The Financial Crisis and the Central Limit Theorem

Probability and Statistics for Data Science

Carlos Fernandez-Granda

NYU COURANT INSTITUTE OF MATHEMATICAL SCIENCES

NYU DATA SCIENCE

These slides are based on the book Probability and Statistics for Data Science by Carlos Fernandez-Granda, available for purchase here. A free preprint, videos, code, slides and solutions to exercises are available at https://www.ps4ds.net

Central limit theorem

If $\tilde{x}_1$, $\tilde{x}_2$, ... are independent random variables with mean μ and variance σ^2

$$\tilde{m}_n := \frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i$$

$$\mathrm{E}\left[\tilde{m}_n\right] = \mu$$

$$\mathrm{Var}\left[\tilde{m}_n\right] = \frac{\sigma^2}{n}$$

As $n \to \infty$ $\tilde{m}_n$ converges in distribution to a Gaussian with mean μ and variance $\frac{\sigma^2}{n}$

Central limit theorem

If $\tilde{x}_1, \tilde{x}_2, \ldots$ are independent random variables with mean μ and variance σ^2

$$\tilde{m}_n := \frac{1}{n} \sum_{i=1}^{n} \tilde{x}_i$$

$$\mathrm{E}\left[\tilde{m}_n\right] = \mu$$

$$\mathrm{Var}\left[\tilde{m}_n\right] = \frac{\sigma^2}{n}$$

As $n \to \infty$ $\tilde{m}_n$ converges in distribution to a Gaussian with mean μ and variance $\frac{\sigma^2}{n}$

Mortgages

We have n subprime mortgages

Probability that each borrower defaults: 2/3

Can we build a low-risk investment out of this?

Collaterized mortgage obligation

Idea: We divide the mortgages into ten *tranches*, that suffer losses sequentially

If $< 10\%$ default: Only tranche 1 is affected

If between 10% and 20% default: Tranches 1 and 2 are affected

Tranche 10 (senior tranche) only loses money if 90% default

Goal: Estimate risk of the senior tranche

Borrowers default independently

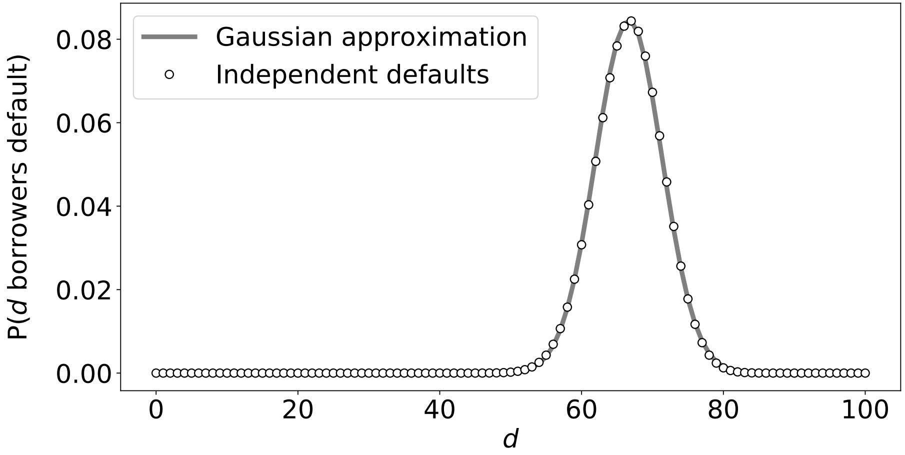
Number of defaults $\tilde{d}$? Binomial with parameters n and $\theta := 2/3$

Probability that senior tranche loses money? Essentially zero

$$\begin{aligned} \mathrm{P}\left(\tilde{d} > 0.9n\right) &= \mathrm{P}\left(\frac{\tilde{d} - \theta n}{\sqrt{\theta(1-\theta)n}} > \frac{0.9n - \theta n}{\sqrt{\theta(1-\theta)n}}\right) \\ &\approx \mathrm{P}\left(\tilde{z} > 0.49\sqrt{n}\right) \\ &< 10^{-7}! \qquad \text{for } n := 100 \end{aligned}$$

where $\tilde{z}$ is Gaussian with zero mean and unit variance

Borrowers default independently



Awesome

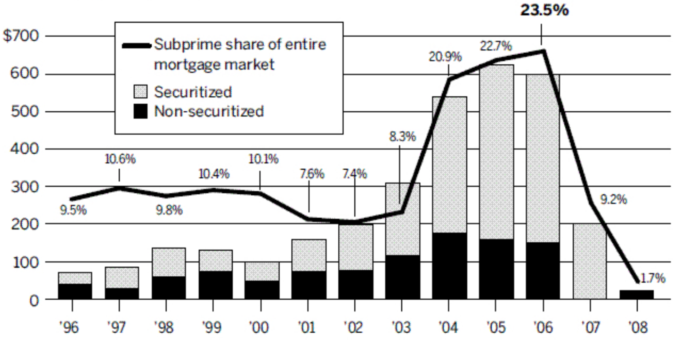

1% Low Start Rate
Stated Income
No Documentation Loans
100% Finance Available
Interest Only Loans
Debt Consolidation
SE HABLA ESPAÑOL

Awesome

Subprime Mortgage Originations

In 2006, $600 billion of subprime loans were originated, most of which were securitized. That year, subprime lending accounted for 23.5% of all mortgage originations.

IN BILLIONS OF DOLLARS



NOTE: Percent securitized is defined as subprime securities issued divided by originations in a given year. In 2007, securities issued exceeded originations.

SOURCE: Inside Mortgage Finance

What could go wrong?

THE WALL STREET JOURNAL.

DOW JONES A NEWS CORPORATION COMPANY

TUESDAY, SEPTEMBER 16, 2008 - VOL. CCLII NO. 65 ★★★★ $2.00

DJIA 10917.51 ▼504.48 -4.4% NASDAQ 2179.91 ▼3.6% NIKKEI Closed(12214.76) DJ STOXX 50 2744.81 ▼4.0% 10-YR TREAS ▲2 3/32, yield 3.482% OIL $95.71 ▼$5.47 GOLD $783.10 ▲$22.80 EURO $1.4310 YEN 104.88

AIG, Lehman Shock Hits World Markets

Focus Moves to Fate of Giant Insurer After U.S. Allows Investment Bank to Fail; Barclays in Talks to Buy Core Lehman Unit

By Susanne Craig, Jeffrey McCracken, Jon Hilsenrath and Deborah Solomon

The convulsions in the U.S. financial system sent markets across the globe tumbling, as two of Wall Street's biggest firms looked set to exit the scene and insurance titan American International Group Inc. turned to the Federal Reserve and the state of New York for assistance.

The U.S. stock market suffered its worst daily point plunge since the first day of trading after the Sept. 11, 2001, terrorist attacks. Financial markets were rattled by the rushed sale Sunday of Merrill Lynch & Co. and the bankruptcy-court filing of Lehman Brothers Holdings Inc., which scrambled Monday to sell its most-prized businesses before too many employees and customers walk out the door. (Please see related article on Page C1.)

All day Monday, top Lehman officials were huddled in Manhattan at their Seventh Avenue headquarters negotiating a sale of the U.S. investment bank—the core part of Lehman—to Barclays PLC of the U.K. People involved in the discussions were increasingly hopeful late Monday ing. For much of the day, the major U.S. market indexes were down 2%, which, while a good-sized decline, was smaller than many had thought would be the case. But in the final hour of trading, a wave of selling hit, driven by concerns about the fate of AIG. The Dow Jones Industrial Average ended down 504.48 points on Monday, off 4.4%, at its daily low of 10917.51, down 18% on the year. Of the Dow industrials' 30 components, all but one—Coca-Cola Co.—fell, led by a 60.8% plunge in AIG.

In Europe, London's FTSE 100 index dropped 3.9%. Several Asian markets, including Japan and China, were closed Monday due to holiday. By Tuesday, Tokyo shares were down 5.1% in early trading, and Hong Kong's Hang Seng index was down 6.1%.

Monday's action was the latest fallout in a widening financial crisis that began a year ago with the fall of American housing prices and is now reordering the U.S. financial system. Steps unveiled by the Federal Reserve to expand its emergency lending arsenal did little to snap the sense of gloom.

Plenty of potential land mines remain. Banks are increasingly hoarding cash, curbing lending at a time when the econ-

AIG Faces Cash Crisis As Stock Dives 61%

By Matthew Karnitschnig, Liam Pleven And Serena Ng

American International Group Inc. was facing a severe cash crunch last night as ratings agencies cut the firm's credit ratings, forcing the giant insurer to raise $14.5 billion to cover its obligations.

With AIG now tottering, a crisis that began with falling home prices and went on to engulf Wall Street has reached one of the world's largest insurance companies, threatening to intensify the financial storm and greatly complicate the government's efforts to contain it. The company, whose stock fell 61% yesterday, is such a big player in insuring risk for institutions around the world that its failure could shake the global financial system.

Alternative model

Defaults depend on the economic context

Random variable $\tilde{r}$ represents whether we are in recession

$\tilde{r} = 0$: economy is strong

$\tilde{r} = 1$: economic disaster

$$f_{\tilde{r}}(r) := 2r \qquad 0 \leq r \leq 1$$

Defaults depend on economic context

Conditioned on $\tilde{r} = r$, each borrower defaults with probability r

Bernoulli random variable $\tilde{b}$ with parameter r

$$\begin{aligned}
\mathrm{P}\left(\text{borrower defaults}\right) = p_{\tilde{b}}(1) &= \int_{r=-\infty}^{\infty} p_{\tilde{b}\,|\,\tilde{r}}(1\,|\,r) f_{\tilde{r}}(r)\,\mathrm{d}r \\
&= \int_{r=0}^{1} 2r^2\,\mathrm{d}r \\
&= \frac{2}{3}
\end{aligned}$$

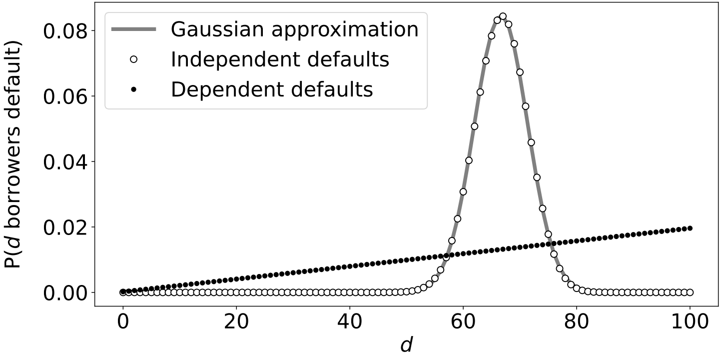
Defaults depend on economic context

Assumption: Defaults are conditionally independent given economic context

Conditioned on $\tilde{r} = r$, number of defaults is binomial with parameter n and r

$$\begin{aligned}
p_{\tilde{d}}(d) &= \int_{r=-\infty}^{\infty} f_{\tilde{r}}(r) p_{\tilde{d} \mid \tilde{r}}(d \mid r) \, \mathrm{d}r \\
&= \int_{r=0}^{1} 2r \binom{n}{d} r^d (1-r)^{n-d} \, \mathrm{d}r \\
&= 2\binom{n}{d} \int_{r=0}^{1} r^{d+1} (1-r)^{n-d} \, \mathrm{d}r \\
&= \frac{2n!}{d!(n-d)!} \frac{(d+1)!(n-d)!}{(n+2)!} \\
&= \frac{2(d+1)}{(n+1)(n+2)}
\end{aligned}$$

Defaults depend on economic context



Probability that senior tranche loses money?

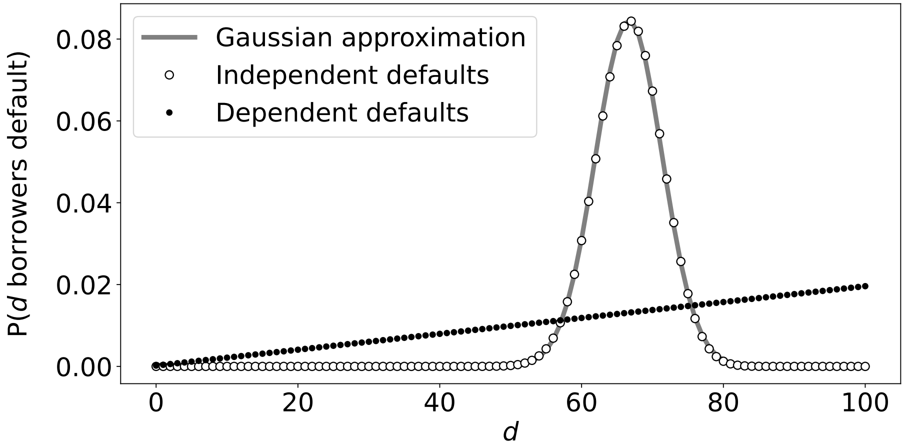
For $n := 100$

$$\mathrm{P}\left(\tilde{d} > 0.9n\right) = 0.187$$

As $n \to \infty$

$$\lim_{n\to\infty} \mathrm{P}\left(\tilde{d} > 0.9n\right) = 0.19$$

Defaults depend on economic context



So 20% of the time...

THE WALL STREET JOURNAL.

DOW JONES A NEWS CORPORATION COMPANY • • • • • TUESDAY, SEPTEMBER 16, 2008 - VOL. CCLII NO. 65 ★★★★ $2.00

DJIA 10917.51 ▼504.48 −4.4% NASDAQ 2179.91 ▼3.6% NIKKEI Closed(12214.76) DJ STOXX 50 2744.81 ▼4.0% 10-YR TREAS ▲2 3/32, yield 3.482% OIL $95.71 ▼$5.47 GOLD $783.10 ▲$22.80 EURO $1.4310 YEN 104.88

AIG, Lehman Shock Hits World Markets

Focus Moves to Fate of Giant Insurer After U.S. Allows Investment Bank to Fail; Barclays in Talks to Buy Core Lehman Unit

By Susanne Craig, Jeffrey McCracken, Jon Hilsenrath and Deborah Solomon

The convulsions in the U.S. financial system sent markets across the globe tumbling, as two of Wall Street's biggest firms looked set to exit the scene and insurance titan American International Group Inc. turned to the Federal Reserve and the state of New York for assistance.

The U.S. stock market suffered its worst daily point plunge since the first day of trading after the Sept. 11, 2001, terrorist attacks. Financial markets were rattled by the rushed sale Sunday of Merrill Lynch & Co. and the bankruptcy-court filing of Lehman Brothers Holdings Inc., which scrambled Monday to sell its most-prized businesses before too many employees and customers walk out the door. (Please see related article on Page C1.)

All day Monday, top Lehman officials were huddled in Manhattan at their Seventh Avenue headquarters negotiating a sale of the U.S. investment bank—the core part of Lehman—to Barclays PLC of the U.K. People involved in the discussions were increasingly hopeful late Monday

ing. For much of the day, the major U.S. market indexes were down 2%, which, while a good-sized decline, was smaller than many had thought would be the case. But in the final hour of trading, a wave of selling hit, driven by concerns about the fate of AIG. The Dow Jones Industrial Average ended down 504.48 points on Monday, off 4.4%, at its daily low of 10917.51, down 18% on the year. Of the Dow industrials' 30 components, all but one—Coca-Cola Co.—fell, led by a 60.8% plunge in AIG.

In Europe, London's FTSE 100 index dropped 3.9%. Several Asian markets, including Japan and China, were closed Monday due to holiday. By Tuesday, Tokyo shares were down 5.1% in early trading, and Hong Kong's Hang Seng index was down 6.1%.

Monday's action was the latest fallout in a widening financial crisis that began a year ago with the fall of American housing prices and is now reordering the U.S. financial system. Steps unveiled by the Federal Reserve to expand its emergency lending arsenal did little to snap the sense of gloom.

Plenty of potential land mines remain. Banks are increasingly hoarding cash, curbing lending at a time when the econ-

DJIA ▼504.48
One-minute intervals
11500 11400 11300 11200 11100 11000 10900
10 11 noon 1 2 3 4
Source: WSJ Market Data Group

AIG Faces Cash Crisis As Stock Dives 61%

BY MATTHEW KARNITSCHNIG, LIAM PLEVEN AND SERENA NG

American International Group Inc. was facing a severe cash crunch last night as ratings agencies cut the firm's credit ratings, forcing the giant insurer to raise $14.5 billion to cover its obligations.

With AIG now tottering, a crisis that began with falling home prices and went on to engulf Wall Street has reached one of the world's largest insurance companies, threatening to intensify the financial storm and greatly complicate the government's efforts to contain it. The company, whose stock fell 61% yesterday, is such a big player in insuring risk for institutions around the world that its failure could shake the global financial system.

What have we learned

Independence assumptions can have enormous influence on risk estimates