Bayesian Models

Probability and Statistics for Data Science

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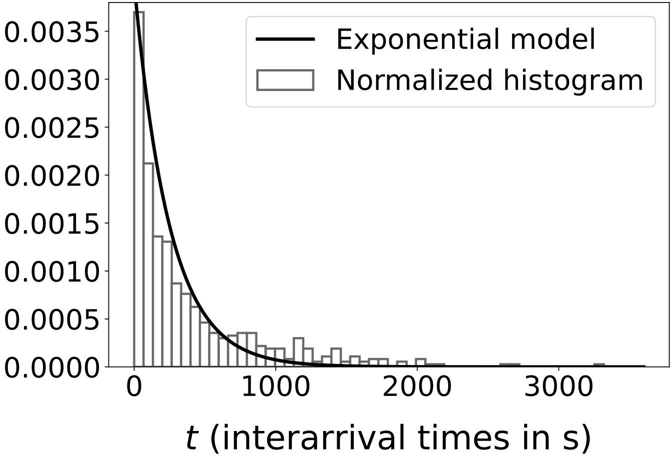
These slides are based on the book Probability and Statistics for Data Science by Carlos Fernandez-Granda, available for purchase here. A free preprint, videos, code, slides and solutions to exercises are available at https://www.ps4ds.net

Goal

Describe the framework of Bayesian modeling

Parametric modeling



Bayesian parametric modeling

Key idea: Interpret parameters as random variables

Modeling decisions

Parameters: $\tilde{\theta}$

Data: $\tilde{x}$

We need to make 2 decisions:

1. Prior distribution of parameters: $f_{\tilde{\theta}}$
2. Conditional distribution or likelihood of the data given the parameters $p_{\tilde{x} \mid \tilde{\theta}}$ or $f_{\tilde{x} \mid \tilde{\theta}}$

Goal: Compute posterior distribution of parameters given data

Coin flip

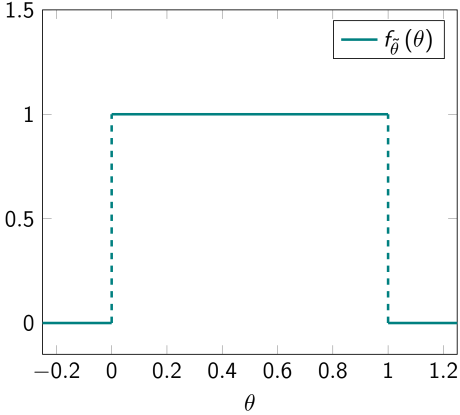
Bet on a coin flip, what is the probability of heads?

We are not sure whether coin flip is fair or not...

How can we encode uncertainty?

Through prior distribution of the probability $\tilde{\theta}$ that coin is heads

Uniform prior



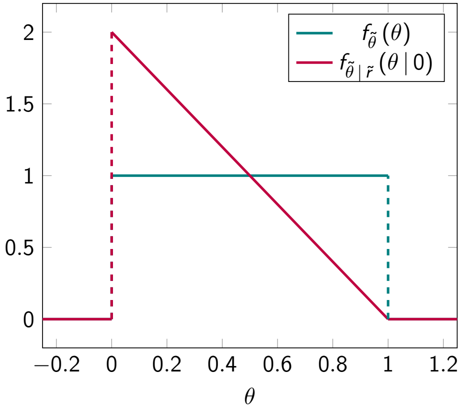
Posterior

Data: Coin lands on tails

Posterior pdf of $\tilde{\theta}$ given this information?

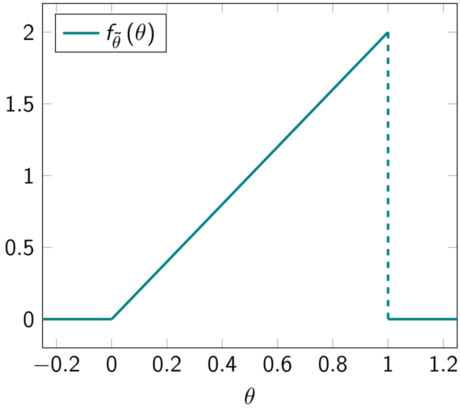
$$
\begin{aligned}
f_{\tilde{\theta}\,|\,\tilde{r}}(\theta\,|\,0) &= \frac{f_{\tilde{\theta}}(\theta)\, p_{\tilde{r}\,|\,\tilde{\theta}}(0\,|\,\theta)}{p_{\tilde{r}}(0)} \\
&= \frac{1-\theta}{\int_{u=-\infty}^{\infty} f_{\tilde{\theta}}(u)\, p_{\tilde{r}\,|\,\tilde{\theta}}(0\,|\,u)\,\mathrm{d}u} \\
&= \frac{1-\theta}{\int_{u=0}^{1}(1-u)\,\mathrm{d}u} \\
&= 2\,(1-\theta)
\end{aligned}
$$

Posterior



$f_{\tilde{\theta}}(\theta)$
$f_{\tilde{\theta}\,|\,\tilde{r}}(\theta\,|\,0)$
2
1.5
1
0.5
0
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
θ

Triangular prior



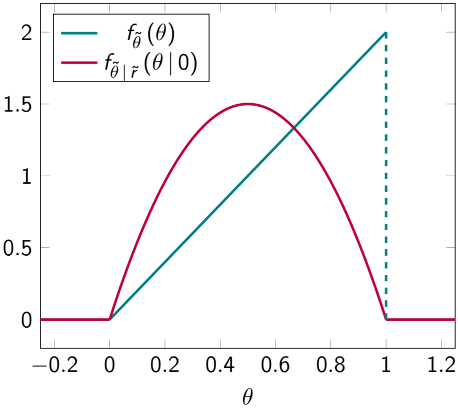
Posterior

Data: Coin lands on tails

Posterior pdf of $\tilde{\theta}$ given this information?

$$
\begin{aligned}
f_{\tilde{\theta}\,|\,\tilde{r}}(\theta\,|\,0) &= \frac{f_{\tilde{\theta}}(\theta)\, p_{\tilde{r}\,|\,\tilde{\theta}}(0\,|\,\theta)}{p_{\tilde{r}}(0)} \\
&= \frac{2\theta(1-\theta)}{\int_{u=-\infty}^{\infty} f_{\tilde{\theta}}(u)\, p_{\tilde{r}\,|\,\tilde{\theta}}(0\,|\,u)\, \mathrm{d}u} \\
&= \frac{2\theta(1-\theta)}{\int_{u=0}^{1} 2u(1-u)\, \mathrm{d}u} \\
&= 6\theta(1-\theta)
\end{aligned}
$$

Posterior



$f_{\tilde{\theta}}(\theta)$
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θ

Beta distribution

Unimodal prior for parameters that represent probabilities

The pdf of a beta random variable $\tilde{t}$ with parameters a and b is

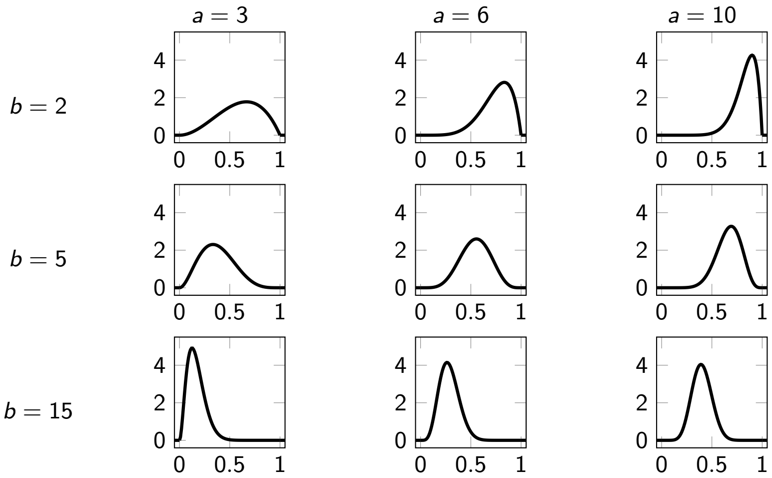
$$f_{\tilde{t}}(t) := \frac{t^{a-1}(1-t)^{b-1}}{\beta(a,b)} \qquad \text{if} \quad 0 \leq t \leq 1$$

and zero otherwise, where

$$\beta(a,b) := \int_u u^{a-1}(1-u)^{b-1}\,\mathrm{d}u$$

is a beta function or Euler integral of the first kind

Beta distribution



Conditional independence

What if we have more data?

Common assumption: Data are conditionally independent given parameters

Same effect as iid assumption: likelihood factorizes

$$p_{\tilde{x} \mid \tilde{\theta}}(x \mid \theta) = \prod_{i=1}^{n} p_{\tilde{x}[i] \mid \tilde{\theta}}(x[i] \mid \theta)$$

$$f_{\tilde{x} \mid \tilde{\theta}}(x \mid \theta) = \prod_{i=1}^{n} f_{\tilde{x}[i] \mid \tilde{\theta}}(x[i] \mid \theta)$$

Estimating parameter of Bernoulli

Data: Sequence of n binary outcomes (0 or 1)

Prior: Parameter $\tilde{\theta}$ modeled as beta with parameters a and b

Likelihood: Given $\tilde{\theta} = \theta$, each data point is Bernoulli with parameter θ

Distribution of number of 1s if data are conditionally independent given $\tilde{\theta}$?

Binomial with parameters n and θ

Posterior distribution

$$
\begin{aligned}
f_{\tilde{\theta} \mid \tilde{x}}(\theta \mid x) &= \frac{f_{\tilde{\theta}}(\theta)\, p_{\tilde{x} \mid \tilde{\theta}}(x \mid \theta)}{p_{\tilde{x}}(x)} \\
&= \frac{f_{\tilde{\theta}}(\theta)\, p_{\tilde{x} \mid \tilde{\theta}}(x \mid \theta)}{\int_u f_{\tilde{\theta}}(u)\, p_{\tilde{x} \mid \tilde{\theta}}(x \mid u)\, \mathrm{d}u} \\
&= \frac{\theta^{a-1}(1-\theta)^{b-1} \binom{n}{x} \theta^{x} (1-\theta)^{n-x}}{\int_u u^{a-1}(1-u)^{b-1} \binom{n}{x} u^{x} (1-u)^{n-x}\, \mathrm{d}u} \\
&= \frac{\theta^{x+a-1}(1-\theta)^{n-x+b-1}}{\int_u u^{x+a-1}(1-u)^{n-x+b-1}\, \mathrm{d}u}
\end{aligned}
$$

Beta random variable with parameters $x + a$ and $n - x + b$

Estimating parameter of Bernoulli

Data: Sequence of n binary outcomes (0 or 1)

Prior: Parameter $\tilde{\theta}$ modeled as beta with parameters a and b

Likelihood: Given $\tilde{\theta} = \theta$, each data point is Bernoulli with parameter θ, so number of 1s is binomial with parameters n and θ

Posterior distribution is beta with parameters $x + a$ and $n - x + b$

The beta distribution is a conjugate prior for the binomial likelihood

Real poll (Pennsylvania)

Data: 281 people intend to vote for Trump, 300 for Biden

Fraction of people that intend to vote for Trump: $\tilde{\theta}$

If n people are chosen independently at random with replacement from the population, likelihood of x voting for Trump?

Conditioned on $\tilde{\theta} = \theta$, binomial with parameters n and θ

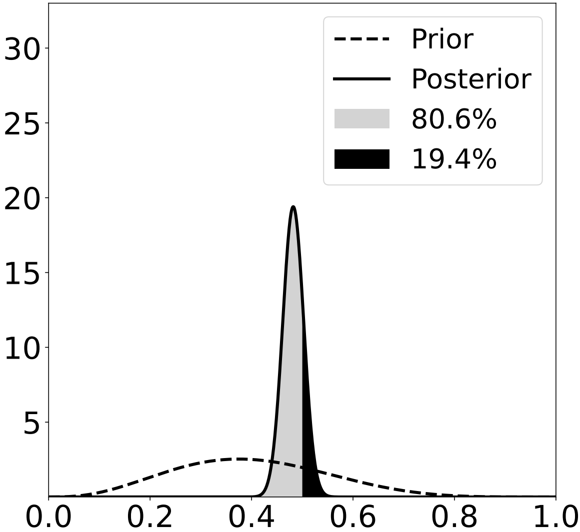
Posterior distribution of $\tilde{\theta}$ depends on prior!

If prior is beta with parameters a and b, posterior is beta with parameters $a + 281$ and $b + 300$

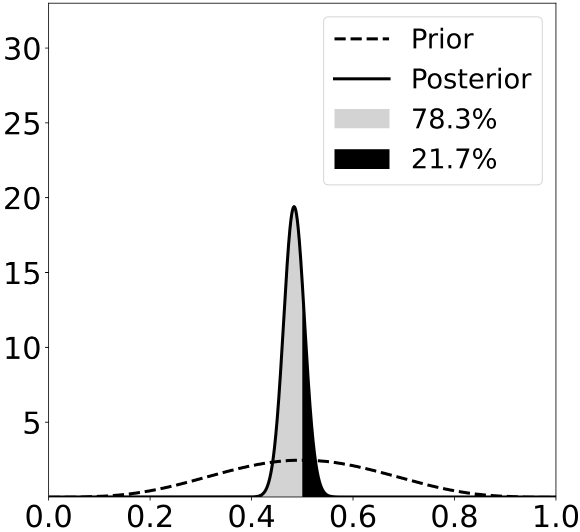
Real poll (Pennsylvania)

Probability that Trump wins in Pennsylvania? $P(\tilde{\theta} > 0.5 \,|\, \tilde{x} = x)$

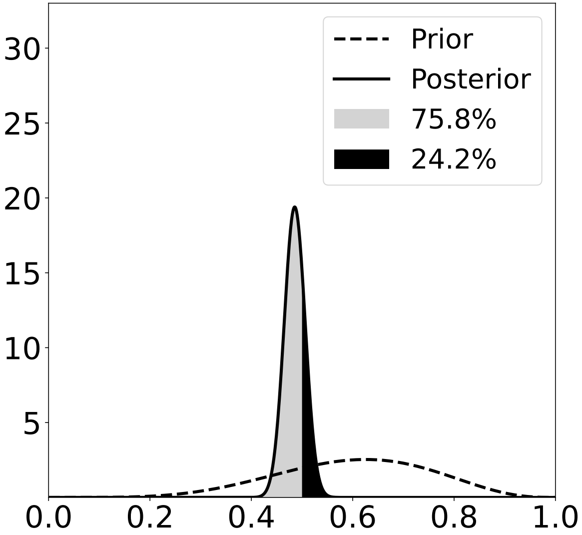
Poll in Pennsylvania



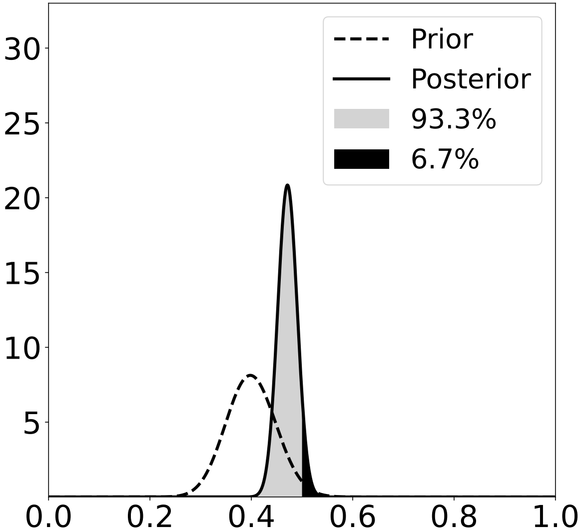
Poll in Pennsylvania



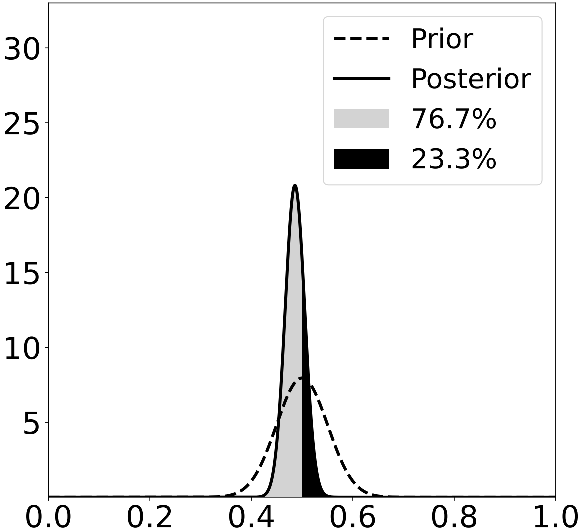
Poll in Pennsylvania



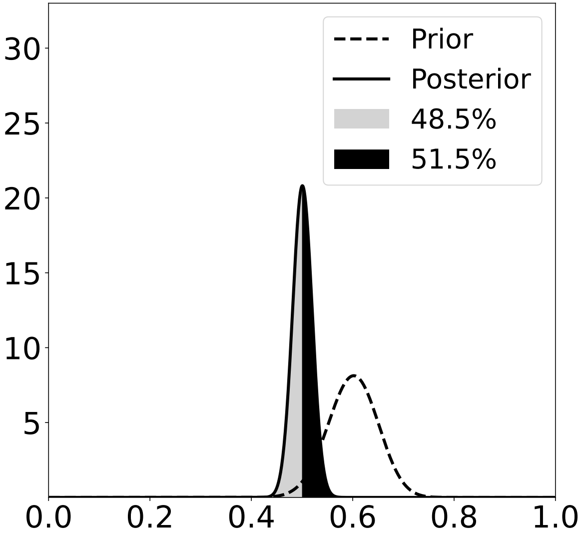
Poll in Pennsylvania



Poll in Pennsylvania



Poll in Pennsylvania



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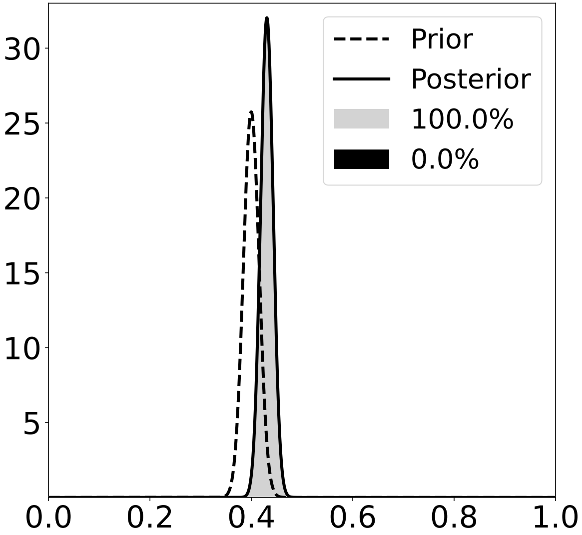

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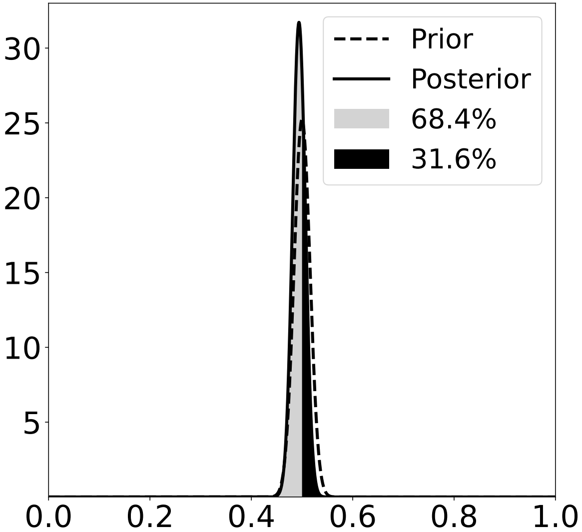
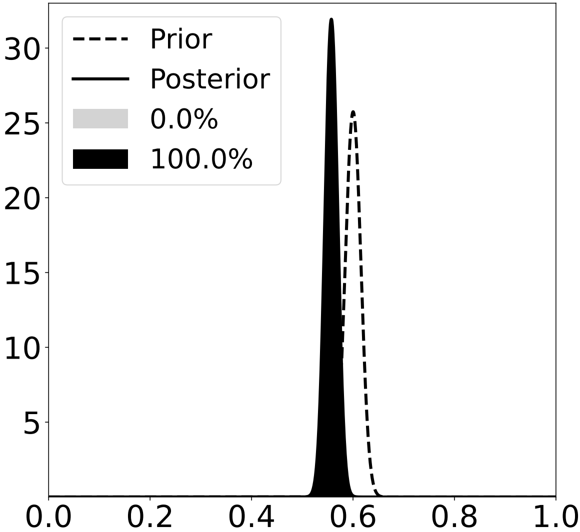


Poll in Pennsylvania



What have we learned?

Bayesian framework for parametric modeling

Conjugate priors