Conditional Independence

Probability and Statistics for Data Science

Carlos Fernandez-Granda

These slides are based on the book Probability and Statistics for Data Science by Carlos Fernandez-Granda, available for purchase here. A free preprint, videos, code, slides and solutions to exercises are available at https://www.ps4ds.net

Plan

Define conditional independence

Show that conditioning can completely change dependence between events

Independence

Two events A, B are independent if

$$\mathrm{P}(A \mid B) = \mathrm{P}(A)$$

or equivalently

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$$

Conditional independence

A, B are conditionally independent given C if

$$\mathrm{P}(A \mid B, C) = \mathrm{P}(A \mid C)$$

or equivalently

$$\mathrm{P}(A \cap B \mid C) = \mathrm{P}(A \mid C)\,\mathrm{P}(B \mid C)$$

Conditional independence

$A_1, A_2, \ldots, A_n \in \mathcal{F}$ are mutually conditionally independent given C if and only if for any $\{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$,

$$\mathrm{P}\left(\cap_{j=1}^{m} A_{i_j} \mid C\right) = \prod_{j=1}^{m} \mathrm{P}\left(A_{i_j} \mid C\right)$$

Important questions

Does conditional independence imply independence?

Does independence imply conditional independence?

Flight delay, rain and taxis

Probabilistic model for flight delay (L), rain (R) and taxi availability (T)

$$
\begin{aligned}
&\mathrm{P}(R) = 0.2 \\
&\mathrm{P}(L \mid R) = 0.75 \quad \mathrm{P}(L \mid R^c) = 0.125 \\
&\mathrm{P}(T \mid R) = 0.1 \quad \mathrm{P}(T \mid R^c) = 0.6
\end{aligned}
$$

Assumption: L and T are conditionally independent given R and also given R^c

Are they also independent? $\mathrm{P}(T) = \mathrm{P}(T \mid L)$?

Flight delay, rain and taxis

$$
\begin{aligned}
\mathrm{P}(T) &= \mathrm{P}(T, R) + \mathrm{P}(T, R^c) \\
&= \mathrm{P}(T \mid R)\,\mathrm{P}(R) + \mathrm{P}(T \mid R^c)\,\mathrm{P}(R^c) \\
&= 0.1 \cdot 0.2 + 0.6 \cdot 0.8 = 0.5
\end{aligned}
$$

$\mathrm{P}(R) = 0.2 \quad \mathrm{P}(L \mid R) = 0.75 \quad \mathrm{P}(L \mid R^c) = 0.125$

$\mathrm{P}(T \mid R) = 0.1 \quad \mathrm{P}(T \mid R^c) = 0.6$

Flight delay, rain and taxis

$$
\begin{aligned}
&\mathrm{P}(T \mid L) \\
&= \frac{\mathrm{P}(T, L)}{\mathrm{P}(L)} \\
&= \frac{\mathrm{P}(T, L, R) + \mathrm{P}(T, L, R^c)}{\mathrm{P}(L)} \\
&= \frac{\mathrm{P}(T \mid L, R)\,\mathrm{P}(L \mid R)\,\mathrm{P}(R) + \mathrm{P}(T \mid L, R^c)\,\mathrm{P}(L \mid R^c)\,\mathrm{P}(R^c)}{\mathrm{P}(L)} \\
&= \frac{\mathrm{P}(T \mid R)\,\mathrm{P}(L \mid R)\,\mathrm{P}(R) + \mathrm{P}(T \mid R^c)\,\mathrm{P}(L \mid R^c)\,\mathrm{P}(R^c)}{\mathrm{P}(L)} \\
&= \frac{0.1 \cdot 0.75 \cdot 0.2 + 0.6 \cdot 0.125 \cdot 0.8}{0.25} = 0.3 \quad \neq \quad \mathrm{P}(T) = 0.5
\end{aligned}
$$

$$\mathrm{P}(R) = 0.2 \quad \mathrm{P}(L \mid R) = 0.75 \quad \mathrm{P}(L \mid R^c) = 0.125$$

$$\mathrm{P}(T \mid R) = 0.1 \quad \mathrm{P}(T \mid R^c) = 0.6$$

Important questions

Does conditional independence imply independence? No!

Does independence imply conditional independence?

Flight delay, rain and mechanical problem

Probabilistic model for flight delay (L), rain (R) and mechanical problem (M)

$$
\begin{aligned}
&\mathrm{P}(R) = 0.2 \\
&\mathrm{P}(L \mid R) = 0.75 \quad \mathrm{P}(L \mid R^c) = 0.125 \\
&\mathrm{P}(M) = 0.1 \\
&\mathrm{P}(L \mid M) = 0.7 \quad \mathrm{P}(L \mid M^c) = 0.2 \\
&\mathrm{P}(L \mid R^c, M) = 0.5
\end{aligned}
$$

Assumption: M and R^c are independent

Are they conditionally independent given L?

$\mathrm{P}(M \mid L) = \mathrm{P}(M \mid L, R^c)$?

Flight delay, rain and mechanical problem

$$\begin{aligned}
\mathrm{P}(M \mid L) &= \frac{\mathrm{P}(L, M)}{\mathrm{P}(L)} \\
&= \frac{\mathrm{P}(L \mid M)\,\mathrm{P}(M)}{\mathrm{P}(L \mid M)\,\mathrm{P}(M) + \mathrm{P}(L \mid M^c)\,\mathrm{P}(M^c)} \\
&= \frac{0.7 \cdot 0.1}{0.7 \cdot 0.1 + 0.2 \cdot 0.9} = 0.28
\end{aligned}$$

$\mathrm{P}(R) = 0.2 \quad \mathrm{P}(L \mid R) = 0.75 \quad \mathrm{P}(L \mid R^c) = 0.125 \quad \mathrm{P}(M) = 0.1$

$\mathrm{P}(L \mid M) = 0.7 \quad \mathrm{P}(L \mid M^c) = 0.2 \quad \mathrm{P}(L \mid R^c, M) = 0.5$

Flight delay, rain and mechanical problem

$$\begin{aligned}
\mathrm{P}(M \mid L, R^c) &= \frac{\mathrm{P}(L, R^c, M)}{\mathrm{P}(L, R^c)} \\
&= \frac{\mathrm{P}(L \mid R^c, M)\,\mathrm{P}(R^c \mid M)\,\mathrm{P}(M)}{\mathrm{P}(L \mid R^c)\,\mathrm{P}(R^c)} \\
&= \frac{\mathrm{P}(L \mid R^c, M)\,\mathrm{P}(R^c)\,\mathrm{P}(M)}{\mathrm{P}(L \mid R^c)\,\mathrm{P}(R^c)} \\
&= \frac{0.5 \cdot 0.1}{0.125} = 0.4 \quad \neq \quad \mathrm{P}(M \mid L) = 0.28
\end{aligned}$$

$\mathrm{P}(R) = 0.2 \quad \mathrm{P}(L \mid R) = 0.75 \quad \mathrm{P}(L \mid R^c) = 0.125 \quad \mathrm{P}(M) = 0.1$

$\mathrm{P}(L \mid M) = 0.7 \quad \mathrm{P}(L \mid M^c) = 0.2 \quad \mathrm{P}(L \mid R^c, M) = 0.5$

Important questions

Does conditional independence imply independence? No!

Does independence imply conditional independence? No!

House of Representatives 1984

		Duty-free exports	
		Yes	No
Budget	Yes	151	88
	No	21	140

$$\mathrm{P}(D) = \frac{172}{400} = 0.43$$

$$\mathrm{P}(D \mid B) = \frac{151}{239} = 0.632$$

Is dependence due to political affiliation?

House of Representatives 1984

<table>
<tr><td colspan="2" rowspan="2">Republicans</td><td colspan="2">Duty-free exports</td></tr>
<tr><td>Yes</td><td>No</td></tr>
<tr><td rowspan="2">Budget</td><td>Yes</td><td>7</td><td>15</td></tr>
<tr><td>No</td><td>7</td><td>126</td></tr>
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$$\mathrm{P}(B, D \mid R) = \frac{7}{155} = 0.045 \quad \neq \quad \mathrm{P}(B \mid R)\mathrm{P}(D \mid R) = 0.013$$

$$\mathrm{P}(B \mid R) = \frac{22}{155} = 0.142 \qquad \mathrm{P}(D \mid R) = \frac{14}{155} = 0.090$$

$$\mathrm{P}(B \mid R, D) = \frac{7}{14} = 0.5$$

House of Representatives 1984

Democrats		Duty-free exports	
		Yes	No
Budget	Yes	144	73
	No	14	14

$$\mathrm{P}(B, D \mid R^c) = \frac{144}{245} = 0.588 \quad \approx \quad \mathrm{P}(B \mid R^c)\mathrm{P}(D \mid R^c) = 0.571$$

$$\mathrm{P}(B \mid R^c) = \frac{217}{245} = 0.886 \qquad \mathrm{P}(D \mid R^c) = \frac{158}{245} = 0.645$$

$$\mathrm{P}(B \mid R^c, D) = \frac{144}{158} = 0.911$$

What have we learned?

Definition of conditional independence

Conditioning can completely change dependence between events