Gaussian Random Vectors:

Marginal and Conditional Distributions

Probability and Statistics for Data Science

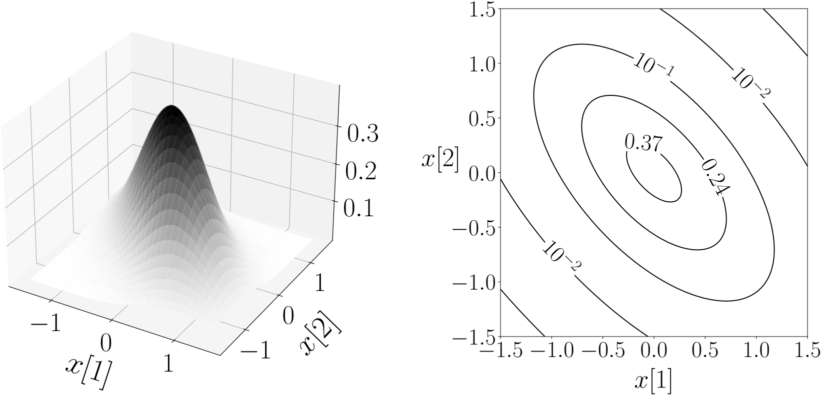
Carlos Fernandez-Granda

These slides are based on the book Probability and Statistics for Data Science by Carlos Fernandez-Granda, available for purchase here. A free preprint, videos, code, slides and solutions to exercises are available at https://www.ps4ds.net

Goal

Study marginal and conditional distributions of Gaussian random vectors

Gaussian random vector



Gaussian random vector

A Gaussian random vector $\tilde{x}$ is a random vector with joint pdf

$$f_{\tilde{x}}(x) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

where $\mu \in \mathbb{R}^d$ is the mean and $\Sigma \in \mathbb{R}^{d \times d}$ the covariance matrix

$\Sigma \in \mathbb{R}^{d \times d}$ is symmetric and positive definite (positive eigenvalues)

2D Gaussian

Gaussian random vector $(\tilde{a}, \tilde{b})$ with zero mean and covariance matrix

$$\Sigma := \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad\qquad -1 < \rho < 1$$

Marginal distribution of $\tilde{a}$?

Conditional distribution of $\tilde{b}$ given $\tilde{a} = a$?

Correlation coefficient

$$\Sigma := \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad \Sigma^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

$$\begin{aligned}
f_{\tilde{a},\tilde{b}}(a,b) &:= \frac{1}{\sqrt{(2\pi)^2 |\Sigma|}} \exp\left(-\frac{1}{2} \begin{bmatrix} a \\ b \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} a \\ b \end{bmatrix} \right) \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{a^2 - 2\rho ab + b^2}{2(1-\rho^2)} \right) \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{(1-\rho^2)a^2 + (b-\rho a)^2}{2(1-\rho^2)} \right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{a^2}{2} \right) \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(b-\rho a)^2}{2(1-\rho^2)} \right) \\
&= f_{\tilde{a}}(a)\, f_{\tilde{b}\,|\,\tilde{a}}(b\,|\,a)
\end{aligned}$$

2D Gaussian

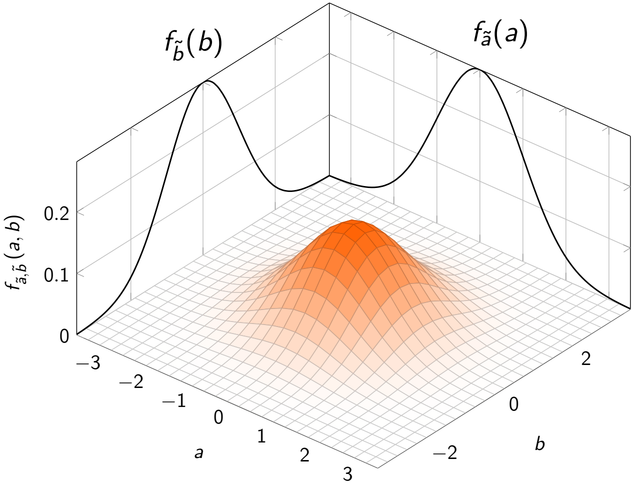
Gaussian random vector $(\tilde{a}, \tilde{b})$ with zero mean and covariance matrix

$$\Sigma := \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad\qquad -1 < \rho < 1$$

Marginal distribution of $\tilde{a}$?

Gaussian with standard deviation 1

Marginals are Gaussian



2D Gaussian

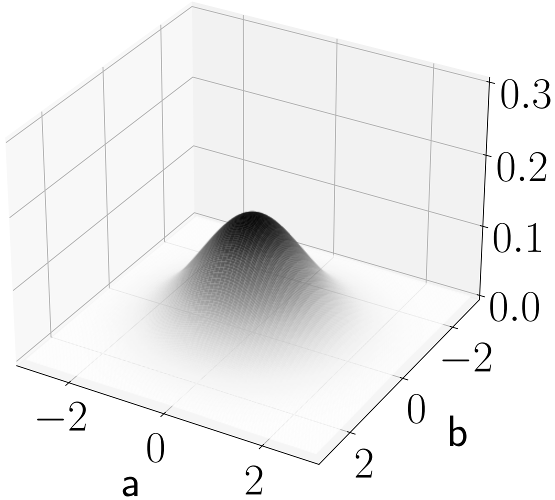
Gaussian random vector $(\tilde{a}, \tilde{b})$ with zero mean and covariance matrix

$$\Sigma := \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad\qquad -1 < \rho < 1$$

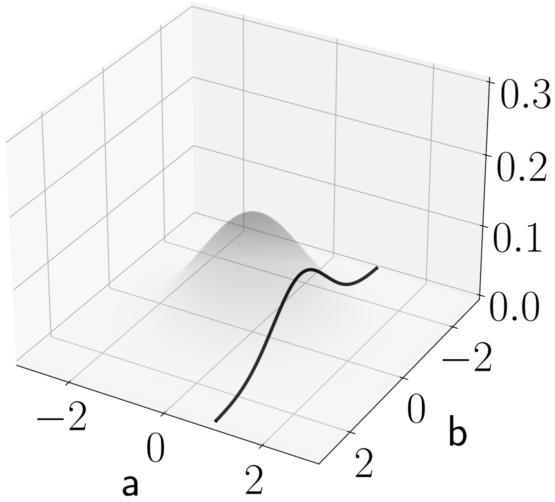
Conditional distribution of $\tilde{b}$ given $\tilde{a} = a$?

Gaussian with mean ρa and standard deviation $\sqrt{1 - \rho^2}$

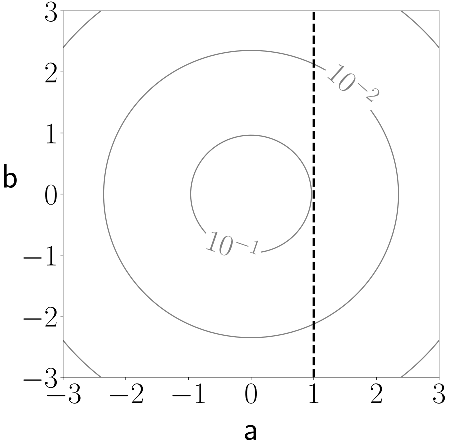
$\rho = 0$



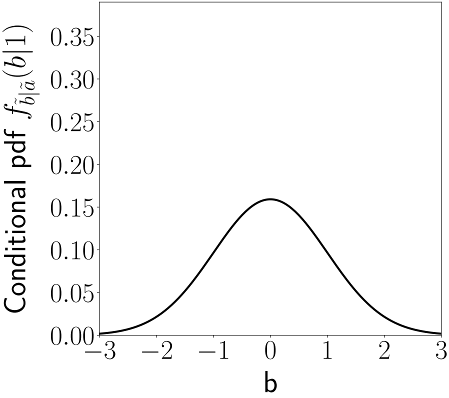
$\mu = \rho a = 0,\ \sigma = \sqrt{1-\rho^2} = 1$

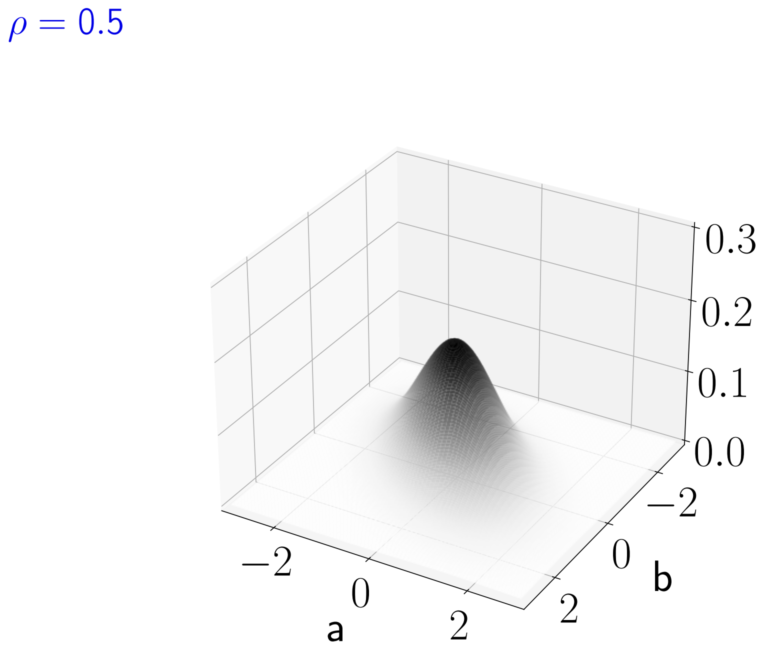


$\mu = \rho a = 0, \; \sigma = \sqrt{1 - \rho^2} = 1$



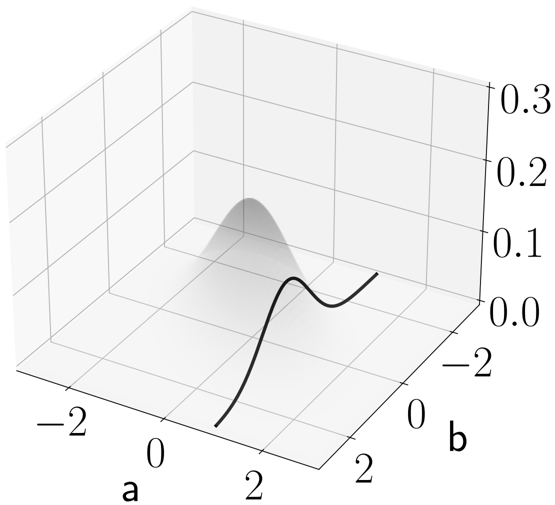
$$\mu = \rho a = 0,\ \sigma = \sqrt{1-\rho^2} = 1$$



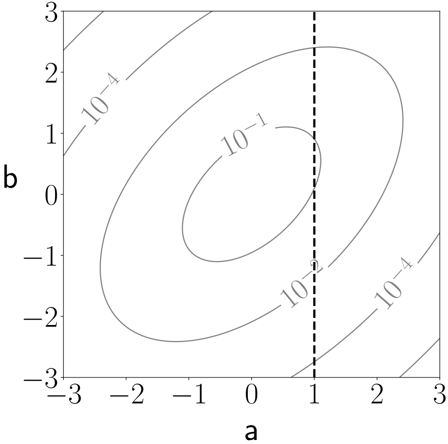


$\rho = 0.5$
0.3
0.2
0.1
0.0
−2
0
2
b
−2
0
2
a

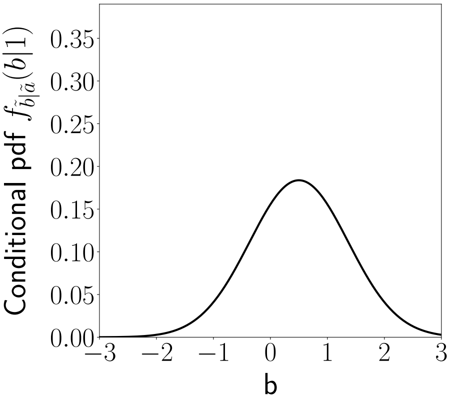
$\mu = 0.5a,\ \sigma = \sqrt{1-\rho^2} = 0.87$



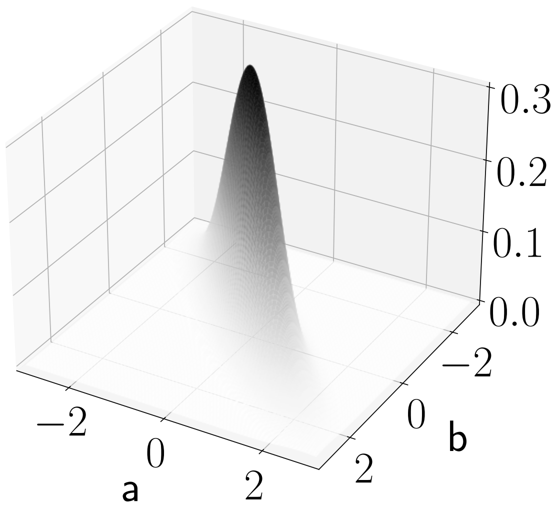
$\mu = 0.5a$, $\sigma = \sqrt{1 - \rho^2} = 0.87$



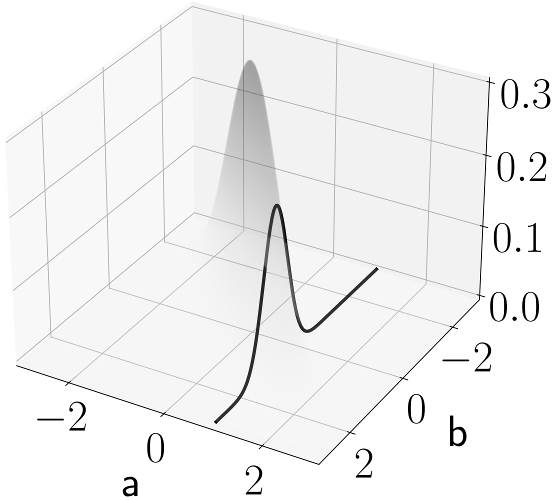
$\mu = 0.5a$, $\sigma = \sqrt{1-\rho^2} = 0.87$



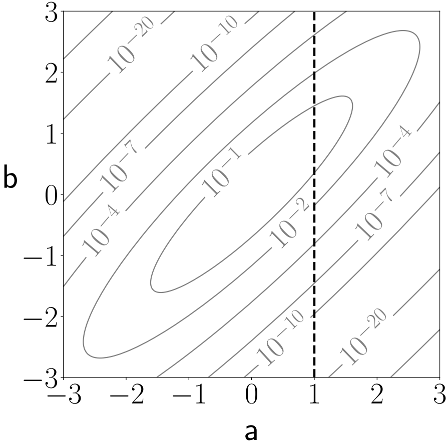
$\rho = 0.9$



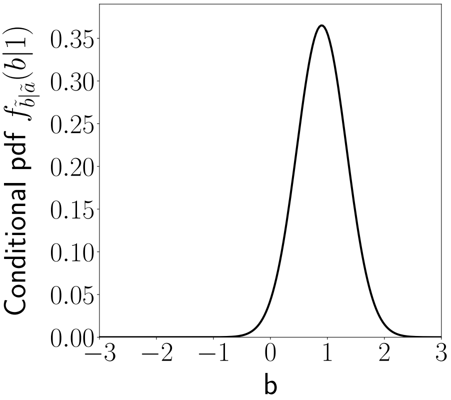
$\mu = 0.9a$, $\sigma = \sqrt{1-\rho^2} = 0.44$



$\mu = 0.9a$, $\sigma = \sqrt{1-\rho^2} = 0.44$



$\mu = 0.9a$, $\sigma = \sqrt{1-\rho^2} = 0.44$



2D Gaussian

Gaussian random vector $(\tilde{a}, \tilde{b})$ with

$$\mu := \begin{bmatrix} \mu_{\tilde{a}} \\ \mu_{\tilde{b}} \end{bmatrix} \qquad \Sigma := \begin{bmatrix} \sigma_{\tilde{a}}^2 & \rho\sigma_{\tilde{a}}\sigma_{\tilde{b}} \\ \rho\sigma_{\tilde{a}}\sigma_{\tilde{b}} & \sigma_{\tilde{b}}^2 \end{bmatrix}$$

Marginal distribution of $\tilde{a}$?

Gaussian with mean $\mu_{\tilde{a}}$ and standard deviation $\sigma_{\tilde{a}}$

2D Gaussian

Gaussian random vector $(\tilde{a}, \tilde{b})$ with

$$\mu := \begin{bmatrix} \mu_{\tilde{a}} \\ \mu_{\tilde{b}} \end{bmatrix} \qquad \Sigma := \begin{bmatrix} \sigma_{\tilde{a}}^2 & \rho\sigma_{\tilde{a}}\sigma_{\tilde{b}} \\ \rho\sigma_{\tilde{a}}\sigma_{\tilde{b}} & \sigma_{\tilde{b}}^2 \end{bmatrix}$$

Conditional distribution of $\tilde{b}$ given $\tilde{a} = a$?

Gaussian with

$$\mu_{\text{cond}} = \mu_{\tilde{b}} + \frac{\rho\,\sigma_{\tilde{b}}(a - \mu_{\tilde{a}})}{\sigma_{\tilde{a}}}$$

$$\sigma_{\text{cond}} = \sigma_{\tilde{b}}\sqrt{1 - \rho^2}$$

d-dimensional Gaussian

Gaussian random vector

$$\tilde{z} := \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \qquad \mu := \begin{bmatrix} \mu_{\tilde{x}} \\ \mu_{\tilde{y}} \end{bmatrix} \qquad \Sigma_{\tilde{z}} = \begin{bmatrix} \Sigma_{\tilde{x}} & \Sigma_{\tilde{x},\tilde{y}} \\ \Sigma_{\tilde{x},\tilde{y}}^T & \Sigma_{\tilde{y}} \end{bmatrix}$$

Marginal distribution of $\tilde{x}$?

Gaussian with mean $\mu_{\tilde{x}}$ and covariance matrix $\Sigma_{\tilde{x}}$

d-dimensional Gaussian

Gaussian random vector

$$\tilde{z} := \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \qquad \mu := \begin{bmatrix} \mu_{\tilde{x}} \\ \mu_{\tilde{y}} \end{bmatrix} \qquad \Sigma_{\tilde{z}} = \begin{bmatrix} \Sigma_{\tilde{x}} & \Sigma_{\tilde{x},\tilde{y}} \\ \Sigma_{\tilde{x},\tilde{y}}^T & \Sigma_{\tilde{y}} \end{bmatrix}$$

Conditional distribution of $\tilde{y}$ given $\tilde{x} = x$?

Gaussian with

$$\mu_{\text{cond}} = \mu_{\tilde{y}} + \Sigma_{\tilde{x},\tilde{y}} \Sigma_{\tilde{x}}^{-1} (x - \mu_{\tilde{x}})$$

$$\Sigma_{\text{cond}} = \Sigma_{\tilde{y}} - \Sigma_{\tilde{x},\tilde{y}} \Sigma_{\tilde{x}}^{-1} \Sigma_{\tilde{x},\tilde{y}}$$

What have we learned?

Marginal and conditional distributions are all Gaussian

For 2D Gaussian dependence is governed by correlation coefficient