Electromagnetism

Shaojun Zan

10 April 2024

Contents

Part I

Electromagnetism 1: EM in Vacuum

1 Fields & Forces

The electric field at a point $\mathbf{r}$ due to a charge that **has always been** at r_1 is

$$\mathbf{E}(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0 \|\mathbf{r}-\mathbf{r}_1\|^3} \left(\mathbf{r}-\mathbf{r}_1\right). \tag{1}$$

Charges do not act on themselves - one computes $\mathbf{E}$ from all charges except q.

When q_1 moves, the equation

$$\mathbf{E}(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0 \|\mathbf{r}-\mathbf{r}_1(t)\|^3} \left[\mathbf{r}-\mathbf{r}_1(t)\right].$$

is not right. According to relativity, nothing can propagate faster than c, and therefore it takes a time at least $\left[\mathbf{r}-\mathbf{r}_1(t)\right]/c$ for $\mathbf{E}$ to respond to changes in the position of q_1. Changes in $\mathbf{E}$ are delayed.

1.1 Maxwell's Equations

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0}\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\
\boldsymbol{\nabla}\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\
\boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}
\end{aligned} \tag{2}$$

Ampere's law was corrected by Maxwell by adding the second term on the RHS. Let's consider how that was derived.

Firstly, the continuity equation relates ρ and $\mathbf{J}$. The rate of charges leaving a volume equals the current on the enclosing surface:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(\mathbf{r})\,\mathrm{d}V = -\oint_S \mathbf{J}\cdot\mathrm{d}\mathbf{S} \quad \Rightarrow \quad \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0. \tag{3}$$

The original Ampere's law,

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}, \tag{4}$$

which is only true in the static case, suggests that charges will never move around:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = 0.$$

This is obviously not true, so Maxwell spared efforts to correct it. Consider adding a vector field $\mathbf{N}$ on the RHS, so

$$\mu_0 \boldsymbol{\nabla} \cdot \mathbf{J} + \boldsymbol{\nabla} \cdot \mathbf{N} = 0 \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \mathbf{N} = \mu_0 \frac{\partial \rho}{\partial t}.$$

By Gauss's law,

$$\boldsymbol{\nabla} \cdot \mathbf{N} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{E}.$$

This suggests

$$\mathbf{N} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{G}, \tag{5}$$

where $\mathbf{G}$ is any vector field. In usual cases, we acquiesce that $\mathbf{G} = 0$.

1.2 Lorentz's Law

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{6}$$

1.3 Divergence Theorem

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V = \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} \tag{7}$$

More generally, by eliminating the vector field $\mathbf{F}$,

$$\int_V \mathrm{d}V \, \boldsymbol{\nabla} = \oint_S \mathrm{d}\mathbf{S} \, . \tag{8}$$

1.4 Stokes' Theorem

$$\int_S \left(\boldsymbol{\nabla} \times \mathbf{F}\right) \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{l} \tag{9}$$

Similarly,

$$\int_S \left(\mathrm{d}\mathbf{S} \times \boldsymbol{\nabla}\right) = \int_C \mathrm{d}\mathbf{l} \, . \tag{10}$$

2 Charge and Current Sheets

2.1 Charge Sheets

2.1.1 Parallel Component

- Symmetry:

$$\mathbf{E} = E(z)\hat{\mathbf{k}}$$

- Pillbox:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{\mathrm{encl}}}{\epsilon_0}$$

$$E(z)A - E(-z)A = \frac{\sigma A}{\epsilon_0}$$

$$\Rightarrow E_{\parallel}(z) - E_{\parallel}(-z) = \frac{\sigma}{\epsilon_0} \qquad \text{discontinuous}$$

We need the symmetry $\mathbf{E}(z) = -\mathbf{E}(-z)$ for further simplification.

2.1.2 Perpendicular Component

- Stokes' Theorem:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_S (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S}$$

- Let $z \to 0$,

(1) The area shrinks to 0, so

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0.$$

(2) The contribution from parallel legs is 0, so

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = E_{\perp}(z)L - E_{\perp}(-z)L = 0$$

This gives

$$E_{\perp}(z) - E_{\perp}(-z) = 0, \;\; z \to 0 \qquad \text{continuous}. \tag{11}$$

2.1.3 Summary

$$\mathbf{E}_+ - \mathbf{E}_- = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \tag{12}$$

2.2 Current Sheets

2.2.1 Parallel Component

- Divergence Theorem:

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{B} \, \mathrm{d}V = \oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$

- Let $z \to 0$, so the contribution from perpendicular surfaces is 0.

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = B_{\|}(z)A - B_{\|}(-z)A = 0$$

  This means

$$B_{\|}(z) - B_{\|}(-z) = 0, \; z \to 0 \qquad \text{continuous}. \tag{13}$$

2.2.2 Perpendicular Component

By Stokes' theorem, in static case,

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}\,.$$

Therefore,

$$B_{\perp\perp}(z)L - B_{\perp\perp}(-z)L = \mu_0 I L,$$

where the second $\perp$ means perpendicular to the direction of current. Thus,

$$B_{\perp\perp}(z) - B_{\perp\perp}(-z) = \mu_0 I, \; z \to 0 \qquad \text{discontinuous}. \tag{14}$$

2.2.3 Summary

$$\mathbf{B}_+ - \mathbf{B}_- = \mu_0 \mathbf{I} \times \hat{\mathbf{n}}. \tag{15}$$

3 Wave Equations

3.1 Wave Equations in Vacuum

Remember that there is a laplacian in wave equations. This means the direct way that leads to wave equations is through taking the curl of $\boldsymbol{\nabla} \times \mathbf{E}$ and $\boldsymbol{\nabla} \times \mathbf{B}$. What we would get is

$$\nabla^2 \mathbf{F} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{F}}{\partial t^2} = 0. \tag{16}$$

3.2 Directions of Electric Field and Magnetic Field

Let the wave propagate in z direction. With z being the only spatial variation, we get

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_z}{\partial z} = 0.$$

By assuming that the fields vanish at infinity, we get

$$F_z(z, t) = 0. \tag{17}$$

$\mathbf{E}$ and $\mathbf{B}$ are perpendicular, or **transverse**, to the direction of propagation.

Let x be the direction of $\mathbf{E}$. By Ampere's law,

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \tag{18}$$

so $\mathbf{B}$ is in y direction.

The plane containing the electric field defines the direction of polarization.

3.3 Complex Representation

There are sinusoidal solutions to the wave equation, e.g.

$$E_x(z, t) = E_0 \cos(kz - \omega t + \psi).$$

Complex numbers allow us to construct a complex field with a complex amplitude,

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 \exp[i(kz - \omega t)]. \tag{19}$$

The linearity of Maxwell's equations suggests that they also apply to the complex fields.

The sinusoidal solutions translate differential operators into simple multiplications:

$$\frac{\partial}{\partial t} \to -i\omega; \quad \boldsymbol{\nabla} \to i\mathbf{k}. \tag{20}$$

Therefore,

$$i\mathbf{k} \times \tilde{\mathbf{E}} = i\omega\tilde{\mathbf{B}}, \tag{21}$$

and

$$\tilde{\mathbf{E}} \cdot \left(\mathbf{k} \times \tilde{\mathbf{E}}\right) = \omega\tilde{\mathbf{E}} \cdot \tilde{\mathbf{B}} = 0, \tag{22}$$

so $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are **orthogonal**.

Taking the curl of Ampere's law gives

$$k^2\tilde{\mathbf{B}} = \frac{\omega^2}{c^2}\tilde{\mathbf{B}}, \tag{23}$$

so this means

$$c = \frac{\omega}{k}. \tag{24}$$

Also, the original Ampere's law shows

$$\left\|\tilde{\mathbf{B}}_0\right\| = \frac{\left\|\tilde{\mathbf{E}}_0\right\|}{c}. \tag{25}$$

3.4 Law of Reflection

- A perfect conductor has no $\mathbf{E}$ inside, as charges will move around to cancel the incoming $\mathbf{E}$. This means we have a charge sheet.

- Phase Dependence:

  A monochromatic wave has dependence $\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$. The boundary conditions at $\mathbf{r} = 0$ for all time suggests

  $$\omega_I = \omega_R \equiv \omega, \qquad \|\mathbf{k}_I\| = \|\mathbf{k}_R\| \equiv k.$$

  Therefore, we write

  $$\mathbf{E}_I = \tilde{E}_I\hat{\mathbf{I}}\exp[i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)]$$

  $$\mathbf{E}_R = \tilde{E}_R\hat{\mathbf{R}}\exp[i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)].$$

- Boundary Conditions:

The boundary is a charge sheet, so the perpendicular component of $\mathbf{E}$ is continuous across the boundary. At $\mathbf{r} = (X, 0, 0)$,

$$(\mathbf{E}_I + \mathbf{E}_R)_\perp = \exp(-i\omega t)\left[\tilde{E}_I \hat{\mathbf{I}} \exp(ikX \sin\theta_I) + \tilde{E}_R \hat{\mathbf{R}} \exp(ikX \sin\theta_R)\right] = 0.$$

To satisfy this for all x, we need

$$\theta_I = \theta_R, \qquad \tilde{E}_I = -\tilde{E}_R. \tag{26}$$

4 Energy Densities and Energy Flux

4.1 Work Done on Charge

The power delivered to the charge is equal to the rate of change of kinetic energy:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right) = q\mathbf{v} \cdot \mathbf{E}. \tag{27}$$

4.2 Ensemble of Particles

For a volume V containinig N charges,

$$\rho(\mathbf{r}, t) = \sum q_i \delta\left[\mathbf{r} - \mathbf{r}_i(t)\right], \tag{28}$$

$$\mathbf{J}(\mathbf{r}, t) = \sum q_i \mathbf{v}_i \delta\left[\mathbf{r} - \mathbf{r}_i(t)\right]. \tag{29}$$

Therefore, the rate of change of the total kinetic energy is

$$\sum \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m_i v_i^2\right) = \sum q_i \mathbf{v}_i \cdot \mathbf{E}(\mathbf{r}_i, t).$$

To write this in terms of $\mathbf{J}$, consider integrating $\mathbf{J} \cdot \mathbf{E}$ over the volume containing all the charges:

$$\int_V \mathbf{J} \cdot \mathbf{E}\,\mathrm{d}V = \sum q_i \mathbf{v}_i \cdot \int_V \mathbf{E}(\mathbf{r}, t)\delta\left[\mathbf{r} - \mathbf{r}_i(t)\right]\mathrm{d}V = \sum q_i \mathbf{v}_i \cdot \mathbf{E}(\mathbf{r}_i, t).$$

So, the rate at which work is done on charges by the field, per unit volume, is $\mathbf{J} \cdot \mathbf{E}$:

$$\sum \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m_i v_i^2\right) = \int_V \mathbf{J} \cdot \mathbf{E}\,\mathrm{d}V. \tag{30}$$

To calculate $\mathbf{J} \cdot \mathbf{E}$, use Ampere's law:

$$\mathbf{J} \cdot \mathbf{E} = \frac{1}{\mu_0}\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2\right). \tag{31}$$

4.3 Poynting's Theorem

Firstly, notice that

$$\boldsymbol{\nabla} \cdot \mathbf{F} \times \mathbf{G} = \mathbf{G} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) - \mathbf{F} \cdot (\boldsymbol{\nabla} \times \mathbf{G}). \tag{32}$$

We can see $\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B})$ above. Is there a way that we can find $\mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E})$? This leads us to

$$\frac{1}{\mu_0} \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) + \frac{\partial}{\partial t} \left(\frac{1}{2\mu_0} B^2 \right) = 0. \tag{33}$$

Therefore,

$$\mathbf{J} \cdot \mathbf{E} = \frac{1}{\mu_0} \left[\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) \right] - \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right).$$

By the identity above, we get the Poynting's theorem:

$$\mathbf{J} \cdot \mathbf{E} = \frac{1}{\mu_0} \boldsymbol{\nabla} \boldsymbol{\cdot} \mathbf{E} \times \mathbf{B} - \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right). \tag{34}$$

- $W = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$ is the electromagnetic energy density per unit volume.
- $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ is the Poynting vector. It suggests **the direction of energy flow**.
- $\mathbf{J} \cdot \mathbf{E}$ is the power delivered to the charges per unit volume.

The above notations lead to Poynting's theorem in differential form:

$$\frac{\partial W}{\partial t} + \boldsymbol{\nabla} \boldsymbol{\cdot} \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}. \tag{35}$$

In integral form,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V W \, \mathrm{d}V = -\oint_S \mathbf{S} \cdot \mathrm{d}\mathbf{S} - \int_V \mathbf{J} \cdot \mathbf{E} \, \mathrm{d}V \,. \tag{36}$$

Quadratic quantities are no longer linear in the use of complex notations. For example,

$$P = \mathbf{J} \cdot \mathbf{E} \quad \Rightarrow \quad \langle P \rangle = \frac{1}{2} \operatorname{Re}\left\{ \tilde{J}^* \cdot \tilde{E} \right\}. \tag{37}$$

Examples where we use poynting vectors include heating a wire and charging a capacitor.

5 Scalar & Vector Potentials

5.1 Electrostatics

By statics we mean solutions to time-independent problems, where $\partial_t = 0$. Therefore, we immediately get

$$\boldsymbol{\nabla} \times \mathbf{E} = 0, \tag{38}$$

and this leads to the definition of a scalar potential (electrostatic potential): not unique

$$\mathbf{E} = -\boldsymbol{\nabla}\phi, \tag{39}$$

where the minus sign is just a convention.

Poisson's equation for ϕ stems from the Gauss's law,

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}, \tag{40}$$

and the solution is

$$\phi(\mathbf{r}) = \int_{V'} \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0\|\mathbf{r}-\mathbf{r}'\|}\, \mathrm{d}^3r'\,. \tag{41}$$

Note that Laplace's equation has unique solutions.

5.2 Magnetostatics

The fact that

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{42}$$

implies the definition of a vector potential (magnetic vector potential): not unique

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{43}$$

5.3 Gauge Transformation

The magnetic vector potential is not unique. This is more important and more subtle than the non-uniqueness of the electrostatic potential. We can add the gradient of any scalar field $\psi(\mathbf{r})$ to $\mathbf{A}$ and it will not change $\mathbf{B}$. This is an example of a gauge transformation:

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\psi. \tag{44}$$

5.3.1 Gauge Condition

If we want to make $\mathbf{A}$ unique, we need to impose an additional condition. This is called a **gauge condition**.

There are two common gauge conditions:

- Coulomb Gauge: Statics

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{45}$$

- Lorenz Gauge: Time-Dependent

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{46}$$

In the Coulomb gauge,

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \tag{47}$$

so the solution is

$$\mathbf{A}(\mathbf{r}) = \int_{V'} \frac{\mu_0 \mathbf{J}(\mathbf{r}')}{4\pi \|\mathbf{r} - \mathbf{r}'\|}\, \mathrm{d}^3 r'\,. \tag{48}$$

We shall verify that the solution satisfies $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

5.4 Magnetic Dipoles

By Stokes' theorem,

$$\begin{aligned}
\mathbf{F} &= I \oint_C \mathrm{d}\mathbf{l} \times \mathbf{B} \\
&= I \int_S (\mathrm{d}\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{B} \\
&= I \int_S \left[\boldsymbol{\nabla}\left(\mathbf{B} \cdot \mathrm{d}\mathbf{S}\right) - \left(\boldsymbol{\nabla} \cdot \mathbf{B}\right) \mathrm{d}\mathbf{S}\right] \\
&= \boldsymbol{\nabla}\left(\mathbf{B} \cdot I \int_S \mathrm{d}\mathbf{S}\right) \\
&= \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}).
\end{aligned}$$

So, we see the magnetic dipole as

$$\mathbf{m} = I \int_S \mathrm{d}\mathbf{S}\,, \tag{49}$$

and the force it feels by an external magnetic field $\mathbf{B}$ is

$$\mathbf{F} = \boldsymbol{\nabla} \left(\mathbf{m} \cdot \mathbf{B} \right). \tag{50}$$

The associated potential is

$$U = -\mathbf{m} \cdot \mathbf{B}. \tag{51}$$

6 Time-dependent Electromagnetism

6.1 Electric Field

By the magnetic vector potential,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \frac{\partial \mathbf{A}}{\partial t},$$

so the time-dependent electric field is

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{52}$$

which contains a curl-free part (electrostatics) and a source-free part (induced electric field).

6.2 Motivation of Lorenz Gauge

By Ampere's law,

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A}\right) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}\right).$$

Rearrangement then gives

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0 \mathbf{J} + \boldsymbol{\nabla}\left(\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A}\right). \tag{53}$$

The Lorenz gauge helps to make the above expression much clearer:

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}. \tag{54}$$

This leaves us with the wave-like equation:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0 \mathbf{J}. \tag{55}$$

We can also apply this Lorenz gauge to the electric potential:

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\frac{\rho}{\epsilon_0}. \tag{56}$$

The above expressions imply the wave nature of electromagnetic waves. All gauge conditions can lead to wave-like equations, and we choose Lorenz gauge only for its simplicity.

6.3 Gauge Transformation

In the realm of electromagnetism, there are multiple sets of potential $(\phi, \mathbf{A})$ that satisfy our needs for $\mathbf{E}$ and $\mathbf{B}$, and each set can be regarded as a gauge. This also leads us to the notion of degree of freedom in gauge transformation, as one set of $\mathbf{E}$ and $\mathbf{B}$ can correspond to multiple sets of $(\phi, \mathbf{A})$.

Let's consider two sets of potentials $(\phi, \mathbf{A})$ and $(\phi', \mathbf{A}')$. Assume they differ by a constant amount:

$$\begin{aligned}\mathbf{A}' &= \mathbf{A} + \underline{\alpha} \\ \phi' &= \phi + \beta.\end{aligned}$$

For $\mathbf{A}'$ and $\mathbf{A}$ to yield the same $\mathbf{B}$, we need

$$\boldsymbol{\nabla} \times \underline{\alpha} = 0 \quad \Rightarrow \quad \underline{\alpha} = \boldsymbol{\nabla}\lambda,$$

and similarly,

$$\boldsymbol{\nabla}\beta + \frac{\partial \underline{\alpha}}{\partial t} = 0.$$

Combining the two results gives us

$$\boldsymbol{\nabla}\left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0. \tag{57}$$

This means the quantity $\beta + \dfrac{\partial \lambda}{\partial t}$ must not have any spatial variation, but also implies the possibility of time dependence. Therefore, we may write

$$\beta = -\frac{\partial \lambda}{\partial t} + \gamma(t).$$

Actually, we may use the above to find a new quantity ψ such that

$$\frac{\partial \psi}{\partial t} = \frac{\partial \lambda}{\partial t} - \gamma(t) \quad \Rightarrow \quad \psi(t) = \lambda(\mathbf{r}, t) + \int^t k(t')\,\mathrm{d}t'\,.$$

This doesn't change the gradient of λ, as the gradient doesn't depend on time. Therefore, we get the gauge transformation:

$$\begin{aligned}\mathbf{A}' &= \mathbf{A} + \boldsymbol{\nabla}\psi \\ \phi' &= \phi - \frac{\partial \psi}{\partial t}.\end{aligned} \tag{58}$$

6.4 The Wave-like Equations

The solutions to the wave-like equations,

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\frac{\rho}{\epsilon_0}, \tag{59}$$

are obtained through the technique of Green's functions:

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\rho(\mathbf{r}', t')\,\mathrm{d}^3r'\,, \tag{60}$$

where t' is the retarded time:

$$t' = t - \frac{1}{c}\|\mathbf{r}-\mathbf{r}'\|. \tag{61}$$

Similarly,

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_V \frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\mathbf{J}(\mathbf{r}', t')\,\mathrm{d}^3r'\,. \tag{62}$$

6.5 Larmor's Formula

An accelerating charge radiates energy. Our derivation goes through the following logic:

$$\mathbf{A} \to \mathbf{B} \to \mathbf{E} \to \mathbf{S} \to \frac{\mathrm{d}E}{\mathrm{d}t}. \tag{63}$$

By defining the electric dipole moment,

$$\mathbf{p} = \int_{V'} \mathbf{r}'\rho(\mathbf{r}', t')\,\mathrm{d}^3r'\,, \tag{64}$$

we can show, for a small confined volume V',

$$\mathbf{A} \simeq \frac{\mu_0}{4\pi r}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\bigg|_{t'}. \tag{65}$$

Then, the magnetic field is

$$\mathbf{B} = -\frac{\mu_0}{4\pi r^2}\hat{\mathbf{r}}\times\dot{\mathbf{p}} - \frac{\mu_0}{4\pi rc}\hat{\mathbf{r}}\times\ddot{\mathbf{p}}. \tag{66}$$

For large r,

$$\mathbf{B} \approx -\frac{\mu_0}{4\pi rc}\hat{\mathbf{r}}\times\ddot{\mathbf{p}}. \tag{67}$$

In vacuum, by Ampere's law,

$$\mathbf{E} = \frac{\mu_0}{4\pi r}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}) = c\mathbf{B}\times\hat{\mathbf{r}}. \tag{68}$$

Then, the Poynting vector is:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \hat{\mathbf{r}}B^2 = \frac{\mu_0}{16\pi^2 r^2 c}\|\ddot{\mathbf{p}}\|^2 \sin^2\theta\hat{\mathbf{r}}. \tag{69}$$

By Poynting's theorem,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\oint_S \mathbf{S} \cdot \mathrm{d}\mathbf{S} = -\frac{\mu_0}{6\pi c}\|\ddot{\mathbf{p}}\|^2. \tag{70}$$

For a hydrogen atom (stationary positive charge),

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\mu_0 e^2}{6\pi c}a^2. \tag{71}$$

Part II

Electromagnetism 2: EM in Matter

7 Introduction

In this ludicrous series of lectures, we're going to investigate mainly 4 kinds of matter: **dielectrics** as opposed to **conductors**, the ephemeral **magnetic materials**, and **plasmas**. Our primary objective is to see how electromagnetic waves traverse through these media and on their boundaries. Through the theoretical formalism and into the specific calculations, we might then discover Snell's law and Fresnel equations.

Firstly, it helps a lot if we clarify the notations that appear ubiquitously in the notes at the very beginning; the most daunting and perplexing ones are the charge and current densities.

	Free Charges	Bound Charges
Charge Density	ρ_f	ρ_p
Current Density	$\mathbf{J}_c$	$\mathbf{J}_p$, $\mathbf{J}_m$

Scandalously, all these quantities can come with their surface counterparts, where we denote them by adding an "s" in the subscript, like ρ_{sf}. These surface quantities carry the dimensions of their counterparts multiplied by that of length.

$$\begin{cases} f: & \text{free} \\ p: & \text{polarization} \\ c: & \text{conduction} \\ m: & \text{magnetization} \\ d: & \text{displacement (current)} \\ s: & \text{surface} \end{cases}$$

8 Requisite Concepts

8.1 Polarization

If we apply an electric field $\mathbf{E}$ to a piece of dielectric, then the atoms become aligned electric dipoles. These many little dipoles constitute a macroscopic effect, and we say the material is polarized. Therefore, we define the **polarization** vector:

$$\begin{aligned} \mathbf{P} &\equiv \text{dipole moment per unit volume} \\ &= N\mathbf{p} = Nq\mathbf{d}, \end{aligned} \tag{72}$$

where N is the number density of dipoles, $\mathbf{p}$ is the atomic dipole moment, and $\mathbf{d}$ is the vector displacement from the negative charge to the positive charge in the dipole.

Some particles inherit permanent dipoles from different electronegativities of their constituent atoms, while others need applied $\mathbf{E}$ field to induce a dipole. Therefore, it's natural for us to expect that the magnitude of the dipole is proportional to the applied $\mathbf{E}$ field:

$$\mathbf{p} = \alpha\mathbf{E}, \tag{73}$$

where α is the atomic polarizability. We would also want to find a relationship between $\mathbf{P}$ and the applied $\mathbf{E}$ field. In fact, if the dielectric is **linear**,

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \tag{74}$$

where χ_e is the susceptibility tensor.

Homogeneous Isotropic Linear (HIL) Dielectrics

We're too young to study the complicated, ill-behaved materials, so we normally assume some properties in our material.

- A **homogeneous** medium has properties do not vary with <u>position</u> (translational symmetry).
- An **isotropic** medium has properties are the same in all <u>directions</u> (rotational symmetry).
- Here, by **linearity**, we mean that at any given point, and for $\mathbf{E}$ in a given direction, the components of $\mathbf{P}$ are <u>proportional</u> to $\mathbf{E}$.

Therefore, equation (74) is created with linearity; homogeneity ensures that χ_e does not depend on position (uniform); isotropicity reduces χ_e from a rank-2 tensor to a simple scalar so that $\mathbf{P} \parallel \mathbf{E}$.

Then, we move on to see the related charge and current densities. To solve problems relevant to dipoles, let us firstly revisit the multipole expansion.

Multipole Expansion

Consider a charge distribution ρ. Its electrostatic potential at point $\mathbf{r}'$, by Green's functions, is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathrm{d}^3\mathbf{r}' \, \frac{\rho(r')}{\|\mathbf{r} - \mathbf{r}'\|}. \tag{75}$$

Note that here we assume that this point is at a large distance from the charge distribution, or $\mathbf{r} \gg \mathbf{r}'$. The denominator, after some variable substitution, gives

$$t \equiv \frac{r'}{r}, \quad x \equiv \cos\theta \quad \Rightarrow \quad \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \frac{1}{r} \frac{1}{\sqrt{1 - 2xt + t^2}} = \frac{1}{r} \sum P_l(x) t^l, \tag{76}$$

where $P_l(x)$ is Legendre's polynomial. This allows us to expand the potential into multiple contributions. The first term,

$$\phi_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r} \int_{V'} \mathrm{d}^3\mathbf{r}' \, \rho(r') = \frac{1}{4\pi\epsilon_0 r} q, \tag{77}$$

is simply the contribution from the charge as a whole (the monopole). The second term is actually from the dipole moment:

$$\phi_2(\mathbf{r}) = \frac{\mathbf{r}}{4\pi\epsilon_0 r^3} \cdot \int_{V'} \mathrm{d}^3\mathbf{r}' \, \mathbf{r}' \rho(\mathbf{r}') = \frac{\mathbf{r} \cdot \mathbf{p}}{4\pi\epsilon_0 r^3}. \tag{78}$$

Therefore, the potential given by a dipole is equation (78) with no net charge. If we consider a distribution of dipoles with $\mathbf{p} = \mathbf{P} \, \mathrm{d}^3\mathbf{r}'$,

$$\phi(\mathbf{r}') = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathrm{d}^3\mathbf{r}' \, \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{r}}}{r^2}.$$

An astute reader would identify $\mathbf{r}$ as the distance between the dipole and the point we evaluate, as we've assumed that the point is so far away. However, to be more general, we should replace $\mathbf{r}$ as the distance between the specific dipole and the point (); moreover,

$$\boldsymbol{\nabla}' \left(\frac{1}{\ } \right) = \frac{\hat{\ }}{\ ^2}.$$

This is redolent of an integration by parts:

$$\phi(\mathbf{r}') = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathrm{d}^3\mathbf{r}' \left[\mathbf{P}(\mathbf{r}') \cdot \boldsymbol{\nabla}' \left(\frac{1}{\ } \right) \right] = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathrm{d}^3\mathbf{r}' \left[\boldsymbol{\nabla}' \cdot \left(\frac{\mathbf{P}(\mathbf{r}')}{\ } \right) - \frac{\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}')}{\ } \right].$$

By invoking divergence theorem, we get

$$\phi(\mathbf{r}') = \frac{1}{4\pi\epsilon_0} \oint \mathrm{d}S' \, \frac{\mathbf{P}\cdot\hat{\mathbf{n}}}{} - \frac{1}{4\pi\epsilon_0} \int \mathrm{d}V' \, \frac{\boldsymbol{\nabla}'\cdot\mathbf{P}}{}. \tag{79}$$

The first term looks like the potential of a surface charge, and the second, the potential of a volume charge. In this context, we recognize ρ_{sp} and ρ_p:

$$\rho_{sp} = \mathbf{P}\cdot\hat{\mathbf{n}}, \qquad \rho_p = -\boldsymbol{\nabla}\cdot\mathbf{P}. \tag{80}$$

One can also check their relationships by conservation of total charge.

Evidently, the charge density can be divided into two parts: free charge density by free charges, and polarization charge density by bound charges. The relevant Maxwell's equation writes

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho_f + \rho_p}{\epsilon_0} \quad \Rightarrow \quad \boldsymbol{\nabla}\cdot(\epsilon_0\mathbf{E} + \mathbf{P}) = \rho_f.$$

This implies the definition of a new quantity. We define $\mathbf{D}$ as the electric displacement:

$$\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P}. \tag{81}$$

Also, we define the displacement and polarization current density:

$$\mathbf{J}_d = \frac{\partial\mathbf{D}}{\partial t}, \qquad \mathbf{J}_p = \frac{\partial\mathbf{P}}{\partial t} = Nq\frac{\partial\mathbf{d}}{\partial t}. \tag{82}$$

In HIL dielectrics,

$$\mathbf{D} = (1+\chi_e)\epsilon_0\mathbf{E} = \epsilon_0\epsilon_r\mathbf{E} = \epsilon\mathbf{E}, \tag{83}$$

where the dimensionless quantity

$$\epsilon_r = 1 + \chi_e \tag{84}$$

is the **relative permittivity**.

8.2 Clausius-Mossotti Equation

The $\mathbf{E}$'s in (73) and (74) are different. The former encompasses all except that due to the particle in interest ($\mathbf{E}_l$, local), while the latter is just the applied macroscopic field in the dielectric ($\mathbf{E}_d$). This difference leads to the Clausius-Mossotti equation.

To calculate $\mathbf{E}_l$, it suffices to calculate the electric field due to a polarized atom. Consider two uniformly charged spheres with centers displaced by $\mathbf{d}$, with $\|\mathbf{d}\| < R$. If we assume the negatively charged sphere is located at the center,

$$4\pi r^2 E = \frac{-\rho}{\epsilon_0}\frac{4}{3}\pi r^3 \quad \Rightarrow \quad \mathbf{E} = \frac{-\rho\mathbf{r}}{3\epsilon_0}.$$

Therefore, the total electric field is

$$\mathbf{E}_n = \frac{-\rho\mathbf{r}}{3\epsilon_0} + \frac{\rho(\mathbf{r}-\mathbf{d})}{3\epsilon_0} = -\frac{\rho\mathbf{d}}{3\epsilon_0} = -\frac{\mathbf{P}}{3\epsilon_0}.$$

This is the electric field due to the atom *if it was there*. In other words, we have

$$\mathbf{E}_d = \mathbf{E}_l - \frac{\mathbf{P}}{3\epsilon_0}. \tag{85}$$

Some algebraic manipulation yields the flamboyant yet useless Clausius-Mossotti equation:

$$\epsilon_r = \frac{\left(1 + \dfrac{2N\alpha}{3\epsilon_0}\right)}{\left(1 - \dfrac{N\alpha}{3\epsilon_0}\right)}. \tag{86}$$

8.3 Magnetization

Similarly, we define the **magnetization** vector:

$$\begin{aligned}\mathbf{M} &\equiv \text{magnetic dipole moment per unit volume} \\ &= N\mathbf{m} = NI\mathbf{a},\end{aligned} \tag{87}$$

where $\mathbf{m}$ is the atomic magnetic dipole moment, and $\mathbf{a}$ is the vector area enclosed by loop (right-hand-rule).

To derive the current densities from magnetization, let us firstly think about magnetic vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3\mathbf{r}' \, \frac{\mathbf{J}(\mathbf{r}')}{\|\mathbf{r}-\mathbf{r}'\|}. \tag{88}$$

For line and surface currents, the above reduces to

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \mathrm{d}l' \, \frac{\mathbf{I}}{} = \frac{\mu_0 I}{4\pi}\int \mathrm{d}\mathbf{l}' \, \frac{1}{}, \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int \mathrm{d}a' \, \frac{\mathbf{K}}{}. \tag{89}$$

For the line current, by multipole expansion and Stokes' theorem, we readily see the magnetic potential of a magnetic dipole:

$$\oint \mathrm{d}\mathbf{l}' \, (\hat{\mathbf{r}}\cdot\mathbf{r}') = -\hat{\mathbf{r}} \times \int \mathrm{d}\mathbf{a}' \quad \Rightarrow \quad \mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\ }}{{}^2}. \tag{90}$$

In the same fashion, with $\mathbf{m} = \mathbf{M}\,\mathrm{d}^3\mathbf{r}'$,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3\mathbf{r}' \left[\frac{\boldsymbol{\nabla}'\times\mathbf{M}}{} - \boldsymbol{\nabla}'\times\left(\frac{\mathbf{M}}{}\right)\right].$$

Using divergence theorem once again, we get

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathrm{d}V' \, \frac{\boldsymbol{\nabla}'\times\mathbf{M}}{} + \frac{\mu_0}{4\pi}\oint \mathrm{d}S' \, \frac{\mathbf{M}\times\hat{\mathbf{n}}}{}. \tag{91}$$

From the above, we identify the surface magnetization current density and volume magnetization current density:

$$\mathbf{J}_{sm} = \mathbf{M}\times\hat{\mathbf{n}}, \qquad \mathbf{J}_m = \boldsymbol{\nabla}\times\mathbf{M}. \tag{92}$$

One can check the relationships between $\mathbf{J}_{sm}$ and $\mathbf{J}_m$ through an analogous argument of conservation of current.

Apart from magnetization, there are current density contributions from direct conduction and polarization. Therefore, Ampere-Maxwell's law reads

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J}_c + \mathbf{J}_m + \mathbf{J}_p \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

By replacing the formulas for $\mathbf{J}_m$ and $\mathbf{J}_p$, we get

$$\boldsymbol{\nabla} \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) = \mathbf{J}_c + \frac{\partial(\epsilon_0 \mathbf{E} + \mathbf{P})}{\partial t}. \tag{93}$$

This seduces the definition of the magnetic field $\mathbf{H}$:

$$\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad \Rightarrow \quad \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}. \tag{94}$$

If we define

$$\mathbf{M} = \chi_m \mathbf{H} \tag{95}$$

with magnetic susceptibility χ_m, which is a scalar constant in HIL media, then we have

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0(1 + \chi_m)} = \frac{\mathbf{B}}{\mu_0 \mu_r} = \frac{\mathbf{B}}{\mu}, \tag{96}$$

where the dimensionless quantity

$$\mu_r = 1 + \chi_m \tag{97}$$

is called the relative permeability.

8.4 Classification of Magnetic Materials

- Diamagnetic: slightly reduce applied **B**, $\mu_r < 1$

  By Lenz's law, the atoms would realign to oppose the change in applied **B**. Their magnetic moments stay modified until **B** is switched off.

  Diamagnetism occurs in all materials.

- Paramagnetic: slightly increase applied **B**, $\mu_r > 1$

  Some particles have permanent dipole moments, and when external **B** applied, the atomic dipole moment align to increase applied **B**.

  In a paramagnetic material, this effect of atomic dipole alignment is stronger than the diamagnetic effect.

- Ferromagnetic: hugely increase applied **B**

  Ferromagnetic materials exhibit nonlinearity and hysterisis (**M** at any instant depends on past history, not just **B** at that instant).

8.5 Conductors

8.5.1 Free Charge Related Quantities

Different from dielectrics, conductors contain mobile free charges. This implies that we can now examine the terms from free charges: ρ_f and $\mathbf{J}_c$.

Assume electron number density N_e and singly charged ion number density N_i. With the ions held fixed, we have

$$\rho_f = N_i e - N_e e, \qquad \mathbf{J}_c = -N_e e \mathbf{v}_e. \tag{98}$$

8.5.2 Ohm's Law

Ohm's law relates the conduction current density with the applied $\mathbf{E}$ field:

$$\mathbf{J}_c = \sigma \mathbf{E}, \tag{99}$$

with the conductivity $\sigma = 1/\eta$ as the reciprocal of resistivity.

Clearly, in an ohmic conductor, the conservation of free charge becomes a differential equation about ρ_f:

$$\frac{\partial \rho_f}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_c = -\frac{\sigma \rho_f}{\epsilon}, \qquad \rho_f = \rho_f(0) \exp\left(-\frac{t}{\tau^*}\right), \tag{100}$$

where

$$\tau^* = \frac{\epsilon}{\sigma} \tag{101}$$

is the **charge rearrangement time**.

> If we apply $\mathbf{E}$ to a conductor, then the free electrons rearrange themselves until $\mathbf{E} = 0$ inside. This process happends on the timescale of charge rearrangement τ^*.
>
> Also, as we see in an ohmic conductor, ρ_f cannot increase and decays exponentially. Therefore, it is safe to assume $\rho_f = 0$ inside a conductor, even when $\mathbf{E} \neq 0$.

8.5.3 Good and Poor Conductors

In an ohmic conductor, we have both conduction and displacement current densities related to $\mathbf{E}$. If $\mathbf{E}$ has time dependence $\exp(-i\omega t)$, then

$$\mathbf{J}_c = \sigma \mathbf{E}, \qquad \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = -i\omega\epsilon \mathbf{E} \quad \Rightarrow \quad \frac{\|\mathbf{J}_d\|}{\|\mathbf{J}_c\|} = \frac{\omega\epsilon}{\sigma} = \omega\tau^*. \tag{102}$$

Intuitively, a good conductor must "conduct" more than "displace". Therefore, the criterion sets on the ratio above:

- Good conductor: $\omega\epsilon/\sigma \ll 1$

  In a good conductor, free charges rearrange themselves faster than $\mathbf{E}$ field changes. As free charges dominate the behavior, we may assume $\rho_p = 0$ and $\epsilon = \epsilon_0$.

- Poor conductor: $\omega\epsilon/\sigma \gg 1$

  In a poor conductor, $\mathbf{E}$ field changes faster then free charges can rearrange themselves. Therefore, polarization might be important.

  We can think of a poor conductor as being basically a dielectric which conducts very slightly. In other words, dielectrics are the poorest conductors.

8.5.4 Drude Model

The Drude model describes the equation of motion of the electrons. If $\mathbf{E} = 0$, then the average velocity of the electrons is 0. If $\mathbf{E} \neq 0$, then the model argues that

$$m_e \frac{\partial \mathbf{v}_e}{\partial t} = -e\mathbf{E} - \frac{m_e \mathbf{v}_e}{\tau_c}, \tag{103}$$

where the second term comes from the collisions among electrons at a frequency of $1/\tau_c$. If we assume $\mathbf{v}_e$ and $\mathbf{E}$ both have time dependence $\exp(-i\omega t)$, then

$$\mathbf{J}_c = -N_e e \mathbf{v}_e = \frac{\sigma \mathbf{E}}{1 - i\omega\tau_c} \quad \text{where} \quad \sigma = \frac{N_e e^2 \tau_c}{m_e}. \tag{104}$$

Therefore,

$$\begin{cases} \omega\tau_c \ll 1: & \mathbf{J}_c = \sigma\mathbf{E} \qquad \text{(high frequency collisions)} \\ \omega\tau_c \gg 1: & \mathbf{J}_c \neq \sigma\mathbf{E} \qquad \text{(low frequency collisions)} \end{cases} \tag{105}$$

8.5.5 Skin Effect

In a good conductor, assume that $\rho_f = 0$, $\mathbf{J}_d = 0$, $\epsilon = \epsilon_0$, and $\mu = \mu_0$ (neglect magnetization). Taking the curl of Faraday's law, we are left with a diffusion equation:

$$\nabla^2 \mathbf{J}_c = \mu_0 \sigma \frac{\partial \mathbf{J}_c}{\partial t}. \tag{106}$$

Second year's platitude asks for a separable solution first. If we have $\mathbf{J}_c = \mathbf{j}_c \exp(-i\omega t)$, then

$$\nabla^2 \mathbf{j}_c = -\kappa^2 \mathbf{j}_c \quad \Rightarrow \quad \kappa = \frac{1+i}{\delta}, \tag{107}$$

where

$$\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}} \tag{108}$$

is called the skin depth. It is the current diffusion lengthscale (current penetration depth). For a wire of radius a,

- $\mathbf{J}_c$ is nearly uniform if $\delta \gg a$, or $\omega \ll \dfrac{2}{\mu_0 \sigma a^2}$ (low frequencies);
- $\mathbf{J}_c$ is confined to layer of width $\sim \delta$ if $\delta \ll a$, or $\omega \gg \dfrac{2}{\mu_0 \sigma a^2}$ (high frequencies).

To explain the skin effect physically, a time varying current leads to a time varying $\mathbf{B}$, and the induced $\mathbf{E}$ induces a new current that opposes the applied I (Lenz's law). As this effect is strongest at the center, the current is reduced inside the wire.

8.6 Plasmas

8.6.1 Introduction

Plasma is an **ionized gas** which exhibits collective behavior.

As plasmas have a lot of mobile free charges, we can assume $\mathbf{P} = 0$, and there is no need to use $\mathbf{D}$. However, typically, $\mathbf{M} \neq 0$, but as it is due to free charges, we can in principle calculate it; therefore, there is no need for $\mathbf{H}$.

8.6.2 Collisionless Plasma

Typically, the effects of collisions in plasmas can be ignored. Drude model implies that

$$\mathbf{J}_c = \frac{N_e e^2}{m_e \omega} i\mathbf{E}, \tag{109}$$

where the imaginary unit just means that $\mathbf{J}_c$ and $\mathbf{E}$ are out of phase by $\pi/2$.

We can also calculate the power:

$$\langle \mathbf{J}_c \cdot \mathbf{E} \rangle = \frac{1}{\tau} \int_0^\tau \mathrm{d}t \, \frac{N_e e^2}{2 m_e \omega} E_0^2 \sin(2\omega t) = 0. \tag{110}$$

The energy goes back and forth between the field and the electrons. Also, there is **no energy loss** (no dissipation).

Also, from equation (109), the behavior of free charge is vastly different from that in ohmic conductors. In fact, it obeys SHM in plasmas with a **plasma frequency** ω_p:

$$\omega_p = \sqrt{\frac{N_e e^2}{m_e \epsilon_0}}. \tag{111}$$

9 Waves

9.1 Theoretical Formalism

We'll only consider plane waves in this atrocious subject:

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_0 \exp[i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)],\\ \mathbf{B} &= \mathbf{B}_0 \exp[i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)],\end{aligned} \tag{112}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are generally vectors with complex amplitudes. We define **wave impedance** as

$$Z = \frac{\mu_0 E_0}{B_0}. \tag{113}$$

Propitiously, from now on, we will only investigate **simple materials**, whose properties take uniform, scalar values for a given ω. Ignoring magnetization ($\mu_r = 1$), it is now easier for us to use the reduced 2-field Maxwell's equations. With the simple temporal and spatial dependence by plane waves, we have

$$\begin{aligned}
&\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_f & &\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho_f}{\epsilon} & &\mathbf{k}\cdot\mathbf{E} = -i\frac{\rho_f}{\epsilon}\\
&\boldsymbol{\nabla}\cdot\mathbf{B} = 0 & &\boldsymbol{\nabla}\cdot\mathbf{B} = 0 & &\mathbf{k}\cdot\mathbf{B} = 0\\
&\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} & &\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} & &\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}\\
&\boldsymbol{\nabla}\times\mathbf{H} = \mathbf{J}_c + \frac{\partial\mathbf{D}}{\partial t} & &\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}_c + \mu_0\epsilon\frac{\partial\mathbf{E}}{\partial t} & &\mathbf{k}\times\mathbf{B} = -i\mu_0\mathbf{J}_c - \mu_0\epsilon\omega\mathbf{E}.
\end{aligned}$$

The simple materials in our consideration are listed in the follwing table:

	Free Charge	Conduction Current	Permittivity
Vacuum	$\rho_f = 0$	$\mathbf{J}_c = 0$	$\epsilon = \epsilon_0$
HIL Dielectric	$\rho_f = 0$	$\mathbf{J}_c = 0$	$\epsilon = \epsilon_0\epsilon_r$
Ohmic Poor Conductor	$\rho_f = 0$	$\mathbf{J}_c = \sigma\mathbf{E}$	$\epsilon = \epsilon_0\epsilon_r$
Ohmic Good Conductor	$\rho_f = 0$	$\mathbf{J}_c = \sigma\mathbf{E}$	$\epsilon = \epsilon_0$
Collisionless, Unmagnetized Plasma	$\rho_f \neq 0$	$\mathbf{J}_c = i\frac{N_e e^2}{m_e\omega}\mathbf{E}$	$\epsilon = \epsilon_0$

Clearly, from the above table, we see that all $\mathbf{J}_c$'s are proportional to $\mathbf{E}$. By taking a curl in Ampere-Maxwell's law, finding the dispersion relations would be as easy as eating a pie. The only thing that comes at odds is the oscillator model in HIL dielectrics.

Complex Wavenumber Vectors

To be general, we normally assume a complex wavenumber vector:

$$\mathbf{k} = \mathbf{k}_R + i\mathbf{k}_I. \tag{114}$$

In the case of plane waves, we see that

- The wave propagates in the direction of $\mathbf{k}_R$.
- The wave decays in the direction of $\mathbf{k}_I$.

If $\mathbf{k}$ is purely imaginary, we say that this is not a propagating wave. Also, note that

$$k^2 = (\mathbf{k}_R + i\mathbf{k}_I)^2 = k_R^2 - k_I^2 + 2i\mathbf{k}_R \cdot \mathbf{k}_I \neq \|\mathbf{k}\|^2. \tag{115}$$

9.2 Waves in Vacuum

In vacuum, we have $\rho_f = \mathbf{J}_c = 0$, and $\epsilon = \epsilon_0$. This implies that $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{B}$ form a right handed set of mutually perpendicular vectors. Also, the dispersion relation lies in the curl of Ampere-Maxwell equation:

$$k^2 = \epsilon_0\mu_0\omega^2 \quad \Rightarrow \quad v_p = c = \frac{1}{\sqrt{\mu_0\epsilon_0}}. \tag{116}$$

Also, the impedance

$$Z = \sqrt{\frac{\mu_0}{\epsilon_0}} \tag{117}$$

is real and positive. This means that $\mathbf{E}$ and $\mathbf{B}$ are in phase.

9.3 Waves in HIL Dielectrics

By a similar argument, we find

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon}} = \frac{c}{\sqrt{\epsilon_r}}. \tag{118}$$

Therefore, we define the refractive index n such that

$$k = \frac{\omega n}{c} \quad \Rightarrow \quad n = \sqrt{\epsilon_r}. \tag{119}$$

Also,

$$Z = \frac{1}{n}\sqrt{\frac{\mu_0}{\epsilon_0}}. \tag{120}$$

To understand waves in HIL dielectrics more carefully, we apply the oscillator model, where we think of an atom as a driven damped harmonic oscillator.

Let the electron be oscillating ($x\hat{\mathbf{x}}$). Newton's law of motion leads us to

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\hat{\mathbf{x}} = -m\omega_0^2x\hat{\mathbf{x}} - m\gamma\frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{x}} + qE\hat{\mathbf{x}} \quad \Rightarrow \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2x = \frac{q}{m}E, \tag{121}$$

where the electric field $\mathbf{E}$ is the applied macroscopic field instead of the Coulomb forces generated by the positive charged nucleus. (We've neglected the correction to $\mathbf{E}$ due to adjacent dipoles.) As we are considering forced oscillations at ω being the frequency of wave, we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + Nq\mathbf{d} = \left[1 - \frac{Nq^2}{\epsilon_0 m(\omega^2 - \omega_0^2 + i\omega\gamma)}\right]\mathbf{E}. \tag{122}$$

Usually, the imaginary terms can be ignored, but this is not always the case. Some people are just stupid enough to call the result of this ignorance as "anomalous dispersion". This is not professional and this is not ethical.

$$\begin{cases} \text{Neglecting } i\omega\gamma: & \epsilon_r = 1 - \dfrac{Nq^2}{\epsilon_0 m(\omega^2 - \omega_0^2)} \\ \text{Anomalous Dispersion:} & n \simeq 1 - \dfrac{Nq^2}{2\epsilon_0 m}\dfrac{1}{(\omega^2 - \omega_0^2 + i\omega\gamma)}. \end{cases} \tag{123}$$

A medium in which the phase velocity of a wave depends on its frequency is said to be **dispersive**.

9.4 Waves in Collisionless, Unmagnetized Plasmas

In plasmas, we get

$$\mathbf{k} \times \mathbf{B} = \frac{\omega}{c^2}\left(\frac{\omega_p^2}{\omega^2} - 1\right)\mathbf{E}. \tag{124}$$

Firstly, if $\mathbf{B} = 0$, then we have $\omega = \omega_p$. ω is independent of k, and this means there is no dispersion.

If $\mathbf{B} \neq 0$, we have

$$k = \frac{n\omega}{c} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}. \tag{125}$$

For $\omega > \omega_p$, k is real. We can verify that the group velocity travels at a velocity $v_g < c$.

For $\omega < \omega_p$, k is imaginary. This means that the wave decays exponentially and is not propagating.

Also, we see that the plasma does not dissipate energy, and this means that the wave is reflected. The critical density of plasma where the wave gets reflected is:

$$\omega_p = \sqrt{\frac{N_e e^2}{m_e\epsilon_0}} = \omega \quad \Rightarrow \quad N = \frac{m_e\epsilon_0\omega^2}{e^2}. \tag{126}$$

9.5 Waves in Ohmic Conductors

The dispersion relation now reads

$$k^2 = \mu_0 \epsilon \omega^2 \left(1 + \frac{i\sigma}{\epsilon\omega}\right). \tag{127}$$

This means that $\mathbf{k}$ is complex, where the wave propagates in the direction of $\mathbf{k}_R$ and decays in the direction of $\mathbf{k}_I$.

Here, we assume $\mathbf{k}_R \parallel \mathbf{k}_I$. In a poor conductor, $\omega\epsilon/\sigma \gg 1$, so we have

$$k = \omega\sqrt{\mu_0\epsilon}\left(1 + \frac{i\sigma}{\epsilon\omega}\right)^{1/2} \simeq \omega\sqrt{\mu_0\epsilon}\left(1 + \frac{i\sigma}{2\epsilon\omega}\right). \tag{128}$$

This further implies that $k_I/k_R \ll 1$. Most of the electromagnetic wave propagates through the conductor with a slight attenuation.

In a good conductor, $\epsilon \to \epsilon_0$ and $\omega\epsilon_0/\sigma \ll 1$. Therefore,

$$k^2 = \mu_0 \epsilon \omega^2 \left(1 + \frac{i\sigma}{\epsilon\omega}\right) \simeq i\mu_0\sigma\omega. \tag{129}$$

Some calculations give

$$k_R = k_I = \sqrt{\frac{\mu_0\sigma\omega}{2}} = \frac{1}{\delta}. \tag{130}$$

This implies strong attenuation, and also

$$Z = \frac{\mu_0\omega\delta}{1+i}. \tag{131}$$

Metals can be conductors or plasmas depending on the wave frequency.

10 Boundaries

10.1 Theoretical Formalism

By Maxwell's equations, we can derive the general conditions for any materials:

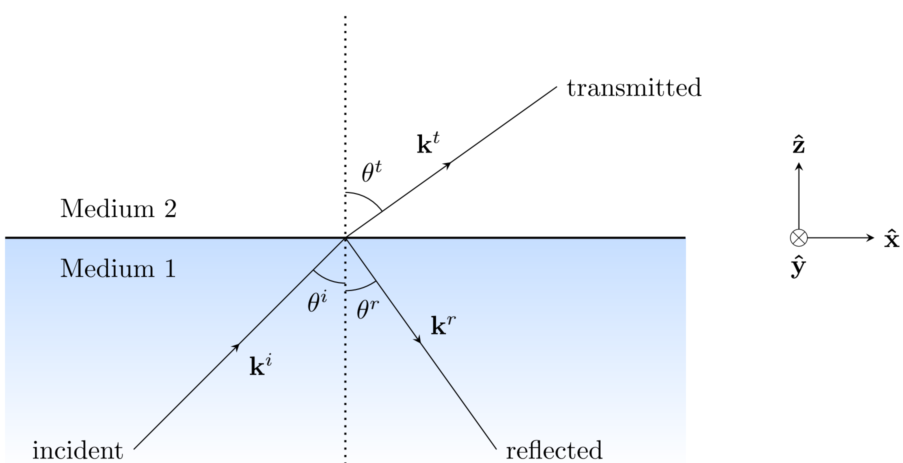
$$\begin{aligned} D_{1\perp} - D_{2\perp} &= \rho_{sf} \\ B_{1\perp} - B_{2\perp} &= 0 \\ E_{1\parallel} - E_{2\parallel} &= 0 \\ H_{1\parallel} - H_{2\parallel} &= J_{sc}. \end{aligned} \tag{132}$$

For simple materials with $\mu_r = 1$:

$$\begin{aligned} \epsilon_{r1} E_{1\perp} - \epsilon_{r2} E_{2\perp} &= \frac{\rho_{sf}}{\epsilon_0} \\ B_{1\perp} - B_{2\perp} &= 0 \\ E_{1\parallel} - E_{2\parallel} &= 0 \\ B_{1\parallel} - B_{2\parallel} &= \mu_0 J_{sc}. \end{aligned} \tag{133}$$

We're then going to investigate optics. To begin with, consider a plane electromagnetic wave in medium 1 encounters an interface with medium 2. Assume:

- the radius of curvature of the interface $\gg \lambda$;
- the thickness of the transition region from 1 to 2 $\ll \lambda$;
- the three waves have the same ω (set by the source);
- each wave has its own $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{B}$.



Here, we use superscripts

$$\begin{cases} i: & \text{incident} \\ r: & \text{reflected} \\ t: & \text{transmitted.} \end{cases} \tag{134}$$

Also, we assume $\mathbf{k}^i$ is real: the wave propagates in medium 1 without attenuation.

10.2 Law of Reflection

The electric fields in the two media are:

$$\mathbf{E}_1 = \mathbf{E}_0^i \exp\big[i\left(\mathbf{k}^i \cdot \mathbf{r} - \omega t\right)\big] + \mathbf{E}_0^r \exp[i\left(\mathbf{k}^r \cdot \mathbf{r} - \omega t\right)], \qquad \mathbf{E}_2 = \mathbf{E}_0^t \exp\big[i\left(\mathbf{k}^t \cdot \mathbf{r} - \omega t\right)\big].$$

By Faraday's law, we know that $E_\parallel$ is continuous across the boundary. We assume that $k_y^i = 0$ and the interface lies on the $z = 0$ plane. This means

$$E_{0\parallel}^i + E_{0\parallel}^r \exp\big[i\left(k_x^r - k_x^i\right)x\big] \exp\big(ik_y^r y\big) - E_{0\parallel}^t \exp\big[i\left(k_x^t - k_x^i\right)x\big] \exp\big(ik_y^t y\big) = 0.$$

The first term has no spatial dependence, and this suggests the same for the other terms:

$$\begin{cases} k_y^r = k_y^t = 0 \\ k_x^r = k_x^t = k_x^i. \end{cases} \tag{135}$$

For a given medium and a given ω, it is possible for us to solve the dispersion relations. Therefore, $k_x^i = k_x^r$ gives the law of reflection:

$$\theta^i = \theta^r. \tag{136}$$

In fact, $\mathbf{k}^t$ need not be real. The above restrictions means that

$$\mathbf{k}^t = k_x^t \hat{\mathbf{x}} + (k_{zR}^t + ik_{zI}^t)\hat{\mathbf{z}}. \tag{137}$$

10.3 Snell's Law

Assuming both media as HIL dielectrics, we have the dispersion relation

$$k = \frac{n\omega}{c}. \tag{138}$$

Therefore, equating $k_x^t = k_x^i$ gives Snell's law of refraction:

$$n_1 \sin\theta^i = n_2 \sin\theta^t. \tag{139}$$

10.4 Fresnel Equations

The Fresnel equations describe

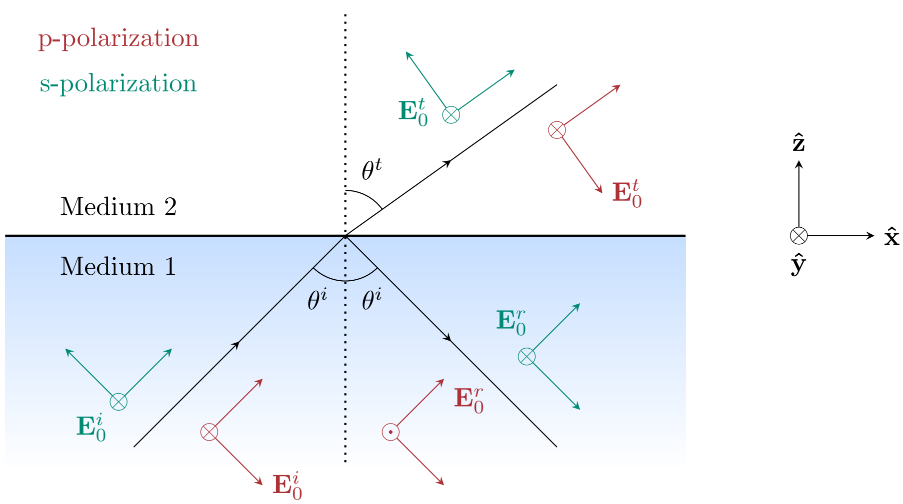
$$\begin{cases} t = \text{amplitude transmission coefficient} = \dfrac{E_0^t}{E_0^i} \\ r = \text{amplitude reflection coefficient} = \dfrac{E_0^r}{E_0^i}. \end{cases} \tag{140}$$

By assuming medium 1 as a dielectric and medium 2 as either another dielectric or ohmic conductor, we have

$$B_0 = \frac{\mu_0 E_0}{Z}, \qquad J_{sc} = 0. \tag{141}$$

By the boundary conditions where parallel components of the field are continuous across the interface, it is possible for us to solve the coefficients. There are two types of independent polarizations:

$$\begin{cases} \text{p-polarization}: & \mathbf{E}^i \text{ parallel to plane of incidence } (\mathbf{B}^i \text{ parallel to interface}) \\ \text{s-polarization}: & \mathbf{E}^i \text{ perpendicular to plane of incidence } (\mathbf{E}^i \text{ parallel to interface}) \end{cases} \tag{142}$$



Accordingly,

$$\begin{aligned} t_p &= \frac{2\cos\theta^i}{\cos\theta^t + (Z_1/Z_2)\cos\theta^i} \\ r_p &= \frac{\cos\theta^t - (Z_1/Z_2)\cos\theta^i}{\cos\theta^t + (Z_1/Z_2)\cos\theta^i} \\ t_s &= \frac{2\cos\theta^i}{\cos\theta^i + (Z_1/Z_2)\cos\theta^t} \\ r_s &= \frac{\cos\theta^i - (Z_1/Z_2)\cos\theta^t}{\cos\theta^i + (Z_1/Z_2)\cos\theta^t}. \end{aligned} \tag{143}$$

10.5 Both Media as Dielectrics

With the formula of impedance in dielectrics, we can change all the Z's into corresponding n's. The formulas suggest that r_p and r_s can be negative. If so, then E_0^r and E_0^i have a phase difference of π.

For r_p, we define the Brewster angle as the incident angle at which $r_p = 0$:

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right). \tag{144}$$

Consider both $n_1 < n_2$ and $n_1 > n_2$ and observe when the half wave loss happens for s and p polarization.

If $n_1 > n_2$, after the critical angle, the wave propagates in the x direction and decays in the z direction (evanescent wave). This is total internal reflection (TIR). By placing two interfaces very close together, it is possible to construct frustrated TIR, which is exactly analogous to quantum tunneling.

10.6 From Vaccum to Good Conductor

10.6.1 Normal Incidence

With normal incidence, we have $\theta^i = \theta^t$. This means

$$r_p = r_s = r = \frac{1 - Z_1/Z_2}{1 + Z_1/Z_2}. \tag{145}$$

For a good conductor, we have

$$r \simeq -1, \tag{146}$$

so the phase change on reflection is about π. Moreover, the fraction of energy reflected is (Hagen-Rubens relation)

$$R = |r|^2 \simeq 1 - \sqrt{\frac{8\epsilon_0\omega}{\sigma}} \approx 1, \tag{147}$$

which means that almost all of the wave energy is reflected.

Furthermore, we can work out the radiation pressure. To see this, first derive the force per volume on the conductor:

$$\begin{aligned} \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}V} &= N_i\left[e\mathbf{E} + (e\mathbf{v}_i) \times \mathbf{B}\right] + N_e\left[-e\mathbf{E} + (-e\mathbf{v}_e) \times \mathbf{B}\right] \\ &= (N_i - N_e)e\mathbf{E} - N_e e\mathbf{v}_e \times \mathbf{B} \\ &= \mathbf{J}_c \times \mathbf{B}. \end{aligned} \tag{148}$$

The radiation pressure is:

$$P_{rad} = \left\langle \frac{\|\mathbf{F}\|}{A} \right\rangle \simeq \frac{2}{c}\left\langle S^i \right\rangle, \tag{149}$$

where $\left\langle S^i \right\rangle$ is the intensity of the incident wave.

10.6.2 General

We know from equation (129) that

$$k^{t^2} = k_2^2 = i\mu_0\sigma\omega. \tag{150}$$

If we define θ^* as the angle at which the wave propagates in the good conductor, then

$$\frac{\sin^2\theta^*}{\cos\theta^*} = \frac{2\omega\epsilon_0}{\sigma}\sin^2\theta^i \ll 1 \quad \Rightarrow \quad \theta^* \text{ is very small.} \tag{151}$$

Waves in a good conductor propagates approximately normal to the surface for arbitrary angle of incidence.