The Mean of Parametric Distributions

Probability and Statistics for Data Science

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These slides are based on the book Probability and Statistics for Data Science by Carlos Fernandez-Granda, available for purchase here. A free preprint, videos, code, slides and solutions to exercises are available at https://www.ps4ds.net

Parametric distributions

Distribution	**Parameter**	**Maximum-likelihood estimator**
Bernoulli	θ	$\frac{1}{n}\sum_{i=1}^{n} x_i$
Geometric	α	$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{-1}$
Poisson	λ	$\frac{1}{n}\sum_{i=1}^{n} x_i$
Exponential	λ	$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{-1}$
Gaussian	μ	$\frac{1}{n}\sum_{i=1}^{n} x_i$

Discrete random variable

The mean of a discrete random variable $\tilde{a}$ with range A is

$$\mathrm{E}\left[\tilde{a}\right] := \sum_{a \in A} a \, p_{\tilde{a}}\left(a\right)$$

if the sum converges

Bernoulli

$$\begin{aligned} \mathrm{E}\left[\tilde{a}\right] &= 0 \cdot p_{\tilde{a}}\left(0\right) + 1 \cdot p_{\tilde{a}}\left(1\right) \\ &= \theta \end{aligned}$$

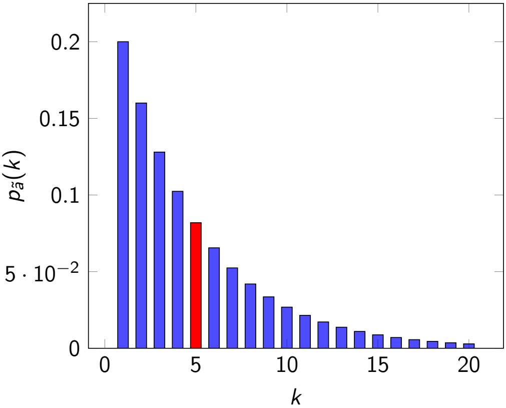
Geometric

$$\sum_{k=0}^{\infty} \alpha^{k} = \frac{1}{1-\alpha}$$

$$\sum_{k=0}^{\infty} k\alpha^{k-1} = \frac{1}{(1-\alpha)^{2}}$$

$$\begin{aligned} \mathrm{E}\left[\tilde{a}\right] &= \sum_{k=1}^{\infty} k\, p_{\tilde{a}}\left(k\right) \\ &= \sum_{k=1}^{\infty} k\,\theta\left(1-\theta\right)^{k-1} \\ &= \frac{1}{\theta} \end{aligned}$$

Geometric, $\theta := 0.2$

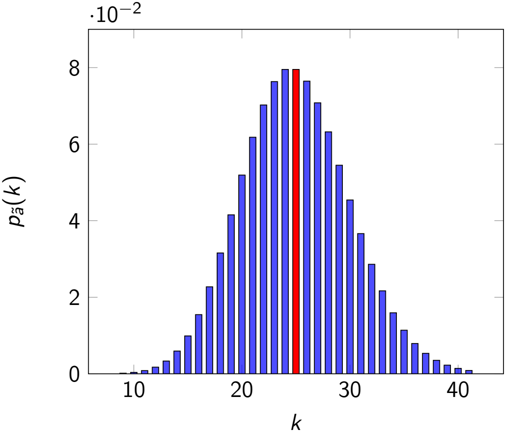


Poisson

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$\begin{aligned}
\mathrm{E}\left[\tilde{a}\right] &= \sum_{k=1}^{\infty} k \, p_{\tilde{a}}\left(k\right) \\
&= \sum_{k=1}^{\infty} \frac{k \lambda^k e^{-\lambda}}{k!} \\
&= \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-1)!} \\
&= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} \\
&= \lambda
\end{aligned}$$

Poisson, $\lambda := 25$



Continuous random variable

The mean of a continuous random variable $\tilde{a}$ is

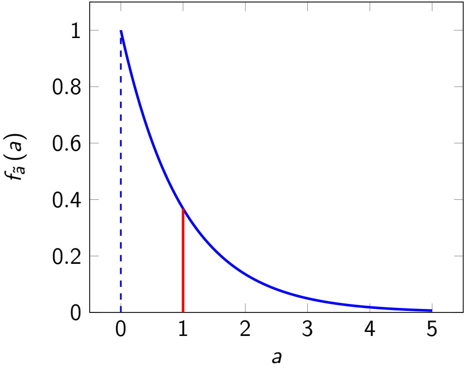
$$\mathrm{E}\left[\tilde{a}\right] := \int_{a=-\infty}^{\infty} a f_{\tilde{a}}\left(a\right) \, \mathrm{d}a$$

if the integral converges

Exponential

$$\begin{aligned}
\mathrm{E}\left[\tilde{a}\right] &= \int_{a=-\infty}^{\infty} a f_{\tilde{a}}\left(a\right)\,\mathrm{d}a \\
&= \int_{a=0}^{\infty} a\lambda e^{-\lambda a}\,\mathrm{d}a \\
&= ae^{-\lambda a}]_0^{\infty} + \frac{1}{\lambda}\int_0^{\infty} \lambda e^{-\lambda a}\,\mathrm{d}a \\
&= \frac{1}{\lambda}
\end{aligned}$$

Exponential, $\lambda := 1$

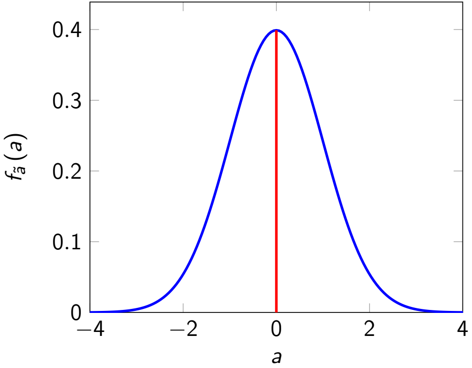


Gaussian

Change of variables $t = (a - \mu) / \sigma$

$$
\begin{aligned}
\mathrm{E}[\tilde{a}] &= \int_{a=-\infty}^{\infty} a f_{\tilde{a}}(a) \, \mathrm{d}a \\
&= \int_{a=-\infty}^{\infty} \frac{a}{\sqrt{2\pi}\sigma} e^{-\frac{(a-\mu)^2}{2\sigma^2}} \, \mathrm{d}a \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{t=-\infty}^{\infty} t e^{-\frac{t^2}{2}} \mathrm{d}t + \frac{\mu}{\sqrt{2\pi}} \int_{t=-\infty}^{\infty} e^{-\frac{t^2}{2}} \, \mathrm{d}t \\
&= \mu
\end{aligned}
$$

Gaussian, $\mu := 0$



Sample mean

The sample mean of a dataset $X := \{x_1, x_2, \ldots, x_n\}$ is

$$m(X) := \frac{\sum_{i=1}^{n} x_i}{n}$$

Method of moments

Distribution	Parameter	Maximum-likelihood estimator	Mean
Bernoulli	θ	$\frac{1}{n}\sum_{i=1}^{n} x_i = m(X)$	θ
Geometric	α	$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{-1} = m(X)^{-1}$	α^{-1}
Poisson	λ	$\frac{1}{n}\sum_{i=1}^{n} x_i = m(X)$	λ
Exponential	λ	$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{-1} = m(X)^{-1}$	λ^{-1}
Gaussian	μ	$\frac{1}{n}\sum_{i=1}^{n} x_i = m(X)$	μ

What have we learned?

Mean of parametric distributions

Method of moments