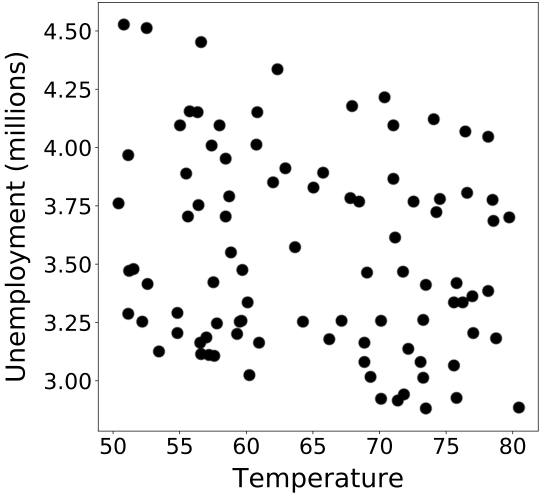
Causal Inference via Linear Regression

Probability and Statistics for Data Science

Carlos Fernandez-Granda

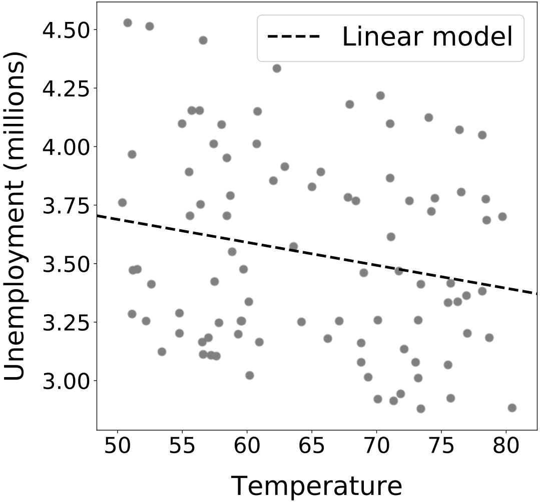
These slides are based on the book Probability and Statistics for Data Science by Carlos Fernandez-Granda, available for purchase here. A free preprint, videos, code, slides and solutions to exercises are available at https://www.ps4ds.net

Unemployment and temperature in Spain (2015-2022)



Linear relationship between unemployment and temperature?

Linear coefficient: -0.01



Would an increase in temperature decrease unemployment?

Causal inference

Key question: Does a treatment $\tilde{t}$ cause a certain outcome?

Potential outcome: $\widetilde{\text{po}}_t$ (defined for observed and unobserved t)

Observed data:

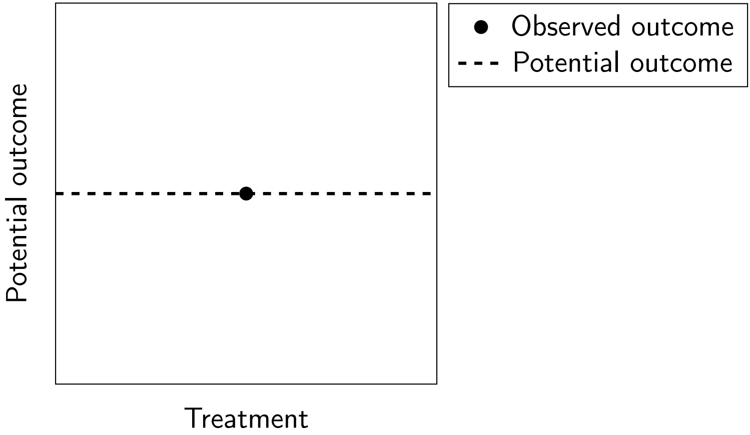
$$\tilde{y} := \widetilde{\text{po}}_t \qquad \text{if} \qquad \tilde{t} = t$$

Fundamental problem of causal inference: Incomplete data

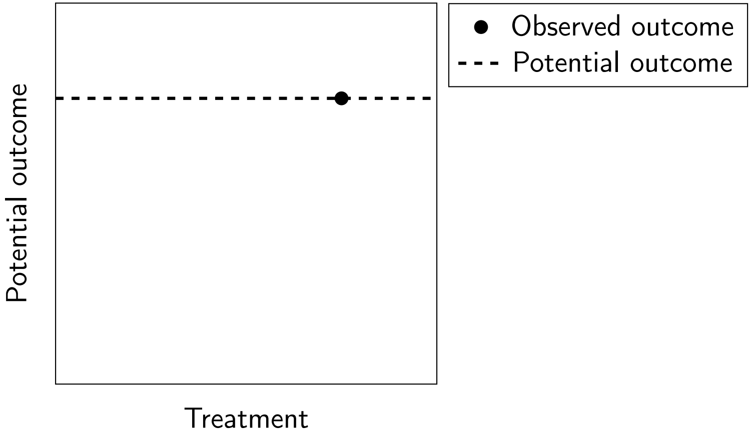
If temperature is 40°, we only observe $\tilde{y} := \widetilde{\text{po}}_{40}$

$\widetilde{\text{po}}_{30}$, $\widetilde{\text{po}}_{45}$, $\widetilde{\text{po}}_{63}$, $\widetilde{\text{po}}_{75}$... are all counterfactuals

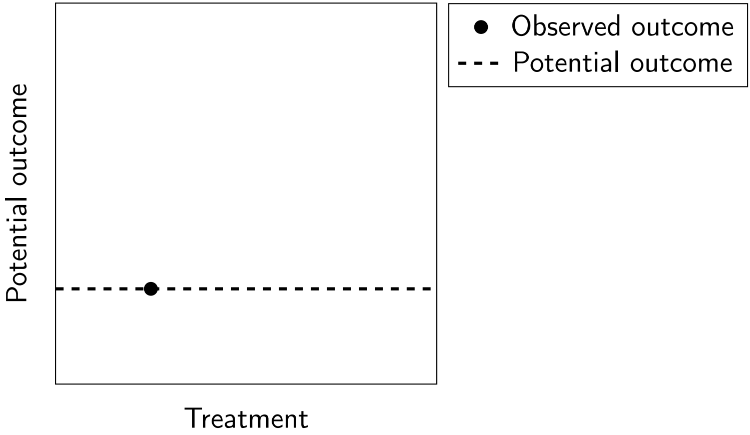
Potential outcomes



Potential outcomes



Potential outcomes

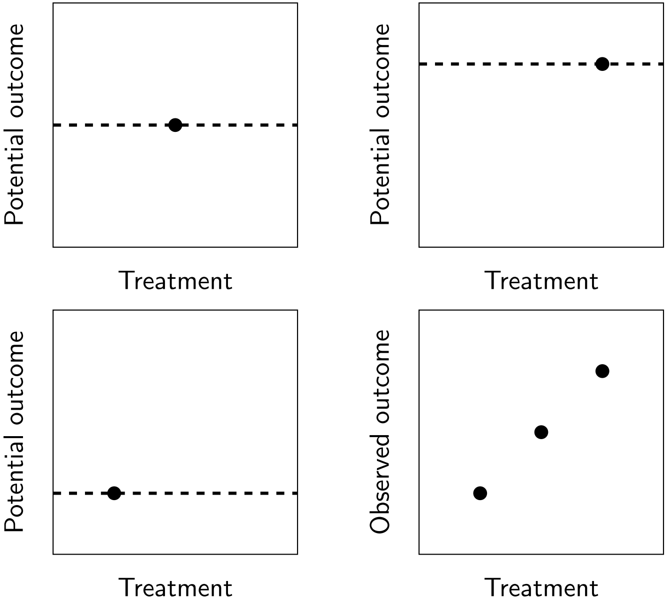


Linear causal effect

For some constant $\beta \in \mathbb{R}$

$$\mathrm{E}\left[\widetilde{\mathsf{po}}_{t}\right] = \beta t$$

Are observed linear effects always causal?



Idealized model

Potential outcome $\widetilde{\mathrm{po}}_t$ depends linearly on treatment t, but also on confounder $\tilde{c}$

$$\widetilde{\mathrm{po}}_t := \beta_{\text{treat}} t + \beta_{\text{conf}} \tilde{c} + \tilde{z}$$

$$\tilde{y} = \beta_{\text{treat}} \tilde{t} + \beta_{\text{conf}} \tilde{c} + \tilde{z}$$

Assumptions:

- $\tilde{t}$ and $\tilde{c}$ are standardized
- $\tilde{z}$ is independent from $\tilde{t}$ and $\tilde{c}$

Estimation of linear effect

Goal: Estimate β_{treat}

$$\tilde{y} = \beta_{\text{treat}} \tilde{t} + \beta_{\text{conf}} \tilde{c} + \tilde{z}$$

Idea: Apply linear regression

- Response: $\tilde{y}$
- Feature: $\tilde{t}$

Simple linear regression

$$\tilde{y} = \beta_{\text{treat}} \tilde{t} + \beta_{\text{conf}} \tilde{c} + \tilde{z}$$

Minimum mean-squared error estimator of $\tilde{y}$ given $\tilde{t} = t$

$$\ell_{\text{MMSE}}(t) := \beta_{\text{MMSE}} t$$

$$\begin{aligned} \beta_{\text{MMSE}} &= \frac{\operatorname{Cov}[\tilde{y}, \tilde{t}\,]}{\operatorname{Var}[\tilde{t}]} \\ &= \operatorname{Cov}[\tilde{y}, \tilde{t}\,] \\ &= \beta_{\text{treat}} + \beta_{\text{conf}} \sigma_{\tilde{t},\tilde{c}} \end{aligned}$$

Coefficient distorted by confounder

Guinea-pig rescue

Goal: Fatten the guinea pigs

Question: Does a nutritional supplement increase weight?

Guinea pigs

Potential outcome $\widetilde{\text{po}}_t$: Weight change

$$\widetilde{\text{po}}_t := \tilde{c} + \tilde{z}$$

$$\tilde{y} = \tilde{c} + \tilde{z}$$

Treatment $\tilde{t}$: Supplement intake, $\beta_{\text{treat}} := 0$

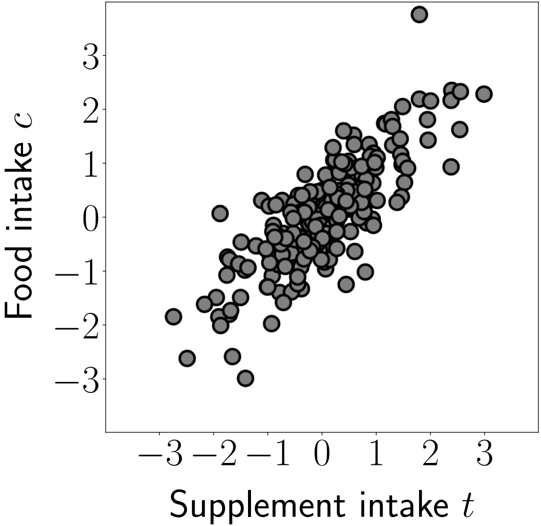
Confounder $\tilde{c}$: Food intake, $\beta_{\text{conf}} := 1$

$$\begin{aligned} \beta_{\text{MMSE}} &= \beta_{\text{treat}} + \beta_{\text{conf}} \sigma_{\tilde{t},\tilde{c}} \\ &= \sigma_{\tilde{t},\tilde{c}} \end{aligned}$$

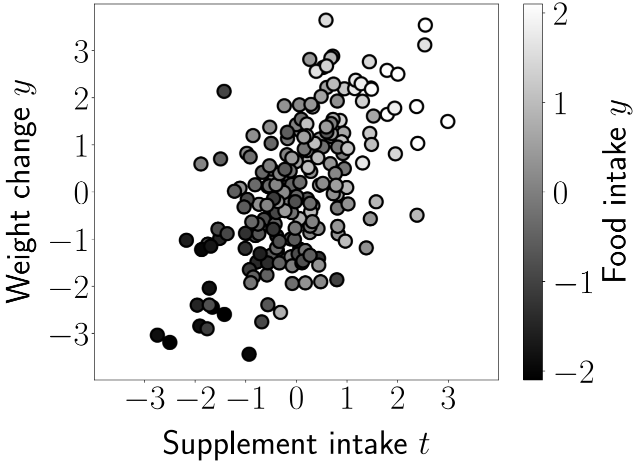
Week 1

Supplement mixed with food

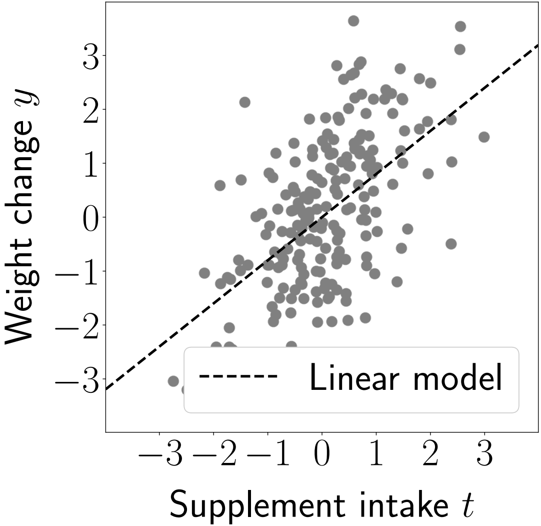
Supplement mixed with food: $\sigma_{\tilde{t},\tilde{c}} := 0.8$



Supplement mixed with food: $\mathrm{Cov}[\tilde{y}, \tilde{t}] = 0.8$



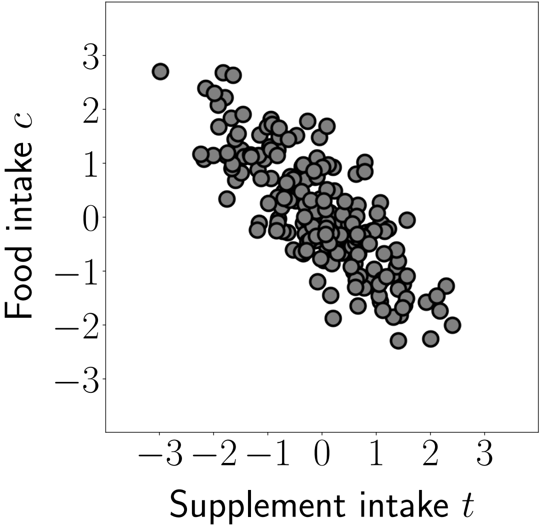
Supplement mixed with food: $\beta_{\mathsf{MMSE}} = 0.8$



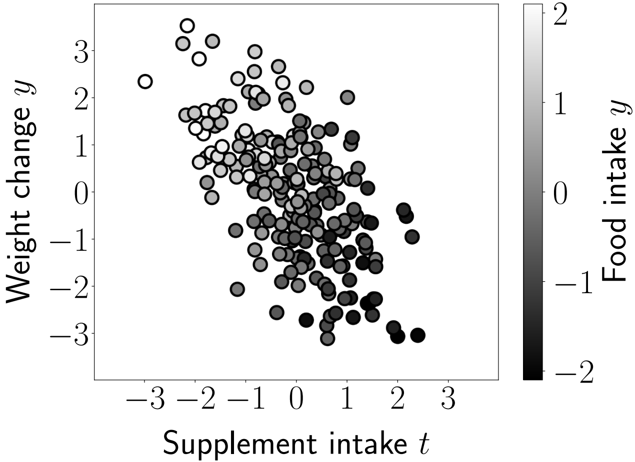
Week 2

Supplement provided after food

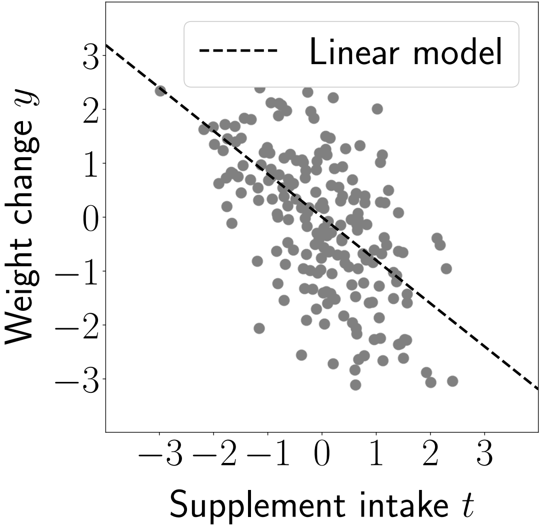
Supplement after food: $\sigma_{\tilde{t},\tilde{c}} := -0.8$



Supplement after food: $\mathrm{Cov}\left[\tilde{y}, \tilde{t}\,\right] = -0.8$



Supplement after food: $\beta_{\mathsf{MMSE}} = -0.8$



Short regression

$$\tilde{y} = \beta_{\text{treat}} \tilde{t} + \beta_{\text{conf}} \tilde{c} + \tilde{z}$$

Linear coefficient distorted by confounder

$$\beta_{\text{MMSE}} = \beta_{\text{treat}} + \beta_{\text{conf}} \sigma_{\tilde{t},\tilde{c}}$$

Solution: Randomization, so that $\sigma_{\tilde{t},\tilde{c}} = 0$

Often costly, or infeasible...

Alternative: **Adjust for confounder** including it in the model

Long regression

$$\tilde{y} = \beta_{\text{treat}}\tilde{t} + \beta_{\text{conf}}\tilde{c} + \tilde{z}$$

- Response $\tilde{y}$
- Feature vector $\tilde{x} := \begin{bmatrix} \tilde{t} \\ \tilde{c} \end{bmatrix}$

Long regression controls for confounders

$$
\begin{aligned}
\tilde{y} &:= \beta_{\text{treat}} \tilde{t} + \beta_{\text{conf}} \tilde{c} + \tilde{z} \\
&= \tilde{x}^T \begin{bmatrix} \beta_{\text{treat}} \\ \beta_{\text{conf}} \end{bmatrix} + \tilde{z} \\
\beta_{\text{MMSE}} &= \Sigma_{\tilde{x}}^{-1} \Sigma_{\tilde{x}\tilde{y}} = \begin{bmatrix} \beta_{\text{treat}} \\ \beta_{\text{conf}} \end{bmatrix} \qquad \text{It works!} \\
\Sigma_{\tilde{x}\tilde{y}} &= \mathrm{E}\left[\tilde{x}\tilde{y}\right] \\
&= \mathrm{E}\left[\tilde{x}\left(\tilde{x}^T \begin{bmatrix} \beta_{\text{treat}} \\ \beta_{\text{conf}} \end{bmatrix} + \tilde{z}\right)\right] \\
&= \mathrm{E}\left[\tilde{x}\tilde{x}^T\right] \begin{bmatrix} \beta_{\text{treat}} \\ \beta_{\text{conf}} \end{bmatrix} + \mathrm{E}\left[\tilde{x}\tilde{z}\right] \\
&= \Sigma_{\tilde{x}} \begin{bmatrix} \beta_{\text{treat}} \\ \beta_{\text{conf}} \end{bmatrix} + \mathrm{E}\left[\tilde{x}\right] \mathrm{E}\left[\tilde{z}\right] \\
&= \Sigma_{\tilde{x}} \begin{bmatrix} \beta_{\text{treat}} \\ \beta_{\text{conf}} \end{bmatrix}
\end{aligned}
$$

Guinea pigs

Guinea pigs

$$\tilde{y} = \tilde{c} + \tilde{z}$$

Treatment $\tilde{t}$: Supplement intake, $\beta_{\text{treat}} := 0$

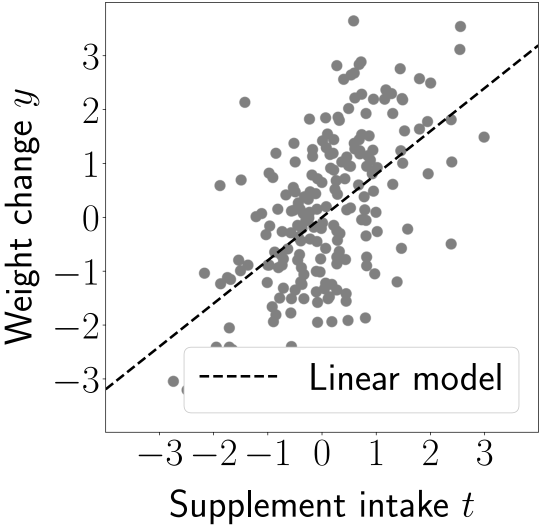
Confounder $\tilde{c}$: Food intake, $\beta_{\text{conf}} := 1$

Week 1

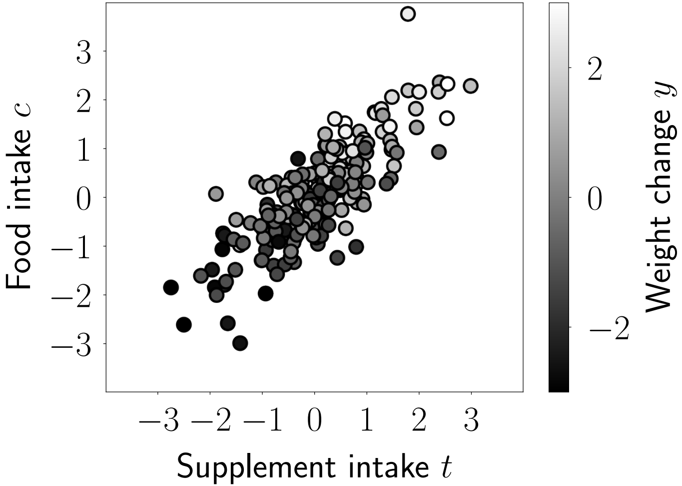
Supplement mixed with food

Supplement mixed with food

Short regression coefficient: 0.8

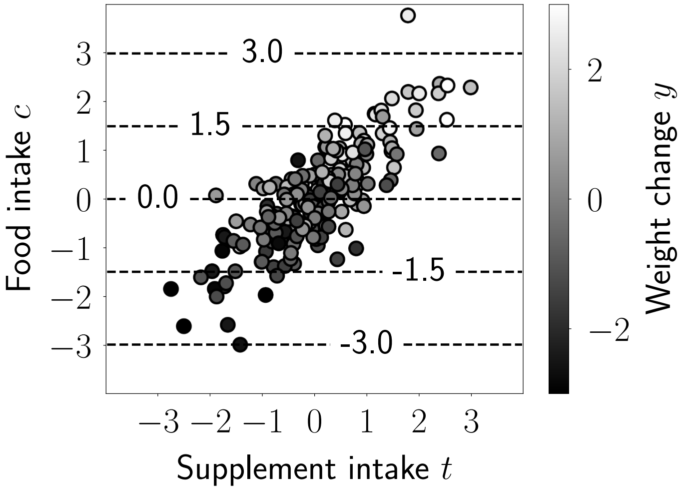


Incorporating the confounder



Incorporating the confounder

Long regression coefficient: 0

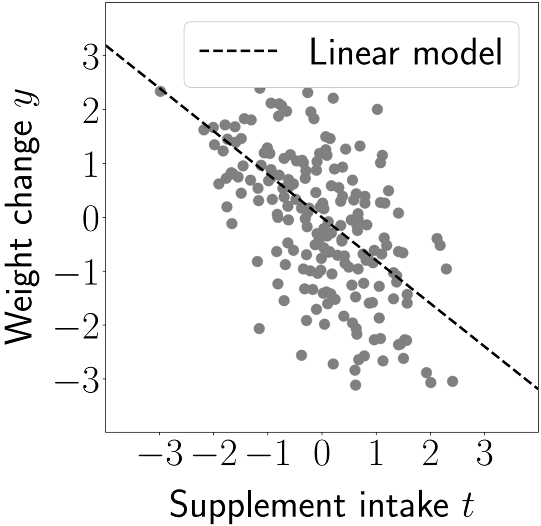


Week 2

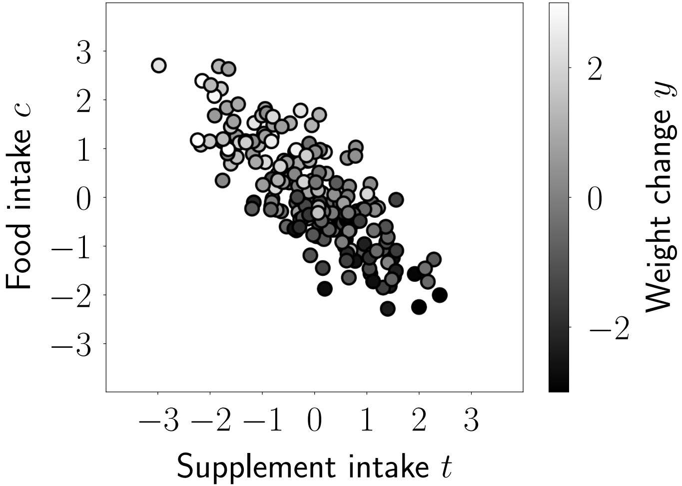
Supplement provided after food

Supplement after food

Short regression coefficient: -0.8

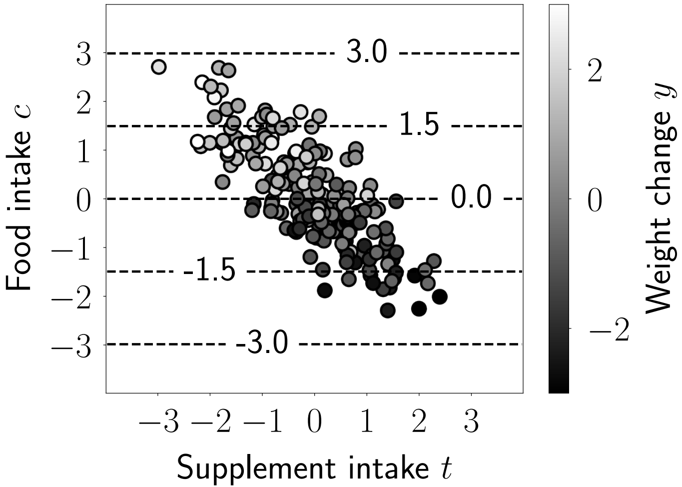


Incorporating the confounder

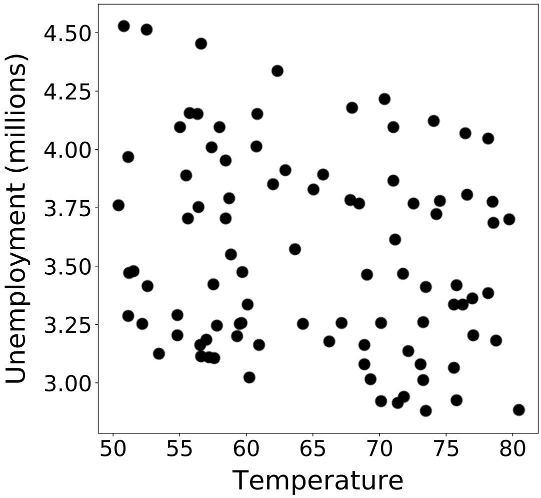


Incorporating the confounder

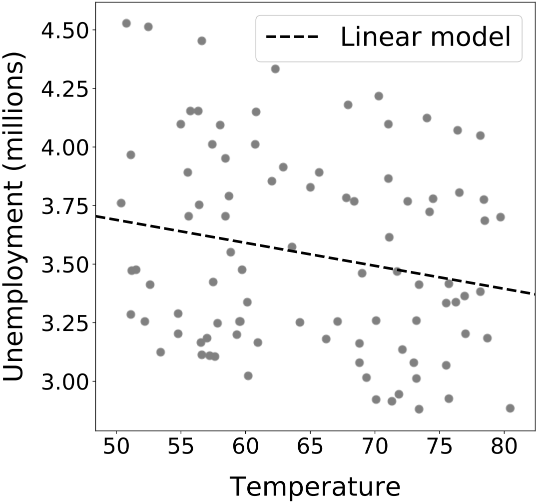
Long regression coefficient: 0



Unemployment and temperature in Spain

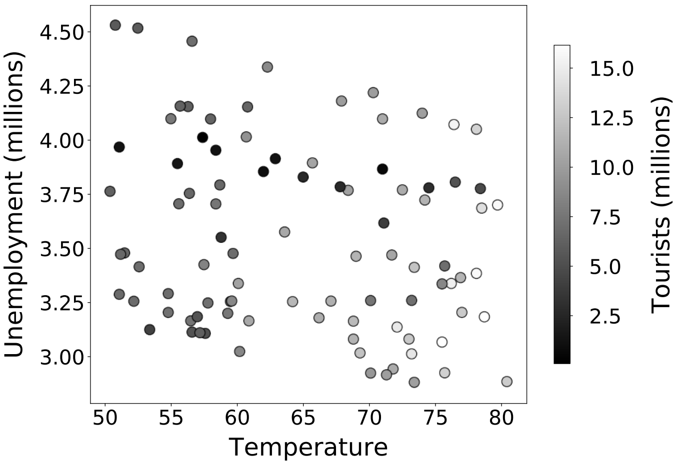


Short regression coefficient: -0.01



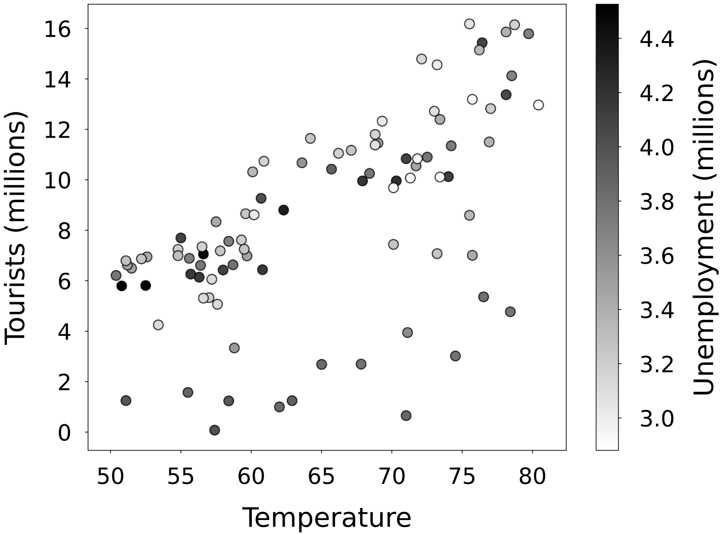
Confounder?

Tourists!

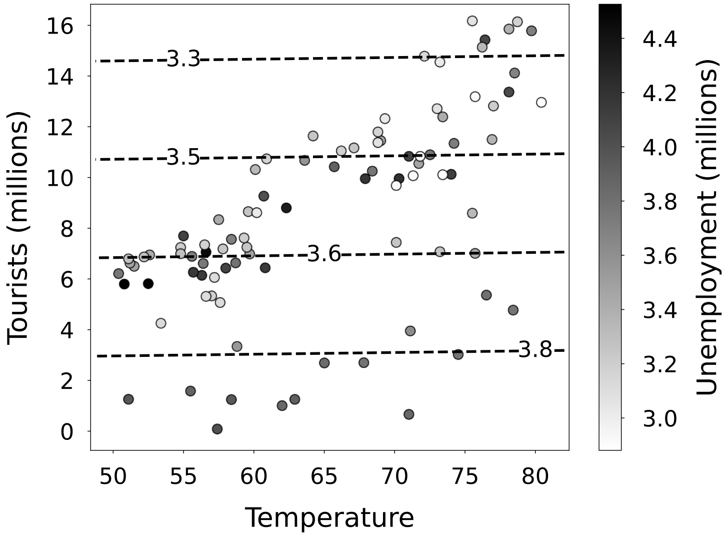


Unemployment (millions)
4.50
4.25
4.00
3.75
3.50
3.25
3.00
50
55
60
65
70
75
80
Temperature
15.0
12.5
10.0
7.5
5.0
2.5
Tourists (millions)

Incorporating the confounder



Long regression coefficient: 0.0003



Variance analysis

Fraction of variance explained by:

- Only temperature: $R^2 = 0.042$
- Only tourism: $R^2 = 0.12024$
- Temperature and tourism: $R^2 = 0.12026$

What have we learned

Unobserved confounders distort linear coefficients in short regression models

We can adjust for the confounders by including them as features in long regression models

This works (under linear assumptions) as long as there are no additional confounders