu\nu} - \frac{\mu}{2c}\left[\frac{5}{4}(F_{\mu\nu}F^{\mu\nu})^2 - \frac{7}{2}F_{\mu\nu}F_{\sigma\rho}F^{\mu\sigma}F^{\nu\rho}\right]
\end{aligned}
$$

2 Basic Equations

In this section, we stick to Heaviside-Lorentz Units.

2.1 Einstein field Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}^{\text{em}}$$

$$
\begin{aligned}
T_{\mu\nu}^{\text{em}} &= -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}^{\text{em}})}{\delta g^{\mu\nu}} \\
&= -2\frac{\delta\mathcal{L}^{\text{em}}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}^{\text{em}} \\
&= g_{\mu\nu}\mathcal{L}^{\text{em}} - 2\left(\mathcal{L}_{\mathcal{F}}^{\text{em}}\frac{\delta\mathcal{F}}{\delta g^{\mu\nu}} + \mathcal{L}_{\mathcal{G}}^{\text{em}}\frac{\delta\mathcal{G}}{\delta g^{\mu\nu}}\right) \\
&= g_{\mu\nu}(\mathcal{L}^{\text{em}} - \mathcal{G}\mathcal{L}_{\mathcal{G}}^{\text{em}}) - 4\mathcal{L}_{\mathcal{F}}^{\text{em}}F_{\mu}{}^{\alpha}F_{\nu\alpha}
\end{aligned}
$$

$$\mathrm{Tr}[T_{\mu\nu}^{\text{em}}] = 4(\mathcal{L}^{\text{em}} - \mathcal{F}\mathcal{L}_{\mathcal{F}}^{\text{em}} - \mathcal{G}\mathcal{L}_{\mathcal{G}}^{\text{em}})$$

$$\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}\sqrt{-g}, \frac{\delta(\varepsilon_{\alpha\beta\gamma\delta})}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}\varepsilon_{\alpha\beta\gamma\delta}$$

$$\frac{\delta\mathcal{F}}{\delta g^{\mu\nu}} = 2F_{\mu}{}^{\alpha}F_{\nu\alpha}, \frac{\delta\mathcal{G}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}\mathcal{G} + 4F_{\mu}{}^{\alpha}(^{*}F)_{\nu\alpha} = \frac{1}{2}g_{\mu\nu}\mathcal{G}$$

In our case,

$$
\begin{aligned}
T_{\mu\nu}^{\text{em}} &= g_{\mu\nu}(\mathcal{L}^{\text{em}} - \mathcal{G}\mathcal{L}_{\mathcal{G}}^{\text{em}}) - 4\mathcal{L}_{\mathcal{F}}^{\text{em}}F_{\mu}{}^{\alpha}F_{\nu\alpha} \\
&= g_{\mu\nu}\frac{1}{c}\left[-\frac{1}{4}\mathcal{F} + \frac{\mu}{4}\left(\mathcal{F}^2 - \frac{7}{4}\mathcal{G}^2\right)\right] - 4\frac{1}{c}\left[-\frac{1}{4} + \frac{\mu}{4}2\mathcal{F}\right]F_{\mu}{}^{\alpha}F_{\nu\alpha} \\
&= \frac{1}{c}\left[F_{\mu}{}^{\alpha}F_{\nu\alpha} - \frac{1}{4}\mathcal{F}g_{\mu\nu}\right] - \frac{\mu}{4c}\left[8\mathcal{F}F_{\mu}{}^{\alpha}F_{\nu\alpha} - \left(\mathcal{F}^2 - \frac{7}{4}\mathcal{G}^2\right)g_{\mu\nu}\right]
\end{aligned}
$$

By the way, in Gaussian Units,

$$T_{\mu\nu}^{\text{em}} = \frac{1}{4\pi c}\left[F_{\mu}{}^{\alpha}F_{\nu\alpha} - \frac{1}{4}\mathcal{F}g_{\mu\nu}\right] - \frac{1}{(4\pi)^2}\frac{\mu}{4c}\left[8\mathcal{F}F_{\mu}{}^{\alpha}F_{\nu\alpha} - \left(\mathcal{F}^2 - \frac{7}{4}\mathcal{G}^2\right)g_{\mu\nu}\right]$$

2.2 Maxwell's Equations

$$
\begin{aligned}
\frac{\delta S}{\delta A_{\mu}} &= \frac{\partial\mathcal{L}^{\text{em}}}{\partial A_{\mu}} - \partial_{\nu}\left[\frac{\partial\mathcal{L}^{\text{em}}}{\partial(\partial_{\nu}A_{\mu})}\right] \\
&= 0 - \partial_{\nu}\left(\mathcal{L}_{\mathcal{F}}^{\text{em}}\frac{\partial\mathcal{F}}{\partial(\partial_{\nu}A_{\mu})} + \mathcal{L}_{\mathcal{G}}^{\text{em}}\frac{\partial\mathcal{G}}{\partial(\partial_{\nu}A_{\mu})}\right) \\
&= -2\partial_{\nu}(\mathcal{L}_{\mathcal{F}}^{\text{em}}F^{\nu\mu} - \mathcal{L}_{\mathcal{F}}^{\text{em}}F^{\mu\nu} + \mathcal{L}_{\mathcal{G}}^{\text{em}}(^{*}F)^{\nu\mu} - \mathcal{L}_{\mathcal{G}}^{\text{em}}(^{*}F)^{\mu\nu}) \\
&= 4\partial_{\nu}(\mathcal{L}_{\mathcal{F}}^{\text{em}}F^{\mu\nu} + \mathcal{L}_{\mathcal{G}}^{\text{em}}(^{*}F)^{\mu\nu}) = 0
\end{aligned}
$$

$$\Rightarrow \partial_{\nu}(\mathcal{L}_{\mathcal{F}}^{\text{em}}F^{\mu\nu} + \mathcal{L}_{\mathcal{G}}^{\text{em}}(^{*}F)^{\mu\nu}) = 0$$

2.3 EOM of photon

According to M. Novello's PRD paper, photon's trajectory in nonlinear electrodynamics is a null geodesic of the effective metric $K_{\pm}^{\mu\nu}$, whose general form is

$$
\begin{aligned}
K_{\pm}^{\mu\nu} &\propto \mathcal{L}_{\mathcal{F}}^{\text{em}}g^{\mu\nu} + 4[(\mathcal{L}_{\mathcal{F}\mathcal{F}}^{\text{em}} + \Omega_{\pm}\mathcal{L}_{\mathcal{F}\mathcal{G}}^{\text{em}})F_{\lambda}{}^{\mu}F^{\lambda\nu} + (\mathcal{L}_{\mathcal{F}\mathcal{G}}^{\text{em}} + \Omega_{\pm}\mathcal{L}_{\mathcal{G}\mathcal{G}}^{\text{em}})F_{\lambda}{}^{\mu}(^{*}F)^{\lambda\nu}] \\
&= [\mathcal{L}_{\mathcal{F}}^{\text{em}} + (\mathcal{L}_{\mathcal{F}\mathcal{G}}^{\text{em}} + \Omega_{\pm}\mathcal{L}_{\mathcal{G}\mathcal{G}}^{\text{em}})\mathcal{G}]g^{\mu\nu} + 4(\mathcal{L}_{\mathcal{F}\mathcal{F}}^{\text{em}} + \Omega_{\pm}\mathcal{L}_{\mathcal{F}\mathcal{G}}^{\text{em}})F_{\lambda}{}^{\mu}F^{\lambda\nu}
\end{aligned}
$$

where $\Omega_\pm$ is the solution of equation

$$\Omega^2\Omega_1 + \Omega\Omega_2 + \Omega_3 = 0$$

with

$$\begin{cases} \Omega_1 = -\mathcal{L}_\mathcal{F}^{em}\mathcal{L}_{\mathcal{FG}}^{em} + 2\mathcal{F}\mathcal{L}_{\mathcal{FG}}^{em}\mathcal{L}_{\mathcal{GG}}^{em} + \mathcal{G}[(\mathcal{L}_{\mathcal{GG}}^{em})^2 - (\mathcal{L}_{\mathcal{FG}}^{em})^2] \\ \Omega_2 = (\mathcal{L}_\mathcal{F}^{em} + 2\mathcal{G}\mathcal{L}_{\mathcal{FG}}^{em})(\mathcal{L}_{\mathcal{GG}}^{em} - \mathcal{L}_{\mathcal{FF}}^{em}) + 2\mathcal{F}[\mathcal{L}_{\mathcal{FF}}^{em}\mathcal{L}_{\mathcal{GG}}^{em} + (\mathcal{L}_{\mathcal{FG}}^{em})^2] \\ \Omega_3 = \mathcal{L}_\mathcal{F}^{em}\mathcal{L}_{\mathcal{FG}}^{em} + 2\mathcal{F}\mathcal{L}_{\mathcal{FF}}^{em}\mathcal{L}_{\mathcal{FG}}^{em} + \mathcal{G}[(\mathcal{L}_{\mathcal{FG}}^{em})^2 - (\mathcal{L}_{\mathcal{FF}}^{em})^2] \end{cases}$$

The equations of motion reads

$$H = H(q_\mu, x^\mu) = \frac{1}{2}K^{\mu\nu}(x)q_\mu q_\nu$$

$$\begin{cases} \dot{x}^\mu = \frac{\partial H}{\partial q_\mu} = K^{\mu\nu}q_\nu \\ \dot{q}_\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2}\partial_\mu K^{\alpha\beta}q_\alpha q_\beta \end{cases}$$

In our Eular-Heisenburg Lagrangian case, they read

$$\Omega_1 = \left(\frac{7\mu}{8c}\right)^2\mathcal{G}, \quad \Omega_2 = -\frac{3\mu}{32c^2} + \frac{17\mu^2}{16c^2}\mathcal{F}, \quad \Omega_3 = -\left(\frac{\mu}{2c}\right)^2\mathcal{G}$$

$$\Omega_\pm = \frac{-\Omega_2 \pm \sqrt{\Omega_2^2 - 4\Omega_1\Omega_3}}{2\Omega_1} = -\frac{\Omega_2}{2\Omega_1}\left(1 \mp \sqrt{1 - \frac{4\Omega_1\Omega_3}{\Omega_2^2}}\right)$$

$$K_\pm^{\mu\nu} \propto \left[-\frac{1}{4} + \frac{\mu}{2}\mathcal{F} + \frac{7\mu}{8}\Omega_\pm\mathcal{G}\right]g^{\mu\nu} + 2\mu F_\lambda{}^\mu F^{\lambda\nu} \propto g^{\mu\nu} + \frac{2\mu F_\lambda{}^\mu F^{\lambda\nu}}{-\frac{1}{4} + \frac{\mu}{2}\mathcal{F} + \frac{7\mu}{8}\Omega_\pm\mathcal{G}}$$

Taylor expansion gives

$$\begin{cases} \Omega_+ = -\frac{\Omega_3}{\Omega_2}(1 + O(\mu^2)) = -\frac{8}{3}\mu\mathcal{G} - \frac{8 \times 34}{9}\mu^2\mathcal{F}\mathcal{G} + O(\mu^3) = 0 + O(\mu) \\ \Omega_- = -\frac{\Omega_2}{\Omega_1}(1 + O(\mu^2)) = \frac{6}{49}(\mu\mathcal{G})^{-1} - \frac{68}{49}\frac{\mathcal{F}}{\mathcal{G}} + O(\mu) \end{cases}$$

$$\begin{cases} -\frac{1}{4} + \frac{\mu}{2}\mathcal{F} + \frac{7\mu}{8}\Omega_+\mathcal{G} = -\frac{1}{4} + \frac{\mu}{2}\mathcal{F} + O(\mu^2) \\ -\frac{1}{4} + \frac{\mu}{2}\mathcal{F} + \frac{7\mu}{8}\Omega_-\mathcal{G} = -\frac{1}{7} - \frac{5}{7}\mu\mathcal{F} + O(\mu^2) \end{cases}$$

3 Metric and Electromagnetic Field

Using Gaussian units here. Gürses and Gürsey Metric reads

$$ds^2 = -\left(1 - \frac{2m(r)r}{\rho^2}\right)c^2dt^2 + \frac{\rho^2}{\Delta}dr^2 - \frac{4am(r)r\sin^2\theta}{\rho^2}cdtd\phi + \rho^2d\theta^2 + \frac{\Sigma\sin^2\theta}{\rho^2}d\phi^2$$

where

$$\begin{cases} m(r) = \frac{GM}{c^2}\left(1 - \frac{Q_m^2}{2Mc^2r} + A\frac{Q_m^4}{40Mc^2r^5}\right) \\ \Delta = r^2 + a^2 - 2m(r)r \\ \Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \\ \rho^2 = r^2 + a^2\cos^2\theta \end{cases}$$

In other words,

$$
g_{\mu\nu} =
\begin{bmatrix}
-\left(1 - \frac{2m(r)r}{\rho^2}\right)c^2 & 0 & 0 & -\frac{2a\,m(r)r\sin^2\theta}{\rho^2}c \\
0 & \frac{\rho^2}{\Delta} & 0 & 0 \\
0 & 0 & \rho^2 & 0 \\
-\frac{2a\,m(r)r\sin^2\theta}{\rho^2}c & 0 & 0 & \frac{\Sigma\sin^2\theta}{\rho^2}
\end{bmatrix}
$$

$$
|g| = -\rho^4\sin^2\theta
$$

The electromagnetic field reads

$$
A_\mu = \frac{Q_m}{\rho^2}\cos\theta\left(\; -a, \;\; 0, \;\; 0, \;\; r^2+a^2 \;\right)
$$

$$
F_{\mu\nu} = \frac{Q_m}{\rho^4}\left\{ 2ar\cos\theta
\begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & -a\sin^2\theta \\
0 & 0 & 0 & 0 \\
0 & a\sin^2\theta & 0 & 0
\end{bmatrix}
+ (r^2 - a^2\cos^2\theta)\sin\theta
\begin{bmatrix}
0 & 0 & -a & 0 \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & -(r^2+a^2) \\
0 & 0 & r^2+a^2 & 0
\end{bmatrix}
\right\}
$$

$$
F^{\mu\nu} = \frac{Q_m}{\rho^6}\left\{ 2ar\cos\theta
\begin{bmatrix}
0 & r^2+a^2 & 0 & 0 \\
-(r^2+a^2) & 0 & 0 & -a \\
0 & 0 & 0 & 0 \\
0 & a & 0 & 0
\end{bmatrix}
+ (r^2 - a^2\cos^2\theta)
\begin{bmatrix}
0 & 0 & a\sin\theta & 0 \\
0 & 0 & 0 & 0 \\
-a\sin\theta & 0 & 0 & -\csc\theta \\
0 & 0 & \csc\theta & 0
\end{bmatrix}
\right\}
$$

$$
(^*F)_{\mu\nu} = \frac{Q_m}{\rho^4}\left\{ (r^2 - a^2\cos^2\theta)
\begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & -a\sin^2\theta \\
0 & 0 & 0 & 0 \\
0 & a\sin^2\theta & 0 & 0
\end{bmatrix}
+ ar\sin(2\theta)
\begin{bmatrix}
0 & 0 & a & 0 \\
0 & 0 & 0 & 0 \\
-a & 0 & 0 & r^2+a^2 \\
0 & 0 & -(r^2+a^2) & 0
\end{bmatrix}
\right\}
$$

$$
\begin{aligned}
\mathcal{F} &= F^{\mu\nu}F_{\mu\nu} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2) \\
&= \frac{Q_m^2}{\rho^{10}}[-4a^2r^2\cos^2\theta 2\rho^2 + (r^2 - a^2\cos^2\theta)^2 2\rho^2] \\
&= -2\frac{Q_m^2}{\rho^8}[4a^2r^2\cos^2\theta - (r^2 - a^2\cos^2\theta)^2]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G} &= F^{\mu\nu}(^*F)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma} = 4\boldsymbol{B}\cdot\boldsymbol{E} \\
&= \frac{Q_m^2}{\rho^{10}}(r^2 - a^2\cos^2\theta)2ar\cos\theta[-2\rho^2 - 2\rho^2] \\
&= -4\frac{Q_m^2}{\rho^8}(r^2 - a^2\cos^2\theta)2ar\cos\theta
\end{aligned}
$$

According to calculations by Mathematica,

$$
\lim_{a\to 0}\left(G_{\mu\nu} - \frac{8\pi G}{c^4}T_{\mu\nu}^{\text{em}} \right) = 0
$$

There're the two constrains for symmetric tensor in axisymmetric geometry,

$$
\begin{cases}
a\sin^2\theta G_{t\phi} + G_{\phi\phi} = \frac{(r^2+a^2)\sin^2\theta}{\rho^2}G_{\theta\theta} \\
a^2\sin^4\theta G_{tt} - G_{\phi\phi} = -\frac{(r^2+a^2+a^2\sin^2\theta)\sin^2\theta}{\rho^2}G_{\theta\theta}
\end{cases}
$$

and a trivial constrain for symmetric tensor

$$G_{t\phi} = G_{\phi t}$$

that hold for both Einstein tensor and energy-momentum tensor. Therefore, there are only three independent variables in the whole 4×4 tensor. And we define δ_i for $i = 1, 2, 3$ to represent the relative deviation from Einstein field equations,

$$\delta_1 \equiv \left| \frac{G_{rr} - 8\pi T_{rr}}{G_{rr}} \right|, \delta_2 \equiv \left| \frac{G_{\theta\theta} - 8\pi T_{\theta\theta}}{G_{\theta\theta}} \right|, \delta_3 \equiv \left| \frac{G_{t\phi} - 8\pi T_{t\phi}}{G_{t\phi}} \right|$$

4 Summary

In Gaussian Units, define

$$
\begin{cases}
m(r) = \frac{GM}{c^2}\left(1 - \frac{Q_m^2}{2Mc^2 r} + A\frac{Q_m^4}{40Mc^2 r^5}\right) \\
\Delta = r^2 + a^2 - 2m(r)r \\
\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \\
\rho^2 = r^2 + a^2\cos^2\theta
\end{cases}
$$

where

$$A = \frac{2}{\pi}\mu$$

$$\mu = \frac{2}{45}\alpha^2\left(\frac{\hbar}{m_e c}\right)^3 \frac{1}{m_e c^2}$$

The metric reads

$$
g_{\mu\nu} = \begin{bmatrix}
-\left(1 - \frac{2m(r)r}{\rho^2}\right)c^2 & 0 & 0 & -\frac{2am(r)r\sin^2\theta}{\rho^2}c \\
0 & \frac{\rho^2}{\Delta} & 0 & 0 \\
0 & 0 & \rho^2 & 0 \\
-\frac{2am(r)r\sin^2\theta}{\rho^2}c & 0 & 0 & \frac{\Sigma\sin^2\theta}{\rho^2}
\end{bmatrix}
$$

The electromagnetic field reads

$$A_\mu = \frac{Q_m}{\rho^2}\cos\theta\left(\ -a,\ \ 0,\ \ 0,\ \ r^2 + a^2\ \right)$$

$$F_{\mu\nu} = \frac{Q_m}{\rho^4}\left\{2ar\cos\theta\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & 0 & 0 \\ 0 & a\sin^2\theta & 0 & 0 \end{bmatrix} + (r^2 - a^2\cos^2\theta)\sin\theta\begin{bmatrix} 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & -(r^2 + a^2) \\ 0 & 0 & r^2 + a^2 & 0 \end{bmatrix}\right\}$$

The energy-momentum tensor reads

$$T_{\mu\nu}^{\text{em}} = \frac{1}{4\pi c}\left[F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}\mathcal{F}g_{\mu\nu}\right] - \frac{1}{(4\pi)^2}\frac{\mu}{4c}\left[8\mathcal{F}F_\mu{}^\alpha F_{\nu\alpha} - \left(\mathcal{F}^2 - \frac{7}{4}\mathcal{G}^2\right)g_{\mu\nu}\right]$$

With

$$\mathcal{F} = F^{\mu\nu}F_{\mu\nu} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2) = -2\frac{Q_m^2}{\rho^8}[4a^2 r^2\cos^2\theta - (r^2 - a^2\cos^2\theta)^2]$$

$$\mathcal{G} = F^{\mu\nu}(*F)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma} = 4\boldsymbol{B}\cdot\boldsymbol{E} = -4\frac{Q_m^2}{\rho^8}(r^2 - a^2\cos^2\theta)2ar\cos\theta$$

The metric is only exact when $a = 0$. For $a \neq 0$ cases, define

$$\delta_1 \equiv \left| \frac{G_{rr} - 8\pi T_{rr}}{G_{rr}} \right|, \delta_2 \equiv \left| \frac{G_{\theta\theta} - 8\pi T_{\theta\theta}}{G_{\theta\theta}} \right|, \delta_3 \equiv \left| \frac{G_{t\phi} - 8\pi T_{t\phi}}{G_{t\phi}} \right|$$

To represent the deviation from Einstein field equations.

The motion of the photon is given by the Hamilton equations

$$H = H(q_\mu, x^\mu) = \frac{1}{2} K^{\mu\nu}(x) q_\mu q_\nu$$

$$\begin{cases} \dot{x}^\mu = \frac{\partial H}{\partial q_\mu} = K^{\mu\nu} q_\nu \\ \dot{q}_\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2} \partial_\mu K^{\alpha\beta} q_\alpha q_\beta \end{cases}$$

With the effective metric $K^{\mu\nu}$ defined as

$$\Omega_1 = \left(\frac{7\mu}{8c} \right)^2 \frac{\mathcal{G}}{4\pi}, \Omega_2 = -\frac{3\mu}{32c^2} + \frac{17\mu^2}{16c^2} \frac{\mathcal{F}}{4\pi}, \Omega_3 = -\left(\frac{\mu}{2c} \right)^2 \frac{\mathcal{G}}{4\pi}$$

$$\Omega_\pm = \frac{-\Omega_2 \pm \sqrt{\Omega_2^2 - 4\Omega_1\Omega_3}}{2\Omega_1} = -\frac{\Omega_2}{2\Omega_1} \left(1 \mp \sqrt{1 - \frac{4\Omega_1\Omega_3}{\Omega_2^2}} \right)$$

$$K_\pm^{\mu\nu} \propto \left[-\frac{1}{4} + \frac{\mu}{8\pi}\mathcal{F} + \frac{7\mu}{32\pi}\Omega_\pm\mathcal{G} \right] g^{\mu\nu} + \frac{\mu}{2\pi} F_\lambda{}^\mu F^{\lambda\nu}$$

where $\pm$ indicates the two branches of birefringence.

Appendix

1 Dimensional Analysis and Natural Units

$$[\mathrm{G}][M]^2[r]^{-1} = [Q]^2[r]^{-1} = [M][c]^2 = [\hbar][c][r]^{-1}$$

$$\Rightarrow \begin{cases} [r] = [\mathrm{G}][M][c]^{-2} \\ [Q] = [\mathrm{G}]^{\frac{1}{2}}[M] \\ [\mu] = [\mathrm{G}]^3[M]^2[c]^{-8} \end{cases}$$

Therefore, we may define natural units by the dimensionless quantities

$$r \to r / \left(\frac{\mathrm{G}M}{c^2} \right)$$

$$Q \to Q / (\sqrt{\mathrm{G}}M)$$

$$\begin{aligned} \mu \to \mu / \left(\frac{\mathrm{G}^3 M^2}{c^8} \right) &= \frac{2}{45}\alpha^2 \left(\frac{\hbar c}{\mathrm{G}M m_e} \right)^3 \frac{M}{m_e} \\ &\approx \left(\frac{1.911514991 \times 10^{34}\mathrm{Kg}}{M} \right)^2 \\ &\approx 0.9236 \left(\frac{M_\odot}{M} \right)^2 \times 10^8 \end{aligned}$$