KerrQED Theory

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Abstract

This note clarifies all the notation and unit and equation issues. For direct usage, go to the summary.

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1 Settings

$$S = \int \mathrm{d}^D x \sqrt{-g} \mathcal{L}$$

$$\mathcal{L} = \frac{c^4}{16\pi \mathrm{G}} R + \mathcal{L}^{\mathrm{em}}$$

1.1 Novello's Notation

Define

$$\mathcal{F} = F^{\mu\nu} F_{\mu\nu} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2)$$

$$\mathcal{G} = F^{\mu\nu} ({}^*F)_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} = 4 \boldsymbol{B} \cdot \boldsymbol{E}$$

$$\mu = \frac{2}{45} \alpha^2 \left(\frac{\hbar}{m_e c} \right)^3 \frac{1}{m_e c^2}$$

In Heaviside–Lorentz Units,

$$\mathcal{L}^{\text{em}} = \frac{1}{c}\left[-\frac{1}{4}\mathcal{F} + \frac{\mu}{4}\left(\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2\right)\right]$$

In Gaussian Units,

$$\mathcal{L}^{\text{em}} = \frac{1}{4\pi c}\left[-\frac{1}{4}\mathcal{F} + \frac{\mu}{16\pi}\left(\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2\right)\right]$$

1.2 Breton's Notation

$$X = \frac{1}{4}\mathcal{F}, Y = \frac{1}{4}\mathcal{G}, A = \frac{2}{\pi}\mu, B = \frac{7}{2\pi}\mu$$

In Heaviside–Lorentz Units

$$\mathcal{L}^{\text{em}} = \frac{1}{c}[-X + 2\pi A X^2 + 2\pi B Y^2]$$

In Gaussian Units,

$$\mathcal{L}^{\text{em}} = \frac{1}{4\pi c}\left[-X + \frac{A}{2}X^2 + \frac{B}{2}Y^2\right]$$

1.3 Hu and Zhong's Notation

$$\begin{aligned}
\mathcal{G}^2 &= (F^{\mu\nu}\tilde{F}_{\mu\nu})^2 \\
&= \frac{1}{4}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\mu\nu\sigma\rho}F^{\alpha\beta}F^{\gamma\delta}F_{\mu\nu}F_{\sigma\rho} \\
&= -\frac{1}{4}\begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\alpha & \delta^\sigma_\alpha & \delta^\rho_\alpha \\ \delta^\mu_\beta & \delta^\nu_\beta & \delta^\sigma_\beta & \delta^\rho_\beta \\ \delta^\mu_\gamma & \delta^\nu_\gamma & \delta^\sigma_\gamma & \delta^\rho_\gamma \\ \delta^\mu_\delta & \delta^\nu_\delta & \delta^\sigma_\delta & \delta^\rho_\delta \end{vmatrix} F^{\alpha\beta}F^{\gamma\delta}F_{\mu\nu}F_{\sigma\rho} \\
&= -2(F_{\mu\nu}F^{\mu\nu})^2 + 4F_{\mu\nu}F_{\sigma\rho}F^{\mu\sigma}F^{\nu\rho}
\end{aligned}$$

$$\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2 = -\frac{5}{2}(F_{\mu\nu}F^{\mu\nu})^2 + 7F_{\mu\nu}F_{\sigma\rho}F^{\mu\sigma}F^{\nu\rho}$$

In Heaviside–Lorentz Units,

$$\begin{aligned}
\mathcal{L}^{\text{em}} &= \frac{1}{c}\left[-\frac{1}{4}\mathcal{F} + \frac{\mu}{4}\left(\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2\right)\right] \\
&= -\frac{1}{4c}F^{\mu\nu}F_{\mu\nu} - \frac{\mu}{2c}\left[\frac{5}{4}(F_{\mu\nu}F^{\mu\nu})^2 - \frac{7}{2}F_{\mu\nu}F_{\sigma\rho}F^{\mu\sigma}F^{\nu\rho}\right]
\end{aligned}$$

2 Basic Equations

In this section, we stick to Heaviside-Lorentz Units.

2.1 Einstein Equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi \mathrm{G}}{c^4} T_{\mu\nu}^{\mathrm{em}}$$

$$\begin{aligned} T_{\mu\nu}^{\mathrm{em}} &= -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}^{\mathrm{em}})}{\delta g^{\mu\nu}} \\ &= -2\frac{\delta \mathcal{L}^{\mathrm{em}}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}^{\mathrm{em}} \\ &= g_{\mu\nu}\mathcal{L}^{\mathrm{em}} - 2\left(\mathcal{L}_{\mathcal{F}}^{\mathrm{em}} \frac{\delta \mathcal{F}}{\delta g^{\mu\nu}} + \mathcal{L}_{\mathcal{G}}^{\mathrm{em}} \frac{\delta \mathcal{G}}{\delta g^{\mu\nu}} \right) \\ &= g_{\mu\nu}(\mathcal{L}^{\mathrm{em}} - \mathcal{G}\mathcal{L}_{\mathcal{G}}^{\mathrm{em}}) - 4\mathcal{L}_{\mathcal{F}}^{\mathrm{em}} F_{\mu}{}^{\alpha} F_{\nu\alpha} \end{aligned}$$

$$\mathrm{Tr}[T_{\mu\nu}^{\mathrm{em}}] = 4(\mathcal{L}^{\mathrm{em}} - \mathcal{F}\mathcal{L}_{\mathcal{F}}^{\mathrm{em}} - \mathcal{G}\mathcal{L}_{\mathcal{G}}^{\mathrm{em}})$$

$$\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu}\sqrt{-g}, \frac{\delta(\varepsilon_{\alpha\beta\gamma\delta})}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu}\varepsilon_{\alpha\beta\gamma\delta}$$

$$\frac{\delta \mathcal{F}}{\delta g^{\mu\nu}} = 2F_{\mu}{}^{\alpha} F_{\nu\alpha}, \frac{\delta \mathcal{G}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu}\mathcal{G} + 4F_{\mu}{}^{\alpha}({}^{*}F)_{\nu\alpha} = \frac{1}{2} g_{\mu\nu}\mathcal{G}$$

In our case,

$$\begin{aligned} T_{\mu\nu}^{\mathrm{em}} &= g_{\mu\nu}(\mathcal{L}^{\mathrm{em}} - \mathcal{G}\mathcal{L}_{\mathcal{G}}^{\mathrm{em}}) - 4\mathcal{L}_{\mathcal{F}}^{\mathrm{em}} F_{\mu}{}^{\alpha} F_{\nu\alpha} \\ &= g_{\mu\nu}\frac{1}{c}\left[-\frac{1}{4}\mathcal{F} + \frac{\mu}{4}\left(\mathcal{F}^2 - \frac{7}{4}\mathcal{G}^2 \right) \right] - 4\frac{1}{c}\left[-\frac{1}{4} + \frac{\mu}{4} 2\mathcal{F} \right] F_{\mu}{}^{\alpha} F_{\nu\alpha} \\ &= \frac{1}{c}\left[F_{\mu}{}^{\alpha} F_{\nu\alpha} - \frac{1}{4}\mathcal{F} g_{\mu\nu} \right] - \frac{\mu}{4c}\left[8\mathcal{F} F_{\mu}{}^{\alpha} F_{\nu\alpha} - \left(\mathcal{F}^2 - \frac{7}{4}\mathcal{G}^2 \right) g_{\mu\nu} \right] \end{aligned}$$

By the way, in Gaussian Units,

$$T_{\mu\nu}^{\mathrm{em}} = \frac{1}{4\pi c}\left[F_{\mu}{}^{\alpha} F_{\nu\alpha} - \frac{1}{4}\mathcal{F} g_{\mu\nu} \right] - \frac{1}{(4\pi)^2}\frac{\mu}{4c}\left[8\mathcal{F} F_{\mu}{}^{\alpha} F_{\nu\alpha} - \left(\mathcal{F}^2 - \frac{7}{4}\mathcal{G}^2 \right) g_{\mu\nu} \right]$$

2.2 Maxwell Equation

$$\begin{aligned} \frac{\delta S}{\delta A_{\mu}} &= \frac{\partial \mathcal{L}^{\mathrm{em}}}{\partial A_{\mu}} - \partial_{\nu}\left[\frac{\partial \mathcal{L}^{\mathrm{em}}}{\partial(\partial_{\nu} A_{\mu})} \right] \\ &= 0 - \partial_{\nu}\left(\mathcal{L}_{\mathcal{F}}^{\mathrm{em}} \frac{\partial \mathcal{F}}{\partial(\partial_{\nu} A_{\mu})} + \mathcal{L}_{\mathcal{G}}^{\mathrm{em}} \frac{\partial \mathcal{G}}{\partial(\partial_{\nu} A_{\mu})} \right) \\ &= -2\partial_{\nu}(\mathcal{L}_{\mathcal{F}}^{\mathrm{em}} F^{\nu\mu} - \mathcal{L}_{\mathcal{F}}^{\mathrm{em}} F^{\mu\nu} + \mathcal{L}_{\mathcal{G}}^{\mathrm{em}}({}^{*}F)^{\nu\mu} - \mathcal{L}_{\mathcal{G}}^{\mathrm{em}}({}^{*}F)^{\mu\nu}) \\ &= 4\partial_{\nu}(\mathcal{L}_{\mathcal{F}}^{\mathrm{em}} F^{\mu\nu} + \mathcal{L}_{\mathcal{G}}^{\mathrm{em}}({}^{*}F)^{\mu\nu}) = 0 \end{aligned}$$

$$\Rightarrow \partial_{\nu}(\mathcal{L}_{\mathcal{F}}^{\mathrm{em}} F^{\mu\nu} + \mathcal{L}_{\mathcal{G}}^{\mathrm{em}}({}^{*}F)^{\mu\nu}) = 0$$

2.3 EOM of photon

According to M. Novello's PRD paper, photon's trajectory in nonlinear electrodynamics is a null geodesic of the effective metric $K_{\pm}^{\mu\nu}$, whose general form is

$$\begin{aligned} K_{\pm}^{\mu\nu} &\propto \mathcal{L}_{\mathcal{F}}^{\mathrm{em}} g^{\mu\nu} + 4[(\mathcal{L}_{\mathcal{F}\mathcal{F}}^{\mathrm{em}} + \Omega_{\pm}\mathcal{L}_{\mathcal{F}\mathcal{G}}^{\mathrm{em}}) F_{\lambda}{}^{\mu} F^{\lambda\nu} + (\mathcal{L}_{\mathcal{F}\mathcal{G}}^{\mathrm{em}} + \Omega_{\pm}\mathcal{L}_{\mathcal{G}\mathcal{G}}^{\mathrm{em}}) F_{\lambda}{}^{\mu}({}^{*}F)^{\lambda\nu}] \\ &= [\mathcal{L}_{\mathcal{F}}^{\mathrm{em}} + (\mathcal{L}_{\mathcal{F}\mathcal{G}}^{\mathrm{em}} + \Omega_{\pm}\mathcal{L}_{\mathcal{G}\mathcal{G}}^{\mathrm{em}})\mathcal{G}] g^{\mu\nu} + 4(\mathcal{L}_{\mathcal{F}\mathcal{F}}^{\mathrm{em}} + \Omega_{\pm}\mathcal{L}_{\mathcal{F}\mathcal{G}}^{\mathrm{em}}) F_{\lambda}{}^{\mu} F^{\lambda\nu} \end{aligned}$$

where $\Omega_\pm$ is the solution of equation

$$\Omega^2\Omega_1+\Omega\Omega_2+\Omega_3=0$$

with

$$\begin{cases}\Omega_1=-\mathcal{L}_{\mathcal{F}}^{\text{em}}\mathcal{L}_{\mathcal{FG}}^{\text{em}}+2\mathcal{F}\mathcal{L}_{\mathcal{FG}}^{\text{em}}\mathcal{L}_{\mathcal{GG}}^{\text{em}}+\mathcal{G}[(\mathcal{L}_{\mathcal{GG}}^{\text{em}})^2-(\mathcal{L}_{\mathcal{FG}}^{\text{em}})^2]\\ \Omega_2=(\mathcal{L}_{\mathcal{F}}^{\text{em}}+2\mathcal{G}\mathcal{L}_{\mathcal{FG}}^{\text{em}})(\mathcal{L}_{\mathcal{GG}}^{\text{em}}-\mathcal{L}_{\mathcal{FF}}^{\text{em}})+2\mathcal{F}[\mathcal{L}_{\mathcal{FF}}^{\text{em}}\mathcal{L}_{\mathcal{GG}}^{\text{em}}+(\mathcal{L}_{\mathcal{FG}}^{\text{em}})^2]\\ \Omega_3=\mathcal{L}_{\mathcal{F}}^{\text{em}}\mathcal{L}_{\mathcal{FG}}^{\text{em}}+2\mathcal{F}\mathcal{L}_{\mathcal{FF}}^{\text{em}}\mathcal{L}_{\mathcal{FG}}^{\text{em}}+\mathcal{G}[(\mathcal{L}_{\mathcal{FG}}^{\text{em}})^2-(\mathcal{L}_{\mathcal{FF}}^{\text{em}})^2]\end{cases}$$

The equation of motion reads

$$H=H(q_\mu,x^\mu)=\frac{1}{2}K^{\mu\nu}(x)q_\mu q_\nu$$

$$\begin{cases}\dot{x}^\mu=\frac{\partial H}{\partial q_\mu}=K^{\mu\nu}q_\nu\\ \dot{q}_\mu=-\frac{\partial H}{\partial x^\mu}=-\frac{1}{2}\partial_\mu K^{\alpha\beta}q_\alpha q_\beta\end{cases}$$

In our Eular-Heisenburg Lagrangian case, they read

$$\Omega_1=\left(\frac{7\mu}{8c}\right)^2\mathcal{G},\Omega_2=-\frac{3\mu}{32c^2}+\frac{17\mu^2}{16c^2}\mathcal{F},\Omega_3=-\left(\frac{\mu}{2c}\right)^2\mathcal{G}$$

$$\Omega_\pm=\frac{-\Omega_2\pm\sqrt{\Omega_2^2-4\Omega_1\Omega_3}}{2\Omega_1}=-\frac{\Omega_2}{2\Omega_1}\left(1\mp\sqrt{1-\frac{4\Omega_1\Omega_3}{\Omega_2^2}}\right)$$

$$K_\pm^{\mu\nu}\propto\left[-\frac{1}{4}+\frac{\mu}{2}\mathcal{F}+\frac{7\mu}{8}\Omega_\pm\mathcal{G}\right]g^{\mu\nu}+2\mu F_\lambda{}^\mu F^{\lambda\nu}\propto g^{\mu\nu}+\frac{2\mu F_\lambda{}^\mu F^{\lambda\nu}}{-\frac{1}{4}+\frac{\mu}{2}\mathcal{F}+\frac{7\mu}{8}\Omega_\pm\mathcal{G}}$$

Taylor expansion gives

$$\begin{cases}\Omega_+=-\frac{\Omega_3}{\Omega_2}(1+O(\mu^2))=-\frac{8}{3}\mu\mathcal{G}-\frac{8\times34}{9}\mu^2\mathcal{F}\mathcal{G}+O(\mu^3)=0+O(\mu)\\ \Omega_-=-\frac{\Omega_2}{\Omega_1}(1+O(\mu^2))=\frac{6}{49}(\mu\mathcal{G})^{-1}-\frac{68}{49}\frac{\mathcal{F}}{\mathcal{G}}+O(\mu)\end{cases}$$

$$\begin{cases}-\frac{1}{4}+\frac{\mu}{2}\mathcal{F}+\frac{7\mu}{8}\Omega_+\mathcal{G}=-\frac{1}{4}+\frac{\mu}{2}\mathcal{F}+O(\mu^2)\\ -\frac{1}{4}+\frac{\mu}{2}\mathcal{F}+\frac{7\mu}{8}\Omega_-\mathcal{G}=-\frac{1}{7}-\frac{5}{7}\mu\mathcal{F}+O(\mu^2)\end{cases}$$

3 Metric and Electromagnetic Field

Using Gaussian units here. Gürses and Gürsey Metric reads

$$\mathrm{d}s^2=-\left(1-\frac{2m(r)r}{\rho^2}\right)c^2\mathrm{d}t^2+\frac{\rho^2}{\Delta}\mathrm{d}r^2-\frac{4am(r)r\sin^2\theta}{\rho^2}c\mathrm{d}t\mathrm{d}\phi+\rho^2\mathrm{d}\theta^2+\frac{\Sigma\sin^2\theta}{\rho^2}\mathrm{d}\phi^2$$

where

$$\begin{cases}m(r)=\frac{GM}{c^2}\left(1-\frac{Q_m^2}{2Mc^2r}+A\frac{Q_m^4}{40Mc^2r^5}\right)\\ \Delta=r^2+a^2-2m(r)r\\ \Sigma=(r^2+a^2)^2-a^2\Delta\sin^2\theta\\ \rho^2=r^2+a^2\cos^2\theta\end{cases}$$

In other words,

$$g_{\mu\nu}=\begin{bmatrix} -\left(1-\frac{2m(r)r}{\rho^2}\right)c^2 & 0 & 0 & -\frac{2am(r)r\sin^2\theta}{\rho^2}c \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2am(r)r\sin^2\theta}{\rho^2}c & 0 & 0 & \frac{\Sigma\sin^2\theta}{\rho^2} \end{bmatrix}$$

$$|g|=-\rho^4\sin^2\theta$$

The electromagnetic field reads

$$A_\mu=\frac{Q_m}{\rho^2}\cos\theta\left(\ -a,\ \ 0,\ \ 0,\ \ r^2+a^2\ \right)$$

$$F_{\mu\nu}=\frac{Q_m}{\rho^4}\left\{2ar\cos\theta\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & 0 & 0 \\ 0 & a\sin^2\theta & 0 & 0 \end{bmatrix}+(r^2-a^2\cos^2\theta)\sin\theta\begin{bmatrix} 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & -(r^2+a^2) \\ 0 & 0 & r^2+a^2 & 0 \end{bmatrix}\right\}$$

$$F^{\mu\nu}=\frac{Q_m}{\rho^6}\left\{2ar\cos\theta\begin{bmatrix} 0 & r^2+a^2 & 0 & 0 \\ -(r^2+a^2) & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{bmatrix}+(r^2-a^2\cos^2\theta)\begin{bmatrix} 0 & 0 & a\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ -a\sin\theta & 0 & 0 & -\csc\theta \\ 0 & 0 & \csc\theta & 0 \end{bmatrix}\right\}$$

$$({}^*F)_{\mu\nu}=\frac{Q_m}{\rho^4}\left\{(r^2-a^2\cos^2\theta)\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & 0 & 0 \\ 0 & a\sin^2\theta & 0 & 0 \end{bmatrix}+ar\sin2\theta\begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ -a & 0 & 0 & r^2+a^2 \\ 0 & 0 & -(r^2+a^2) & 0 \end{bmatrix}\right\}$$

$$\begin{aligned}\mathcal{F} &= F^{\mu\nu}F_{\mu\nu}=2(\boldsymbol{B}^2-\boldsymbol{E}^2)\\ &= \frac{Q_m^2}{\rho^{10}}[-4a^2r^2\cos^2\theta 2\rho^2+(r^2-a^2\cos^2\theta)^2 2\rho^2]\\ &= -2\frac{Q_m^2}{\rho^8}[4a^2r^2\cos^2\theta-(r^2-a^2\cos^2\theta)^2]\end{aligned}$$

$$\begin{aligned}\mathcal{G} &= F^{\mu\nu}({}^*F)_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma}=4\boldsymbol{B}\cdot\boldsymbol{E}\\ &= \frac{Q_m^2}{\rho^{10}}(r^2-a^2\cos^2\theta)2ar\cos\theta[-2\rho^2-2\rho^2]\\ &= -4\frac{Q_m^2}{\rho^8}(r^2-a^2\cos^2\theta)2ar\cos\theta\end{aligned}$$

According to calculations by Mathematica,

$$\lim_{a\to0}\left(G_{\mu\nu}-\frac{8\pi\mathrm{G}}{c^4}T^{\mathrm{em}}_{\mu\nu}\right)=0$$

There're the two constrains for symmetric tensor in axisymmetric geometry,

$$\begin{cases} a\sin^2\theta G_{t\phi}+G_{\phi\phi}=\frac{(r^2+a^2)\sin^2\theta}{\rho^2}G_{\theta\theta} \\ a^2\sin^4\theta G_{tt}-G_{\phi\phi}=-\frac{(r^2+a^2+a^2\sin^2\theta)\sin^2\theta}{\rho^2}G_{\theta\theta} \end{cases}$$

and a trivial constrain for symmetric tensor

$$G_{t\phi}=G_{\phi t}$$

that hold for both Einstein tensor and energy-momentum tensor. Therefore, there are only three independent variables in the whole 4×4 tensor. And we define δ_i for $i=1,2,3$ to represent the relative deviation from Einstein equation,

$$\delta_1\equiv\left|\frac{G_{rr}-8\pi T_{rr}}{G_{rr}}\right|, \delta_2\equiv\left|\frac{G_{\theta\theta}-8\pi T_{\theta\theta}}{G_{\theta\theta}}\right|, \delta_3\equiv\left|\frac{G_{t\phi}-8\pi T_{t\phi}}{G_{t\phi}}\right|$$

4 Summary

In Gaussian Units, define

$$\begin{cases} m(r)=\frac{GM}{c^2}\left(1-\frac{Q_m^2}{2Mc^2r}+A\frac{Q_m^4}{40Mc^2r^5}\right) \\ \Delta=r^2+a^2-2m(r)r \\ \Sigma=(r^2+a^2)^2-a^2\Delta\sin^2\theta \\ \rho^2=r^2+a^2\cos^2\theta \end{cases}$$

where

$$A=\frac{2}{\pi}\mu$$

$$\mu=\frac{2}{45}\alpha^2\left(\frac{\hbar}{m_ec}\right)^3\frac{1}{m_ec^2}$$

The metric reads

$$g_{\mu\nu}=\begin{bmatrix} -\left(1-\frac{2m(r)r}{\rho^2}\right)c^2 & 0 & 0 & -\frac{2am(r)r\sin^2\theta}{\rho^2}c \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2am(r)r\sin^2\theta}{\rho^2}c & 0 & 0 & \frac{\Sigma\sin^2\theta}{\rho^2} \end{bmatrix}$$

The electromagnetic field reads

$$A_\mu=\frac{Q_m}{\rho^2}\cos\theta\left(\ -a,\ \ 0,\ \ 0,\ \ r^2+a^2\ \right)$$

$$F_{\mu\nu}=\frac{Q_m}{\rho^4}\left\{2ar\cos\theta\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & 0 & 0 \\ 0 & a\sin^2\theta & 0 & 0 \end{bmatrix}+(r^2-a^2\cos^2\theta)\sin\theta\begin{bmatrix} 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & -(r^2+a^2) \\ 0 & 0 & r^2+a^2 & 0 \end{bmatrix}\right\}$$

The energy-momentum tensor reads

$$T_{\mu\nu}^{\text{em}}=\frac{1}{4\pi c}\left[F_\mu{}^\alpha F_{\nu\alpha}-\frac{1}{4}\mathcal{F}g_{\mu\nu}\right]-\frac{1}{(4\pi)^2}\frac{\mu}{4c}\left[8\mathcal{F}F_\mu{}^\alpha F_{\nu\alpha}-\left(\mathcal{F}^2-\frac{7}{4}\mathcal{G}^2\right)g_{\mu\nu}\right]$$

With

$$\mathcal{F}=F^{\mu\nu}F_{\mu\nu}=2(\boldsymbol{B}^2-\boldsymbol{E}^2)=-2\frac{Q_m^2}{\rho^8}[4a^2r^2\cos^2\theta-(r^2-a^2\cos^2\theta)^2]$$

$$\mathcal{G}=F^{\mu\nu}({}^*F)_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma}=4\boldsymbol{B}\cdot\boldsymbol{E}=-4\frac{Q_m^2}{\rho^8}(r^2-a^2\cos^2\theta)2ar\cos\theta$$

The metric is only exact when $a=0$. For $a\neq 0$ cases, define

$$\delta_1\equiv\left|\frac{G_{rr}-8\pi T_{rr}}{G_{rr}}\right|,\delta_2\equiv\left|\frac{G_{\theta\theta}-8\pi T_{\theta\theta}}{G_{\theta\theta}}\right|,\delta_3\equiv\left|\frac{G_{t\phi}-8\pi T_{t\phi}}{G_{t\phi}}\right|$$

To represent the deviation from Einstein equation.

The motion of the photon is given by the Hamilton equations

$$H=H(q_\mu,x^\mu)=\frac{1}{2}K^{\mu\nu}(x)q_\mu q_\nu$$

$$\begin{cases}\dot{x}^\mu=\frac{\partial H}{\partial q_\mu}=K^{\mu\nu}q_\nu\\ \dot{q}_\mu=-\frac{\partial H}{\partial x^\mu}=-\frac{1}{2}\partial_\mu K^{\alpha\beta}q_\alpha q_\beta\end{cases}$$

With the effective metric $K^{\mu\nu}$ defined as

$$\Omega_1=\left(\frac{7\mu}{8c}\right)^2\frac{\mathcal{G}}{4\pi},\Omega_2=-\frac{3\mu}{32c^2}+\frac{17\mu^2}{16c^2}\frac{\mathcal{F}}{4\pi},\Omega_3=-\left(\frac{\mu}{2c}\right)^2\frac{\mathcal{G}}{4\pi}$$

$$\Omega_\pm=\frac{-\Omega_2\pm\sqrt{\Omega_2^2-4\Omega_1\Omega_3}}{2\Omega_1}=-\frac{\Omega_2}{2\Omega_1}\left(1\mp\sqrt{1-\frac{4\Omega_1\Omega_3}{\Omega_2^2}}\right)$$

$$K_\pm^{\mu\nu}\propto\left[-\frac{1}{4}+\frac{\mu}{8\pi}\mathcal{F}+\frac{7\mu}{32\pi}\Omega_\pm\mathcal{G}\right]g^{\mu\nu}+\frac{\mu}{2\pi}F_\lambda{}^\mu F^{\lambda\nu}$$

where $\pm$ indicates the two branches of birefringence.

Appendix

1 Dimensional Analysis and Natural Units

$$[\mathrm{G}][M]^2[r]^{-1}=[Q]^2[r]^{-1}=[M][c]^2=[\hbar][c][r]^{-1}$$

$$\Rightarrow\begin{cases}[r]=[\mathrm{G}][M][c]^{-2}\\ [Q]=[\mathrm{G}]^{\frac{1}{2}}[M]\\ [\mu]=[\mathrm{G}]^3[M]^2[c]^{-8}\end{cases}$$

Therefore, we may define natural units by the dimensionless quantities

$$r\rightarrow r/\left(\frac{\mathrm{G}M}{c^2}\right)$$

$$Q\rightarrow Q/(\sqrt{\mathrm{G}}M)$$

$$\begin{aligned}\mu\rightarrow\mu/\left(\frac{\mathrm{G}^3M^2}{c^8}\right)&=\frac{2}{45}\alpha^2\left(\frac{\hbar c}{\mathrm{G}Mm_e}\right)^3\frac{M}{m_e}\\ &\approx\left(\frac{1.911514991\times10^{34}\mathrm{Kg}}{M}\right)^2\\ &\approx 0.9236\left(\frac{M_\odot}{M}\right)^2\times10^8\end{aligned}$$