Causal Inference (overview)

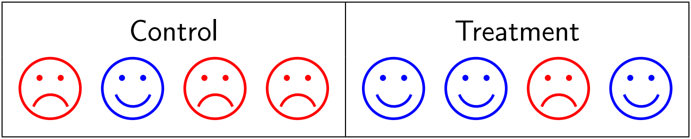
Probability and Statistics for Data Science

Carlos Fernandez-Granda

Key question

Does an observed *statistical* effect imply a causal effect?

New drug



Control group recovery rate: 25%

Treatment group recovery rate: **75%**

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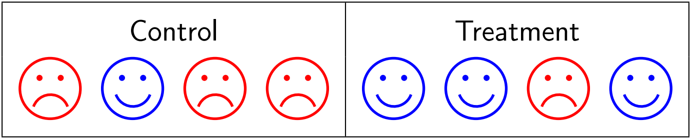

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Treatment group recovery rate: **75%**

Is a new patient more likely to recover if treated?

3-point shooting

Stephen Curry: 41.7%

Courtney Lee: **43.9%**

3-point shooting

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Courtney Lee: **43.9%**

Was Lee a better shooter?

Grades in a Portuguese school

With private classes, average grade: **10.94 / 20**

Without private classes, average grade: 9.98 / 20

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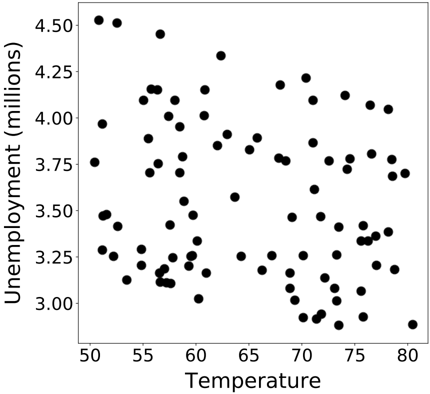
Are the classes useful?

Unemployment and temperature in Spain (2015-2022)



Correlation coefficient: -0.21

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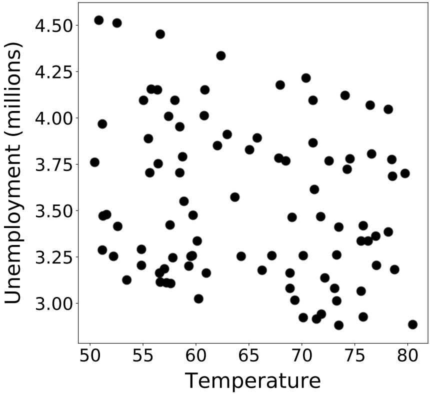

Correlation coefficient: -0.21

Would an increase in temperature decrease unemployment?

Plan

Potential outcomes

Confounding factors

Adjusting for confounders

Potential outcomes

Confounding factors

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Probabilistic modeling

Probabilistic modeling

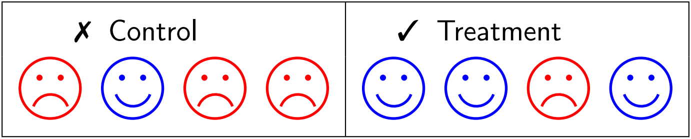
Outcome: $\tilde{y}$

Probabilistic modeling

Outcome: $\tilde{y}$

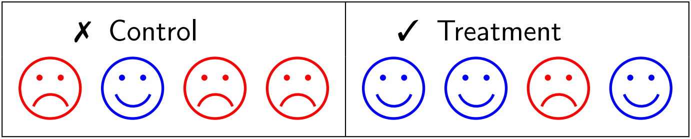
Treatment: $\tilde{t}$

New drug



New drug

Outcome $\tilde{y}$: If patient recovered $\tilde{y} = 1$, if not $\tilde{y} = 0$



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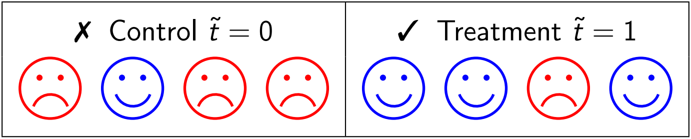

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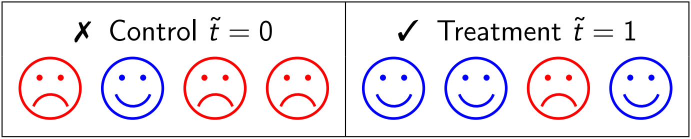

$$0.25 = p_{\tilde{y}\,|\,\tilde{t}}(1\,|\,0) < p_{\tilde{y}\,|\,\tilde{t}}(1\,|\,1) = 0.75$$

3-point shooting

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Outcome $\tilde{y}$: If shot goes in $\tilde{y} = 1$, if not $\tilde{y} = 0$

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Treatment $\tilde{t}$: Player who shoots

3-point shooting

Outcome $\tilde{y}$: If shot goes in $\tilde{y} = 1$, if not $\tilde{y} = 0$

Treatment $\tilde{t}$: Player who shoots

$$P\left(\tilde{y} = 1 \,|\, \tilde{t} = \text{Curry}\right) = 0.417$$

$$P\left(\tilde{y} = 1 \,|\, \tilde{t} = \text{Lee}\right) = 0.439$$

Grades in a Portuguese school

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Outcome $\tilde{y}$: Grades

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$$\mu_{\tilde{y} \,|\, \tilde{t}}(1) = 10.94 \qquad \mu_{\tilde{y} \,|\, \tilde{t}}(0) = 9.98$$

Grades in a Portuguese school

Outcome $\tilde{y}$: Grades

Treatment $\tilde{t}$: If student receives private classes $\tilde{t} = 1$, if not $\tilde{t} = 0$

Outcome is not binary, so we use *conditional mean* to compute average treatment effect (ATE)

$$\mu_{\tilde{y} \mid \tilde{t}}(1) = 10.94 \qquad \mu_{\tilde{y} \mid \tilde{t}}(0) = 9.98$$

$$\text{observed ATE} := \mu_{\tilde{y} \mid \tilde{t}}(1) - \mu_{\tilde{y} \mid \tilde{t}}(0) = 0.96$$

Unemployment in Spain

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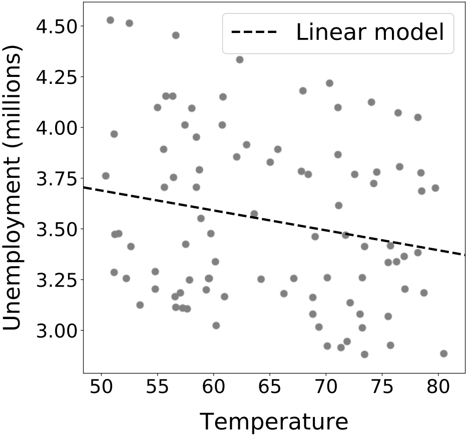
Equivalently, *linear model* of outcome given treatment

Linear coefficient: -0.01



Potential outcomes

Alternative scenarios tied to different values of the treatment

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Alternative scenarios tied to different values of the treatment

They are always defined regardless of the value of the treatment

New drug

$\widetilde{\mathsf{po}}_0$: Outcome if patient is untreated

$\widetilde{\mathsf{po}}_0 = 1$

$\widetilde{\mathsf{po}}_0 = 0$

New drug

$\widetilde{\mathrm{po}}_0$: Outcome if patient is untreated

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Causal effect: $\mathrm{P}\left(\widetilde{\text{po}}_0 = 1\right)$ compared to $\mathrm{P}\left(\widetilde{\text{po}}_1 = 1\right)$

3-point shooting

$\widetilde{\text{po}}_{\text{Curry}}$: Outcome if Curry shoots

$\widetilde{\text{po}}_{\text{Curry}} = 1$ shot made, $\widetilde{\text{po}}_{\text{Curry}} = 0$ shot missed

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$\widetilde{\text{po}}_{\text{Lee}}$: Outcome if Lee shoots

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$\widetilde{\text{po}}_0$: Grade without private classes

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Causal ATE $= \mathrm{E}\left[\widetilde{\text{po}}_1\right] - \mathrm{E}\left[\widetilde{\text{po}}_0\right]$

Unemployment in Spain

$\widetilde{\text{po}}_t$: Unemployment when temperature is t

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Defined regardless of temperature value

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Causal linear effect: $\mathrm{E}\left[\widetilde{\text{po}}_t\right] = \beta_{\text{treat}} t$

Fundamental problem of causal inference

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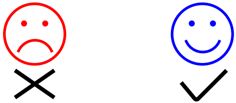
All potential outcomes are defined regardless of treatment value



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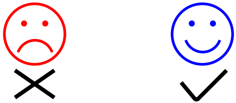

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If $\tilde{t} = 1$

All other potential outcomes are unobserved counterfactuals

Key question

When do observed statistics reflect causal effects?

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- Binary outcomes / treatment

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- Nonbinary outcomes and binary treatment

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- Nonbinary outcomes / treatment, such that $\mathrm{E}\left[\widetilde{\mathsf{po}}_t\right] = \beta_{\mathsf{treat}}\, t$

$$\beta_{\mathsf{treat}} = \mathrm{Cov}\left[\tilde{y}, \tilde{t}\,\right] = \beta_{\mathsf{MMSE}} \quad ?$$

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When treatment and potential outcomes are independent

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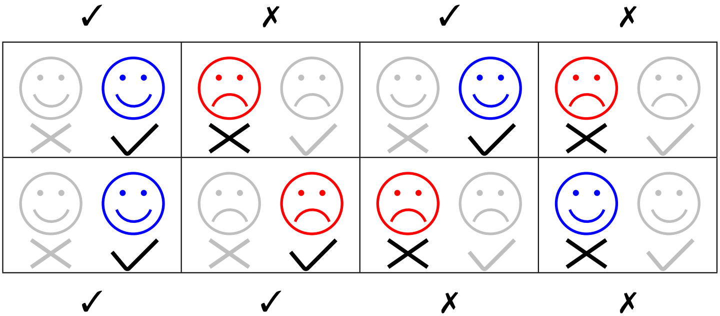
When treatment and potential outcomes are independent

Can be ensured by randomizing the treatment

Independence is essential

Avoids systematic differences between control and treatment groups



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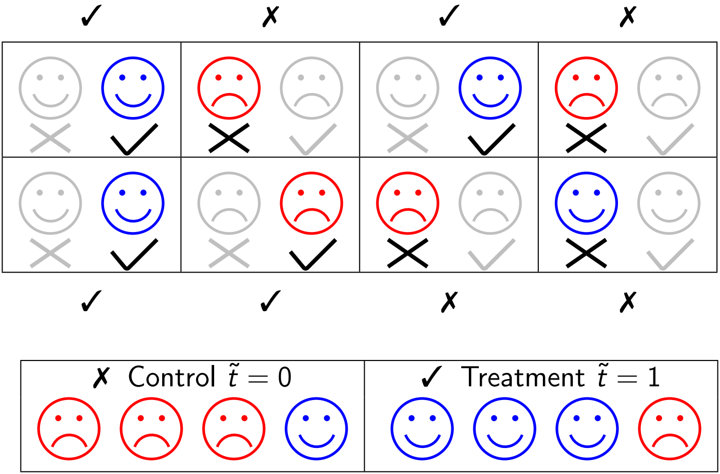

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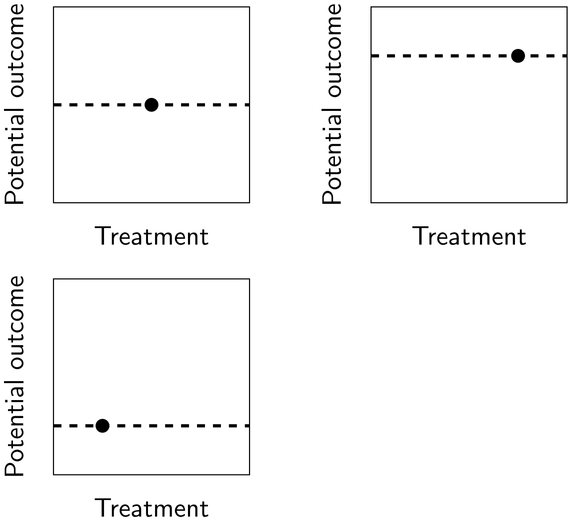

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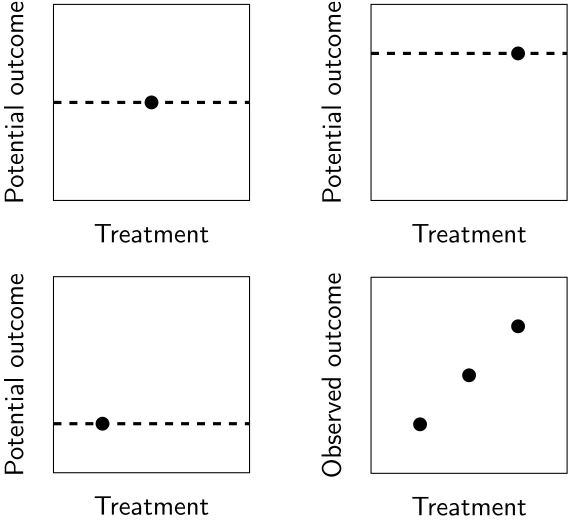

Potential outcomes

Confounding factors

Adjusting for confounders

Confounding factor (a.k.a. confounder)

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Induces dependence between treatment and potential outcomes

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Assumption: Conditional independence of treatment and potential outcomes given confounder

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Confounder completely governs dependence

Binary outcome / treatment

Confounder $\tilde{c}$ with C values

$$\mathrm{P}\left(\tilde{y} = 1 \,|\, \tilde{t} = 1\right)$$

Binary outcome / treatment

Confounder $\tilde{c}$ with C values

$$\begin{aligned}
&\mathrm{P}\left(\tilde{y} = 1 \,|\, \tilde{t} = 1\right) \\
&= \sum_{c=1}^{C} \mathrm{P}(\tilde{c} = c \,|\, \tilde{t} = 1) \mathrm{P}(\widetilde{\mathsf{po}}_1 = 1 \,|\, \tilde{c} = c)
\end{aligned}$$

Binary outcome / treatment

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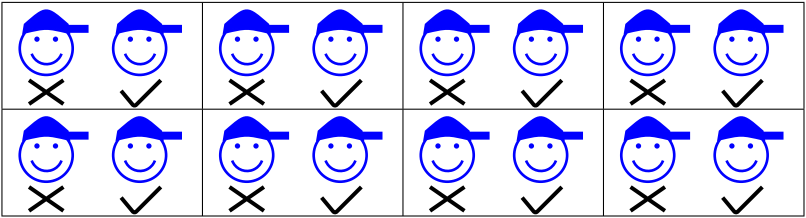
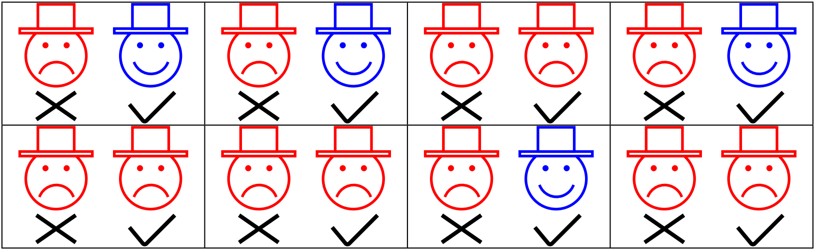
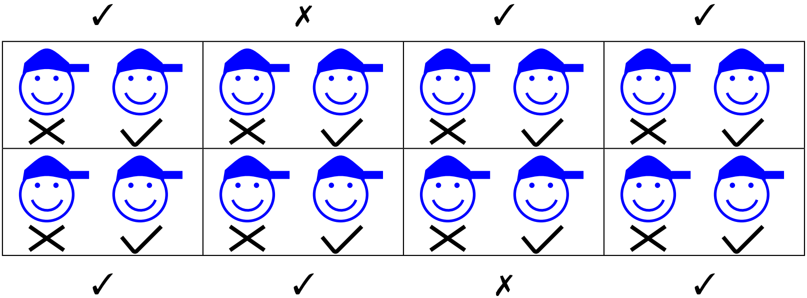
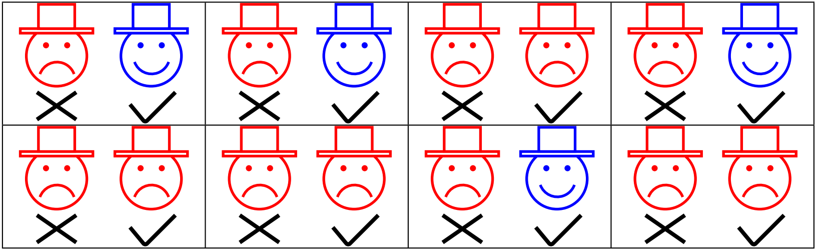
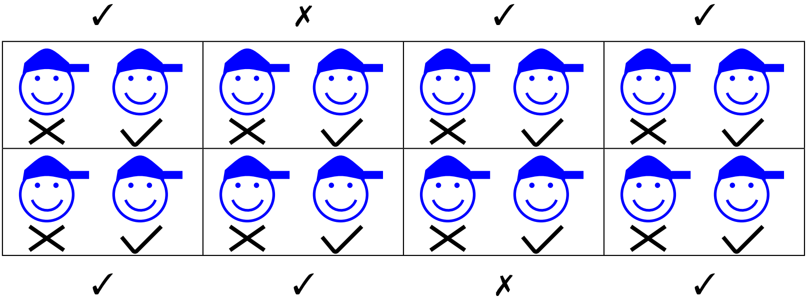
Example: Age as a confounder in new drug evaluation

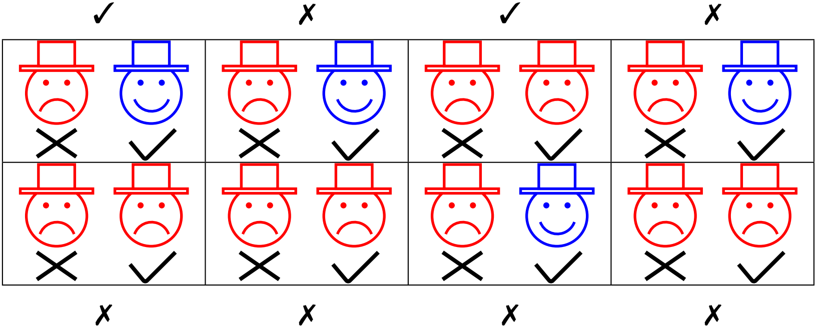


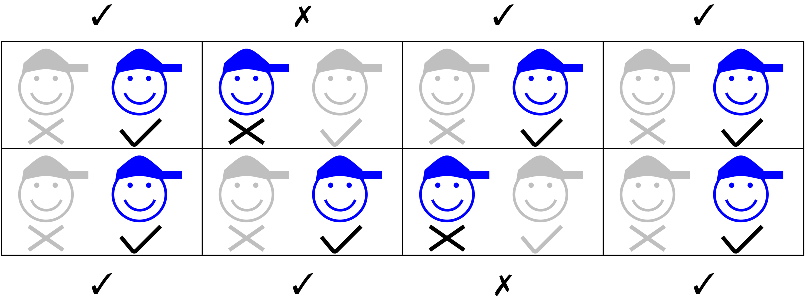


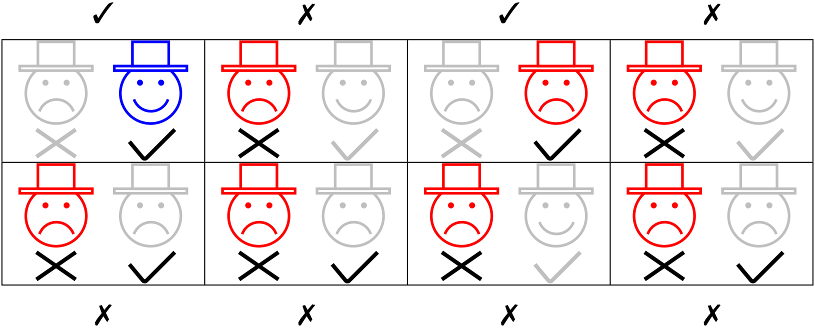


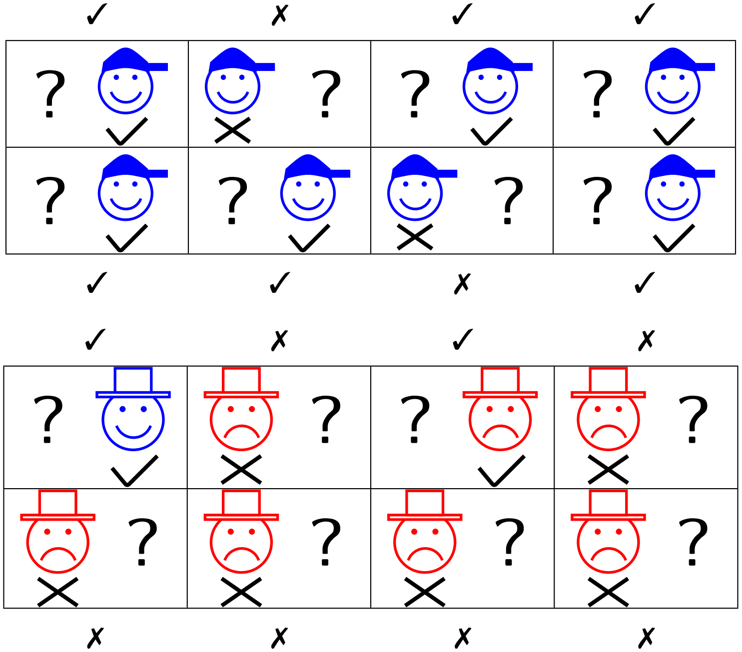








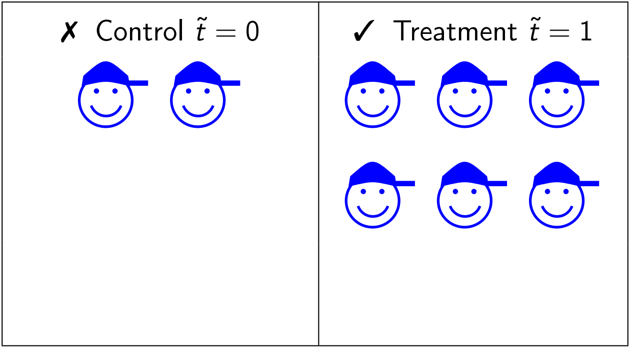




Data

✗ Control $\tilde{t} = 0$	✓ Treatment $\tilde{t} = 1$

Data



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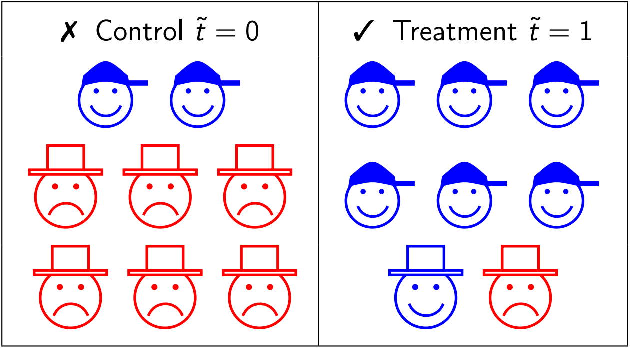

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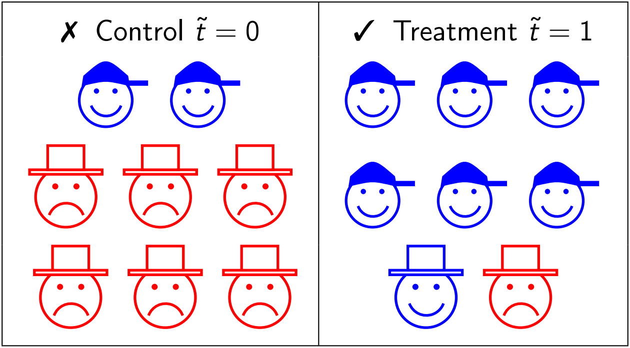

$$0.25 = \mathrm{P}\left(\tilde{y} \mid \tilde{t} = 0\right) < \mathrm{P}\left(\tilde{y} \mid \tilde{t} = 1\right) = 0.875$$

3-point shooting

Confounder?

	Stephen Curry	Courtney Lee
Total	$190/456 = 41.7\%$	$75/171 = \mathbf{43.9\%}$

3-point shooting

Confounder? Shot distance $\tilde{d}$

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Simpson's paradox

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Simpson's paradox

$$\mathrm{P}(\tilde{d} = \text{short} \mid \tilde{t} = \text{Curry}) = 0.197$$
$$\mathrm{P}(\tilde{d} = \text{short} \mid \tilde{t} = \text{Lee}) = 0.678$$

Nonbinary outcome binary treatment

Confounder $\tilde{c}$ with C values

$$\mu_{\tilde{y} \mid \tilde{t}}(1)$$

Nonbinary outcome binary treatment

Confounder $\tilde{c}$ with C values

$$\begin{aligned}
&\mu_{\tilde{y} \mid \tilde{t}}(1) \\
&= \sum_{c=1}^{C} \mathrm{P}(\tilde{c} = c \mid \tilde{t} = 1) \mu_{\widetilde{\mathbf{po}_1} \mid \tilde{c}}(c)
\end{aligned}$$

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$$
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&\neq \sum_{c=1}^{C} \mathrm{P}(\tilde{c} = c) \mu_{\widetilde{\mathbf{po}_1} \mid \tilde{c}}(c)
\end{aligned}
$$

Nonbinary outcome binary treatment

Confounder $\tilde{c}$ with C values

$$\begin{aligned}
&\mu_{\tilde{y}\,|\,\tilde{t}}(1) \\
&= \sum_{c=1}^{C} \mathrm{P}(\tilde{c} = c \,|\, \tilde{t} = 1)\, \mu_{\widetilde{\mathrm{po}}_1\,|\,\tilde{c}}(c) \\
&\neq \sum_{c=1}^{C} \mathrm{P}(\tilde{c} = c)\, \mu_{\widetilde{\mathrm{po}}_1\,|\,\tilde{c}}(c) \\
&= \mathrm{E}\left[\widetilde{\mathrm{po}}_1\right]
\end{aligned}$$

Grades in a Portuguese school

Confounder: Whether students previously failed the course ($\tilde{c} = 1$) or not ($\tilde{c} = 0$)

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Previously failed	Private classes	No private classes
Yes		
No		

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No	11.20	11.31

$$\mathrm{P}\left(\text{Previously failed} \,|\, \text{Private classes}\right) = 0.122$$

$$\mathrm{P}\left(\text{Previously failed} \,|\, \text{No private classes}\right) = 0.285$$

Nonbinary outcome nonbinary treatment

Assuming linear model given confounder $\tilde{c}$ and treatment $\tilde{t}$ (both standardized)

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Assuming linear model given confounder $\tilde{c}$ and treatment $\tilde{t}$ (both standardized)

$$\widetilde{\mathrm{po}}_t := \beta_{\mathrm{treat}} t + \beta_{\mathrm{conf}} \tilde{c} + \tilde{z}$$

$\tilde{z}$ independent from $\tilde{t}$ and $\tilde{c}$

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$$\beta_{\mathsf{MMSE}} = \mathrm{Cov}\left[\tilde{y}, \tilde{t}\,\right]$$

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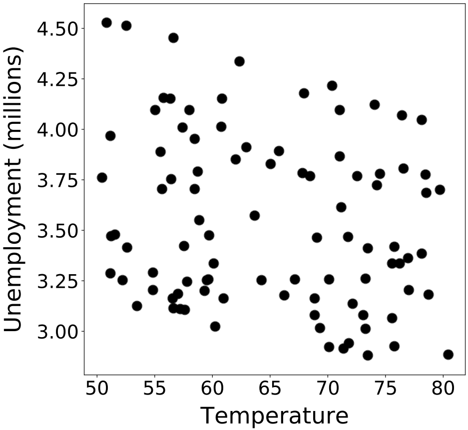
$\tilde{z}$ independent from $\tilde{t}$ and $\tilde{c}$

"Short" linear regression model of $\tilde{y}$ given $\tilde{t}$

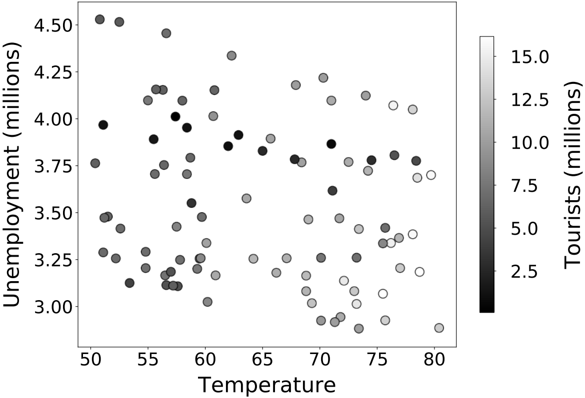
$$\begin{aligned}\beta_{\mathsf{MMSE}} &= \mathrm{Cov}\left[\tilde{y}, \tilde{t}\,\right] \\ &= \beta_{\mathsf{treat}} + \beta_{\mathsf{conf}} \sigma_{\tilde{t},\tilde{c}}\end{aligned}$$

Unemployment and temperature in Spain



Confounder?

Tourists!



4.50
4.25
4.00
3.75
3.50
3.25
3.00
Unemployment (millions)
50
55
60
65
70
75
80
Temperature
15.0
12.5
10.0
7.5
5.0
2.5
Tourists (millions)

Potential outcomes

Confounding factors

Adjusting for confounders

Motivation

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Assumption: Treatment is conditionally independent of the potential outcomes given the confounder

Adjusting for a confounder

If treatment is conditionally independent of the potential outcomes given the confounder

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(1) Conditional statistics given confounder are OK

Adjusting for a confounder

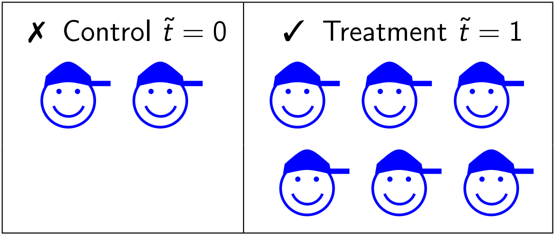
If treatment is conditionally independent of the potential outcomes given the confounder

(1) Conditional statistics given confounder are OK

(2) They can be combined to estimate true causal effect

New drug: Young subjects

Confounder: Age $\tilde{a}$



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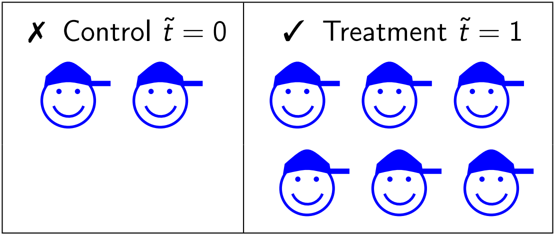
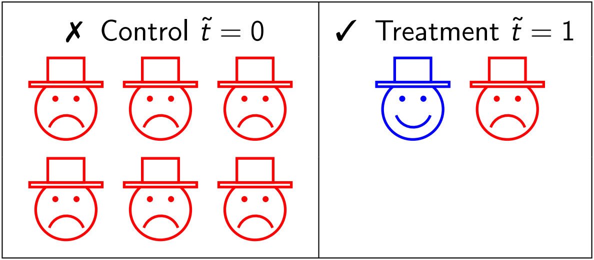


$$1 = \mathrm{P}\left(\tilde{y} \mid \tilde{t} = 0, \tilde{a} = \text{young}\right) = \mathrm{P}\left(\tilde{y} \mid \tilde{t} = 1, \tilde{a} = \text{young}\right) = 1$$

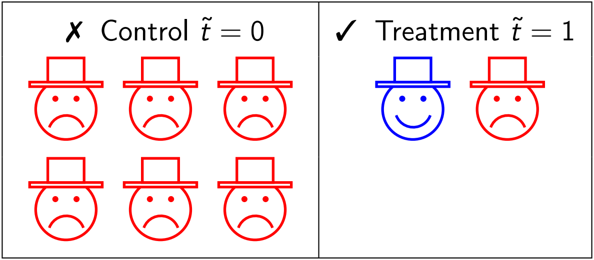
New drug: Old subjects

Confounder: Age $\tilde{a}$



New drug: Old subjects

Confounder: Age $\tilde{a}$



$$0 = \mathrm{P}\left(\tilde{y} \mid \tilde{t} = 0, \tilde{a} = \text{old}\right) < \mathrm{P}\left(\tilde{y} \mid \tilde{t} = 1, \tilde{a} = \text{old}\right) = 0.5$$

Adjusted causal effect

$$p_{\widetilde{\text{po}_0}}(1) = \sum_{a \in \{\text{young},\text{old}\}} p_{\tilde{a}}(a) \mathrm{P}\left(\tilde{y} \mid \tilde{t} = 0, \tilde{a} = a\right)$$

$$p_{\widetilde{\text{po}_1}}(1) = \sum_{a \in \{\text{young},\text{old}\}} p_{\tilde{a}}(a) \mathrm{P}\left(\tilde{y} \mid \tilde{t} = 1, \tilde{a} = a\right)$$

Adjusted causal effect

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$$= 0.5 \cdot 1 + 0.5 \cdot 0 = \mathbf{0.5}$$

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$$= 0.5 \cdot 1 + 0.5 \cdot 0.5 = \mathbf{0.75}$$

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$$= 0.5 \cdot 1 + 0.5 \cdot 0 = \mathbf{0.5}$$

Observed: 0.25

$$p_{\widetilde{\mathrm{po}}_1}(1) = \sum_{a \in \{\text{young},\text{old}\}} p_{\tilde{a}}(a) \mathrm{P}\left(\tilde{y} \mid \tilde{t} = 1, \tilde{a} = a\right)$$

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$$= 0.5 \cdot 1 + 0.5 \cdot 0.5 = \mathbf{0.75}$$

Observed: 0.875

3-point shooting

Confounder: Shot distance $\tilde{d}$

$$p_{\widetilde{\text{po}}_{\text{Curry}}}(1) = \sum_{d \in \{\text{short}, \text{long}\}} p_{\tilde{d}}(d) p_{\tilde{y} \mid \tilde{d}, \tilde{t}}(1 \mid d, \text{Curry})$$

$$p_{\widetilde{\text{po}}_{\text{Lee}}}(1) = \sum_{d \in \{\text{short}, \text{long}\}} p_{\tilde{d}}(d) p_{\tilde{y} \mid \tilde{d}, \tilde{t}}(1 \mid d, \text{Lee})$$

3-point shooting

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$$= 0.329 \cdot 0.5 + 0.671 \cdot 0.396 = \mathbf{0.430}$$

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$$= 0.329 \cdot 0.483 + 0.671 \cdot 0.345 = \mathbf{0.390}$$

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Observed: 0.417

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$$= 0.329 \cdot 0.483 + 0.671 \cdot 0.345 = \mathbf{0.390}$$

Observed: 0.439

Grades in a Portuguese school

Confounder: Whether student previously failed

$$\mathrm{E}\left[\widetilde{\mathsf{po}}_1\right] = \sum_{c \in \{0,1\}} p_{\tilde{c}}(c)\mu_{\tilde{y}\,|\,\tilde{c},\tilde{t}}(c,1)$$

$$\mathrm{E}\left[\widetilde{\mathsf{po}}_0\right] = \sum_{c \in C} p_{\tilde{c}}(c)\mu_{\tilde{y}\,|\,\tilde{c},\tilde{t}}(c,0)$$

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$$\mathrm{E}\left[\widetilde{\mathsf{po}}_1\right] = \sum_{c \in \{0,1\}} p_{\tilde{c}}(c)\mu_{\tilde{y}\,|\,\tilde{c},\tilde{t}}(c,1)$$

$$= 0.79 \cdot 11.20 + 0.21 \cdot 8.95 = 10.73$$

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$$\text{adjusted ATE} = 10.73 - 10.33 = \mathbf{0.4}$$

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adjusted ATE $= 10.73 - 10.33 = \mathbf{0.4}$ Observed: 0.93

Nonbinary outcome nonbinary treatment

Assuming linear model given confounder $\tilde{c}$ and treatment $\tilde{t}$ (both standardized)

$$\tilde{y} = \beta_{\text{treat}}\tilde{t} + \beta_{\text{conf}}\tilde{c} + \tilde{z}$$

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We adjust for the confounder including it in the model

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- Response $\tilde{y}$
- Feature vector $\tilde{x} := \begin{bmatrix} \tilde{t} \\ \tilde{c} \end{bmatrix}$

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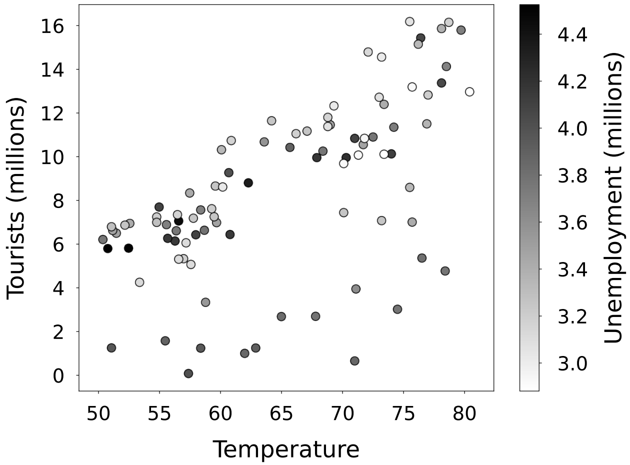
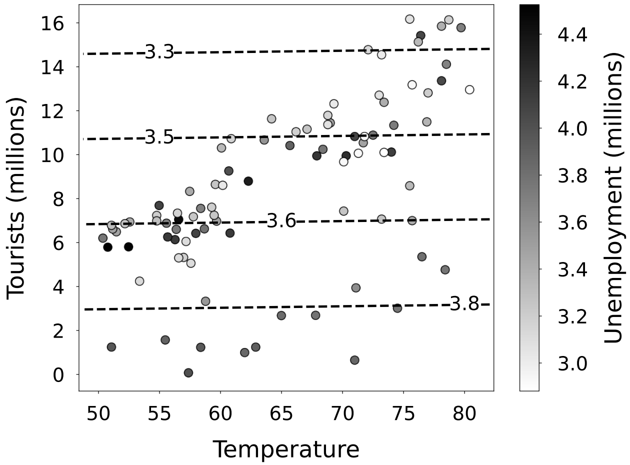
- Feature vector $\tilde{x} := \begin{bmatrix} \tilde{t} \\ \tilde{c} \end{bmatrix}$

$$\beta_{\text{MMSE}} = \begin{bmatrix} \beta_{\text{treat}} \\ \beta_{\text{conf}} \end{bmatrix}$$

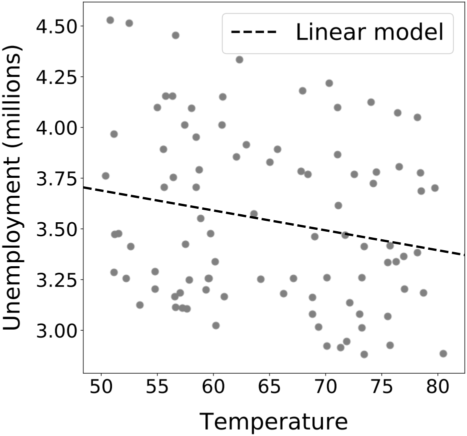
Incorporating the confounder



With confounder, temperature coefficient: 0.0003



Without confounder, temperature coefficient: -0.01



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Otherwise confounders can completely distort observed effects

Under conditional independence assumptions, we can adjust for *known* confounders