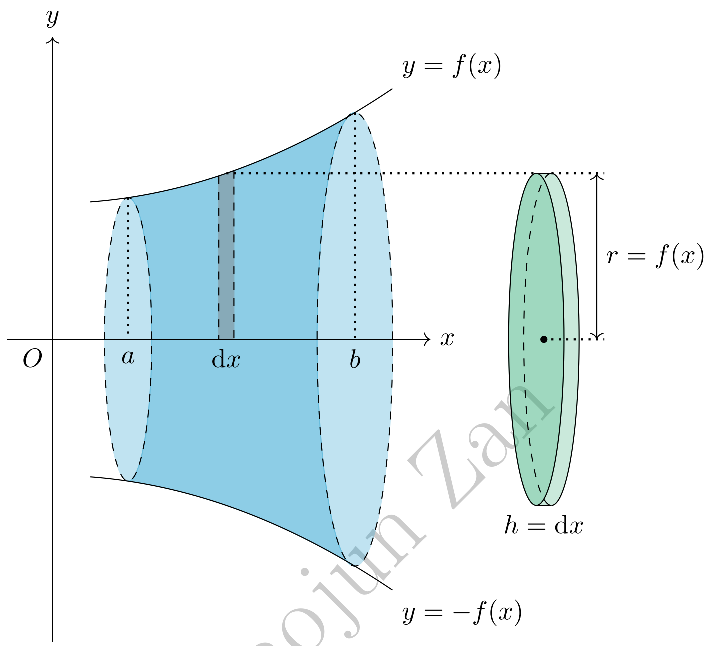
1 Volume of Revolution

1.1 Disks

Take function $y = f(x)$ in the 2-dimensional plane and rotate it about the x axis. what we get is a 3-dimensional shape.



Consider a small strip with x coordinate x and strip width $\mathrm{d}x$. The width is so small that we can assume the height $f(x)$ is unchanged with this change of $\mathrm{d}x$. When rotated about the x axis, this strip generates a thin disk volume of $\mathrm{d}V$:

$$\begin{aligned} \mathrm{d}V &= (\text{area of disk}) \cdot (\text{height of disk}) \\ &= \pi y^2 \mathrm{d}x. \end{aligned}$$

Therefore,

$$V = \int \mathrm{d}V = \int_a^b \pi y^2 \mathrm{d}x \tag{1}$$

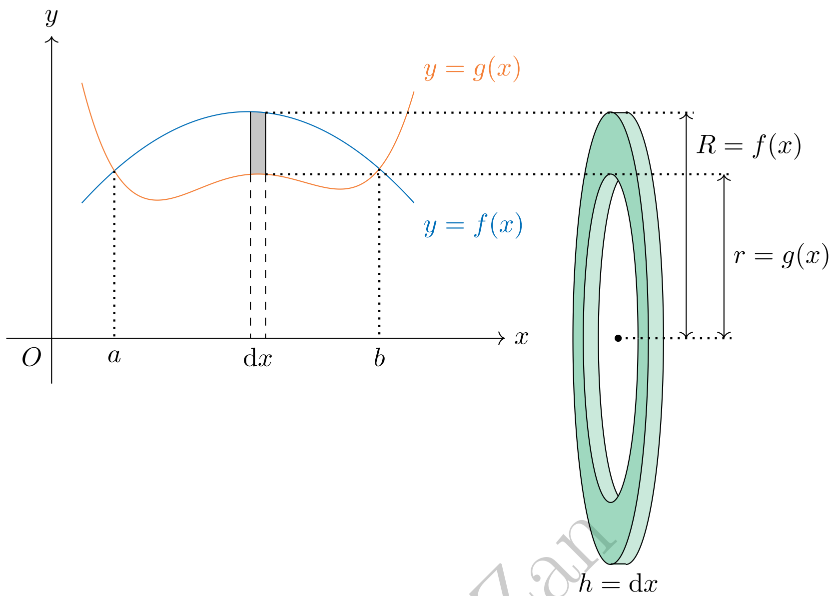
is the volume of revolution when a curve of $y = f(x)$ is rotated about the x axis.

When a curve is rotated about the y axis, we write:

$$V = \int \mathrm{d}V = \int_a^b \pi x^2 \mathrm{d}y. \tag{2}$$

1.2 Washers

The method of washers is used to deal with volume of revolution between two curves.



Consider a small rectangle bounded between $y = f(x)$ and $y = g(x)$ with width $\mathrm{d}x$. When rotated about the x axis, the corresponding volume $\mathrm{d}V$ is:

$$\begin{aligned}\mathrm{d}V &= (\text{area of washer}) \cdot (\text{height of washer}) \\ &= \pi\left(R^2 - r^2\right)\mathrm{d}x \\ &= \pi\left([f(x)]^2 - [g(x)]^2\right)\mathrm{d}x.\end{aligned}$$

Therefore,

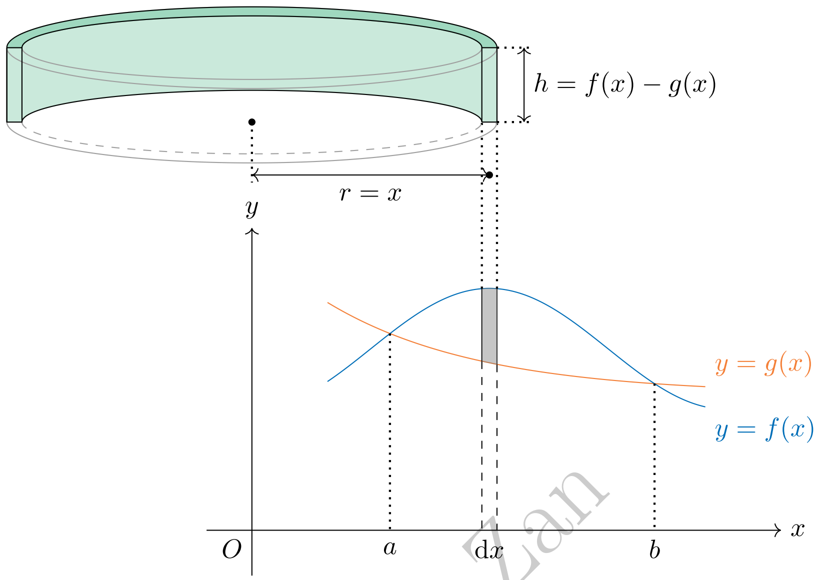
$$V = \int \mathrm{d}V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right)\mathrm{d}x. \tag{3}$$

When rotated about the y axis, we write:

$$V = \int \mathrm{d}V = \int_a^b \pi\left([f(y)]^2 - [g(y)]^2\right)\mathrm{d}y. \tag{4}$$

1.3 Shell

One should pay attention to the difference between the shell method and the washer method. The washer has its height $\mathrm{d}x$, while the shell has its thickness $\mathrm{d}x$.



Consider again a small rectangle bounded between $y = f(x)$ and $y = g(x)$ with width $\mathrm{d}x$. However, this time, this rectangle is rotated about the y axis. What we get this time is a very thin cylindrical shell. Its height is $h = f(x) - g(x)$, radius is $r = x$, and the thickness is $\mathrm{d}x$. As this shell is very thin, we can approximate its volume by thinking it as a long cuboid with length $2\pi r$, height h, and width $\mathrm{d}x$.

$$\begin{aligned}\mathrm{d}V &= (\text{length of cuboid}) \cdot (\text{height of cuboid}) \cdot (\text{width of cuboid}) \\ &= 2\pi x[f(x) - g(x)]\mathrm{d}x.\end{aligned}$$

Therefore,

$$V = \int \mathrm{d}V = \int_a^b 2\pi x[f(x) - g(x)]\mathrm{d}x. \tag{5}$$

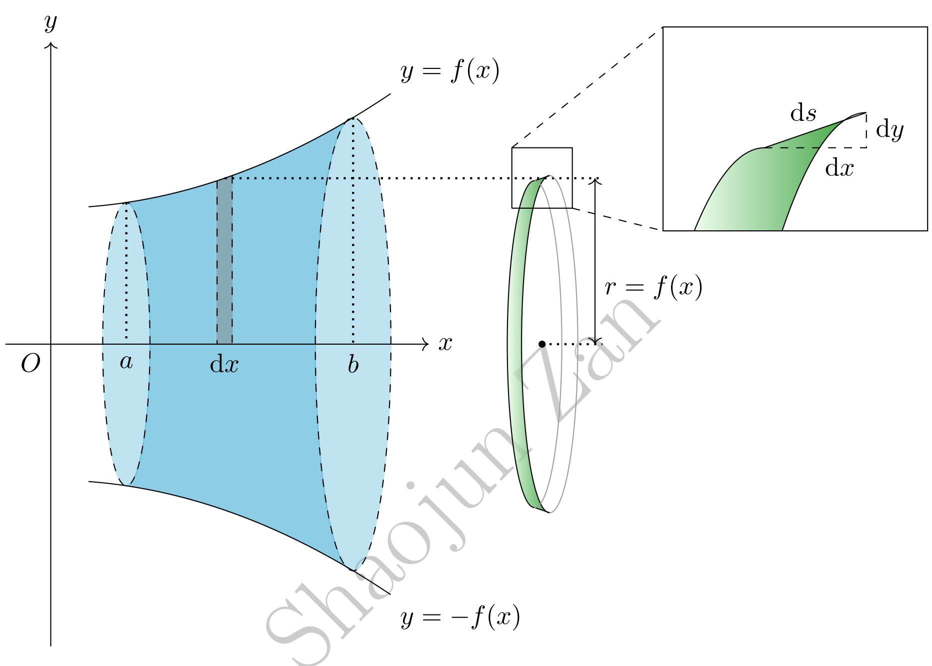
When rotated about the x axis, we write:

$$V = \int \mathrm{d}V = \int_a^b 2\pi y[f(y) - g(y)]\mathrm{d}y. \tag{6}$$

To sum up,

$$
\begin{cases}
\text{disk: } \mathrm{d}V = \pi r^2 \mathrm{d}x & V = \pi \displaystyle\int_a^b r^2 \mathrm{d}x \\
\text{washer: } \mathrm{d}V = \pi (R^2 - r^2) \mathrm{d}x & V = \pi \displaystyle\int_a^b (R^2 - r^2) \mathrm{d}x \\
\text{shell: } \mathrm{d}V = 2\pi r h \mathrm{d}x & V = 2\pi \displaystyle\int_a^b r h \mathrm{d}x.
\end{cases}
\tag{7}
$$

2 Surface Area of Revolution



Consider $\mathrm{d}x$. The curve length associated with this $\mathrm{d}x$ is $\mathrm{d}s$. As $\mathrm{d}x$ is so small that we can neglect the change in $f(x)$. Therefore, we can approximate the surface area of this surface of revolution by a rectangle of length $2\pi r$ and width $\mathrm{d}s$.

$$
\begin{aligned}
\mathrm{d}S &= (\text{length of rectangle}) \cdot (\text{width of rectangle}) \\
&= 2\pi r \mathrm{d}s \\
&= 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x.
\end{aligned}
$$

Therefore, the surface area of revolution is given by the formula:

$$
S = \int \mathrm{d}S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x. \tag{8}
$$