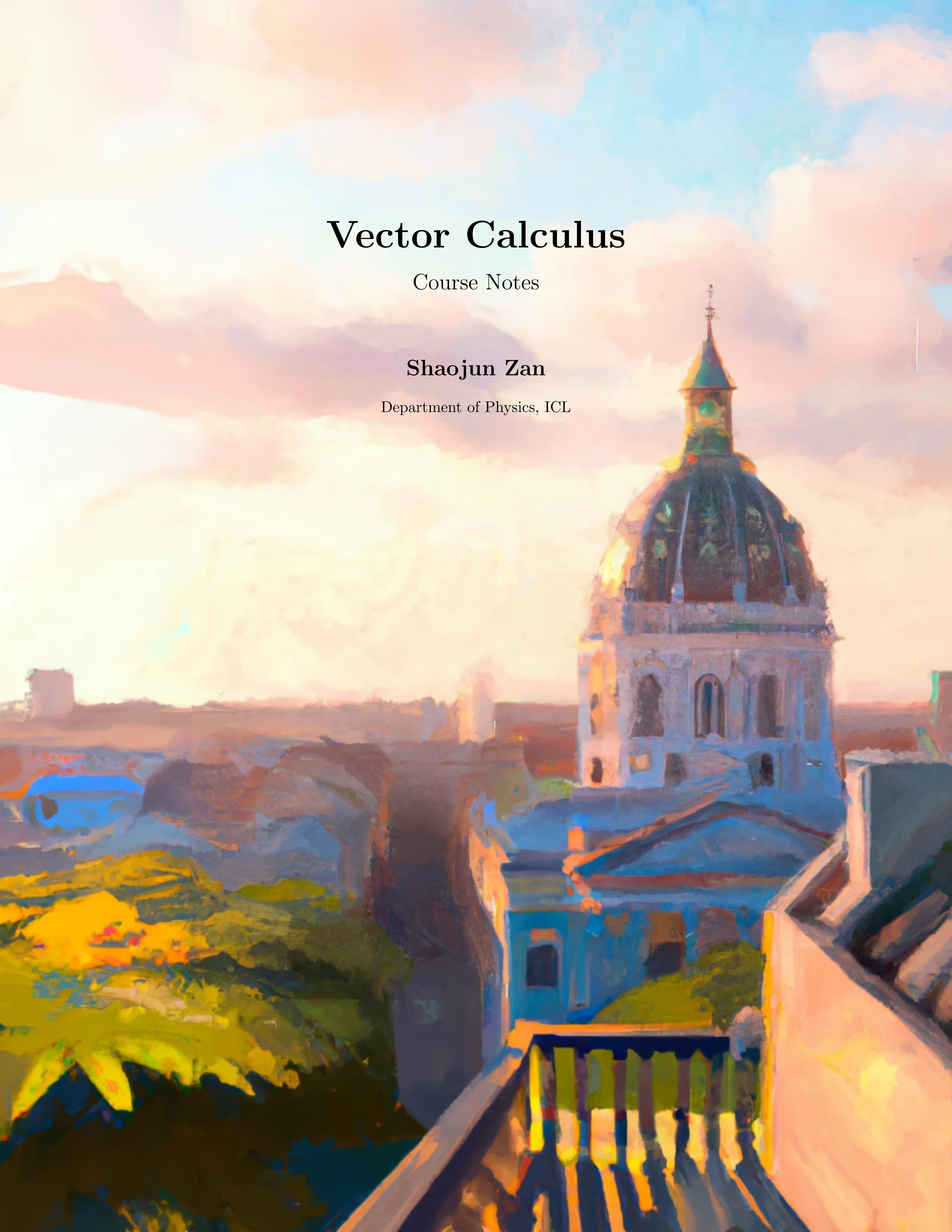
Vector Calculus

Course Notes

Shaojun Zan

Department of Physics, ICL



Contents

Chapter 1

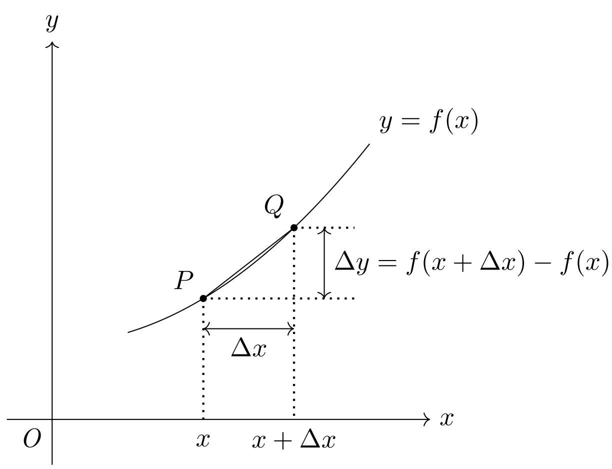
Differential Calculus

1.1 Ordinary Differentiation

For a function, we write

$$f = f(x),$$

where f is the dependent variable, $f(x)$ is the function, in which x is the independent variable.



(1) Derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{1.1}$$

(2) Differential:

$$\mathrm{d}f = \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x. \tag{1.2}$$

Note that differential has two different meanings. $\mathrm{d}f$ is the tangent approximation to the change in f; alternatively, we say $\mathrm{d}x$ is infinitesimal, and then we say $\mathrm{d}f$ is the change in f.

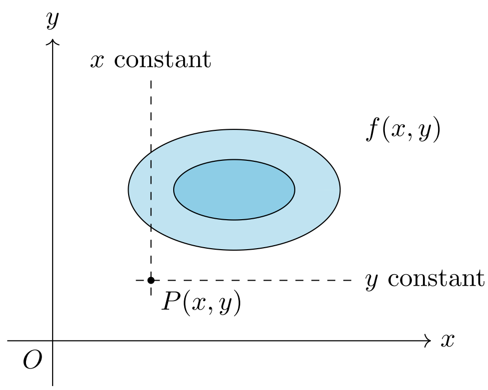
(3) Chain Rule:

$$f = f(x(t))$$
$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{1.3}$$

1.2 Partial Differentiation of Scalar Fields

Partial differentiation is employed when there are more than one independent variable, say the function

$$f = f(x, y).$$



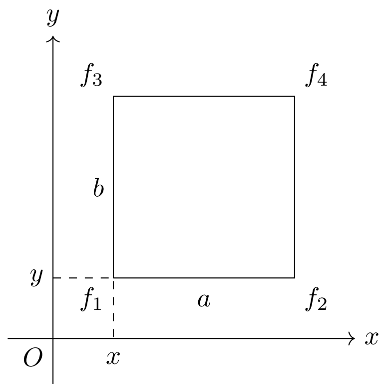
(1) Partial Derivative:
$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_y = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}. \tag{1.4}$$

When we take the partial derivative of $f(x, y)$ with respect to x, it is implicit that we assume y is constant. We write the second expression when clarity is needed.

(2) Total Differential:
$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \tag{1.5}$$

is the "tangent-plane" approximation to Δf (or the change in f for infinitesimal changes $\mathrm{d}x$ and $\mathrm{d}y$). It is important to note that we never assume that x and y are orthogonal; the conditions are satisfied only if they are independent.

(3) Clairaut's Theorem: Clairaut's theorem says that if the second partial derivatives of a function are continuous, then the order of differentiation is immaterial. Here comes a very rough method to show the idea.



From the graph, we can write:

$$\frac{\partial f}{\partial x} \simeq \frac{f_2 - f_1}{a}$$
$$\frac{\partial f}{\partial y} \simeq \frac{f_3 - f_1}{b},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \simeq \frac{1}{b}\left(\frac{f_4 - f_3}{a} - \frac{f_2 - f_1}{a}\right) = \frac{f_1 + f_4 - (f_2 + f_3)}{ab}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \simeq \frac{1}{a}\left(\frac{f_4 - f_2}{b} - \frac{f_3 - f_1}{b}\right) = \frac{f_1 + f_4 - (f_2 + f_3)}{ab}.$$

Therefore, we say that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

1.3 Partial Differentiation of Vector Fields

A 2D vector field defines a vector at every x, y:

$$\boldsymbol{A}(x,y) = A_x(x,y)\hat{\mathbf{i}} + A_y(x,y)\hat{\mathbf{j}},$$

where the two components A_x and A_y are defined by two scalar fields.

By analogy, we define the partial derivative:

$$\begin{aligned}\frac{\partial \boldsymbol{A}}{\partial x} &= \lim_{\Delta x \to 0} \frac{\boldsymbol{A}(x+\Delta x, y) - \boldsymbol{A}(x,y)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{A_x(x+\Delta x,y)\hat{\mathbf{i}} + A_y(x+\Delta x,y)\hat{\mathbf{j}} - A_x(x,y)\hat{\mathbf{i}} - A_y(x,y)\hat{\mathbf{j}}}{\Delta x}.\end{aligned}$$

By rearranging, we can see that

$$\frac{\partial \boldsymbol{A}}{\partial x} = \frac{\partial A_x}{\partial x}\hat{\mathbf{i}} + \frac{\partial A_y}{\partial x}\hat{\mathbf{j}}. \tag{1.6}$$

That is to say, the partial derivative of a vector field is the sum of partial derivatives of each component.

Also, don't confuse this derivative with the total differential:

$$\mathrm{d}\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial x}\mathrm{d}x + \frac{\partial \boldsymbol{A}}{\partial y}\mathrm{d}y. \tag{1.7}$$

1.4 Exact Differential

An exact differential is really a total differential that needs to be recognized. From before, the total differential is

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y.$$

Now, a particular differential

$$P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$$

is a total differential of some parent function f if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \left(\text{because } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}\right).$$

This is a sufficient condition. Then, we call the differential form $P\mathrm{d}x + Q\mathrm{d}y$ an exact differential.

Finding the parent function: take $2xy\mathrm{d}x + x^2\mathrm{d}y$ as an example.

This is an exact differential, as

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}.$$

Therefore, we can identify the function $P(x,y) \equiv \frac{\partial f}{\partial x}$:

$$\frac{\partial f}{\partial x} = 2xy \Rightarrow f = x^2 y + g(y).$$

Therefore, from this expression, we can write the partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = x^2 + \frac{\mathrm{d}g}{\mathrm{d}y} \equiv Q(x,y) = x^2.$$

So, we have $\frac{\mathrm{d}g}{\mathrm{d}y} = 0 \Rightarrow g(y) = C$ where C is an arbitrary constant, and the parent function is:

$$f = x^2 y + C.$$

Interestingly, the integral of an exact differential between two end states is independent of the path:

$$\int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}f = \int_{\mathrm{A}}^{\mathrm{B}} A\mathrm{d}x + \int_{\mathrm{A}}^{\mathrm{B}} B\mathrm{d}y = f_{\mathrm{A}} - f_{\mathrm{B}}. \tag{1.8}$$

1.5 Chain Rule with Partial Differentiation

For a function $f = f(x, y)$, look at three cases:

(1) $y = y(x)$ - only one independent variable (x or y);

First, we may write the total differential $\mathrm{d}f$ in terms of $\mathrm{d}x$ and $\mathrm{d}y$:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y.$$

However, as we know that x and y are not independent, we can relate $\mathrm{d}y$ to $\mathrm{d}x$:

$$\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x.$$

Therefore,

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \\ &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x,\end{aligned}$$

and finally,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{1.9}$$

(2) $x = x(t)$ and $y = y(t)$ - only one independent variable (t);

In this case, we may write:

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t, \quad \mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t.$$

Therefore,

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \\ &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t,\end{aligned}$$

and finally,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{1.10}$$

(3) $x = x(u, v)$ and $y = y(u, v)$.

This time, we can write:

$$\begin{aligned}
\mathrm{d}x &= \frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v \\
\mathrm{d}y &= \frac{\partial y}{\partial u}\mathrm{d}u + \frac{\partial y}{\partial v}\mathrm{d}v.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \\
&= \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u}\mathrm{d}u + \frac{\partial y}{\partial v}\mathrm{d}v\right) \\
&= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right)\mathrm{d}u + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right)\mathrm{d}v,
\end{aligned}$$

and we say

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.
\end{aligned} \tag{1.11}$$

Chapter 2

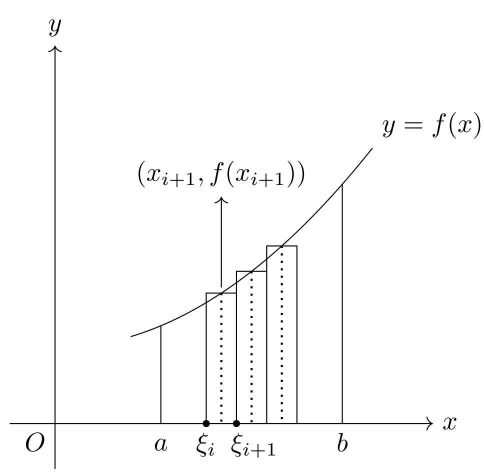
Integration

2.1 1D Integration

Integral over interval a to b of $f(x)$ is the limit of the Riemann sum:

$$S = \sum_{i=1}^{n} f(x_i)(\xi_i - \xi_{i-1}), \tag{2.1}$$

where x_i is ant point between ξ_{i-1} and ξ_i. Then, $\int_a^b f(x)\mathrm{d}x$ is the limit of S as $n \to \infty$ and width of each strip $\xi_i - \xi_{i-1} \to 0$ (if sum exists). Note that the widths do not have to be uniform.



1D integral is the "area under curve". It is better to consider it as a "weighted sum" because of many different uses of integration.

Consider a rod of uniform density ρ, width b, length l, and varying thickness $t(x)$.

(1) Mass per Unit Length:

$$\lambda(x) = \rho b t(x).$$

(2) Mass: 0th moment of mass

$$m = \int_0^l \lambda \mathrm{d}x.$$

(3) Centre of Mass: 1st moment of mass

$$\bar{x} = \frac{1}{m}\int_0^l \lambda x \mathrm{d}x.$$

(4) Radius of Gyration: 2nd moment of mass

$$x_r^2 = \frac{1}{m}\int_0^l \lambda x^2 \mathrm{d}x.$$

Radius of gyration, or gyradius, of a body about the axis of rotation is defined as the radial distance to a point which would have a moment of inertia the same as the body's actual distribution of mass, if the total mass of the body were concentrated there.

2.2 Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** states that

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a), \text{ where } \frac{\mathrm{d}F(x)}{\mathrm{d}x} = f(x).$$

Substituting b by x, we get:

$$\int_a^x f(x)\mathrm{d}x = F(x) - F(a).$$

Taking the derivative of x at both sides gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(x)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}x}[F(x) - F(a)] = \frac{\mathrm{d}F(x)}{\mathrm{d}x} = f(x),$$

and that is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(x)\mathrm{d}x = f(x). \tag{2.2}$$

Furthermore, substituting x by $u = g(x)$ gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^u f(x)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}x}[F(u) - F(a)] = \frac{\mathrm{d}F(u)}{\mathrm{d}x} = \frac{\mathrm{d}F(u)}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}.$$

In general, differentiating an integral that has variable upper and lower limits gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x)\mathrm{d}x = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x). \tag{2.3}$$

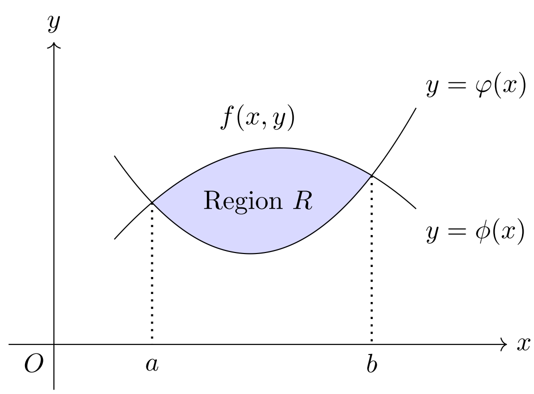
2.3 2D Integrals

The 2D integral of $f(x, y)$ over a region R, $\iint_R f(x,y)\mathrm{d}A$, is the limit of Riemann sum

$$S = \sum_{p=1}^{n} f(x_p, y_p)\Delta A_p$$

as $n \to \infty$ and $\Delta A_p \to 0$. The shape and relative sizes of ΔA_p do not matter.

2D integral is like "volume under surface", but generally is a weighted sum. We can use uniform grid with Δx and Δy. Then, as $n \to \infty$, area elements are infinitesimal $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$.



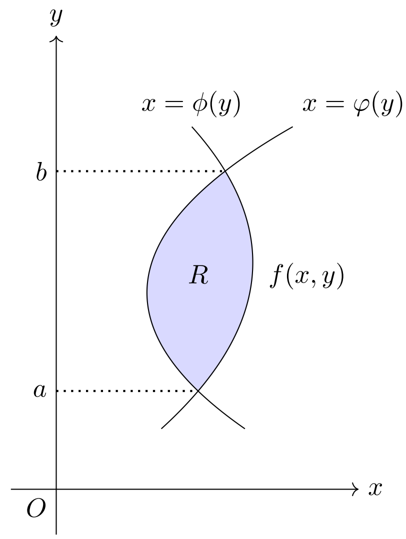
For this function, we write:

$$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y = \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\phi(x)} f(x,y)\mathrm{d}y, \tag{2.4}$$

where, as a reminder,

$$\int_a^b \mathrm{d}x \int_{\varphi(x)}^{\phi(x)} f(x,y)\mathrm{d}y \equiv \int_a^b \left(\int_{\varphi(x)}^{\phi(x)} f(x,y)\mathrm{d}y\right)\mathrm{d}x.$$

We can also integrate first with respect to y. Consider an alternative plot as follows.



In this case,

$$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y = \int_a^b \mathrm{d}y \int_{\varphi(y)}^{\phi(y)} f(x,y)\mathrm{d}x \tag{2.5}$$

is the better way to integrate $f(x, y)$ over the region R. Choosing the right integration order can really make life easier.

2.4 Change of Variables in 2D Integrals - The Jacobian

2.4.1 Change of Variables in 1D Integration

Remember to change 3 things:

(1) The Integrand: express $f(x)$ as $f(x(\theta))$

e.g. $x = \sin\theta$, $f(x(\theta)) = f(\sin\theta)$.

(2) The Limits:

e.g. With $x = \sin\theta$, $0 < x < 1 \Rightarrow 0 < \theta < \pi/2$.

(3) The Differential:

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta$$

e.g. $x = \sin\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta$.

It is important for us to see the “1D Jacobian”. In this case,

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta,$$

and $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ is "how much x changes as θ changes by $\mathrm{d}\theta$", and it is just the "1D Jacobian".

2.4.2 Change of Variables in 2D Integrals

$$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y \equiv \iint_{R'} f(x(u,v),y(u,v))|J|\mathrm{d}u\mathrm{d}v, \tag{2.6}$$

where $|J|$ is the Jacobian.

Remember also to change 3 things in 2D integration:

(1) The Integrand: express $f(x,y)$ as $f(x(u,v),y(u,v))$

(2) The Limits:

Draw (yes!) R_{xy} as R_{uv}, and find the upper and lower limits.

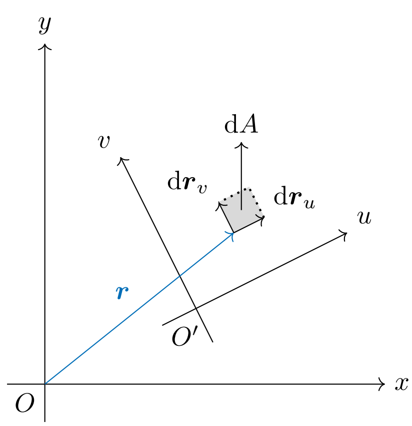
Note that in R_{xy} or R_{uv} we may need to split into more than one integral.

(3) Transform the Differential:

$$\mathrm{d}x\mathrm{d}y \equiv |J|\mathrm{d}u\mathrm{d}v$$

2.4.3 The Jacobian

Think about the question: what area in the xy plane does infinitesimal changes $\mathrm{d}u\mathrm{d}v$ cover?



Consider the position vector and the field $\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$. Then, the total differential of $\boldsymbol{r}$ in the uv plane can be written as:

$$\mathrm{d}\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial u}\mathrm{d}u + \frac{\partial \boldsymbol{r}}{\partial v}\mathrm{d}v = \mathrm{d}\boldsymbol{r}_u + \mathrm{d}\boldsymbol{r}_v.$$

The area of the parallelogram $\mathrm{d}A = |\mathrm{d}\boldsymbol{A}|$ is:

$$\mathrm{d}\boldsymbol{A} = \mathrm{d}\boldsymbol{r}_u \times \mathrm{d}\boldsymbol{r}_v = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right)\mathrm{d}u\mathrm{d}v.$$

As $\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$,

$$\frac{\partial \boldsymbol{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}}$$
$$\frac{\partial \boldsymbol{r}}{\partial v} = \frac{\partial x}{\partial v}\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\hat{\mathbf{j}}.$$

Therefore,

$$\mathrm{d}\boldsymbol{A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \hat{\mathbf{k}} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathrm{d}u\mathrm{d}v.$$

The absolute value of the determinant is the 2D Jacobian.

$$\mathrm{d}A = |J|\mathrm{d}u\mathrm{d}v$$

Sometimes, we write:

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}. \tag{2.7}$$

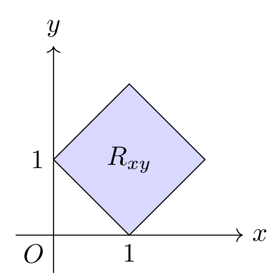
Example: Integrate $f(x,y) = x + y$ over the region with boundaries

(1) $x + y = 1$;

(2) $x + y = 3$;

(3) $y - x = -1$;

(4) $y - x = 1$.



Without changing the variables, we can find the integral:

$$\begin{aligned} I &= \iint_{R_{xy}} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \mathrm{d}x \int_{1-x}^{1+x} (x+y)\mathrm{d}y + \int_1^2 \mathrm{d}x \int_{x-1}^{3-x} (x+y)\mathrm{d}y \\ &= \frac{5}{3} + \frac{7}{3} = 4. \end{aligned}$$

Now, consider the new variables u and v, with

$$u = x + y$$

$$v = y - x.$$

Recall the three steps when transforming the integral:

Firstly, the integrand now is:

$$f(x, y) = x + y \equiv u.$$

Then, it is easy to see that the boundaries now become:

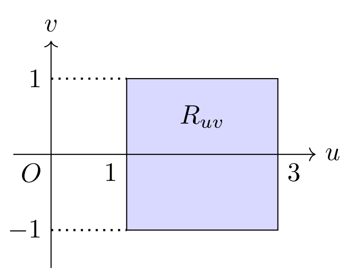
(1) $x + y \equiv u = 1$;

(2) $x + y \equiv u = 3$;

(3) $y - x \equiv v = -1$;

(4) $y - x \equiv v = 1$.

So, we can draw the region R_{uv} after transformation:



Finally, we need to transform the differential:

$$\mathrm{d}x\mathrm{d}y = |J|\mathrm{d}u\mathrm{d}v.$$

By expressing x and y using u and v, we can find the Jacobian:

$$x = \frac{u - v}{2}$$
$$y = \frac{u + v}{2}$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

Therefore,

$$
\begin{aligned}
I &= \iint_{R_{uv}} f(x(u,v), y(u,v))|J|\mathrm{d}u\mathrm{d}v \\
&= \int_{-1}^{1} \mathrm{d}v \int_{1}^{3} \frac{u}{2}\mathrm{d}u \\
&= \int_{-1}^{1} \left(\frac{u^2}{4}\right)\Bigg|_{1}^{3} \mathrm{d}v \\
&= \int_{-1}^{1} 2\mathrm{d}v \\
&= 4.
\end{aligned}
$$

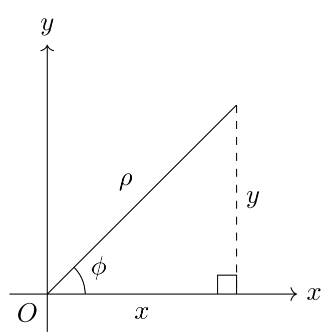
Clearly, this method of integration is easier than directly integrating with x and y.

We can also think of the Jacobian in a geometric perspective. The region R_{xy} has an area of 2 in the xy plane, but an area of 4 in the uv plane. That is to say, the area in xy plane is scaled by a factor of 2 by transforming into the uv plane. Therefore, $\mathrm{d}u\mathrm{d}v = 2\mathrm{d}x\mathrm{d}y$, or

$$\mathrm{d}x\mathrm{d}y = \frac{1}{2}\mathrm{d}u\mathrm{d}v.$$

2.5 Plane Polar Coordinates

The plane polar coordinates are very useful for problems with circular symmetry.



Note that the polar coordinates are also orthogonal, with

$$
\begin{aligned}
x(\rho,\phi) &= \rho\cos\phi \\
y(\rho,\phi) &= \rho\sin\phi \\
\rho(x,y) &= \sqrt{x^2+y^2} \\
\phi(x,y) &= \arctan\left(\frac{y}{x}\right).
\end{aligned}
$$

Geometrically, a change in the angle ϕ ($\mathrm{d}\phi$) would lead to a change in the path length by $\rho\mathrm{d}\phi$, so

$$\mathrm{d}A = \rho\mathrm{d}\rho\mathrm{d}\phi \equiv |J|\mathrm{d}\rho\mathrm{d}\phi,$$

and $|J| = \rho$.

Analytically, by the determinant,

$$J = \frac{\partial(x,y)}{\partial(\rho,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{vmatrix} = \rho.$$

For example, consider a disk of radius R with surface mass density $\sigma(x,y) = \dfrac{B}{\sqrt{x^2+y^2}}$. What is the mass of the disk?

By the three steps, we get:

(1) $\sigma(x,y) = \dfrac{B}{\sqrt{x^2+y^2}} \equiv \dfrac{B}{\rho}$.

(2) The limits:

We know R_{xy} is a circle, where ρ changes from 0 to R continuously, and ϕ changes from 0 to 2π continuously. Therefore, $R_{\rho\phi}$ is a rectangle.

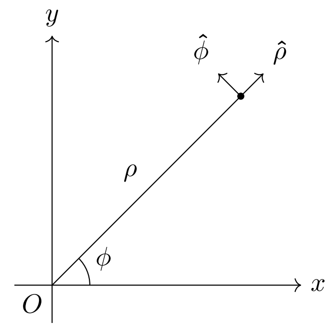
(3) $|J| = \rho$.

Therefore,

$$\begin{aligned} I &= \iint_{R_{xy}} \sigma(x,y)\mathrm{d}x\mathrm{d}y \\ &= \iint_{R_{\rho\phi}} \sigma(\rho,\phi)|J|\mathrm{d}\rho\mathrm{d}\phi \\ &= \int_0^{2\pi} \mathrm{d}\phi \int_0^R \frac{B}{\rho}\rho\mathrm{d}\rho \\ &= \int_0^{2\pi} BR\mathrm{d}\phi \\ &= 2\pi BR. \end{aligned}$$

2.6 Unit Vectors in Plane Polar Coordinates

Differentiating vector fields in plane polar coordinates is more difficult. It is useful to define the unit vectors in radial $\hat{\rho}$ and tangential $\hat{\phi}$ directions. They both depend on ϕ!



The unit vectors $\hat{\rho}$ and $\hat{\phi}$ are mutually orthogonal, and they are both orthogonal to $\hat{\mathbf{k}}$. To differentiate them, write them in Cartesian coordinates:

$$\begin{aligned}\hat{\rho} &= \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \\ \hat{\phi} &= -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}.\end{aligned}$$

Then, we can get:

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial\phi} &= -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \equiv \hat{\phi} \\ \frac{\partial\hat{\phi}}{\partial\phi} &= -\cos\phi\hat{\mathbf{i}} - \sin\phi\hat{\mathbf{j}} \equiv -\hat{\rho},\end{aligned} \tag{2.8}$$

and their partial derivatives with respect to ρ are 0.

2.6.1 Differentiating Vector Fields Using Plane Polar Coordinates

Generally, we can write the vector field as follows:

$$\boldsymbol{A} = A_\rho(\rho,\phi)\hat{\rho}(\phi) + A_\phi(\rho,\phi)\hat{\phi}(\phi).$$

Then, we can see

$$\frac{\partial\boldsymbol{A}}{\partial\phi} = \frac{\partial}{\partial\phi}(A_\rho\hat{\rho}) + \frac{\partial}{\partial\phi}\left(A_\phi\hat{\phi}\right) \tag{2.9}$$

and use the product rule.

We can differentiate the position vector $\boldsymbol{r} = \rho\hat{\rho}$ to derive the Jacobian for plane polar coordinates.

Generally,

$$\mathrm{d}\boldsymbol{r} = \frac{\partial\boldsymbol{r}}{\partial\rho}\mathrm{d}\rho + \frac{\partial\boldsymbol{r}}{\partial\phi}\mathrm{d}\phi,$$

and the Jacobian is:

$$J = \left|\frac{\partial\boldsymbol{r}}{\partial\rho} \times \frac{\partial\boldsymbol{r}}{\partial\phi}\right|.$$

Using the product rule, we can easily find that

$$\begin{aligned}\frac{\partial \boldsymbol{r}}{\partial \rho} &= \hat{\rho} \\ \frac{\partial \boldsymbol{r}}{\partial \phi} &= \rho\hat{\phi}.\end{aligned}$$

Therefore,

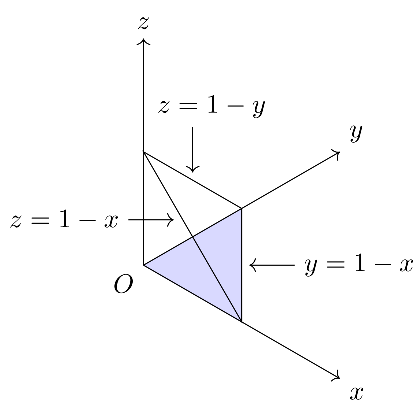
$$\frac{\partial \boldsymbol{r}}{\partial \rho} \times \frac{\partial \boldsymbol{r}}{\partial \phi} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} = \rho\hat{\mathbf{k}} \Rightarrow |J| = \rho.$$

Chapter 3

3D Integrals

3.1 Cartesian Coordinates

Consider a simple example: integrate $f(x, y, z) = \alpha x$. The volume over which we may integrate is bound by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.



Let's integrate in order of z, y, and x.

$$
\begin{aligned}
I &= \iiint_{R_{xyz}} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_{\min(x)}^{\max(x)} \mathrm{d}x \int_{\min(y)}^{\max(y)} \mathrm{d}y \int_{\min(z)}^{\max(z)} \alpha x \mathrm{d}z
\end{aligned}
$$

We first integrate with respect to z. Therefore, in the xyz space, we can easily see that $\min(z) = 0$, and $\max(z) = 1 - x - y$.

After integrating z, we need to integrate with y. In the xy plane, we see that $\min(y) = 0$, and $\max(y) = 1 - x$.

And finally, we know that $\min(x) = 0$ and $\max(x) = 1$. Therefore,

$$\begin{aligned}
I &= \int_{\min(x)}^{\max(x)} \mathrm{d}x \int_{\min(y)}^{\max(y)} \mathrm{d}y \int_{\min(z)}^{\max(z)} \alpha x \mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{1-x-y} \alpha x \mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_0^{1-x} (\alpha x z)\Big|_0^{1-x-y} \mathrm{d}y \\
&= \alpha \int_0^1 \mathrm{d}x \int_0^{1-x} (x - x^2 - xy)\mathrm{d}y \\
&= \alpha \int_0^1 \left(xy - x^2 y - \frac{xy^2}{2} \right)\Big|_0^{1-x} \mathrm{d}x \\
&= \alpha \int_0^1 \frac{x(1-x)^2}{2} \mathrm{d}x \\
&= \frac{\alpha}{24}.
\end{aligned}$$

3.2 Jacobian in 3D

$$I = \iiint_{R_{xyz}} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_{R_{uvw}} f(x(u,v,w), y(u,v,w), z(u,v,w))|J|\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

Consider the total differential of the position vector $\boldsymbol{r}$:

$$\mathrm{d}\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial u}\mathrm{d}u + \frac{\partial \boldsymbol{r}}{\partial v}\mathrm{d}v + \frac{\partial \boldsymbol{r}}{\partial w}\mathrm{d}w = \boldsymbol{r}_u + \boldsymbol{r}_v + \boldsymbol{r}_w.$$

Since $\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, we know the partial derivatives are given by, for example,

$$\mathrm{d}\boldsymbol{r}_u = \left(\frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}} + \frac{\partial z}{\partial u}\hat{\mathbf{k}} \right) \mathrm{d}u.$$

We want the volume of the little element defined by the three little vectors $\mathrm{d}\boldsymbol{r}_u$, $\mathrm{d}\boldsymbol{r}_v$, and $\mathrm{d}\boldsymbol{r}_w$.

Recall that the volume of parallelipiped defined by vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$. So, the volume is

$$\begin{aligned}
\mathrm{d}V_{uvw} &= \mathrm{d}\boldsymbol{r}_u \cdot (\mathrm{d}\boldsymbol{r}_v \times \mathrm{d}\boldsymbol{r}_w) \\
&= \left(\frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}} + \frac{\partial z}{\partial u}\hat{\mathbf{k}}\right) \cdot \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix} \mathrm{d}u\mathrm{d}v\mathrm{d}w \\
&= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix} \mathrm{d}u\mathrm{d}v\mathrm{d}w.
\end{aligned}$$

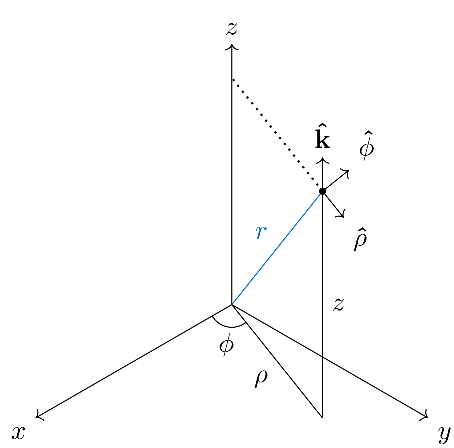
Therefore, we have got what is called the 3D Jacobian:

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}. \tag{3.1}$$

Remember that what we need is the absolute value of J.

3.3 Cylindrical Coordinates

The cylindrical coordinates is an extension of the plane polar coordinates. It is the plane polar coordinates plus a z axis. It is very useful for problems with circular geometry and variation in z, like the pipe flow or conical geometry.



The transformations in this case are:

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \quad \Rightarrow \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \arctan\left(\dfrac{y}{x}\right) \\ z = z. \end{cases} \tag{3.2}$$

The position vector:

$$\boldsymbol{r}(\rho,\phi,z) = \rho\hat{\rho}(\phi) + z\hat{\mathbf{k}}.$$

The unit vectors $\hat{\rho}(\phi)$, $\hat{\phi}(\phi)$, and $\hat{\mathbf{k}}$ are orthogonal, and they have no dependence on ρ or z.

When differentiating a vector field in cylindrical coordinates, remember that $\hat{\rho}$ and $\hat{\phi}$ are dependent on ϕ:

$$\frac{\partial\hat{\rho}}{\partial\phi} = \hat{\phi}$$
$$\frac{\partial\hat{\phi}}{\partial\phi} = -\hat{\rho}.$$

We can derive the Jacobian using the same methods as in plane polar coordinates.

Geometrically, the volume produced by small changes $\mathrm{d}\rho$, $\mathrm{d}\phi$, and $\mathrm{d}z$ is:

$$\mathrm{d}V = \rho\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}z \Rightarrow |J| = \rho.$$

Analytically, using the determinant,

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

More elegantly, using the differential of position vector $\boldsymbol{r}(\rho,\phi,z) = \rho\hat{\rho}(\phi) + z\hat{\mathbf{k}}$, we get

$$\mathrm{d}\boldsymbol{r} = \frac{\partial\boldsymbol{r}}{\partial\rho}\mathrm{d}\rho + \frac{\partial\boldsymbol{r}}{\partial\phi}\mathrm{d}\phi + \frac{\partial\boldsymbol{r}}{\partial z}\mathrm{d}z.$$

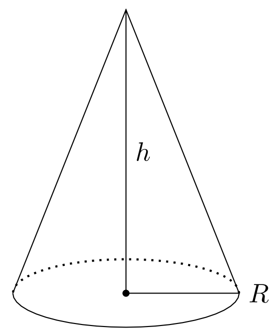
Each change gives a different change in position vector:

$$\mathrm{d}\boldsymbol{r}_\rho = \frac{\partial\boldsymbol{r}}{\partial\rho}\mathrm{d}\rho = \mathrm{d}\rho\hat{\rho}$$
$$\mathrm{d}\boldsymbol{r}_\phi = \frac{\partial\boldsymbol{r}}{\partial\phi}\mathrm{d}\phi = \rho\mathrm{d}\phi\hat{\phi}$$
$$\mathrm{d}\boldsymbol{r}_z = \frac{\partial\boldsymbol{r}}{\partial z}\mathrm{d}z = \mathrm{d}z\hat{\mathbf{k}}.$$

As these vectors are mutually orthogonal, the volume is:

$$\mathrm{d}V = \mathrm{d}\boldsymbol{r}_\rho \cdot (\mathrm{d}\boldsymbol{r}_\phi \times \mathrm{d}\boldsymbol{r}_z) = \rho \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z \Rightarrow |J| = \rho.$$

Example: centre of gravity of a vertical solid cone



We can see that the relationship between the radius and the height is:

$$\rho = R\left(1 - \frac{z}{h}\right).$$

Assume a constant density D, we can get:

$$\bar{z} = \frac{\sum m_i z_i}{\sum m_i} = \frac{\iiint z D \mathrm{d}V}{\iiint D \mathrm{d}V} \equiv \frac{\iiint z \mathrm{d}V}{\iiint \mathrm{d}V}.$$

The denominator is simply the volume of the cone: $V = \frac{1}{3}\pi R^2 h$.

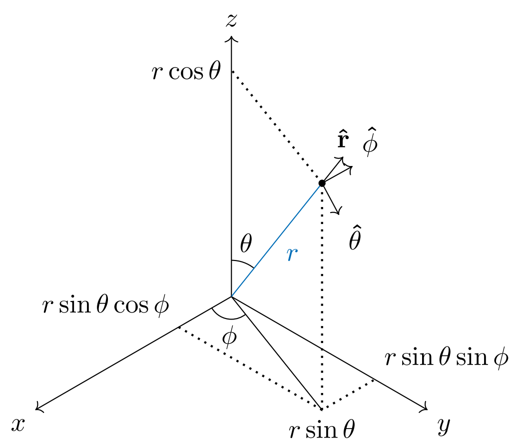
$$\begin{aligned}
I &= \iiint z \mathrm{d}V \\
&= \iiint z \rho \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z \\
&= \int_0^h \mathrm{d}z \int_0^{R(1-z/h)} \mathrm{d}\rho \int_0^{2\pi} \rho z \mathrm{d}\phi \\
&= 2\pi \int_0^h \mathrm{d}z \int_0^{R(1-z/h)} z\rho \mathrm{d}\rho \\
&= \pi \int_0^h (z\rho^2)\Big|_0^{R(1-z/h)} \mathrm{d}z \\
&= \pi R^2 \int_0^h z\left(1 - \frac{z}{h}\right)^2 \mathrm{d}z \\
&= \frac{\pi R^2 h^2}{12}.
\end{aligned}$$

Therefore,

$$\hat{z} = \frac{I}{V} = \frac{\pi R^2 h^2}{12} \cdot \frac{3}{\pi R^2 h} = \frac{h}{4}.$$

3.4 Spherical Coordinates

The spherical coordinates describe the position using r, θ (“latitude”), and ϕ (“longitude”). These unit vectors are orthogonal. The angle θ is the angle between $\boldsymbol{r}$ and the positive z axis ($0 \leq \theta \leq \pi$), and ϕ is the angle between $\boldsymbol{r}$ and the positive x axis ($0 \leq \theta \leq 2\pi$).



The transformations are:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta. \end{cases} \tag{3.3}$$

The unit vectors, when expressed in Cartesian coordinates, are

$$\begin{aligned} \hat{\mathbf{r}}(\theta,\phi) &= \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \\ \hat{\theta}(\theta,\phi) &= \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \\ \hat{\phi}(\phi) &= -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}. \end{aligned} \tag{3.4}$$

Note that if you find it hard to remember $\hat{\theta}$, you can try $\hat{\theta} = \hat{\phi} \times \hat{\mathbf{r}}$. The other two are much easier to derive. The position vector, in this case, is:

$$\boldsymbol{r} = r\hat{\mathbf{r}}(\theta,\phi).$$

It is the function of two angles, so we should figure out the two partial derivatives:

$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} = \hat{\theta}$$
$$\frac{\partial \hat{\mathbf{r}}}{\partial \phi} = -\sin\theta\sin\phi\hat{\mathbf{i}} + \sin\theta\cos\phi\hat{\mathbf{j}} = \sin\theta\hat{\phi}.$$

Then, we can derive the volume element from total differential:

$$\mathrm{d}\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial r}\mathrm{d}r + \frac{\partial \boldsymbol{r}}{\partial \theta}\mathrm{d}\theta + \frac{\partial \boldsymbol{r}}{\partial \phi}\mathrm{d}\phi.$$

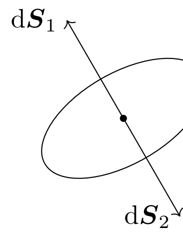
$$\mathrm{d}\boldsymbol{r}_r = \frac{\partial \boldsymbol{r}}{\partial r}\mathrm{d}r = \mathrm{d}r\hat{\mathbf{r}}$$
$$\mathrm{d}\boldsymbol{r}_\theta = \frac{\partial \boldsymbol{r}}{\partial \theta}\mathrm{d}\theta = r\mathrm{d}\theta\hat{\theta}$$
$$\mathrm{d}\boldsymbol{r}_\phi = \frac{\partial \boldsymbol{r}}{\partial \phi}\mathrm{d}\phi = r\sin\theta\mathrm{d}\phi\hat{\phi}.$$

Therefore,

$$\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \Rightarrow |J| = r^2\sin\theta.$$

3.5 Surface Integrals

On a surface in 3D, we define an infinitesimal element of area $\mathrm{d}\boldsymbol{S}$, a vector of area $|\mathrm{d}\boldsymbol{S}|$, with direction normal to the surface.



It is important to notice that the direction of the vector is not fixed. It is just a matter of choice and, sometimes, convention. For example, both $\mathrm{d}\boldsymbol{S}_1$ and $\mathrm{d}\boldsymbol{S}_2$ are the correct normal vectors for the surface.

The vector $\mathrm{d}\boldsymbol{S}$ can be used for computing a number of quantities.

(1) Area:

$$\iint_S |\mathrm{d}\boldsymbol{S}|.$$

Note that the S here is actually a 3D surface. It is a 2D thing, but it lies in a 3D space, just like a 1D curve in the 2D plane.

(2) Total Scalar:

For example, we may assume the surface charge density σ on the surface S. Therefore, the total charge carried by the surface S is given by the integral:

$$\iint_S \sigma |\mathrm{d}\boldsymbol{S}|.$$

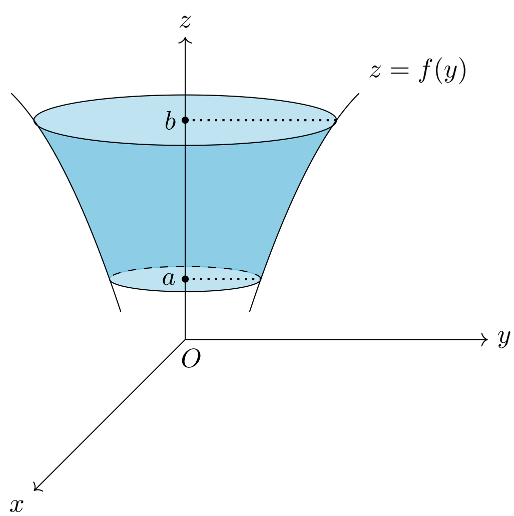
(3) Flux: definition

The flux of any vector field $\boldsymbol{F}$ through the surface S is given by the integral:

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

3.5.1 Surface of Revolution

We generate a surface of revolution using a curve rotated about an axis; for example, consider a curve rotated about the z axis.



Obviously, with "circular geometry and variation in z", this surface can be expressed easily with cylindrical coordinates:

$$z = f(y) \Rightarrow z = f(\rho), \text{ or } \rho = g(z).$$

Consider a thin slice, and it would be trivial to see that:

$$\mathrm{d}V = \pi\rho^2 \mathrm{d}z \Rightarrow V = \pi \int \rho^2 \mathrm{d}z$$

$$\mathrm{d}A = 2\pi\rho \mathrm{d}s \Rightarrow A = 2\pi \int \rho \sqrt{1 + \left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^2} \mathrm{d}z.$$

3.5.2 General Method to Obtain the Surface Area Element

The position vector $\boldsymbol{r}$ of a point on a surface may be expressed using only 2 coordinates.

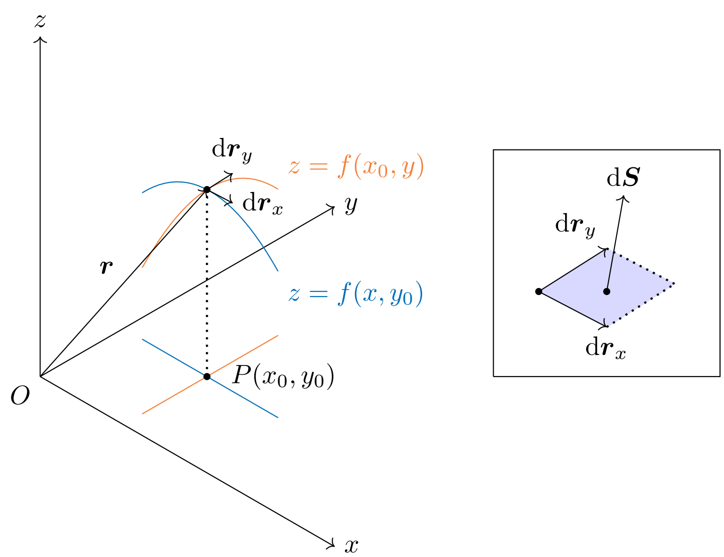
For example, in Cartesian coordinates, the position vector can be expressed as:

$$\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

With the equation of the surface $z = f(x, y)$, we can easily know that the vector pointing from origin to a point on the surface is given by the expression:

$$\boldsymbol{r}(x, y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + f(x, y)\hat{\mathbf{k}}.$$

Now consider the surface area element $\mathrm{d}\boldsymbol{S}$.



We can write

$$\mathrm{d}\boldsymbol{r}_x = \frac{\partial \boldsymbol{r}}{\partial x}\mathrm{d}x$$
$$\mathrm{d}\boldsymbol{r}_y = \frac{\partial \boldsymbol{r}}{\partial y}\mathrm{d}y.$$

Both these two vectors are tangent lines to the surface, in different directions. Obviously, $\mathrm{d}\boldsymbol{r}_x$ and $\mathrm{d}\boldsymbol{r}_y$ define a little tangent plane near the point $(x_0, y_0, f(x_0, y_0))$, and the corresponding area element vector $\mathrm{d}\boldsymbol{S}$ is:

$$\mathrm{d}\boldsymbol{S} = \mathrm{d}\boldsymbol{r}_x \times \mathrm{d}\boldsymbol{r}_y = \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \boldsymbol{N}\mathrm{d}x\mathrm{d}y.$$

Note that sometimes we may want $-\mathrm{d}\boldsymbol{S}$, depending on the question. Generally, we choose the coordinates we use for convenience.

More generally, we can express the vector to the surface in terms of 2 variables u and v:

$$\mathrm{d}\boldsymbol{S} = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) \mathrm{d}u\mathrm{d}v. \tag{3.5}$$

Consider a simple example in cylindrical coordinates. In cylindrical coordinates, the position vector is given by:

$$\boldsymbol{r} = \rho\hat{\rho}(\phi) + z\hat{\mathbf{k}}.$$

The equation of the surface can be given in the form of $z = f(\rho, \phi)$. Then, the vector to the surface is:

$$\boldsymbol{r}(\rho, \phi) = \rho\hat{\rho}(\phi) + f(\rho, \phi)\hat{\mathbf{k}}.$$

And thus, the area element $\mathrm{d}\boldsymbol{S}$ is:

$$\mathrm{d}\boldsymbol{S} = \left(\frac{\partial \boldsymbol{r}}{\partial \rho} \times \frac{\partial \boldsymbol{r}}{\partial \phi}\right) \mathrm{d}\rho\mathrm{d}\phi.$$

Alternatively, if the equation of the surface is given in the form of $\phi = f(\rho, z)$ or $\rho = f(\phi, z)$, we can choose the vector to the surface to be expressed as $\boldsymbol{r}(\rho, z)$ or $\boldsymbol{r}(\phi, z)$.

Example 1: Flux

The surface is defined as $z = f(x, y) = xy$, and the vector field is $\boldsymbol{F} = x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}}$. Find the flux into the surface over the region $0 < x < 1$ and $0 < y < 1$.

So, we are here to find $\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$.

Recalling the key step, we should express the vector to the surface $\boldsymbol{r}$ in terms of only two variables out of x, y, and z. As the surface is given by explicitly expressing z as the function of x and y, we can choose x and y to express $\boldsymbol{r}$:

$$\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \equiv x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + xy\hat{\mathbf{k}}.$$

Therefore, to find the surface area element $\mathrm{d}\boldsymbol{S}$, we find the partial derivatives of $\boldsymbol{r}$ with x and y:

$$\frac{\partial \boldsymbol{r}}{\partial x} = \hat{\mathbf{i}} + y\hat{\mathbf{k}}$$
$$\frac{\partial \boldsymbol{r}}{\partial y} = \hat{\mathbf{j}} + x\hat{\mathbf{k}}.$$

So,

$$\boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = -y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}},$$

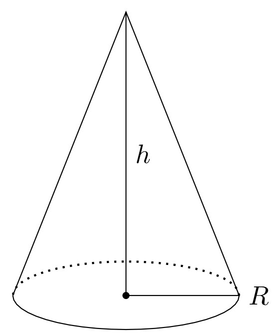
$$\mathrm{d}\boldsymbol{S} = \boldsymbol{N}\mathrm{d}x\mathrm{d}y = (-y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}})\mathrm{d}x\mathrm{d}y.$$

After checking the sign, we know that the flux into the surface is actually given by $-\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = (x^2y + y^2x)\mathrm{d}x\mathrm{d}y$, and finally,

$$I = \iint_S (-\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}) = \int_0^1 \mathrm{d}y \int_0^1 (x^2y + y^2x)\mathrm{d}x = \frac{1}{3}.$$

Example 2: Surface Area of Cone

Consider the same cone as before. We are here to calculate the sloping part of the surface area through the use of the area element.



We know that the surface of the cone may be given by the function

$$\rho = R\left(1 - \frac{z}{h}\right),$$

and the general position vector in cylindrical coordinates is

$$\boldsymbol{r} = \rho\hat{\rho}(\phi) + z\hat{\mathbf{k}},$$

so we may express the vector to the surface as:

$$\boldsymbol{r}(\phi, z) = R\left(1 - \frac{z}{h}\right)\hat{\rho}(\phi) + z\hat{\mathbf{k}}.$$

The surface area element asks for two partial derivatives:

$$\frac{\partial \boldsymbol{r}}{\partial \phi} = R\left(1 - \frac{z}{h}\right)\hat{\phi}$$
$$\frac{\partial \boldsymbol{r}}{\partial z} = -\frac{R}{h}\hat{\rho} + \hat{\mathbf{k}}.$$

$$\boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial \phi} \times \frac{\partial \boldsymbol{r}}{\partial z} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{\mathbf{k}} \\ 0 & R\left(1 - \frac{z}{h}\right) & 0 \\ -\frac{R}{h} & 0 & 1 \end{vmatrix} = R\left(1 - \frac{z}{h}\right)\hat{\rho} + \frac{R^2}{h}\left(1 - \frac{z}{h}\right)\hat{\mathbf{k}}.$$

And thus,

$$|\mathrm{d}\boldsymbol{S}| = |\boldsymbol{N}|\mathrm{d}\phi\mathrm{d}z = R\left(1 - \frac{z}{h}\right)\sqrt{1 + \frac{R^2}{h^2}}\mathrm{d}\phi\mathrm{d}z,$$

$$\begin{aligned} S &= \iint_S |\mathrm{d}\boldsymbol{S}| \\ &= \int_0^{2\pi} \mathrm{d}\phi \int_0^h R\left(1 - \frac{z}{h}\right)\sqrt{1 + \frac{R^2}{h^2}}\mathrm{d}z \\ &= R\sqrt{1 + \frac{R^2}{h^2}} \int_0^{2\pi} \mathrm{d}\phi \int_0^h \left(1 - \frac{z}{h}\right)\mathrm{d}z \\ &= R\sqrt{1 + \frac{R^2}{h^2}} \cdot 2\pi \cdot \frac{h}{2} \\ &= \pi R\sqrt{R^2 + h^2}. \end{aligned}$$

Alternatively, we can choose to express z in terms of ρ and ϕ and still get the same result.

Chapter 4

Line Integrals

4.1 Definition

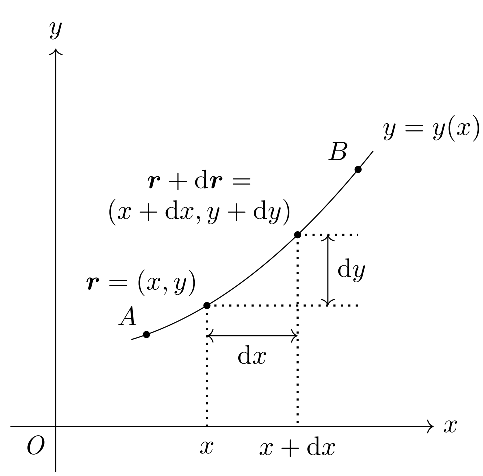
Line integrals are integrals that go along a path, a curve c, in the space. There are different forms of line integrals, like

$$\int_c \Omega \mathrm{d}\boldsymbol{r}, \quad \int_c \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}, \quad \text{and} \int_c \boldsymbol{F} \times \mathrm{d}\boldsymbol{r},$$

where Ω is a scalar and $\boldsymbol{F}$ is a vector. The second integral $\displaystyle\int_c \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ is of particular interest, as this can be interpreted as the work done along a path.

4.2 Methods to Calculate

Say we have a curve $y = y(x)$, or equivalently $x = x(y)$, and a vector field $\boldsymbol{F}(x, y) = F_x(x, y)\hat{\mathbf{i}} + F_y(x, y)\hat{\mathbf{j}}$.



Clearly, we can see from the graph that an infinitesimal increment in the position vector $\boldsymbol{r}$ is given by $\mathrm{d}\boldsymbol{r} = \mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}}$.

Therefore, the integral we are to calculate can be written as:

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = (F_x(x,y)\hat{\mathbf{i}} + F_y(x,y)\hat{\mathbf{j}}) \cdot (\mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}}) = F_x(x,y)\mathrm{d}x + F_y(x,y)\mathrm{d}y.$$

That is to say,

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{x_A}^{x_B} F_x \mathrm{d}x + \int_{y_A}^{y_B} F_y \mathrm{d}y.$$

Note that using the curve we can write each term entirely using only x or y. This is also true in 3D. For example,

$$\begin{aligned}\int_{x_A}^{x_B} F_x \mathrm{d}x &= \int_{x_A}^{x_B} F_x(x, y(x))\mathrm{d}x = \int_{y_A}^{y_B} F_x(x(y), y)\frac{\mathrm{d}x}{\mathrm{d}y}\mathrm{d}y \\ \int_{y_A}^{y_B} F_y \mathrm{d}y &= \int_{y_A}^{y_B} F_y(x(y), y)\mathrm{d}y = \int_{x_A}^{x_B} F_y(x, y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x.\end{aligned}$$

4.3 Four Examples

4.3.1 General Method

Firstly, express each integral with one parameter using the curve. That is to say, express the dot product as several common integrals.

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = (F_x(x,y)\hat{\mathbf{i}} + F_y(x,y)\hat{\mathbf{j}}) \cdot (\mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}}) = F_x(x,y)\mathrm{d}x + F_y(x,y)\mathrm{d}y.$$

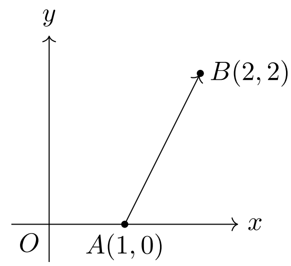
Then, set the limits for that parameter, finding out the x_A, x_B, y_A, and y_B of the curve.

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{x_A}^{x_B} F_x \mathrm{d}x + \int_{y_A}^{y_B} F_y \mathrm{d}y.$$

4.3.2 Example 1: Ordinary Explicit Paths

Consider the vector field $\boldsymbol{F} = 2xy\hat{\mathbf{i}} + x^2\hat{\mathbf{j}}$. The path is the straight line connecting the points $A(1,0)$ to $B(2,2)$. Easily, we can derive the equation of the path:

$$y = 2x - 2 \Leftrightarrow x = \frac{y}{2} + 1.$$



Then, we can write

$$\begin{aligned}\int_A^B \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} &= \int_A^B F_x \mathrm{d}x + \int_A^B F_y \mathrm{d}y \\ &= \int_1^2 2xy(x)\mathrm{d}x + \int_0^2 x^2(y)\mathrm{d}y \\ &= \int_1^2 2x(2x-2)\mathrm{d}x + \int_0^2 \left(\frac{y}{2}+1\right)^2 \mathrm{d}y \\ &= \frac{10}{3} + \frac{14}{3} \\ &= 8.\end{aligned}$$

Alternatively, we can do the second integral in a different way:

$$\begin{aligned}\int_0^2 x^2(y)\mathrm{d}y &= \int_1^2 x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x \\ &= \int_1^2 2x^2\mathrm{d}x \\ &= \frac{14}{3}.\end{aligned}$$

4.3.3 Example 2: Parametric Paths

Sometimes, the path may not be expressed explicitly. It may be given parametrically (where we cannot write $y = f(x)$). For example,

$$\begin{cases} x(t) = 2t+1 \\ y(t) = 4t \end{cases} \qquad \text{from } t=0 \text{ to } t=\frac{1}{2}.$$

This is actually the line $y = 2x - 2$, from A to B, as before! However, in other cases, we might not be able to recognize the parametric curves as explicit functions. This is just a simple example to convey a line of thought.

In order to do parametric paths, we can write, generally,

$$\begin{aligned}
\int_A^B \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} &= \int_A^B F_x(x,y)\mathrm{d}x + \int_A^B F_y(x,y)\mathrm{d}y \\
&= \int_A^B F_x(x(t),y(t))\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t + \int_A^B F_y(x(t),y(t))\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t.
\end{aligned}$$

In this specific example, we can get:

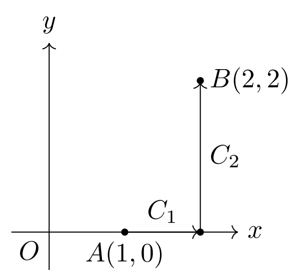
$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 2 \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 4.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_A^B \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} &= \int_A^B 2xy\mathrm{d}x + \int_A^B x^2\mathrm{d}y \\
&= \int_A^B 2xy\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t + \int_A^B x^2\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t \\
&= \int_0^{1/2} 2(2t+1)4t2\mathrm{d}t + \int_0^{1/2} (2t+1)^2 4\mathrm{d}t \\
&= \frac{10}{3} + \frac{14}{3} \\
&= 8.
\end{aligned}$$

4.3.4 Example 3: A Different Path

Consider the same endpoints $A(1,0)$ and $B(2,2)$, but with a different path:



So, the path is firstly C_1, and then C_2.

$$\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_{C_1} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} + \int_{C_2} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$$

Consider each integral separately.

$$\int_{C_1} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_{C_1} 2xy\mathrm{d}x + \int_{C_1} x^2\mathrm{d}y$$

For integral along C_1, y is kept constant at $y = 0$ in the whole path, so we can see $y = 0$ and $\mathrm{d}y = 0$. This gives

$$\begin{aligned}\int_{C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &= \int_{C_1} 2xy\mathrm{d}x + \int_{C_1} x^2\mathrm{d}y \\ &= \int_{C_1} 2x \cdot 0\mathrm{d}x + \int_{C_1} x^2 \cdot 0 \\ &= 0.\end{aligned}$$

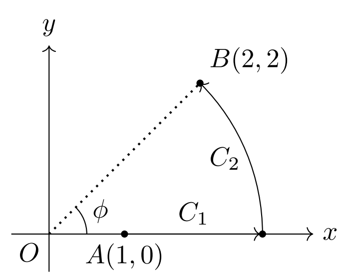
For integral along path C_2, x is kept constant at $x = 2$ in the whole path, so we can write $x = 2$ and $\mathrm{d}x = 0$:

$$\begin{aligned}\int_{C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &= \int_{C_2} 2xy\mathrm{d}x + \int_{C_2} x^2\mathrm{d}y \\ &= \int_{C_2} 4y \cdot 0 + \int_{C_2} 4\mathrm{d}y \\ &= \int_0^2 4\mathrm{d}y \\ &= 8.\end{aligned}$$

Therefore, the whole line integral gives $I = I_1 + I_2 = 0 + 0 + 0 + 8 = 8$.

4.3.5 Example 4: Plane Polar Coordinates with Different Paths

For convenience, we choose the same endpoints $A(1, 0)$ and $B(2, 2)$, and the paths are divided into angular and radial parts:



The first path is done with $\phi = 0$ and $1 \le \rho \le 2\sqrt{2}$, while the second is done with $\rho = 2\sqrt{2}$ and $0 \le \phi \le \dfrac{\pi}{4}$.

The example has become a little awkward, as the vector field is defined very concisely in Cartesian coordinates. In plane polar coordinates,

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \end{cases} \quad \Rightarrow \begin{cases} \mathrm{d}x = \dfrac{\partial x}{\partial \rho}\mathrm{d}\rho + \dfrac{\partial x}{\partial \phi}\mathrm{d}\phi = \cos\phi\mathrm{d}\rho - \rho\sin\phi\mathrm{d}\phi \\ \mathrm{d}y = \dfrac{\partial y}{\partial \rho}\mathrm{d}\rho + \dfrac{\partial y}{\partial \phi}\mathrm{d}\phi = \sin\phi\mathrm{d}\rho + \rho\cos\phi\mathrm{d}\phi \end{cases}$$

And after some algebraic manipulation, we can get

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_C 3\cos^2\phi \sin\phi \rho^2 \mathrm{d}\rho + \int_C \cos\phi (1 - 3\sin^2\phi)\rho^3 \mathrm{d}\phi,$$

and this integral finally is evaluated to be 8.

From all the above examples, we can spot that this integral of $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ is actually independent of the path chosen, as all the paths with the same endpoints yield the same result.

4.4 Conditions for Path Independence

The vector field in the previous examples is $\boldsymbol{F} = 2xy\hat{\mathbf{i}} + x^2\hat{\mathbf{j}}$, and the dot product is

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 2xy\mathrm{d}x + x^2\mathrm{d}y.$$

This integral, with the same endpoints, has the same value for any path. However, for the vector field $\boldsymbol{F} = 2xy\hat{\mathbf{i}} - x^2\hat{\mathbf{j}}$, the integral is now dependent on the path chosen.

Actually, the first dot product, $2xy\mathrm{d}x + x^2\mathrm{d}y$, is an exact differential; the mixed partial derivatives prove this fact:

$$\frac{\partial}{\partial y}(2xy) = 2x, \quad \frac{\partial}{\partial x}\left(x^2\right) = 2x.$$

This means that the dot product is in fact the differential of a parent function $\Omega(x, y)$:

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_A^B \mathrm{d}\Omega(x, y) = \Omega(x_B, y_B) - \Omega(x_A, y_A).$$

In this case, the parent function is $\Omega(x, y) = x^2 y + C$. To check,

$$\mathrm{d}\Omega = \frac{\partial\Omega}{\partial x}\mathrm{d}x + \frac{\partial\Omega}{\partial y}\mathrm{d}y = 2xy\mathrm{d}x + x^2\mathrm{d}y$$

$$\Omega(x_B, y_B) - \Omega(x_A, y_A) = \Omega(2, 2) - \Omega(1, 0) = 8.$$

With the dot product being an exact differential, we are actually calculating the "change in height" on a contour map, or simply the difference of values on the endpoints. However, if the dot product is not an exact differential, there is no parent function, and the line integral may be dependent on the path chosen.

Now consider the situation of an exact differential. With the same endpoints and different path, the integral has the same value. That is to say, if we go around a closed loop, the integral would always be zero:

$$\int_{C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

$$\int_{C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} + \int_{-C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \equiv \oint_{C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0.$$

The integral of an exact differential around a closed path is 0. Physically, we call the vector field $\boldsymbol{F}$ a conservative field, as no work is done around a loop.

To summarize, the following five statements are equivalent (if one is true, then all are true):

$$\begin{cases} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \text{ is an exact differential;} \\ \boldsymbol{F} = \dfrac{\partial \Omega}{\partial x}\hat{\mathbf{i}} + \dfrac{\partial \Omega}{\partial y}\hat{\mathbf{j}} + \dfrac{\partial \Omega}{\partial z}\hat{\mathbf{k}} \text{ for some function } \Omega \\ \displaystyle\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \text{ from } A \text{ to } B \text{ does not depend on path;} \\ \displaystyle\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0 \\ \text{The vector field } \boldsymbol{F} \text{ is conservative.} \end{cases}$$

Chapter 5

Gradient

5.1 Definition

For a scalar field $\Omega(x, y, z)$, the vector field $\boldsymbol{F} = \frac{\partial \Omega}{\partial x}\hat{\mathbf{i}} + \frac{\partial \Omega}{\partial y}\hat{\mathbf{j}} + \frac{\partial \Omega}{\partial z}\hat{\mathbf{k}} = \text{grad}(\Omega) = \boldsymbol{\nabla}\Omega$ is conservative, i.e. $\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$.

The symbol $\boldsymbol{\nabla}$, called "del" or "nabla", is an operator:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}. \tag{5.1}$$

Therefore, operating on a scalar field, it produces a vector field. Here, Ω (often ϕ in physics) is the potential associated with the vector field $\boldsymbol{F}$.

5.2 Directional Derivative

Using $\boldsymbol{\nabla}\Omega$, we can calculate the derivative of Ω in any direction. First of all, we can see that

$$\mathrm{d}\Omega = \frac{\partial \Omega}{\partial x}\mathrm{d}x + \frac{\partial \Omega}{\partial y}\mathrm{d}y + \frac{\partial \Omega}{\partial z}\mathrm{d}z = \boldsymbol{\nabla}\Omega \cdot \mathrm{d}\boldsymbol{r}. \tag{5.2}$$

So, how can we get the derivative in a particular direction, say $\hat{\mathbf{a}}$ (a unit vector)? Moving a distance $\mathrm{d}s$ in the direction, we get the total differential of Ω:

$$\mathrm{d}\Omega = \boldsymbol{\nabla}\Omega \cdot \hat{\mathbf{a}}\mathrm{d}s.$$

Therefore, the rate of change of Ω in that particular direction $\hat{\mathbf{a}}$ is:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}s} = \boldsymbol{\nabla}\Omega \cdot \hat{\mathbf{a}} = \text{directional derivative.} \tag{5.3}$$

For example, we can choose $\hat{\mathbf{a}} = \hat{\mathbf{k}}$, and what we get is

$$\frac{\mathrm{d}\Omega}{\mathrm{d}s} = \left(\frac{\partial\Omega}{\partial x}\hat{\mathbf{i}} + \frac{\partial\Omega}{\partial y}\hat{\mathbf{j}} + \frac{\partial\Omega}{\partial z}\hat{\mathbf{k}}\right) \cdot \hat{\mathbf{k}} = \frac{\partial\Omega}{\partial z}$$

as expected.

With the directional derivative, we can naturally come up with a second question: at a particular point (x, y, z), what direction $\hat{\mathbf{a}}_{\text{max}}$ would maximize $\dfrac{\mathrm{d}\Omega}{\mathrm{d}s}$?

We can return to the expression of the directional derivative:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}s} = \boldsymbol{\nabla}\Omega \cdot \hat{\mathbf{a}}.$$

Evidently, when $\boldsymbol{\nabla}\Omega$ and $\hat{\mathbf{a}}$ are aligned to be parallel, the dot product would have the greatest value. That is to say,

$$\hat{\mathbf{a}}_{\text{max}} = \frac{\boldsymbol{\nabla}\Omega}{|\boldsymbol{\nabla}\Omega|},$$

and this gives:

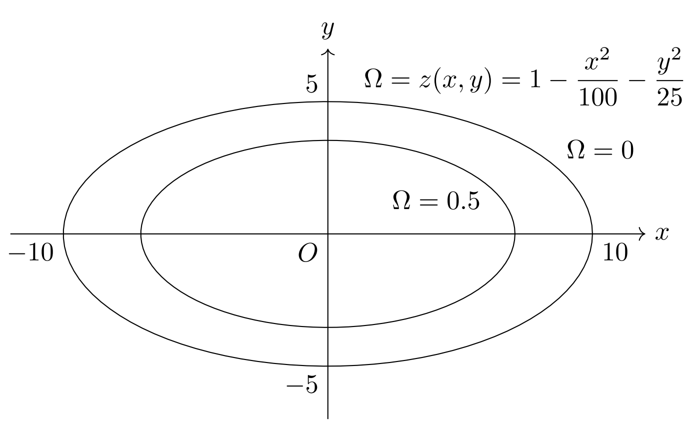
$$\max\left(\frac{\mathrm{d}\Omega}{\mathrm{d}s}\right) = \boldsymbol{\nabla}\Omega \cdot \hat{\mathbf{a}}_{\text{max}} = \boldsymbol{\nabla}\Omega \cdot \frac{\boldsymbol{\nabla}\Omega}{|\boldsymbol{\nabla}\Omega|} = |\boldsymbol{\nabla}\Omega|.$$

Also, by noting that $\hat{\mathbf{a}}_{\text{max}}$ parallel to $\boldsymbol{\nabla}\Omega$ gives the maximum directional derivative, $\boldsymbol{\nabla}\Omega$ itself marks the maximum direction of increase for the scalar field Ω at point (x, y, z).

Now consider an example:

$$\Omega = z(x, y) = 1 - \frac{x^2}{100} - \frac{y^2}{25},$$

which is actually an elliptical hill.



(1) Consider the direction of maximum increase at the point $(3, 2)$.

From the above discussion, we know that the direction of maximum increase is exactly the gradient of the scalar function Ω:

$$\boldsymbol{\nabla} z = \frac{\partial z}{\partial x}\hat{\mathbf{i}} + \frac{\partial z}{\partial y}\hat{\mathbf{j}} = -\frac{x}{50}\hat{\mathbf{i}} - \frac{2y}{25}\hat{\mathbf{j}}.$$

Plugging in $(x, y) = (3, 2)$, we get:

$$\boldsymbol{\nabla} z(3,2) = -\frac{3}{50}\hat{\mathbf{i}} - \frac{4}{25}\hat{\mathbf{j}} = -0.06\hat{\mathbf{i}} - 0.16\hat{\mathbf{j}}.$$

So, the magnitude of the maximum increase is:

$$|\boldsymbol{\nabla} z| = \sqrt{0.06^2 + 0.16^2} = 0.17,$$

and the direction (unit vector) of maximum increase is:

$$\hat{\mathbf{a}}_{\max} = \frac{\boldsymbol{\nabla} z}{|\boldsymbol{\nabla} z|} = -\frac{0.06}{0.17}\hat{\mathbf{i}} - \frac{0.16}{0.17}\hat{\mathbf{j}}.$$

(2) At point $(3, 2)$, consider the derivative in the direction of $\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

The direction unit vector is:

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{i}}}{\sqrt{2}} + \frac{\hat{\mathbf{j}}}{\sqrt{2}}.$$

So, the directional derivative is:

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \boldsymbol{\nabla} z \cdot \hat{\mathbf{a}} = (-0.06\hat{\mathbf{i}} - 0.16\hat{\mathbf{j}}) \cdot \left(\frac{\hat{\mathbf{i}}}{\sqrt{2}} + \frac{\hat{\mathbf{j}}}{\sqrt{2}}\right) = -0.155.$$

The negative value of the directional derivative indicates that the direction $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ is down the hill.

5.3 Gradient in Other Coordinate Systems

Generally, we know the total differential of a scalar function Ω can be expressed as follows:

$$\mathrm{d}\Omega = \boldsymbol{\nabla}\Omega \cdot \mathrm{d}\boldsymbol{r}. \tag{5.4}$$

This expression helps us to quickly figure out (but it is not a formal proof) the expression of gradient in other coordinates.

5.3.1 Cylindrical Polar Coordinates

The total differential is

$$\mathrm{d}\Omega = \frac{\partial \Omega}{\partial \rho}\mathrm{d}\rho + \frac{\partial \Omega}{\partial \phi}\mathrm{d}\phi + \frac{\partial \Omega}{\partial z}\mathrm{d}z,$$

and the infinitesimal increment of position vector is

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}\rho\hat{\rho} + \rho\mathrm{d}\phi\hat{\phi} + \mathrm{d}z\hat{\mathbf{k}}.$$

Therefore, by equating each component in their respective direction, we get:

$$\boldsymbol{\nabla}\Omega(\rho, \phi, z) = \frac{\partial \Omega}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial \Omega}{\partial \phi}\hat{\phi} + \frac{\partial \Omega}{\partial z}\hat{\mathbf{k}}.$$

That is to say, the gradient operator in cylindrical coordinates is:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{\mathbf{k}}.$$

5.3.2 Spherical Polar Coordinates

Similarly, we can lay out the expressions for $\mathrm{d}\Omega$ and $\mathrm{d}\boldsymbol{r}$:

$$\mathrm{d}\Omega = \frac{\partial \Omega}{\partial r}\mathrm{d}r + \frac{\partial \Omega}{\partial \theta}\mathrm{d}\theta + \frac{\partial \Omega}{\partial \phi}\mathrm{d}\phi$$
$$\mathrm{d}\boldsymbol{r} = \mathrm{d}r\hat{\mathbf{r}} + r\mathrm{d}\theta\hat{\theta} + r\sin\theta\hat{\phi}.$$

Then, we can easily derive the expression of gradient in spherical coordinates:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}.$$

5.4 Gradient and the Normal to a Surface

Consider the gradient on a contour line (2D situation). As the gradient marks out the direction of maximum increase, it should be perpendicular to the contour line. By referring to the expression of directional derivative,

$$\frac{\mathrm{d}\Omega}{\mathrm{d}s} = \boldsymbol{\nabla}\Omega \cdot \hat{\mathbf{a}},$$

we know that along a contour, the directional derivative is zero. We can also think in a reverse direction. As the value of the function is kept constant on the contour line, the directional derivative along the contour line is zero.

In conclusion, the gradient is perpendicular to the contour line, and the directional derivative along any contour line is zero.

In a similar fashion, in 3D, the gradient $\boldsymbol{\nabla}\Omega$ is perpendicular to the surface of constant Ω.

A quick example to show the idea is the surface $\Omega = x^2 + y^2 + z^2$. The surfaces of constant Ω are spheres centered at the origin. The gradient is

$$\boldsymbol{\nabla}\Omega = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}} \equiv 2r\hat{\mathbf{r}},$$

which is obviously perpendicular to the spheres.

Now, consider a 3D surface defined by the equation $z = f(x, y)$. Suppose we are to find the direction normal to a point on the surface.

The key step here is to define a new function $\Omega(x, y, z) = f(x, y) - z$. Then, the equation $z = f(x, y)$ is now actually the surface of constant $\Omega = 0$.

Therefore, a vector normal to the surface at any point (x, y, z) is

$$\boldsymbol{n} = \boldsymbol{\nabla}\Omega(x, y, z),$$

and, trivially, the corresponding unit vector is given by

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\nabla}\Omega}{|\boldsymbol{\nabla}\Omega|}.$$

It is important to note that the gradient is yet another way to find out the area element $\mathrm{d}\boldsymbol{S}$ perpendicular to a surface.

Recall that we find out the area element $\mathrm{d}\boldsymbol{S}$ by defining the position vector $\boldsymbol{r}$. For a surface with equation $z = f(x, y)$, we write

$$\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + f(x, y)\hat{\mathbf{k}}.$$

At point (x, y, z), according to the definition of partial derivative, the infinitesimal increments in x and y directions are tangent to the slice of constant y and x. So, the cross product of these two are perpendicular to the surface.

$$\mathrm{d}\boldsymbol{S} = \boldsymbol{N}\mathrm{d}x\mathrm{d}y$$

$$\boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\hat{\mathbf{i}} - \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

Now consider the same function $\Omega = f(x, y) - z$. Clearly, we can recognize the vector $\boldsymbol{N}$ as

$$\boldsymbol{N} = -\boldsymbol{\nabla}\Omega.$$

As there is always an ambiguity about the direction of the vector $\boldsymbol{N}$, which may be determined specifically by the question, we can determine the vector $\boldsymbol{N}$ simply by calculating the gradient of the function $\Omega = f(x, y) - z$.

Chapter 6

Divergence

6.1 Definition

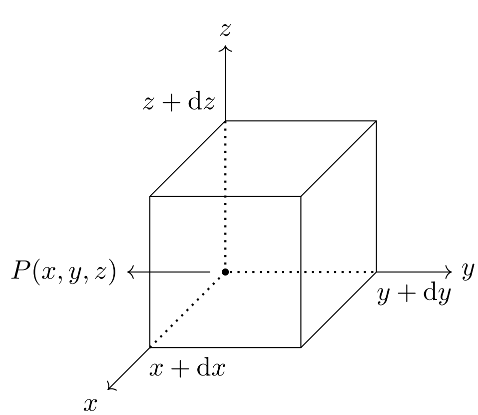
The divergence of a vector field $\boldsymbol{B}$ at a point is defined as the limit of the ratio of the surface integral of $\boldsymbol{B}$ out of the closed surface of a volume V enclosing the point to the volume of V, as V shrinks to zero:

$$\operatorname{div}(\boldsymbol{B}) = \boldsymbol{\nabla} \cdot \boldsymbol{B} = \lim_{V \to 0} \left(\frac{1}{V} \oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \right). \tag{6.1}$$

This integral is done over a closed surface, as marked by the closed circle in the double integral symbol, and the surface area element $\mathrm{d}\boldsymbol{S}$ is pointing outwards. From the definition of the divergence, we can see that it is a "volume density of flux", or simply "flux density".

6.2 Derivation

Consider, in Cartesian coordinates, the vector field $\boldsymbol{B}(x, y, z) = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$.



The volume element here is of widths $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$. There are a total of six faces, and we can number them through as follows.

Face Number	Position	$\mathrm{d}\boldsymbol{S}$
1	$\Delta x = 0$	$-\hat{\mathbf{i}}\mathrm{d}y\mathrm{d}z$
2	$\Delta x = \mathrm{d}x$	$\hat{\mathbf{i}}\mathrm{d}y\mathrm{d}z$
3	$\Delta y = 0$	$-\hat{\mathbf{j}}\mathrm{d}x\mathrm{d}z$
4	$\Delta y = \mathrm{d}y$	$\hat{\mathbf{j}}\mathrm{d}x\mathrm{d}z$
5	$\Delta z = 0$	$-\hat{\mathbf{k}}\mathrm{d}x\mathrm{d}y$
6	$\Delta z = \mathrm{d}z$	$\hat{\mathbf{k}}\mathrm{d}x\mathrm{d}y$

We want $\sum_{i=1}^{6} F_i$ where F_i is flux out over the face i. To compute F_i, we will use the tangent-plane approximation to $\boldsymbol{B}$ over each face. Then, F_i would be arbitrarily accurate as $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ goes to 0.

Now consider face 1, which is the face on the yz plane. Firstly, consider the dot product. As $\mathrm{d}\boldsymbol{S}$ only has a component in $\hat{\mathbf{i}}$ direction, the dot product only picks up the x component of the vector field $\boldsymbol{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$, or B_x.

B_x is generally a function of three variables, x, y, and z. So, the total differential of B_x should be:

$$\mathrm{d}B_x = \frac{\partial B_x}{\partial x}\mathrm{d}x + \frac{\partial B_x}{\partial y}\mathrm{d}y + \frac{\partial B_x}{\partial z}\mathrm{d}z.$$

And as x is kept constant in this plane, we know that $\mathrm{d}x = 0$. That is to say,

$$\begin{aligned} B_x(x, y, z) &= B_x \\ B_x(x, y + \mathrm{d}y, z + \mathrm{d}z) &= B_x + \mathrm{d}B_x \\ &= B_x + \frac{\partial B_x}{\partial y}\mathrm{d}y + \frac{\partial B_x}{\partial z}\mathrm{d}z. \end{aligned}$$

As $\mathrm{d}y$ and $\mathrm{d}z$ are small enough for us to assume that B_x increases linearly across the plane, the average value is at the centre of the plane:

$$\bar{B}_x = B_x + \frac{\partial B_x}{\partial y}\frac{\mathrm{d}y}{2} + \frac{\partial B_x}{\partial z}\frac{\mathrm{d}z}{2}.$$

The average value on face 2 can be figured out in the same way. For convenience, we can spot that the average value is at $y = y + \dfrac{\mathrm{d}y}{2}$ and $z = z + \dfrac{\mathrm{d}z}{2}$. To find out the average value on face 2, we say that we are moving the average value from $x = x$ to $x = x + \mathrm{d}x$ without changing the y and z positions:

$$\begin{aligned}
\mathrm{d}\bar{B}_x &= \frac{\partial \bar{B}_x}{\partial x}\mathrm{d}x + \frac{\partial \bar{B}_x}{\partial y}\mathrm{d}y + \frac{\partial \bar{B}_x}{\partial z}\mathrm{d}z \\
&= \frac{\partial \bar{B}_x}{\partial x}\mathrm{d}x.
\end{aligned}$$

Therefore, the total flux across the faces 1 and 2 is:

$$\begin{aligned}
F_{12} &= F_1 + F_2 \\
&= \iint_{S_1} \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} + \iint_{S_2} \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_{S_1} (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})\cdot(-\hat{\mathbf{i}}\mathrm{d}y\mathrm{d}z) + \iint_{S_2} (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}\mathrm{d}y\mathrm{d}z) \\
&= \iint_{S_1} -B_x\mathrm{d}y\mathrm{d}z + \iint_{S_2} B_x\mathrm{d}y\mathrm{d}z \\
&= -\bar{B}_x\mathrm{d}y\mathrm{d}z + \left(\bar{B}_x + \frac{\partial \bar{B}_x}{\partial x}\mathrm{d}x\right)\mathrm{d}y\mathrm{d}z \\
&= \frac{\partial \bar{B}_x}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{\partial}{\partial x}\left(B_x + \frac{\partial B_x}{\partial y}\frac{\mathrm{d}y}{2} + \frac{\partial B_x}{\partial z}\frac{\mathrm{d}z}{2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned}$$

Then, we can figure out the fluxes for the other faces, and the total flux is:

$$\begin{aligned}
F &= F_{12} + F_{34} + F_{56} \\
&= \left[\frac{\partial}{\partial x}\left(B_x + \frac{\partial B_x}{\partial y}\frac{\mathrm{d}y}{2} + \frac{\partial B_x}{\partial z}\frac{\mathrm{d}z}{2}\right) + \frac{\partial}{\partial y}\left(B_y + \frac{\partial B_y}{\partial x}\frac{\mathrm{d}x}{2} + \frac{\partial B_y}{\partial z}\frac{\mathrm{d}z}{2}\right)\right. \\
&\qquad \left. + \frac{\partial}{\partial z}\left(B_z + \frac{\partial B_z}{\partial x}\frac{\mathrm{d}x}{2} + \frac{\partial B_z}{\partial y}\frac{\mathrm{d}y}{2}\right)\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z
\end{aligned}$$

Therefore, we can calculate the divergence through the definition:

$$\begin{aligned}
\operatorname{div}(\boldsymbol{B}) &= \lim_{V\to 0}\left(\frac{1}{V}\oiint_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}\right) \\
&= \lim_{\mathrm{d}x,\mathrm{d}y,\mathrm{d}z\to 0}\left(\frac{F}{\mathrm{d}x\mathrm{d}y\mathrm{d}z}\right) \\
&= \lim_{\mathrm{d}x,\mathrm{d}y,\mathrm{d}z\to 0}\left[\frac{\partial}{\partial x}\left(B_x + \frac{\partial B_x}{\partial y}\frac{\mathrm{d}y}{2} + \frac{\partial B_x}{\partial z}\frac{\mathrm{d}z}{2}\right) + \frac{\partial}{\partial y}\left(B_y + \frac{\partial B_y}{\partial x}\frac{\mathrm{d}x}{2} + \frac{\partial B_y}{\partial z}\frac{\mathrm{d}z}{2}\right)\right. \\
&\qquad \left. + \frac{\partial}{\partial z}\left(B_z + \frac{\partial B_z}{\partial x}\frac{\mathrm{d}x}{2} + \frac{\partial B_z}{\partial y}\frac{\mathrm{d}y}{2}\right)\right].
\end{aligned}$$

With the limit approaching 0, all the terms involving $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ vanish, and what we are left is

$$\begin{aligned}\operatorname{div}(\boldsymbol{B}) &= \lim_{\mathrm{d}x,\mathrm{d}y,\mathrm{d}z\to 0}\left[\frac{\partial}{\partial x}\left(B_x + \frac{\partial B_x}{\partial y}\frac{\mathrm{d}y}{2} + \frac{\partial B_x}{\partial z}\frac{\mathrm{d}z}{2}\right) + \frac{\partial}{\partial y}\left(B_y + \frac{\partial B_y}{\partial x}\frac{\mathrm{d}x}{2} + \frac{\partial B_y}{\partial z}\frac{\mathrm{d}z}{2}\right)\right.\\ &\qquad \left. + \frac{\partial}{\partial z}\left(B_z + \frac{\partial B_z}{\partial x}\frac{\mathrm{d}x}{2} + \frac{\partial B_z}{\partial y}\frac{\mathrm{d}y}{2}\right)\right]\\ &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\\ &= \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)\cdot(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})\\ &\equiv \boldsymbol{\nabla}\cdot\boldsymbol{B}.\end{aligned}$$

The above derivation is just for information; the only essential takeaway is:

$$\operatorname{div}(\boldsymbol{B}) = \boldsymbol{\nabla}\cdot\boldsymbol{B} = \sum_{i=1}^{N}\frac{\partial B_{x_i}}{\partial x_i}. \tag{6.2}$$

6.3 Divergence in Other Coordinate Systems

It is important to note that divergence in other coordinates are not so simple as merely adding the partial derivatives together. In this sense, using $\boldsymbol{\nabla}\cdot\boldsymbol{B}$ to denote the divergence is like an abuse of the dot product. However, we can still approach the divergence in other coordinates by "differentiate first, dot product second". This way of thinking is also useful in the calculation of curl, where we "differentiate first, cross product second".

6.3.1 Cylindrical Polar Coordinates

We can still derive the expression of divergence through the definition, and calculating the flux through the respective six faces. Nevertheless, that process can be exhausting, though it gives physical intuition of divergence as the "flux density". We apply the method of "differentiate first, dot product second" here.

Firstly, we can derive the del operator in cylindrical coordinates:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{\mathbf{k}}.$$

Then, we know that the divergence is the dot product of the del operator and the vector field $\boldsymbol{B} = B_\rho\hat{\rho} + B_\phi\hat{\phi} + B_z\hat{\mathbf{k}}$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = \left(\frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)\cdot(B_\rho\hat{\rho} + B_\phi\hat{\phi} + B_z\hat{\mathbf{k}}).$$

It is important to point out that we cannot simply multiply and add the terms with the same unit vectors.

This is because the unit vectors $\hat{\rho}$ and $\hat{\phi}$ show dependence on the phase angle ϕ. To put it clearly, we may get help from Cartesian coordinates:

$$\begin{aligned}\hat{\rho}(\phi) &= \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}\\ \hat{\phi}(\phi) &= -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}.\end{aligned}$$

This dependence shows that the partial derivative operator, $\frac{\partial}{\partial\phi}$, may have a different effect from what we expect. In fact,

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial\phi} &= \hat{\phi}\\ \frac{\partial\hat{\phi}}{\partial\phi} &= -\hat{\rho}.\end{aligned}$$

We can consider the term involving $\frac{\partial}{\partial\phi}$ separately in the dot product:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi}\right)\cdot(B_\rho\hat{\rho} + B_\phi\hat{\phi} + B_z\hat{\mathbf{k}}) = \frac{1}{\rho}\hat{\phi}\cdot\left[\frac{\partial}{\partial\phi}\Big(B_\rho\hat{\rho}(\phi) + B_\phi\hat{\phi}(\phi) + B_z\hat{\mathbf{k}}\Big)\right].$$

The unit vector $\hat{\mathbf{k}}$ is constant all the time, so the dot product with the term $B_z\hat{\mathbf{k}}$ is definitely 0. Therefore, with the product rule,

$$\begin{aligned}&\frac{1}{\rho}\hat{\phi}\cdot\left[\frac{\partial}{\partial\phi}\Big(B_\rho\hat{\rho}(\phi) + B_\phi\hat{\phi}(\phi) + B_z\hat{\mathbf{k}}\Big)\right]\\ &= \frac{1}{\rho}\hat{\phi}\cdot\left[\frac{\partial}{\partial\phi}\Big(B_\rho\hat{\rho}(\phi) + B_\phi\hat{\phi}(\phi)\Big)\right]\\ &= \frac{1}{\rho}\hat{\phi}\cdot\left(\frac{\partial B_\rho}{\partial\phi}\hat{\rho} + B_\rho\frac{\partial\hat{\rho}}{\partial\phi} + \frac{\partial B_\phi}{\partial\phi}\hat{\phi} + B_\phi\frac{\partial\hat{\phi}}{\partial\phi}\right)\\ &= \frac{1}{\rho}\hat{\phi}\cdot\left(\frac{\partial B_\rho}{\partial\phi}\hat{\rho} + B_\rho\hat{\phi} + \frac{\partial B_\phi}{\partial\phi}\hat{\phi} - B_\phi\hat{\rho}\right)\\ &= \frac{B_\rho}{\rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi}.\end{aligned}$$

As the unit vectors are not dependent on other coordinates, we can do the dot product by simply matching the same unit vectors. So, the divergence of vector field $\boldsymbol{B}$ is:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{B} &= \left(\frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)\cdot(B_\rho\hat{\rho} + B_\phi\hat{\phi} + B_z\hat{\mathbf{k}})\\ &= \frac{\partial B_\rho}{\partial\rho} + \left(\frac{B_\rho}{\rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi}\right) + \frac{\partial B_z}{\partial z}\\ &= \left(\frac{\partial B_\rho}{\partial\rho} + \frac{B_\rho}{\rho}\right) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z}\\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z}.\end{aligned}$$

The expression of divergence for a vector field $\boldsymbol{B}$ in cylindrical polar coordinates is:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}. \tag{6.3}$$

6.3.2 Spherical Polar Coordinates

The divergence in spherical coordinates can be derived in a similar fashion, but the dependence of the unit vectors on the angles are more complicated. As a reminder,

$$\begin{aligned}
\boldsymbol{\nabla} &= \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi} \\
\hat{\mathbf{r}}(\theta,\phi) &= \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \\
\hat{\theta}(\theta,\phi) &= \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \\
\hat{\phi}(\phi) &= -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}.
\end{aligned}$$

We can calculate and list the partial derivatives for later use:

$$\begin{aligned}
\frac{\partial\hat{\mathbf{r}}}{\partial\theta} &= \hat{\theta} & \frac{\partial\hat{\mathbf{r}}}{\partial\phi} &= \sin\theta\hat{\phi} \\
\frac{\partial\hat{\theta}}{\partial\theta} &= -\hat{\mathbf{r}} & \frac{\partial\hat{\theta}}{\partial\phi} &= \cos\theta\hat{\phi}.
\end{aligned}$$

So, the divergence for the vector field $\boldsymbol{B} = B_r\hat{\mathbf{r}} + B_\theta\hat{\theta} + B_\phi\hat{\phi}$ is:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= \left(\frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}\right)\cdot(B_r\hat{\mathbf{r}} + B_\theta\hat{\theta} + B_\phi\hat{\phi}) \\
&= \frac{\partial B_r}{\partial r} + \frac{1}{r}\hat{\theta}\cdot\frac{\partial\boldsymbol{B}}{\partial\theta} + \frac{1}{r\sin\theta}\hat{\phi}\cdot\frac{\partial\boldsymbol{B}}{\partial\phi} \\
&= \frac{\partial B_r}{\partial r} + \frac{1}{r}\hat{\theta}\cdot\left(\frac{\partial B_r}{\partial\theta}\hat{\mathbf{r}} + B_r\frac{\partial\hat{\mathbf{r}}}{\partial\theta} + \frac{\partial B_\theta}{\partial\theta}\hat{\theta} + B_\theta\frac{\partial\hat{\theta}}{\partial\theta}\right) + \frac{1}{r\sin\theta}\hat{\phi}\cdot\frac{\partial\boldsymbol{B}}{\partial\phi} \\
&= \frac{\partial B_r}{\partial r} + \frac{1}{r}\hat{\theta}\cdot\left(\frac{\partial B_r}{\partial\theta}\hat{\mathbf{r}} + B_r\hat{\theta} + \frac{\partial B_\theta}{\partial\theta}\hat{\theta} - B_\theta\hat{\mathbf{r}}\right) + \frac{1}{r\sin\theta}\hat{\phi}\cdot\frac{\partial\boldsymbol{B}}{\partial\phi} \\
&= \frac{\partial B_r}{\partial r} + \frac{1}{r}\left(B_r + \frac{\partial B_\theta}{\partial\theta}\right) + \frac{1}{r\sin\theta}\hat{\phi}\cdot\left(\frac{\partial B_r}{\partial\phi}\hat{\mathbf{r}} + B_r\frac{\partial\hat{\mathbf{r}}}{\partial\phi} + \frac{\partial B_\theta}{\partial\phi}\hat{\theta} + B_\theta\frac{\partial\hat{\theta}}{\partial\phi} + \frac{\partial B_\phi}{\partial\phi}\hat{\phi} + B_\phi\frac{\partial\hat{\phi}}{\partial\phi}\right) \\
&= \frac{\partial B_r}{\partial r} + \frac{1}{r}\left(B_r + \frac{\partial B_\theta}{\partial\theta}\right) + \frac{1}{r\sin\theta}\hat{\phi}\cdot\left(\frac{\partial B_r}{\partial\phi}\hat{\mathbf{r}} + B_r\sin\theta\hat{\phi} + \frac{\partial B_\theta}{\partial\phi}\hat{\theta} + B_\theta\cos\theta\hat{\phi} + \frac{\partial B_\phi}{\partial\phi}\hat{\phi} + B_\phi\frac{\partial\hat{\phi}}{\partial\phi}\right) \\
&= \frac{\partial B_r}{\partial r} + \frac{1}{r}\left(B_r + \frac{\partial B_\theta}{\partial\theta}\right) + \frac{1}{r\sin\theta}\left(B_r\sin\theta + B_\theta\cos\theta + \frac{\partial B_\phi}{\partial\phi}\right) \\
&= \left(\frac{\partial B_r}{\partial r} + \frac{2B_r}{r}\right) + \left(\frac{1}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{B_\theta\cos\theta}{r\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(B_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}.
\end{aligned}$$

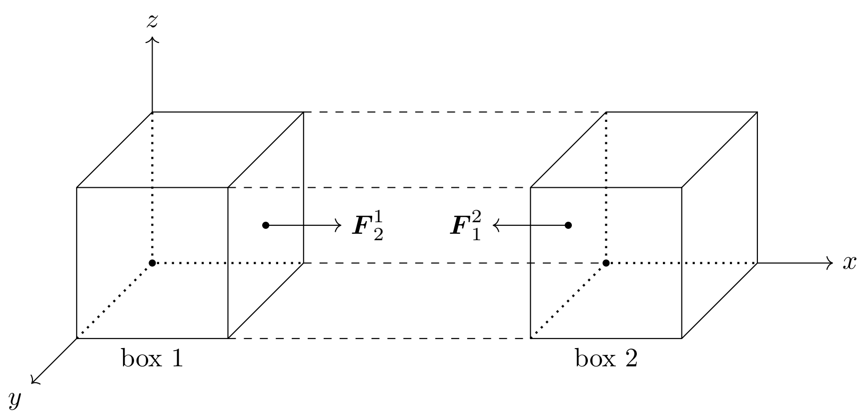
So, the expression of divergence in spherical polar coordinates is:

$$\nabla \cdot \boldsymbol{B} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(B_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}. \tag{6.4}$$

6.4 Divergence Theorem

To begin with, the divergence theorem is sometimes called Gauss's theorem, but never Gauss's law, which is the first equation in Maxwell's equations. It is an integration theorem which relates the flux of a vector field through a closed surface to the divergence of the field in the volume enclosed.

To approach the theorem, consider two infinitesimal boxes adjacent along the x axis. The graph below did not show the two boxes adjacent to each other, in order to show the flux on the faces they meet.



As before, we find six faces and label the flux on each face.

$$\begin{cases} \sum_{i=1}^{6} \boldsymbol{F}_i^1 = \nabla \cdot \boldsymbol{B}_1 \mathrm{d}V \\ \sum_{i=1}^{6} \boldsymbol{F}_i^2 = \nabla \cdot \boldsymbol{B}_2 \mathrm{d}V. \end{cases}$$

According to the graph above, we can see that $\boldsymbol{F}_2^1 = -\boldsymbol{F}_1^2$. Therefore, if we put the boxes together and consider the total flux over the combined surface, we get:

$$\begin{aligned} \sum \boldsymbol{F} &= \sum_{i=1}^{6} \boldsymbol{F}_i^1 + \sum_{i=1}^{6} \boldsymbol{F}_i^2 - \boldsymbol{F}_2^1 - \boldsymbol{F}_1^2 \\ &= \sum_{i=1}^{6} \boldsymbol{F}_i^1 + \sum_{i=1}^{6} \boldsymbol{F}_i^2. \end{aligned}$$

That is to say, the flux through the new combined box is simply the sum of fluxes through the component boxes:

$$\sum \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \sum_{j=1}^{2} \boldsymbol{\nabla} \cdot \boldsymbol{B}_j \mathrm{d}V.$$

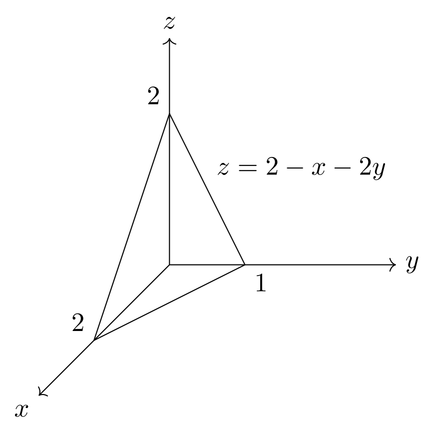
Now, we can add more boxes to create a macroscopic volume, and what we get is the divergence theorem, where S is a closed surface, and V is the enclosed volume by the surface:

$$\oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{B} \mathrm{d}V. \tag{6.5}$$

When some integrals are hard to directly compute, we can use the divergence theorem to convert them into other integrals. Consider the vector field $\boldsymbol{B} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. We are to compute the integral

$$\oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S},$$

where S is the tetrahedron bound by the surfaces $x = 0$, $y = 0$, $z = 0$, and $z = 2 - x - 2y$.



Through the divergence theorem, this integral can be done in two ways.

To evaluate the surface integral, first find out the area element on each surface. The three surfaces on the coordinate planes are easy to figure out, and it is a good practice to calculate the area element on the slanted surface.

With $z = 2 - x - 2y$, we define $\Omega = 2 - x - 2y - z$. So, the surface is the locus of points where $\Omega = 0$. The gradient of Ω points to the direction of fastest increase, so the gradient is perpendicular to the plane.

$$\begin{aligned} \boldsymbol{\nabla}\Omega &= \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \right) \Omega \\ &= -\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}. \end{aligned}$$

The correct direction of $\mathrm{d}\boldsymbol{S}$ in this case is the opposite, so

$$\mathrm{d}\boldsymbol{S} = \boldsymbol{N}\mathrm{d}x\mathrm{d}y = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})\mathrm{d}x\mathrm{d}y.$$

On the xz plane,

$$\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\Big|_{y=0} \cdot (-\hat{\mathbf{j}}\mathrm{d}x\mathrm{d}z) = 0 \Rightarrow \iint_{S_1} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0.$$

On the xy plane,

$$\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\Big|_{z=0} \cdot (-\hat{\mathbf{k}}\mathrm{d}x\mathrm{d}y) = 0 \Rightarrow \iint_{S_2} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0.$$

On the yz plane,

$$\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\Big|_{x=0} \cdot (-\hat{\mathbf{i}}\mathrm{d}y\mathrm{d}z) = 0 \Rightarrow \iint_{S_3} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0.$$

On the plane $z = 2 - x - 2y$ (S_4),

$$\begin{aligned}
\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} &= (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})\mathrm{d}x\mathrm{d}y \\
&= (x + 2y + z)\mathrm{d}x\mathrm{d}y \\
&= (x + 2y + (2 - x - 2y))\mathrm{d}x\mathrm{d}y \\
&= 2\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Therefore,

$$\oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \iint_{S_4} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} 2 \iint_{S_4} \mathrm{d}x\mathrm{d}y.$$

We can recognize that the integration with respect to $\mathrm{d}x$ and $\mathrm{d}y$ over S_4 is simply the integration with the projection of S_4 onto the xy plane, or S_2. Also, we know that when the integrand is 1, the double integral asks for the area. Therefore,

$$\oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 2 \iint_{S_4} \mathrm{d}x\mathrm{d}y = 2 \iint_{S_2} \mathrm{d}x\mathrm{d}y = 2S_2 = 2 \times \frac{1}{2} \times 2 \times 1 = 2.$$

According to the divergence theorem, we can also do the volume integral:

$$\oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{B}\mathrm{d}V.$$

In this case,

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=\left(\frac{\partial}{\partial x}\hat{\mathbf{i}}+\frac{\partial}{\partial y}\hat{\mathbf{j}}+\frac{\partial}{\partial z}\hat{\mathbf{k}}\right)\cdot(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})=1+1+1=3.$$

So,

$$\begin{aligned}\iiint_V \boldsymbol{\nabla}\cdot\boldsymbol{B}\mathrm{d}V &= \iiint_V 3\mathrm{d}V\\ &= 3\iint_V \mathrm{d}V\\ &= 3V\\ &= 3\times\frac{1}{3}\times\text{base}\times\text{height}\\ &= 3\times\frac{1}{3}\times\left(\frac{1}{2}\times 2\times 1\right)\times 2\\ &= 2.\end{aligned}$$

These two methods to do the closed surface integral yield the same answer as expected.

6.5 Laplacian

The Laplace operator, or Laplacian, is a differential operator given by the divergence of the gradient of a scalar function on Euclidean space.

$$\begin{aligned}\boldsymbol{\nabla}\Omega &= \frac{\partial\Omega}{\partial x}\hat{\mathbf{i}}+\frac{\partial\Omega}{\partial y}\hat{\mathbf{j}}+\frac{\partial\Omega}{\partial z}\hat{\mathbf{k}}\\ \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\Omega) &\equiv \nabla^2\Omega\\ &= \frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}+\frac{\partial^2\Omega}{\partial z^2}.\end{aligned}$$

The Laplace Equation of a scalar field u is:

$$\nabla^2 u=0.$$

The wave equation for a scalar field u is:

$$\nabla^2 u=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

6.5.1 Cylindrical Coordinates

The derivation of Laplacian in other coordinates follows the same path of the divergence. According to the definition of the Laplacian,

$$\begin{aligned}\nabla^2 u &= \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}u)\\ &= \left(\frac{\partial}{\partial\rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi}+\frac{\partial}{\partial z}\hat{\mathbf{k}}\right)\cdot\left(\frac{\partial u}{\partial\rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial u}{\partial\phi}\hat{\phi}+\frac{\partial u}{\partial z}\hat{\mathbf{k}}\right).\end{aligned}$$

That is to say, the expression of Laplacian is simply the divergence of the vector field $\boldsymbol{\nabla}u$. According to what we have done in divergence,

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho)+\frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi}+\frac{\partial B_z}{\partial z}.$$

Identifying $\boldsymbol{B}=\boldsymbol{\nabla}u=\frac{\partial u}{\partial\rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial u}{\partial\phi}\hat{\phi}+\frac{\partial u}{\partial z}\hat{\mathbf{k}}$ gives:

$$\begin{cases} B_\rho=\dfrac{\partial u}{\partial\rho}\\ B_\phi=\dfrac{1}{\rho}\dfrac{\partial u}{\partial\phi}\\ B_z=\dfrac{\partial u}{\partial z},\end{cases}$$

$$\begin{aligned}\nabla^2 u &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho)+\frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi}+\frac{\partial B_z}{\partial z}\\ &\equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2 u}{\partial z^2}.\end{aligned}$$

So, the Laplacian in cylindrical coordinates is:

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2 u}{\partial z^2}. \tag{6.6}$$

6.5.2 Spherical Coordinates

Similarly,

$$\begin{aligned}\nabla^2 u &= \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}u)\\ &= \left(\frac{\partial}{\partial r}\hat{\mathbf{r}}+\frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}\right)\cdot\left(\frac{\partial u}{\partial r}\hat{\mathbf{r}}+\frac{1}{r}\frac{\partial u}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}\hat{\phi}\right),\end{aligned}$$

and we know that

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2B_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(B_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}.$$

Identifying that $\boldsymbol{B} = \boldsymbol{\nabla} u = \frac{\partial u}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\hat{\phi}$ gives:

$$\begin{cases} B_r = \dfrac{\partial u}{\partial r} \\ B_\theta = \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \\ B_\phi = \dfrac{1}{r\sin\theta}\dfrac{\partial u}{\partial \phi}, \end{cases}$$

$$\begin{aligned} \nabla^2 u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(B_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}. \end{aligned}$$

So, the Laplacian in spherical coordinates is:

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}. \tag{6.7}$$

Chapter 7

Curl

7.1 Green's Theorem in the Plane

In vector calculus, Green's theorem relates a line integral around a simple closed curve C to a double integral over the plane region R bounded by C. It is the two-dimensional special case of Stokes' theorem.

$$\oint_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y. \tag{7.1}$$

In the above expression, the line integral is taken counterclockwise (when you look down from the above on the xy plane). The double integral is calculated over the region enclosed by the line.

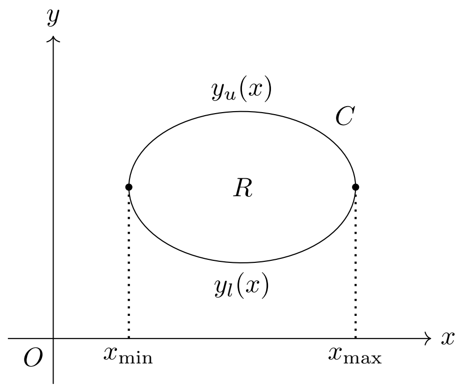
Proof. Firstly, consider part of the integral on the RHS:

$$I_1 = -\iint_R \frac{\partial P}{\partial y}\mathrm{d}y\mathrm{d}x.$$

The curve may be divided into two parts according to their x coordinates. The upper curve is denoted $y_u(x)$, and the lower one is denoted $y_l(x)$. These two curves meet at the endpoints $x_{\min}$ and $x_{\max}$. The curve shown below is a very simple ellipse, but the theorem works as well for loops of more complicates shapes.

To do this 2D integral, we can integrate with respect to y first. That is to say,

$$\begin{aligned} I_1 &= -\iint \frac{\partial P}{\partial y}\mathrm{d}y\mathrm{d}x \\ &= -\int_{x_{\min}}^{x_{\max}} \mathrm{d}x \int_{y_l(x)}^{y_u(x)} \frac{\partial P(x,y)}{\partial y}\mathrm{d}y. \end{aligned}$$



Integrating the derivative simply yields the difference in the values of the function on the two endpoints. So,

$$
\begin{aligned}
I_1 &= -\int_{x_{\min}}^{x_{\max}} \mathrm{d}x \int_{y_l(x)}^{y_u(x)} \frac{\partial P(x,y)}{\partial y}\mathrm{d}y \\
&= -\int_{x_{\min}}^{x_{\max}} \mathrm{d}x[P(x,y)]\Big|_{y=y_l(x)}^{y=y_u(x)} \\
&= -\int_{x_{\min}}^{x_{\max}} [P(x,y_u) - P(x,y_l)]\mathrm{d}x \\
&= -\int_{x_{\min}}^{x_{\max}} P(x,y_u)\mathrm{d}x + \int_{x_{\min}}^{x_{\max}} P(x,y_l)\mathrm{d}x \\
&= \int_{x_{\max}}^{x_{\min}} P(x,y_u)\mathrm{d}x + \int_{x_{\min}}^{x_{\max}} P(x,y_l)\mathrm{d}x \\
&= \oint_C P\mathrm{d}x.
\end{aligned}
$$

Therefore, we get

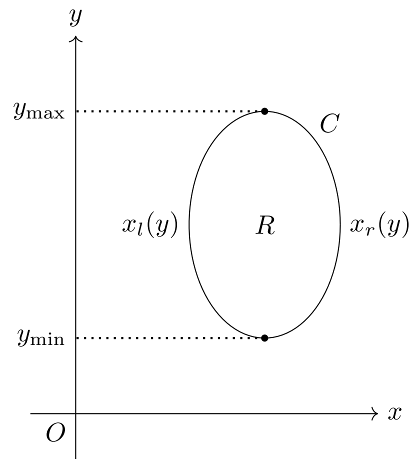
$$
-\iint_R \frac{\partial P}{\partial y}\mathrm{d}y\mathrm{d}x = \oint_C P\mathrm{d}x.
$$

We can follow a very similar process to get the second part of the theorem. Consider

$$
I_2 = \iint_R \frac{\partial Q}{\partial x}\mathrm{d}x\mathrm{d}y.
$$

We would finish the proof with a second simple ellipse, and divide the region through the y coordinates. The curve would then be divided into left and right parts, denoted $x_l(y)$ and $x_r(y)$. Therefore,

$$
\begin{aligned}
I_2 &= \iint \frac{\partial Q}{\partial x}\mathrm{d}x\mathrm{d}y \\
&= \int_{y_{\min}}^{y_{\max}} \mathrm{d}y \int_{x_l(y)}^{x_r(y)} \frac{\partial Q}{\partial x}\mathrm{d}x \\
&= \int_{y_{\min}}^{y_{\max}} \mathrm{d}y[Q(x,y)]\Big|_{x=x_l(y)}^{x=x_r(y)}.
\end{aligned}
$$



With the same algebraic manipulation, we get:

$$
\begin{aligned}
I_2 &= \int_{y_{\min}}^{y_{\max}} [Q(x_r, y) - Q(x_l, y)]\mathrm{d}y \\
&= \int_{y_{\min}}^{y_{\max}} Q(x_r, y)\mathrm{d}y + \int_{y_{\max}}^{y_{\min}} Q(x_l, y)\mathrm{d}y \\
&= \oint_C Q\mathrm{d}y.
\end{aligned}
$$

So, we end up with

$$\iint_R \frac{\partial Q}{\partial x}\mathrm{d}x\mathrm{d}y = \oint_C Q\mathrm{d}y.$$

Adding the two conclusions together, we get the Green's Theorem in the space:

$$\oint_C P\mathrm{d}x + Q\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y. \tag{7.2}$$

□

It is important to note that P and Q can be any continuous functions. This theorem is thus very general and has very wide applications. Some particular examples are of special interest in the field of electromagnetism.

7.2 Four Examples of Green's Theorem

7.2.1 Example 1: Exact Differential

Let's consider the differential form

$$P\mathrm{d}x + Q\mathrm{d}y$$

when it is an exact differential. Then, we know that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Therefore, we get

$$\oint P \mathrm{d}x + Q \mathrm{d}y = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \mathrm{d}y = 0$$

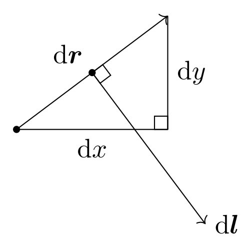
as expected.

7.2.2 Example 2: Flux out of a Loop

We are here to consider the "flux" out of a closed loop in 2D. This is the 2D equivalent of the flux $\oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$.

For a point on the loop, we define $\mathrm{d}\boldsymbol{r} \equiv \mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}}$. Then, the "normal vector" $\mathrm{d}\boldsymbol{l}$ pointing outwards (with anticlockwise loop) should be:

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}y\hat{\mathbf{i}} - \mathrm{d}x\hat{\mathbf{j}}.$$



So, the flux through an infinitesimal line segment is:

$$\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}) \cdot (\mathrm{d}y\hat{\mathbf{i}} - \mathrm{d}x\hat{\mathbf{j}}) = -B_y\mathrm{d}x + B_x\mathrm{d}y.$$

The closed loop integral should be:

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \oint -B_y\mathrm{d}x + B_x\mathrm{d}y.$$

According to Green's Theorem, identifying $P \equiv -B_y$ and $Q \equiv B_x$, we get:

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \iint \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right) \mathrm{d}x\mathrm{d}y.$$

This is also known as the divergence theorem in 2D:

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \iint \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right) \mathrm{d}x\mathrm{d}y. \tag{7.3}$$

7.2.3 Example 3: Work Done over a Loop

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \oint F_x \mathrm{d}x + F_y \mathrm{d}y.$$

By identifying $P \equiv F_x$ and $Q \equiv F_y$, we get:

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \oint F_x \mathrm{d}x + F_y \mathrm{d}y = \iint \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathrm{d}x\mathrm{d}y.$$

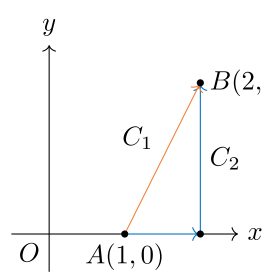
This is the Stokes' Theorem in 2D:

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathrm{d}x\mathrm{d}y. \tag{7.4}$$

7.2.4 Example 4: Interchanging Line Integrals with 2D Integrals

Sometimes, a 2D integral may be easier to do than a line integral, or vice versa.

Recall the line integral we calculated before. The vector field is $\boldsymbol{F} = 2xy\hat{\mathbf{i}} - x^2\hat{\mathbf{j}}$.



According to the calculations before, we get:

$$\begin{aligned}
\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &= \int_{C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} + \int_{-C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \\
&= \int_{C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} - \int_{C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \\
&= -8 - \left(-\frac{4}{3} \right) \\
&= -\frac{20}{3}.
\end{aligned}$$

That is to say,

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \oint 2xy\mathrm{d}x - x^2\mathrm{d}y = -\frac{20}{3}.$$

We can do this line integral in another way. According to Green's Theorem, identifying $P \equiv 2xy$ and

$Q \equiv -x^2$, we get:

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_R \left(\frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(2xy) \right) = -\iint_R 4x\mathrm{d}x\mathrm{d}y.$$

This double integral is somewhat easier to do:

$$\begin{aligned}
I &= -\iint_R 4x\mathrm{d}x\mathrm{d}y \\
&= -4\int_1^2 x\mathrm{d}x \int_0^{2x-2} \mathrm{d}y \\
&= -4\int_1^2 x(2x-2)\mathrm{d}x \\
&= -4\left[\frac{2}{3}x^3 - x^2\right]\Bigg|_1^2 \\
&= -\frac{20}{3}.
\end{aligned}$$

When some line integrals or double integrals are hard to evaluate directly, we can try Green's Theorem to simplify the problem.

7.3 Definition of Curl

Curl is a vector that quantifies the "circulation surface density". It is important to remember that curl is a vector, and the following expression gives its magnitude, but not its direction:

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \hat{\mathbf{n}} = \lim_{A \to 0} \left(\frac{1}{A} \oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} \right). \tag{7.5}$$

In the above equation, $\hat{\mathbf{n}}$ is a vector that is the unit vector that is in the same direction as the curl: perpendicular to the area by the right hand rule.

In Cartesian coordinates, in order to derive one of the components of the curl, consider a loop of finite size perpendicular to the component that shrinks to zero. Now take the $\hat{\mathbf{k}}$ component as an example. By Green's theorem,

$$\begin{aligned}
\frac{1}{A}\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} &= \frac{1}{A}\oint B_x\mathrm{d}x + B_y\mathrm{d}y \\
&= \frac{1}{A}\iint_R \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

By shrinking the area to zero, we get:

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \hat{\mathbf{k}} = \lim_{A \to 0} \frac{\iint_R \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \mathrm{d}x\mathrm{d}y}{A} = \lim_{A \to 0} \frac{\iint_R \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \mathrm{d}x\mathrm{d}y}{\iint_R \mathrm{d}x\mathrm{d}y}.$$

As the area shrinks to zero, we can assume that the integrand $\dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y}$ is actually constant over the infinitesimal area A. Therefore,

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \hat{\mathbf{k}} = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}.$$

Similarly, we can get the component of the curl in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ directions:

$$\begin{aligned}(\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \hat{\mathbf{i}} &= \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\\ (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \hat{\mathbf{j}} &= \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}.\end{aligned}$$

A nice expression of the curl is as follows:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}, \tag{7.6}$$

where

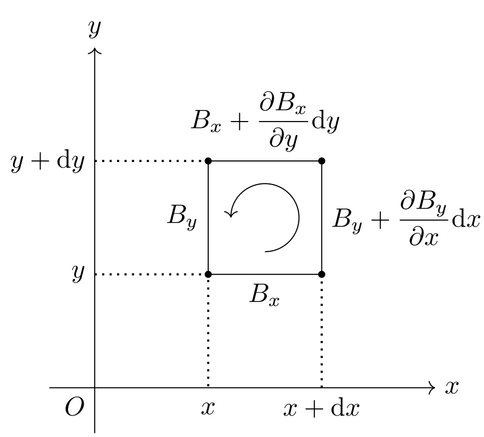
$$\begin{aligned}\boldsymbol{\nabla} &= \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\\ \boldsymbol{B} &= B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}.\end{aligned}$$

In this way, $\boldsymbol{\nabla} \times \boldsymbol{B}$ looks like a cross product between two vectors. More coherently, it should be written as:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \hat{\mathbf{i}} \times \frac{\partial \boldsymbol{B}}{\partial x} + \hat{\mathbf{j}} \times \frac{\partial \boldsymbol{B}}{\partial y} + \hat{\mathbf{k}} \times \frac{\partial \boldsymbol{B}}{\partial z}.$$

7.4 Understanding the "Circulation Surface Density"

Look again at the $\hat{\mathbf{k}}$ component and consider an infinitesimal area $\mathrm{d}x\mathrm{d}y$.



Now we can do the closed path integral over the infinitesimal area:

$$\begin{aligned}\frac{1}{A}\oint \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{r} &= \frac{1}{\mathrm{d}x\mathrm{d}y}\left[B_x\mathrm{d}x + \left(B_y + \frac{\partial B_y}{\partial x}\mathrm{d}x\right)\mathrm{d}y - \left(B_x + \frac{\partial B_x}{\partial y}\mathrm{d}y\right)\mathrm{d}x - B_y\mathrm{d}y\right]\\ &= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}.\end{aligned}$$

By assuming positive partial derivatives, we know that the vertical paths cause a counterclockwise rotation, while the horizontal paths cause a clockwise rotation. This explains the minus sign in the expression.

$$\begin{cases}\dfrac{\partial B_y}{\partial x} & \text{causes CCW rotation}\\ \dfrac{\partial B_x}{\partial y} & \text{causes CW rotation}\end{cases}$$

Also, as a reminder, we talk about the circulation, which is integrating the vector field along a path. For a conservative vector field, we can always write

$$\oint \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{r} = 0,$$

as it does no net work along a closed path. That is to say, a conservative field has zero curl.

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\Omega) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial\Omega}{\partial x} & \dfrac{\partial\Omega}{\partial y} & \dfrac{\partial\Omega}{\partial z}\end{vmatrix}\\ &= \left(\frac{\partial^2\Omega}{\partial y\partial z} - \frac{\partial^2\Omega}{\partial z\partial y}\right)\hat{\mathbf{i}} + \left(\frac{\partial^2\Omega}{\partial z\partial x} - \frac{\partial^2\Omega}{\partial x\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial^2\Omega}{\partial x\partial y} - \frac{\partial^2\Omega}{\partial y\partial x}\right)\hat{\mathbf{k}}\\ &= 0.\end{aligned}$$

$\boldsymbol{\nabla}\times\boldsymbol{B} = 0$ implies that $\boldsymbol{B}$ is a conservative field, or an "irrotational" field.

7.5 Curl in Other Coordinate Systems

We can follow the procedure of "differentiate first, cross product second" to get the correct answer.

7.5.1 Cylindrical Polar Coordinates

As a reminder, in cylindrical polar coordinates,

$$\begin{aligned}\boldsymbol{\nabla} &= \frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \\ \boldsymbol{B} &= B_\rho\hat{\rho} + B_\phi\hat{\phi} + B_z\hat{\mathbf{k}}.\end{aligned}$$

Therefore, the expression of $\boldsymbol{\nabla}\times\boldsymbol{B}$ is:

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \hat{\rho}\times\frac{\partial\boldsymbol{B}}{\partial\rho} + \frac{1}{\rho}\hat{\phi}\times\frac{\partial\boldsymbol{B}}{\partial\phi} + \hat{\mathbf{k}}\times\frac{\partial\boldsymbol{B}}{\partial z}.$$

After some manipulation, we get:

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\phi} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z\end{vmatrix}. \tag{7.7}$$

7.5.2 Spherical Polar Coordinates

Following the same procedure, we can end up with:

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{r}} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ B_r & rB_\theta & r\sin\theta B_\phi\end{vmatrix}. \tag{7.8}$$

7.6 Stokes' Theorem

Consider two adjacent infinitesimal square loops, which are not necessarily in the same plane. Recalling

$$(\boldsymbol{\nabla}\times\boldsymbol{B})\cdot\hat{\mathbf{n}} = \lim_{A\to 0}\left(\frac{1}{A}\oint\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{r}\right)$$

and identifying

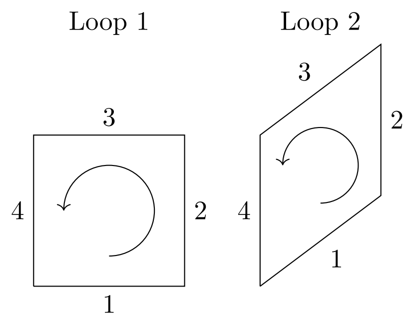
$$\mathrm{d}\boldsymbol{S} = A\hat{\mathbf{n}},$$

we can get

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S} = \oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r}$$

for an infinitesimal loop.

The following graph does not show that loops 1 and 2 are adjacent in order to mark the line integrals on each segment. A direct physical example of the line integral is the work done on a path, so the line integrals on each segment is denoted by w.



Then, we can calculate the circulations through the loops.

$$\begin{aligned}
\text{Loop 1:} \quad & \sum_{i=1}^{4} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \sum_{i=1}^{4} w_i^1 = (\boldsymbol{\nabla} \times \boldsymbol{B}_1) \cdot \mathrm{d}\boldsymbol{S}_1 \\
\text{Loop 2:} \quad & \sum_{i=1}^{4} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \sum_{i=1}^{4} w_i^2 = (\boldsymbol{\nabla} \times \boldsymbol{B}_2) \cdot \mathrm{d}\boldsymbol{S}_2.
\end{aligned}$$

When we join the two loops together, we get:

$$\sum \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \sum_{i=1}^{4} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} + \sum_{i=1}^{4} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} - w_2^1 - w_4^2.$$

As we know that $w_2^1 = -w_4^2$,

$$\sum \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = (\boldsymbol{\nabla} \times \boldsymbol{B}_1) \cdot \mathrm{d}\boldsymbol{S}_1 + (\boldsymbol{\nabla} \times \boldsymbol{B}_2) \cdot \mathrm{d}\boldsymbol{S}_2 = \sum_{j=1}^{2} (\boldsymbol{\nabla} \times \boldsymbol{B}_j) \cdot \mathrm{d}\boldsymbol{S}_j.$$

When we add more loops to create a macroscopic surface attached to a large loop, we get the Stokes' Theorem:

$$\oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S}, \tag{7.9}$$

where the right hand rule determines the direction of $\mathrm{d}S$.

Some examples of the surfaces attached to the loop are the butterfly net and a sock. To be clear on the orientation of $\mathrm{d}\boldsymbol{S}$, we can "collapse" the surface to the loop.

When we want a surface integral $\iint_S (\nabla \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S}$, if the line integral is easier, we can convert the surface integral into the line integral, or vice versa. Also, we may choose a different surface, as long as it is attached to the same loop.

Now consider an example. There is a hemisphere of radius a for $z > 0$. The vector field is $\boldsymbol{B} = z\hat{\mathbf{i}} - y\hat{\mathbf{j}} - x\hat{\mathbf{k}}$. The surface area element, in this case, is $\mathrm{d}\boldsymbol{S} = a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \hat{\mathbf{r}}$. We want to evaluate

$$\oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S}.$$

The curl of $\boldsymbol{B}$ is:

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & -y & -x \end{vmatrix} = 2\hat{\mathbf{j}}.$$

(1) In the xy plane,

$$x = a\cos\phi \qquad \mathrm{d}x = -a\sin\phi \mathrm{d}\phi$$
$$y = a\sin\phi \qquad \mathrm{d}y = a\cos\phi \mathrm{d}\phi.$$

Therefore,

$$\begin{aligned} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} &= B_x \mathrm{d}x + B_y \mathrm{d}y + B_z \mathrm{d}z \\ &= B_x \mathrm{d}x + B_y \mathrm{d}y \quad (\mathrm{d}z = 0) \\ &= z\mathrm{d}x - y\mathrm{d}y \\ &= -y\mathrm{d}y \quad (z = 0) \\ &= -a^2 \sin\phi\cos\phi \mathrm{d}\phi, \end{aligned}$$

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = -a^2 \int_0^{2\pi} \sin\phi\cos\phi \mathrm{d}\phi = 0.$$

(2) On the hemisphere,

$$\mathrm{d}\boldsymbol{S} = a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \hat{\mathbf{r}}$$
$$\nabla \times \boldsymbol{B} = 2\hat{\mathbf{j}}$$
$$\hat{\mathbf{j}} \cdot \hat{\mathbf{r}} = \sin\theta\sin\phi.$$

So,

$$\iint (\nabla \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S} = 2a^2 \int_0^{\pi/2} \sin^2\theta \mathrm{d}\theta \int_0^{2\pi} \sin\phi \mathrm{d}\phi = 0.$$

(3) A quicker way:

With the same close loop, we just choose the closed surface in the xy plane:

$$\mathrm{d}\boldsymbol{S} = \hat{\mathbf{k}}\mathrm{d}S.$$

Therefore,

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S} = 2\mathrm{d}S(\hat{\mathbf{j}} \cdot \hat{\mathbf{k}}) = 0.$$

7.7 Vector Identities

There are many vector identities, and three of them are the most important. The first one is:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Omega) = 0. \tag{7.10}$$

That is to say, if a vector field satisfies the equation $\boldsymbol{B} = \boldsymbol{\nabla}\Omega$, where Ω is any parent function, then we say the vector field $\boldsymbol{B}$ is conservative, or "irrotational".

The second vector identity is

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{V}) = 0. \tag{7.11}$$

A simple proof in Cartesian coordinates is as follows.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{V}) &= \boldsymbol{\nabla} \cdot \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \\
&= 0.
\end{aligned}$$

That is to say, if $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{V}$ where $\boldsymbol{V}$ is a parent function, then we say that $\boldsymbol{B}$ is solenoidal, and V is called the vector potential.

The third vector identity is

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \nabla^2\boldsymbol{B}, \tag{7.12}$$

where

$$\nabla^2\boldsymbol{B} \equiv \nabla^2 B_x\hat{\mathbf{i}} + \nabla^2 B_y\hat{\mathbf{j}} + \nabla^2 B_z\hat{\mathbf{k}}. \tag{7.13}$$

Curl of curl is gradient of divergence minus Laplacian.