Nuclear & Particle Physics Notes

Shaojun Zan

23 April 2025

Contents

1 The Standard Model (SM)

1.1 The "Three" Fundamental Forces & Higgs Boson

Force	Force Particles (Spin 1)	Matter Particles (Spin 1/2)
Strong	Gluon (g)	Quarks (u, c, t; d, s, b)
Electromagnetic (EM)	Photon (γ)	Charged leptons (e, μ, τ) and quarks
Weak	W^+, W^-, Z bosons	Quarks, leptons (including neutrinos)

The only other type of fundamental particle in the SM is the Higgs boson, and it has spin 0. It is responsible for the "drag" on particles, even in vacuum, which we perceive as mass.

Name	Symbol	Number	EM Charge	Mass
Higgs	H	1	0	$125\,\mathrm{GeV/c^2}$

1.2 Matter Particles

There are three generations of fundamental particles, where each generation only differs by mass. The matter particles all have spin 1/2 and are thus fermions.

	Generation			Charge (e)	Feels Force		
	1st	2nd	3rd		Strong	EM	Weak
u-type Quarks (3 colors)	u	c	t	$+2/3$	Y	Y	Y
d-type Quarks (3 colors)	d	s	b	$-1/3$	Y	Y	Y
Charged Leptons	e	μ	τ	-1	N	Y	Y
Neutral Leptons (Neutrinos)	ν_e	ν_μ	ν_τ	0	N	N	Y

For every particle of the above, there is a corresponding antiparticle with identical mass but opposite charge (and opposite other quantum numbers). Antiparticles may be perceived as particles going backwards in time.

1.3 Force Particles

The three forces in the SM are the electromagnetic (EM), strong and weak forces. The particles mediating these forces all have spin 1.

Force	Name	Symbol	Number	EM Charge	Mass
Strong	Gluons	g	8	0	0
EM	Photon	γ	1	0	0
Weak	W and Z	$W^{\pm}$, Z^0	3	± 1, 0	80, 91 $\mathrm{GeV/c^2}$

In the SM, the matter particles do not interact with each other directly, but rather with force fields. Only interacting with $W^{\pm}$ allows particles to change their type.

Force particles also have antiparticles. The W^+ and W^- are antiparticles of each other. The photon and the Z are their own antiparticles.

1.4 Composite Particles and Hadrons

Only the leptons (e, μ, τ and their neutrinos) are actually observed as free particles. The quarks are never seen singly, but always occur in bound states, which are called **hadrons**.

$$\text{hadrons}\begin{cases}\text{Quark-antiquark } q\bar{q} & \textbf{mesons} \text{ (bosons)}\\ \text{Three-quark } qqq & \textbf{baryons} \text{ (fermions)}\end{cases} \tag{1}$$

1.5 Units

A frequently used distance is the **femtometer** (nicknamed the fermi):

$$1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}. \tag{2}$$

Areas are often given in **barns**, where

$$1\,\mathrm{barn} = 10^{-28}\,\mathrm{m}^2 = 100\,\mathrm{fm}^2. \tag{3}$$

Natural units here imply

$$\hbar = c = \epsilon_0 = 1. \tag{4}$$

2 Special Relativity

2.1 Definitions

We need to familiarize ourselves with **relativistic velocity** $\boldsymbol{\beta}$ and the **Lorentz factor** γ:

$$\boxed{\boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \qquad \gamma = \frac{1}{\sqrt{1 - |\boldsymbol{\beta}|^2}}.} \tag{5}$$

This gives directly that

$$\gamma^2 - \gamma^2 |\boldsymbol{\beta}|^2 = 1. \tag{6}$$

The **energy-momentum relationship** is

$$\boxed{E^2 - p^2 c^2 = m^2 c^4.} \tag{7}$$

In terms of the velocity and Lorentz factor,

$$E = \gamma m c^2, \qquad \mathbf{p} = \gamma m \mathbf{v}. \tag{8}$$

Therefore,

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c} = \frac{\mathbf{p}c}{E}, \qquad \gamma = \frac{E}{mc^2}. \tag{9}$$

2.2 Lorentz Invariants

We use a mostly minus Minkowski metric $\eta_{\mu\nu} = (1, -1, -1, -1)$; this presumably serves the purpose of making the energy a positive value.

Lorentz invariants are quantities which are the same in all reference frames. With this metric $\eta_{\mu\nu}$ and contravariant vector x^μ, we form Lorentz invariants:

$$I = \mathbf{x}^T \eta \mathbf{x} \equiv (\mathbf{x}')^T \eta \mathbf{x}', \tag{10}$$

where we've used the definition of Lorentz matrices.

$$\begin{cases} I > 0 & \text{timelike, as the time component is larger;} \\ I < 0 & \text{spacelike, as the space component is larger;} \\ I = 0 & \text{lightlike, as the components are equal.} \end{cases} \tag{11}$$

2.3 Conservation v.s. Invariance

Conserved quantities are constant with time and so are the same before and after an interaction. Invariant quantities can change with time, but are the same in any reference frame when compared at equivalent times.

	Conserved	Invariant
Energy	Y	N
Momentum	Y	N
Mass	N	Y

2.4 Relativistic Kinematics

The law of conservation of momentum and energy can be combined together to make the **conservation of four-momentum**:

$$p^{\mu} = (E, \mathbf{p})\,. \tag{12}$$

2.4.1 Invariant Mass

The inner product of four-momentum can be used to define an invariant "total mass":

$$s \equiv m_T^2 c^4 = E_T^2 - p_T^2 c^2, \tag{13}$$

where we must note that

$$m_T \neq \sum_i m_i. \tag{14}$$

While energy and momentum may be added directly (due to their conservation), the same definition of addition does not apply to invariant mass when it comes to combination of systems.

A common frame to be used in collisions is the **center-of-momentum frame** (CoM frame), where

$$\mathbf{p}_T = 0. \tag{15}$$

3 Basic Concepts

3.1 Decays, Lifetimes, and Widths

The decay rate per particle is

$$\lambda = \frac{1}{\tau}, \tag{16}$$

where τ (called the **lifetime**) is the average lifetime of the decaying particle in its center of momentum. We write that

$$\delta N = -N\lambda\,\delta t\,, \tag{17}$$

which means that a proportion of $\lambda\,\delta t$ per particle will decay in infinitesimal time δt. The solution is

$$N(t) = N_0 \exp\left(-\lambda t\right), \tag{18}$$

and we can define **half-life**, $T_{1/2}$, as the time taken for 1/2 of the initial sample to decay:

$$\exp\left(\lambda T_{1/2}\right) = 2 \quad \Rightarrow \quad T_{1/2} = \frac{\ln 2}{\lambda} \approx 0.693\tau. \tag{19}$$

3.2 Particle Widths

By Heisenberg's uncertainty principle,

$$\Delta E \Delta t \sim \Delta E \tau \sim \hbar \quad \Rightarrow \quad \Delta E \sim \frac{\hbar}{\tau} = \hbar\lambda. \tag{20}$$

An unstable particle's width is proportional to the decay rate, and inversely proportional to the lifetime.

A more rigorous treatment of this relation involves quantum mechanics. Consider a physical state created at $t = 0$ with energy E_0 and lifetime τ. The wavefunction may be written as

$$\psi(t) = \psi(0)\exp\left(-iE_0 t/\hbar\right)\exp\left(-t/2\tau\right) \quad \Rightarrow \quad \left|\psi(t)\right|^2 = \left|\psi(0)\right|^2 \exp\left(-t/\tau\right). \tag{21}$$

A Fourier transform leads us to the wavefunction in E space:

$$\begin{aligned}
\tilde{\psi}(E) &= \int_{-\infty}^{\infty} \mathrm{d}t\,\psi(t)\exp\left(-Et/\hbar\right) \\
&= \int_{0}^{\infty} \mathrm{d}t\,\psi(0)\exp\left[\frac{it}{\hbar}\left(E - E_0 + \frac{i\hbar}{2\tau}\right)\right] \\
&= \frac{\psi(0)}{i(E - E_0)/\hbar - 1/2\tau}.
\end{aligned}$$

This gives us the **Breit-Wigner formula**:

$$P(E) = \left|\tilde{\psi}(E)\right|^2 = \hbar^2 \frac{|\psi(0)|^2}{(E - E_0)^2 + \Gamma^2/4}, \tag{22}$$

where $\Gamma = \hbar\lambda$ is the full width at half maximum (FWHM).

3.3 Decay Modes

Particles may decay in several different ways or "**modes**". Each of these decay modes will happen independently of others, so it's natural to consider separate decay rate constants λ_i and the associated widths (**partial widths**):

$$\Gamma_i = \hbar\lambda_i. \tag{23}$$

Clearly, the total decay rate is the sum of individuals:

$$\lambda = \sum_i \lambda_i \quad \Rightarrow \quad \Gamma = \sum_i \Gamma_i. \tag{24}$$

The proportion of decays to a particular mode is called the **branching fraction** (branching ratio):

$$\mathcal{B}_i = \frac{\Gamma_i}{\Gamma} = \frac{\lambda_i}{\lambda}, \qquad \sum_i \mathcal{B}_i = 1. \tag{25}$$

Finally, note that there is only one lifetime:

$$\tau = \frac{\hbar}{\Gamma}. \tag{26}$$

There is no physical meaning to a "partial lifetime".

3.4 Cross Sections & Mean Free Path

The **cross section** σ is a physical quantity that measures the probability of a particle reaction, effectively representing the "target area" seen by the incoming reaction particle.

Consider a very thin material with number density n. The **mean free path** l implies that as particles penetrate the material, they will collide with a target particle in the material. In mathematical terms, this means

$$P = n\ell\sigma = 1 \quad \Rightarrow \quad \ell = \frac{1}{n\sigma}. \tag{27}$$

The mean time between collisions is evident given $\ell = vt$:

$$t = \frac{\ell}{v} = \frac{1}{n\sigma v}. \tag{28}$$

3.5 Range of a Force

The Poisson equation, $\nabla^2 V = 0$, may be viewed as the spatial part of a relativistic quantum mechanics equation, $\left(\hat{E}^2 - \hat{p}^2 c^2\right) V = 0$. With an nonzero mass, we get the equation

$$\left(\nabla^2 - \frac{m^2c^2}{\hbar^2}\right) V = 0. \tag{29}$$

The solution is called the **Yukawa potential**:

$$\phi = \frac{g^2}{4\pi r} \exp\left(-\frac{mcr}{\hbar}\right) \equiv \frac{g^2}{4\pi r} \exp\left(-\frac{r}{r_0}\right), \tag{30}$$

where

$$\boxed{r_0 = \frac{\hbar}{mc}} \tag{31}$$

is a characteristic range. To check this value as a range, consider a width $\Delta E = mc^2$:

$$r_{\max} \sim c\Delta t = \frac{\hbar c}{\Delta E} = \frac{\hbar}{mc}.$$

While photons and gluons have infinite range, the weak interaction bosons, $W^\pm$ and Z, have a short range of 2×10^{-3} fm. This short range is what makes the weak interaction so weak.

3.6 Fermi's Golden Rule

Since decays and reactions are due to QM, they happen randomly in time. Fermi's golden rule tells us how often such a process will happen:

$$R_{fi} = \frac{2\pi}{\hbar} \left|T_{fi}\right|^2 \rho(E_f). \tag{32}$$

This expression gives us the average rate R_{fi}, probability per unit time, for a transition from some initial state i to any of several possible final states f, each of which have energy E_f. ρ is the density of states.

T_{fi} is the transition matrix element and the QM amplitude for the probability per unit time of changing from i to f. For an initial state u_i in a system described by Hamiltonian $\hat{\mathcal{H}}$, the probability amplitude of measuring u_f after time t is

$$A_{fi}(t) = \langle u_f | u_i(t) \rangle = \int \mathrm{d}^3 r \, u_f^* \exp\left(-\frac{i\hat{\mathcal{H}}t}{\hbar}\right) u_i. \tag{33}$$

A_{fi} is often too complicated to calculate, so we consider **Born's approximation**:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_1, \tag{34}$$

where $\hat{\mathcal{H}}_0$ stands for the dominant free particle term with perturbation $\hat{\mathcal{H}}_1$. The orthogonal initial and final states are defined to be eigenstates of $\hat{\mathcal{H}}_0$.

With these assumptions, we may Taylor expand $\exp\left(-i\hat{\mathcal{H}}t/\hbar\right)$:

$$\exp\left(\frac{-i\hat{\mathcal{H}}t}{\hbar}\right) \approx \left(1 - \frac{it}{\hbar}\hat{\mathcal{H}}_1 + \ldots\right) \exp\left(\frac{-i\hat{\mathcal{H}}_0 t}{\hbar}\right).$$

The leading term vanishes due to orthogonality, so the remaining dominant term gives T_{fi}:

$$T_{fi} = \int \mathrm{d}^3 r \, u_f^* \hat{\mathcal{H}}_1 u_i, \tag{35}$$

with all other constants being absorbed in the definition.

3.7 Mandelstam Variables

In the reaction $A + B \to C + D$, we define the invariant **Mandelstam variables** as:

$$\begin{aligned} s &= \left(\mathbf{p}_A + \mathbf{p}_B\right)^2 = \left(\mathbf{p}_C + \mathbf{p}_D\right)^2, \\ t &= \left(\mathbf{p}_A - \mathbf{p}_C\right)^2 = \left(\mathbf{p}_B - \mathbf{p}_D\right)^2, \\ u &= \left(\mathbf{p}_A - \mathbf{p}_D\right)^2 = \left(\mathbf{p}_B + \mathbf{p}_C\right)^2. \end{aligned} \tag{36}$$

s is called the **invariant mass**, and t is called the **four-momentum transfer**. u is only useful if the final state particles are identical (and thus indifferentiable).

	Exchanged Force Particle Type	Channel
Annihilation	Timelike	s-channel
Scattering	Spacelike	t-channel

4 Overview of Fundamental Forces

There are three types of fundamental forces, and they each correspond to one type of charge.

<table>
<tr><th>Type of Particle</th><th>Strong Color Charge (g_S)</th><th>EM Charge (e)</th></tr>
<tr><td>Up-Quarks (u, c, t)</td><td rowspan="2">color r, g, b</td><td>$+2/3$</td></tr>
<tr><td>Down-Quarks (d, s, b)</td><td>$-1/3$</td></tr>
<tr><td>Charged Leptons (e, μ, τ)</td><td rowspan="3">None</td><td>-1</td></tr>
<tr><td>Neutrinos</td><td rowspan="3">None</td></tr>
<tr><td>Photon (γ)</td></tr>
<tr><td>Eight Gluons (g)</td><td>color + anticolor</td></tr>
<tr><td>$W^{\pm}$ Bosons</td><td rowspan="2">None</td><td>± 1</td></tr>
<tr><td>Z Boson</td><td>None</td></tr>
</table>

The weak charge, in units of g_W, are much more complicated and are thus less discussed.

The 9 combinations of color-anticolor yield only 8 gluons, because only 2 out of 3 combinations of $r\bar{r}$, $g\bar{g}$, and $b\bar{b}$ give the result of color $= 1$. This **universality** of the color charge is required by gauge invariance. meaning that all quarks and gluons must have the same total charge unit g_S, not $g_S/3$ or other value.

4.1 Feynman Diagrams

- Time runs from left to right in a Feynman diagram.
- The statement that antiparticles act like particles going backwards in time only really makes sense here.
- The intermediate (virtual) particles are also called **propagators**. Every propagator brings an amplitude factor of $1/(m^2 - q^2)$.

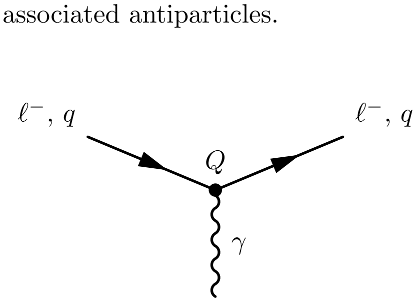
4.2 QED & QCD Vertices

The condition of an EM or a strong interaction is that the associated particles must be charged accordingly. The associated charges are also conserved (each one of three color charges is separately conserved). This means that both quarks and gluons may interact with gluons. Similarly, quarks, charged leptons, and even W bosons can interact with photons (electroweak theory).

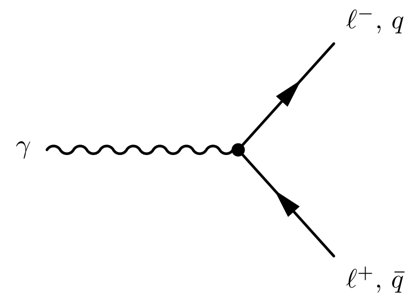
Neither QED nor QCD can change the type of particles (flavor of quarks). Nevertheless, a vertex may consist of a particle and its antiparticle if it represents creation or annihilation.

In QED, W bosons do not interact with photons. Thus, the only particles that may interact with a photon is the charged leptons and quarks (9 fundamental particles in total).

With an actual photon as the input, the output, to conserve charge and flavor, would be particles and their associated antiparticles.



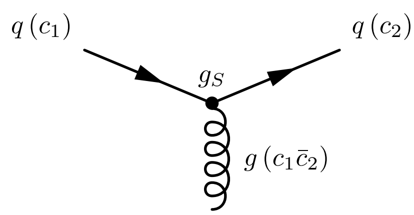
(a) QED scattering vertex.



(b) QED pair creation vertex.

By now, you must have understood the previous statement regarding antiparticles. The two diagrams above are completely equivalent — they represent the same process up to a rotation in spacetime. It is our limited existence as 3D creatures that has led us to perceive time differently from space.

For a QCD vertex, the same applies except that now the gluons also carry color charges.



For a gluon to change a red quark into a blue quark, it must take away the red and bring the blue by taking away the antiblue. This process is described by $q_r \to q_b + g_{r\bar{b}}$. An antiquark carries a unit of anticolor, in case you might wonder.

As gluons carry color charges, self-interaction is possible. However, there is no such vertex with two gluons and one quark — both charges cannot be conserved in this situation. A four-gluon vertex is also observed.

4.3 The Weak Force

While weak charge is seldom considered, we employ EM charge conservation and the accidental **lepton number conservation**. By this accidental we mean that there is no associated symmetry.

Lepton	L_e	L_μ	L_τ
e^-, ν_e	+1	0	0
e^+, $\bar{\nu}_e$	−1	0	0
μ^-, ν_μ	0	+1	0
μ^+, $\bar{\nu}_\mu$	0	−1	0
τ^-, ν_τ	0	0	+1
τ^+, $\bar{\nu}_\tau$	0	0	−1

With $W^\pm$, charged leptons may transform to their respective neutrino, and up-quarks can transform to down-quarks.

The vertices connecting the same generation of quarks are **Cabibbo favored**, while those crossing generations are Cabibbo suppressed (doubly Cabibbo suppressed if two generations are crossed).

There are 12 combinations of this weak vertex. 3 pairs are charged leptons-neutrinos, 3 pairs are same-generation quarks, 4 pairs of quarks cross one generation, and 2 pairs of quarks cross two generations.

The Z vertex is virtually the same as the γ vertex, except that neutrinos can also enter this vertex, giving a total of 12 vertices.

(a) The weak $W^\pm$ vertex.

(b) The weak Z vertex.

4.4 Self-Interaction

Photons cannot self-interact, as they themselves are not electrically charged. However, self-interaction is possible for gluons and weak interaction bosons. W bosons always come in pairs to conserve electric charge.

5 The Electromagnetic Force

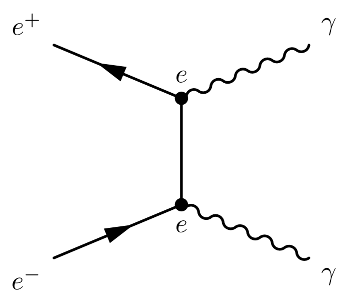
A reaction with only one vertex in the diagram, e.g. $e \to e\gamma$, cannot occur as it cannot conserve energy and momentum. However, if one or more particles are virtual, the mass constraint disappears and such a vertex is allowed. Hence, every vertex must have at least one virtual particle and so an actual reaction needs more than one vertex as any virtual particle has to have both its ends on vertices within the diagram. **Only the particles at the edges of the diagram are observed experimentally.**

Sometimes a real photon is created evanescently with a timescale compatible with the Heisenberg uncertainty principle, and it promptly decays.

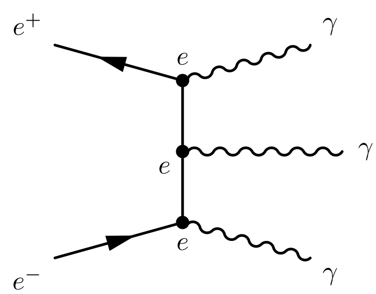
Only composite particles can decay through electromagnetism, and these decays can be of two types:

- from an excited state to a lower state;
- particle-antiparticle annihilation.

The easiest example is the positronium decay. Note that these particles must decay to at least two photons to conserve energy and momentum. Sometimes a third photon may be emitted.



(a) Para-positronium decay ($S = 0$).



(b) Ortho-positronium decay ($S = 1$).

In fact, positronium has its ground state split by a hyperfine (spin-spin) interaction into two very closely spaced states which have different charge conjugation (C) quantum numbers. Both states have $L = 0$ but differ in the combined spin, having $S = 0$ and 1. The exchange symmetry of a fermion-antifermion pair is $(-1)^{(L+S)}$. The charge conjugation operation does such an exchange, so $C(S = 0) = 1$ and $C(S = 1) = -1$.

Additionally,

$$\hat{C} |\gamma\rangle = - |\gamma\rangle , \tag{37}$$

so a system of n photons has an overall C value of $(-1)^n$. Due to conservation of the C quantum number, the $S = 0$ ground state must decay to even numbers of photons (mostly 2), and the $S = 1$, to odd numbers (mostly 3).

A first example of composite particle decay is the decay of the strange baryon Σ^0 to a Λ^0 baryon. Both baryons have quark content uds. The observation of $\Sigma^0 \to \Lambda^0\gamma$ implies that the particles must be composites.

5.1 Reactions

There are basically only two types of reaction: scattering of particles from each other, and particle-antiparticle annihilation to produce something else.

5.1.1 Scattering

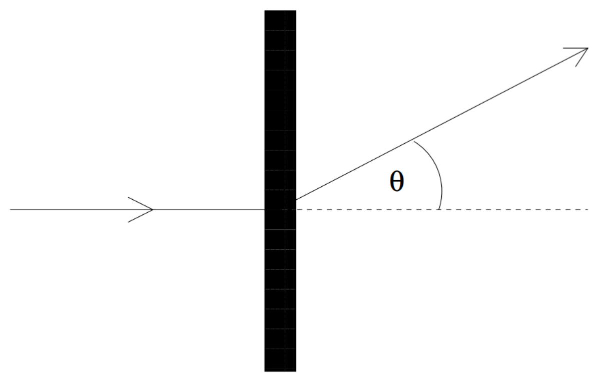


Figure 4: Scattering angle.

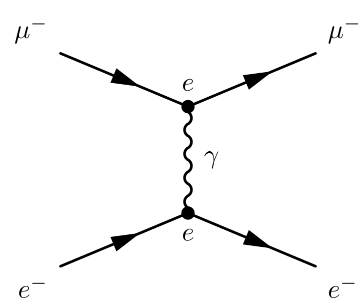
A classical calculation yields the non-isotropic cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Z_1 Z_2 \alpha \hbar c}{2mv^2}\right)^2 \frac{1}{\sin^4(\theta/2)}. \tag{38}$$

By incorporating spin, for $e^-\mu^- \to e^-\mu^-$ scattering, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2\hbar^2 c^2}{2s}\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}. \tag{39}$$

This diagram has two vertices and we get α^2 as expected.



The angular dependence is very similar to Rutherford scattering (scattering of an alpha particle from a larger nucleus), but it has an extra $\cos^4(\theta/2)$ term due to the spin.

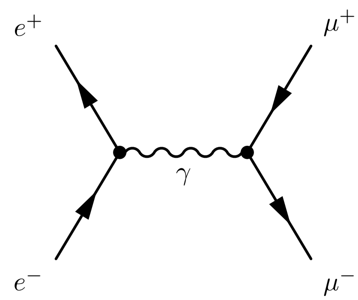
5.1.2 Annihilation

We can collide the electron and the positron at high energies, so more decay modes are allowed than just modes to photons.

The lecture notes never tried to explain the results, and we propose here

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2\hbar^2c^2}{4s}\left(1+\cos^2\theta\right). \tag{40}$$

Again, two vertices give a cross section proportional to α^2.



5.1.3 Electron-Positron Scattering

The reaction

$$e^+e^- \to e^+e^- \tag{41}$$

is known as **Bhabha scattering**. This reaction is actually more complicated, as the diagrams of scattering and annihilation both can contribute to this reaction.

> It turns out that the t-channel Feynman diagram (similar to $e^-\mu^- \to e^-\mu^-$) is the dominant process.

6 The Strong Force

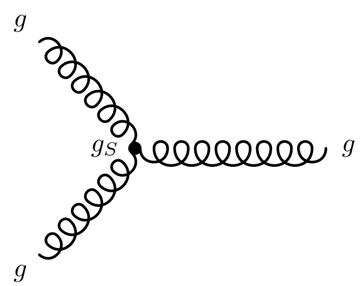
The SM theory of the strong force is called Quantum Chromodynamics (QCD). The strong force equivalent of the fine structure constant in QED is

$$\alpha_S = \frac{g_S^2}{4\pi\hbar c} \sim 0.1 - 1 \gg \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}. \tag{42}$$

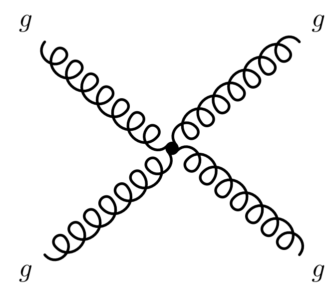
This is why the strong force is called strong.

6.1 Self Interactions: Confinement and Asymptotic Freedom

Anything color charged may participate in a QCD vertex, including the gluons themselves. There is also a four-gluon vertex with amplitude g_S^2.



(a) A three-gluon vertex with amplitude g_S.



(b) A four-gluon vertex with amplitude g_S^2.

Confinement means that we never see single quarks or gluons, but only the bound states called 'hadrons', i.e. the mesons and the baryons. This is why giving an exact value to α_S is not possible: we never have a state with a non-zero strong charge on it.

Asymptotic freedom says that the QCD field has totally the opposite behavior to QED at short distances also, namely the force goes to zero in this case.

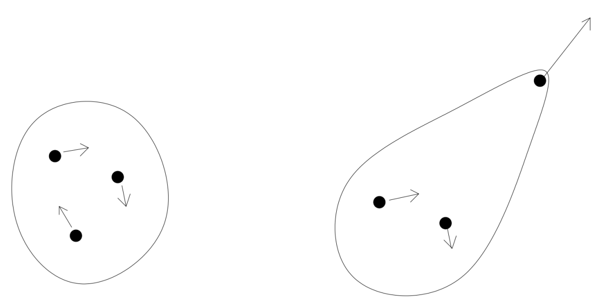


Figure 6: The balloon analogy with a proton.

One analogy often used is the "balloon" model, where the quarks are pictured as being within an infinitely strong balloon. When they are moving around the middle, there is no force on them and they act as if free. However, if given a big kick then they hit the balloon wall and get bounced back in; they can never escape.

Therefore, while the massless gluons seem to have an infinite range, confinement restricts the force field from spreading limitlessly. In effect, the range is $\sim 1\,\mathrm{fm}$.

6.2 Colored and Colorless Combinations

The hadrons themselves must be uncharged with respect to the strong force, or confinement would hold for them also and we would not observe such states as free particles.

The colorless states are given without proof:

$$\frac{1}{\sqrt{3}}\left(r\bar{r} + b\bar{b} + g\bar{g}\right), \qquad \frac{1}{\sqrt{6}}\left(rbg + bgr + grb - rgb - gbr - brg\right). \tag{43}$$

The colored states, meaning one unit of g_S (color = 1), are

$$\frac{1}{\sqrt{6}}\left(r\bar{r} + b\bar{b} - 2g\bar{g}\right), \qquad \frac{1}{\sqrt{2}}\left(r\bar{r} - b\bar{b}\right). \tag{44}$$

6.3 Universality

The **universality** of the color charge is required by gauge invariance. meaning that all quarks and gluons must have the same total charge unit g_S, not $g_S/3$ or other value.

6.4 Mesons

Quark Pair	$J = 0$ Meson	$J = 1$ Meson
$u\bar{d}$, $d\bar{u}$	$\pi^{\pm}$	$\rho^{\pm}$
$u\bar{s}$, $s\bar{u}$	$K^{\pm}$	$K^{*\pm}$
$d\bar{s}$, $s\bar{d}$	K^0, $\bar{K}^0$	K^{*0}, $\bar{K}^{*0}$
$u\bar{u}$	π^0	ρ^0
$d\bar{d}$	η	ω
$s\bar{s}$	η'	ϕ

There are $3 \times 3 = 9$ possible meson flavor combinations. All the nine ground states have the same quantum numbers because all the wavefunctions only differ due to the different masses of the quarks which each meson contains. The first excited states all have $J = 1$ but again, the wavefunction quantum numbers are the same for all nine first excited states. Both these sets of states have $L = 0$ and so only differ in the way the two quark spins add together.

Mainly for historical reasons, we call these ground states and excited states different particles (although they are not fundamental).

6.4.1 Meson Masses

Normally, for a bound state, we would say

$$m_{BS} = m_q + m_{\bar{q}} - \frac{B_E}{c^2}, \tag{45}$$

where B_E is the binding energy, which must be positive to obtain a stable state.

However, as we cannot observe free quarks, there is no way to know their free masses. Also, as they cannot be free, the requirement of a positive binding energy is lifted.

We expect a magnetic-field equivalent in QCD to be responsible for the splitting in energy. This term would look like

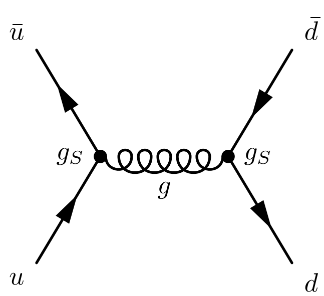
$$\frac{B_E}{c^2} \approx -K\boldsymbol{\mu}_q \cdot \boldsymbol{\mu}_{\bar{q}} \propto -K\frac{\langle \mathbf{s}_q \cdot \mathbf{s}_{\bar{q}} \rangle}{m_q m_{\bar{q}}}. \tag{46}$$

$$\mathbf{J} = (\mathbf{L} = \mathbf{0}) + \mathbf{s}_q + \mathbf{s}_{\bar{q}} \quad \Rightarrow \quad \langle \mathbf{s}_q \cdot \mathbf{s}_{\bar{q}} \rangle = \frac{1}{2}\hbar^2 \left[J(J+1) - s_q(s_q+1) - s_{\bar{q}}(s_{\bar{q}}+1) \right]. \tag{47}$$

The ground state and the first excited state have $J = 0$, 1 respectively, and this yields

$$m_{GS} = m_q + m_{\bar{q}} - \frac{3}{4}\frac{K\hbar^2}{m_q m_{\bar{q}}}, \qquad m_{FES} = m_q + m_{\bar{q}} + \frac{1}{4}\frac{K\hbar^2}{m_q m_{\bar{q}}}. \tag{48}$$

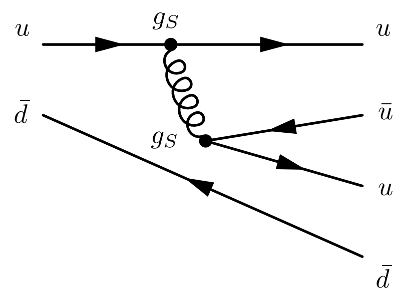
The π^0, ρ^0, ω and ϕ are seen to be in good agreement, but the η and η' are not. This can be understood by the fact that a pure (e.g.) $u\bar{u}$ state can change to (e.g.) a $d\bar{d}$.



Thus, the flavor eigenstates are not necessarily the mass eigenstates. The π^0, η, and η' are in fact mixtures of $u\bar{u}$, $d\bar{d}$, and $s\bar{s}$ in varying proportions. The π^0 happens to have very little $s\bar{s}$ and so still agrees well with the basic model as $m_u \approx m_d$.

6.4.2 Meson Decays

Since the masses of the three colored quarks of one flavor, e.g. u_r, u_g and u_b, have identical masses, then QCD cannot cause the decay of quarks. Thus, like QED, the only strong force decays are the quarks in the composite states being rearranged into lower energy states. For mesons, this means the excited states can decay to ground states.



The ground state mesons by definition have no lower state to decay to, so they would be stable if there was only the strong force. However, the three mesons made of quark-antiquark pairs, namely π^0, η, and η', can decay electromagnetically, mainly to photons in the same way as positronium decays. The other six mesons can only decay through the weak force.

6.5 Baryons

When considering which states are allowed, there is a complication here which is not present in the mesons. Identical fermions cannot appear as one wants in combinations due to Pauli's exclusion. For example, the proton is uud, but some of the other uud states are forbidden.

I feel it necessary to at least scratch the surface of this topic. The total wavefunction is

$$\psi(\text{total}) = \psi(\text{space}) \cdot \psi(\text{spin}) \cdot \psi(\text{flavor}) \cdot \psi(\text{color}), \tag{49}$$

In ground state, we assume that ψ(space) is symmetric.
As we've seen in the three-quark combination, color neutrality of baryons require that ψ(color) is totally antisymmetric.

This being said, we've discovered the condition for the left two wavefunctions:

$$\psi(\text{spin}) \cdot \psi(\text{flavor}) \quad \text{must be effectively symmetric.} \tag{50}$$

The flavor wavefunction is very similar to the color wavefunction, except that we don't necessarily need "flavor neutrality". Yet, it is indeed true that ψ(flavor) for uds is totally antisymmetric:

$$\psi(\text{flavor}) = \frac{1}{\sqrt{6}}\left(uds + dsu + sud - usd - sdu - dus\right). \tag{51}$$

Therefore, the ground state requires that ψ(spin) must be totally antisymmetric as well. As we know $S = 0$ is the only antisymmetric state for two fermions, $S = 1/2$ if we're to add a third fermion.

$S = 1/2$ built on $S = 0$ has two states; one is antisymmetric, and the other is symmetric.

The first excited state of uds is $S = 1/2$, but built on the symmetric $S = 1$ for two fermions. This implies that ψ(spin) is now only partially antisymmetric, and thus ψ(space) is no longer totally symmetric. The second excited state is $S = 3/2$.

For ground-state baryons with two identical quarks, ψ(flavor) is partially antisymmetric again, so we require ψ(spin) to correct the symmetric part by using $S = 0$ in the identical quarks.

For ground-state baryons with three identical quarks, ψ(flavor) is totally symmetric, so ψ(spin) must be totally symmetric as well. This must require $S = 3/2$.

Quark Triplet	$J = 1/2$ Baryon	$J = 3/2$ Baryon
uds	GS: Λ; FES: Σ^0	Σ^{*0}
uud	p	Δ^+
udd	n	Δ^0
uus	Σ^+	Σ^{*+}
uss	Ξ^0	Ξ^{*0}
dds	Σ^-	Σ^{*-}
dss	Ξ^-	Ξ^{*-}
uuu	None	Δ^{++}
ddd		Δ^-
sss		Ω^-

To find baryon masses, similarly, we have

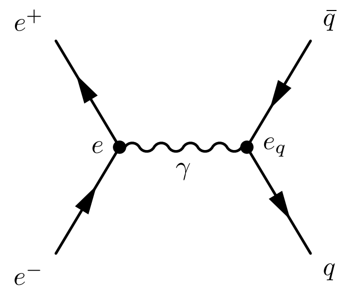
$$M = \sum_i m_i + \sum_{i>j} K \frac{\langle \mathbf{s}_i \cdot \mathbf{s}_j \rangle}{m_i m_j}. \tag{52}$$

For baryons with only u and d, we may use $m_u \approx m_d$ to simplify the expressions.

The masses we find here are actually parameters, which we may call **constituent masses**, as they are appropriate to use when considering the quarks as constituents of hadrons. However, when we interact with quarks using photons or other particles, the masses we measure for the quarks from kinematics, called **current masses**, are quite different.

6.6 Electron-Positron Annihilation into Quarks

This is very similar to the diagram for $e^+e^- \to \mu^+\mu^-$, with the only difference being that the quarks carry different units of charge.



Quantitatively, the cross section scales as

$$\text{Amplitude}(\mu\mu) \propto e^2, \qquad \text{Amplitude}(q\bar{q}) \propto ee_q. \tag{53}$$

Summing over all quarks, the cross section as amplitude squared is

$$\sigma_{\mu\mu} \propto e^4 \propto \alpha^2, \qquad \sigma \propto \alpha^2 \sum_q \left(\frac{e_q}{e}\right)^2. \tag{54}$$

Remember that each flavor of quark comes in 3 color states. This introduces an additional factor of 3.

6.7 Hadronization

Hadronization is how the quarks produced become the hadrons observed experimentally. Comparing opposite-sign electromagnetic charges to QCD charges, then the gluon lines self-interact and tend to collapse to a fixed "radius". This is often called a "string", with roughly constant width $\sim 1\,\mathrm{fm}$. This also implies a constant amount of energy per unit length.

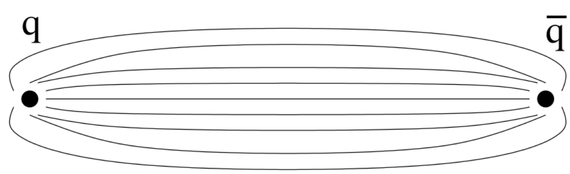


Figure 7: The QCD field lines.

The initial kinetic energy of the quarks is going into energy stored in the flux tube. If a $q\bar{q}$ pair is produced within the tube, so removing a section of the tube, then the remaining energy stored in the field is reduced. The cut-off piece forms a meson (in this case) and the rest of the string can break up further.

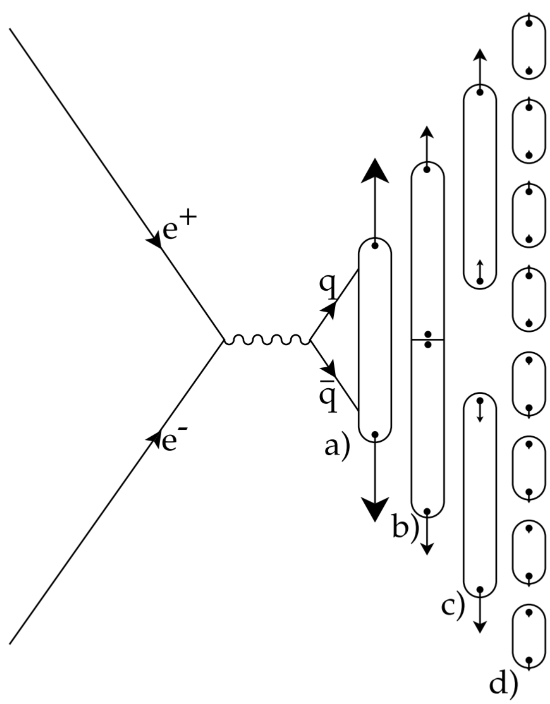


Figure 8: String fragmentation into mesons.

Looking at pictures of these interactions shows that the production is not isotropic. What is actually seen is a narrow "jet" of hadrons which follow the original quark direction.

7 The Weak Force

- The force bosons for weak interaction are not massless.
- The weak interaction $W^{\pm}$ vertex can change the type of fermion which is interacting.
- The weak force does not conserve P and C.

The weak force interacts with all matter, meaning all fundamental fermions carry weak charge.

7.1 The Weak Force Bosons

The $W^{\pm}$ interactions **change the type of particle** as they carry off EM charge. In contrast, the Z vertex has the same particles going in and out. As for gluons, the $W^{\pm}$ and Z carry weak charge (and indeed the $W^{\pm}$ also carry EM charge). Therefore, there are again self-interaction vertices.

We know that

$$m_W \approx 80\,\mathrm{GeV/c^2}, \qquad m_Z \approx 91\,\mathrm{GeV/c^2}, \tag{55}$$

and hence the force is very short-ranged.

The corresponding dimensionless constant for W interactions is

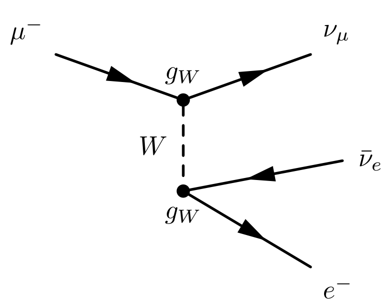
$$\alpha_W = \frac{g_W^2}{4\pi\hbar c} \approx \frac{1}{30} > \alpha. \tag{56}$$

So, actually, the weak force is stronger than the EM force.

7.2 Muon Decay

Muons have only one significant decay mode:

$$\mu^- \to \nu_\mu e^- \bar{\nu}_e. \tag{57}$$



7.3 Tau Decay

Clearly, taus can decay to electrons as muons do, but as they are the heaviest of the three, taus can also decay to muons.

Additionally, the tau is heavier than the pion, so it can decay to hadrons. One example is

$$\tau^- \to \nu_\tau d\bar{u}. \tag{58}$$

If this ever perplexes you, remember that W bosons are capable of changing flavors. While QED or QCD vertices require the creation of a quark and the associated antiquark, this is not the concern of the weak vertex.

Given universality (same amplitude), if we assume that asymptotic freedom holds (by which we mean the quarks are nearly free like electrons and muons), the rate might be expected to be e : μ : hadrons $= 1 : 1 : 1$. However, this ignores color, and just as the e^+e^- cross section gets enhanced by a factor of three, so does the τ decay rate to hadrons. Therefore, e : μ : hadrons $= 1 : 1 : 3$.

7.4 Parity and Charge Conjugation

Helicity is the spin resolved along the momentum direction, usually specified in units of $\hbar$:

$$h = \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{p}|}. \tag{59}$$

Therefore, for a spin-1/2 particle, the helicity is between $-1/2$ and $1/2$.

Handedness, which has no simple classical analogue, is different but in the limit of $v \to c$, handedness and helicity eigenstates coincide. Specifically, as $v \to c$, "right-handed" (RH) becomes helicity $+1/2$, and "left-handed" (LH) becomes helicity $-1/2$.

$$P(RH) = \frac{1}{2}\left(1 + \frac{v}{c}\right), \qquad P(LH) = \frac{1}{2}\left(1 - \frac{v}{c}\right). \tag{60}$$

Slow particles are approximately equally RH and LH. Fast particles are almost always RH.

The W bosons only interact with the LH part of the leptons and quarks, i.e. when $h = -1/2$ in the relativistic limit. The weak force is therefore often called a **left-handed interaction**. Conversely, antifermions only interact if they are right-handed, i.e. $h = +1/2$ in the relativistic limit.

Consider any weak interaction with an incoming LH neutrino. Under a parity operation, all position **r** and momentum **p** change sign, so there is no effect on $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. All angular momentum vectors have the same property, including spin, so the neutrino spin vector is unchanged.

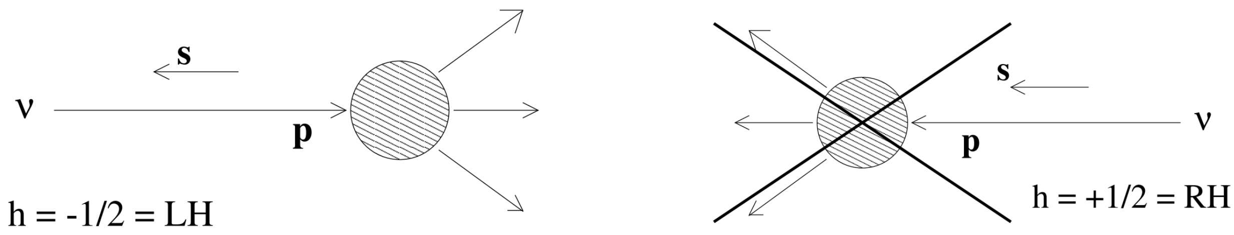


Figure 9: Left: a LH neutrino interaction. Right: the same interaction after a $\hat{P}$ operation, which is not observed as RH neutrinos do not interact.

The same effect is also responsible for $\hat{C}$ violation; applying a $\hat{C}$ operation changes the neutrino to an antineutrino but does not affect the momentum or spin. The transformed reaction does not happen as LH antineutrinos do not interact.

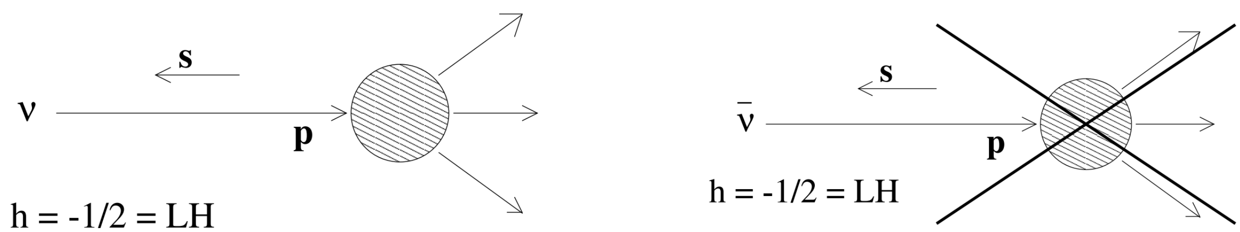


Figure 10: Left: a LH neutrino interaction. Right: the same interaction after a $\hat{C}$ operation, which is not observed as LH antineutrinos do not interact.

Hence, $\hat{P}$ and $\hat{C}$ are violated, but the combined $\hat{P}\hat{C}$ (or $\hat{C}\hat{P}$) operation is not.

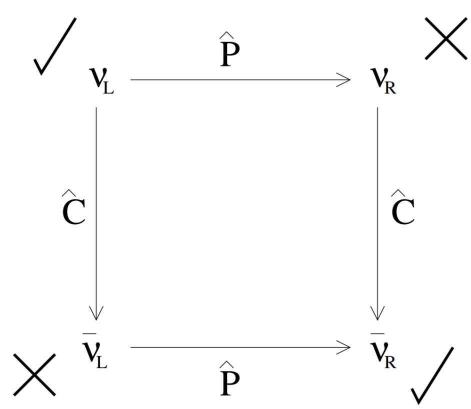
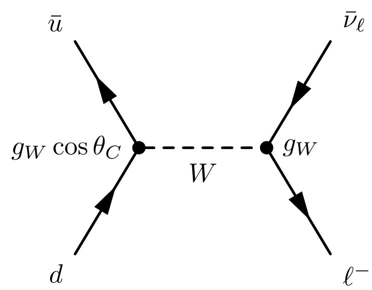


Figure 11: Summary of $\hat{P}$ and $\hat{C}$ operations on neutrinos.

7.5 Charged Pion Decay

The main decays of the charged pion are **leptonic** (i.e. they involve only leptons in the final state) and are

$$\pi^- \to e^- \bar{\nu}_e, \qquad \pi^- \to \mu^- \bar{\nu}_\mu. \tag{61}$$



It seems that there is somewhat limited energy available for the muon decay and it would therefore be expected that the electron phase space, and hence rate, may be higher than the muon rate. However, it turns out that the muon rate is 8000 times bigger than the electron rate. This must have to do with the way the weak force interaction works.

The antineutrino is almost massless compared to the charged leptons, so it must be at the relativistic limit to a very good approximation. Hence, as it is RH, it must have $h = +1/2$. The pion is spin 0, so to balance angular momentum, the charged lepton must also have $h = +1/2$.

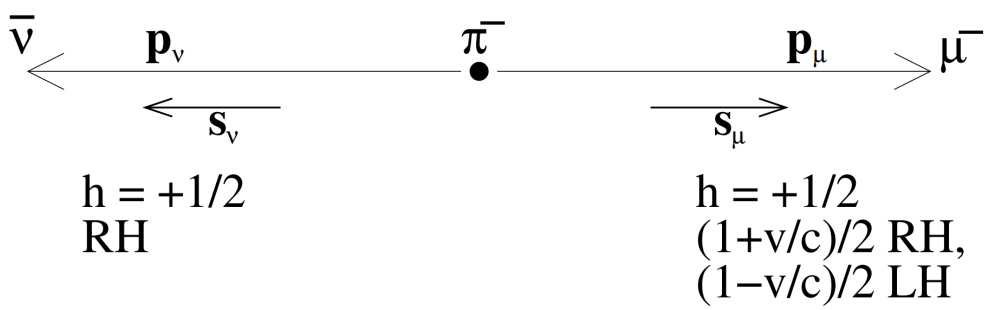


Figure 12: The reaction in the CoM frame of π^-.

If the charged lepton were also at the relativistic limit, this would not be allowed as it would be a RH fermion and so the decay could not occur. However, the charged leptons are not at $v = c$ and so while $h = +1/2$, there is still a small amount of LH state included, namely $(1 - v/c)/2$. Hence, there is a suppression factor which gets larger as the velocity gets closer to c.

Clearly, muons are much heavier than electrons, so they are less relativistic. This gives more LH muons and thus a higher reaction rate.

7.6 The Cabibbo Matrix

The mere fact that kaons (which contain s quarks) decay at all implies that there must be a cross-generational coupling for quarks. There are **semileptonic** (leptons and hadrons produced) and **hadronic** (only hadrons produced) decays.

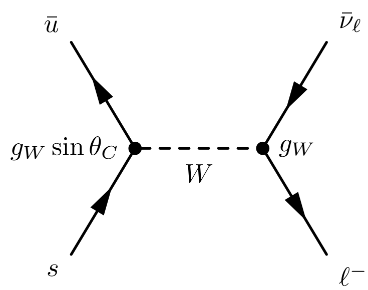
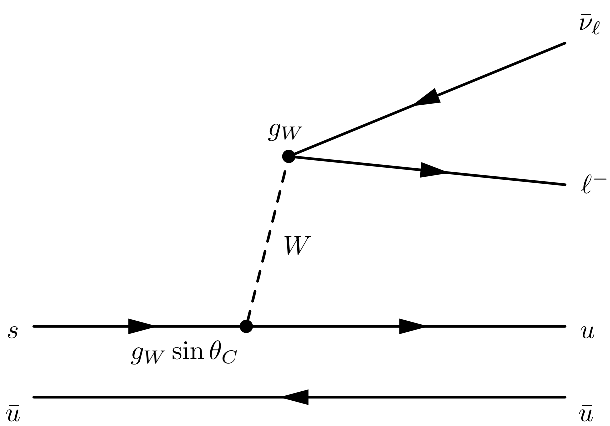
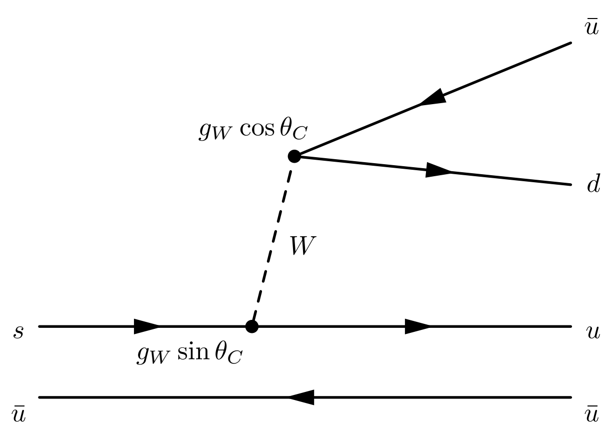


Figure 13: Leptonic decay of the charged kaon.



(a) Semileptonic decay of the charged kaon.



(b) Hadronic decay of the charged kaon.

Indeed, it is also possible for other quarks to interact cross-generationally.

In general, any $+2e/3$ quark can connect to any $-e/3$ quark in a weak interaction. The relative weak force strengths are described by the **Cabibbo matrix** V_{ij}.

For example, the coupling between the first two generations is shown below.

$$\begin{pmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{pmatrix} = \begin{pmatrix} 0.975 & 0.225 \\ -0.225 & 0.975 \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}. \tag{62}$$

This means that every ud vertex has a multiplicative factor of $V_{ud} = \cos\theta_C$, etc. The general 3×3 matrix is called the **CKM matrix** (Cabibbo-Kobayashi-Maskawa).

What is happening is that the flavor eigenstates are not exactly the same as the weak interaction eigenstates. The cross-generational coupling is quite small but not zero.

7.7 Mixing & Oscillations

This is actually standard quantum mechanics and linear algebra. In the CoM system, any state may be expressed as the linear combination of mass eigenstates:

$$\psi(0) = \sum_n a_n u_n. \tag{63}$$

At a later time t, the state will evolve as

$$\psi(t) = \sum_n a_n f_n u_n, \qquad f_n(t) = \exp\left(\frac{-it}{\hbar} m_n c^2\right) \exp\left(-\frac{t}{2\tau_n}\right). \tag{64}$$

Consider the flavor eigenstates as linear combinations of mass eigenstates:

$$\mathbf{v} = A\mathbf{u}, \tag{65}$$

where invertible A may be taken to be a rotation.

Therefore,

$$\psi(t) = \sum_n a_n f_n u_n = \sum_{mn} a_n f_n \left[\left(A^{-1}\right)_{nm} v_m\right] = \sum_m \left(\sum_n a_n f_n \left(A^{-1}\right)_{nm}\right) v_m. \tag{66}$$

So, the probability of measuring a flavor eigenstate is

$$P_m = \left|\sum_n a_n f_n \left(A^{-1}\right)_{nm}\right|^2. \tag{67}$$

The oscillations occur in the amplitudes naturally.

Specifically, for the neutrino mixing, the mixing matrix U is called the **PMNS matrix**:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \tag{68}$$

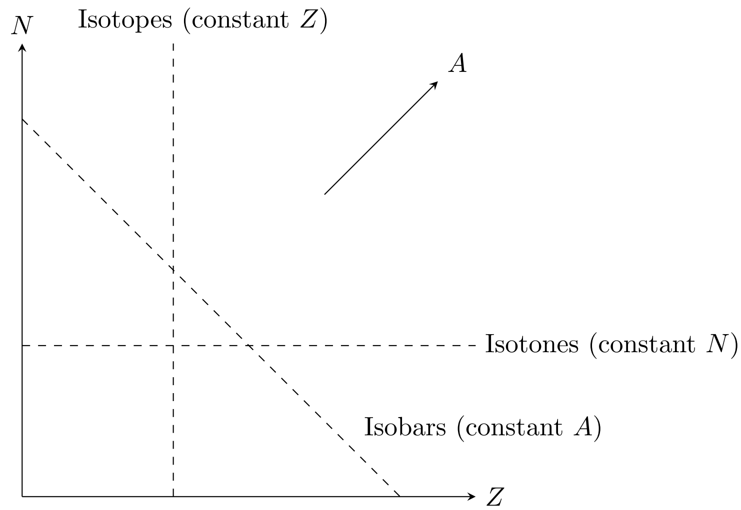
where LHS is the generation eigenstates, and RHS is the mass eigenstates. This mixing requires that the neutrinos have nonzero mass differences.

Neutrino oscillations by definition mean there must be some way to e.g. change a ν_e into a ν_μ. This is a lepton number violating process and so cannot happen under SM. This implies physics beyond the SM.

8 Nuclear Models

8.1 Introduction

A nucleus is described by the number of protons, Z, and the number of neutrons, N. The total number of protons and neutrons (collectively called nucleons) is denoted by the atomic mass number, A, which is, by definition $Z + N$.



Though hadrons are colorless, there are residual multipole terms of the strong force. This residual Van der Waals-like strong force between them which, although much weaker than the full strong force, is still able to overcome the proton EM repulsion and bind nucleons together into nuclei. We will call this force the **nuclear force**.

One important property of the nuclear force is that it is approximately independent of which nucleons are involved. Both the proton and neutron are made of three quarks with very small masses compared to the mass of the QCD field surrounding them, which results in the proton and neutron masses being very close.

The nuclear force can be well described as a Yukawa force resulting from meson exchange, i.e. radiation and absorption of virtual pions, rhos, etc.

Since for any force, the range is $\sim \hbar/mc$, then for all but the shortest distances, **pion exchange** dominates as the pions are the lightest mesons.

We will try to justify quantitatively the binding energy B_E,

$$m = Zm_p + Nm_n - \frac{B_E}{c^2}, \tag{69}$$

Figure 15: Examples of meson exchange diagrams resulting in the nuclear force.

where it is required to be positive, as the constituents can be free particles.

8.2 The Liquid Drop Model

The force between each nucleon and the others is so short range that it is negligible for all but the nearest neighbors. This approximation is good for $A \geq 20$.

This is physically similar to the saturation in a water drop, where the Van der Waals forces between the water molecules effectively make only the nearest-neighbor interactions significant. Because of this analogy, the resulting nuclear model is often called the "liquid drop" model.

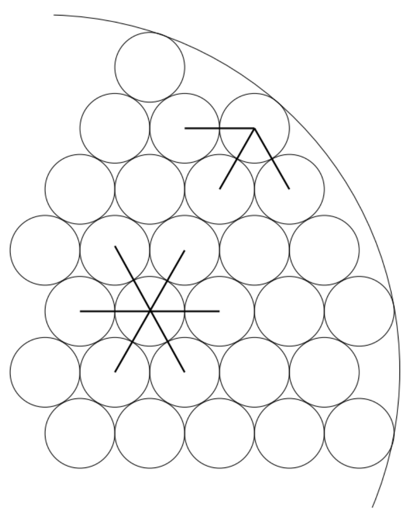


Figure 16: The liquid drop nuclear model.

8.3 The Semi-Empirical Mass Formula (SEMF)

$$B_E = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + a_p \frac{1}{A^{1/2}}. \tag{70}$$

8.3.1 Liquid Drop Model: Volume & Surface Area Term

As the model predicts that the interaction comes from only the nearest neighbors, the binding energy for each nucleon would be expected to be constant.

One may expect that the volume of the nucleus will increase linearly with the number of nucleons. This gives

$$V = \frac{4}{3}\pi r^3 \propto A \quad \Rightarrow \quad r \propto A^{1/3}. \tag{71}$$

However, the nucleons on the surface of the liquid drop have fewer neighbors (overestimate). This requires a correction that scales with the surface area, $4\pi r^2$. So, the simple analysis from the liquid drop gives us

$$B_E = a_v A - a_s A^{2/3}. \tag{72}$$

A simple classical liquid drop would ignore EM and quantum effects, and corrections are thus in place.

8.3.2 Coulomb Repulsion

The protons will on average be spread evenly throughout the nucleus, which means the charge density is uniform.

This standard problem needs no elaboration, and, with Z protons and $r = r_0 A^{1/3}$,

$$\Delta B_E = -\frac{3}{5}\frac{e^2}{4\pi\epsilon_0 r_0}\frac{Z^2}{A^{1/3}} = -a_c \frac{Z^2}{A^{1/3}}.$$

The negative sign is here because this term represents repulsion, making the binding energy "harder to be positive".

However, the energy given by the expression above is that needed to spread all the charge out throughout all space to an infinitely small density. However, the binding energy is defined as the energy need to break the nucleus into its constituent nucleons, i.e. to break it into neutrons and protons, but not to spread the individual proton charges out. Indeed, the equation says even one proton, i.e. $Z = 1$, gives a correction to the binding energy, even though there is nothing to repel it. This means the correction to the binding energy should not be quite as large. Therefore, a more appropriate correction due to Coulomb repulsion is

$$\Delta B_E = -a_c \frac{Z(Z-1)}{A^{1/3}}. \tag{73}$$

8.3.3 Asymmetry Term

The idea here is identical to the concept of a Fermi level in the physics of solids, although instead of just electrons, we now have two types of fermions: protons and neutrons.

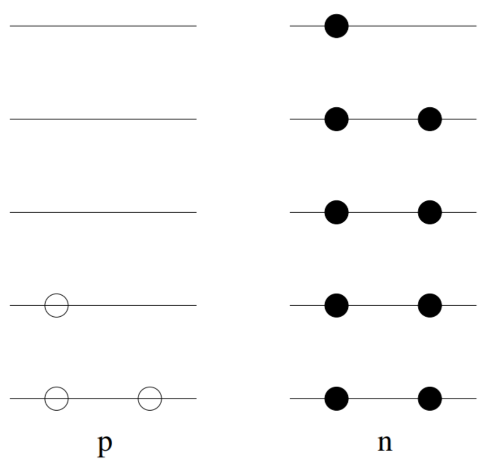


Figure 17: filling up the nuclear states with two sets of identical particles.

If we really tried to form a nucleus purely from neutrons, as implied by the terms we have so far for the binding energy, they would have to be put into higher and higher energy levels and so would be less and less strongly bound, reducing the binding energy. Clearly, putting protons into the nucleus instead would be beneficial for the binding energy as they could go into the deepest empty proton levels. It is clear the best situation is when the two are evenly balanced with $N = Z$.

Therefore, a reasonable parameterization is $\Delta B_E \propto -(N-Z)^2$, i.e. the binding energy is reduced symmetrically for either $N > Z$ or $Z > N$. Also, the spacing between states depends inversely on the size of the nucleus such that larger nuclei have less of a binding energy loss if $N \neq Z$. So,

$$\Delta B_E = -a_a \frac{(N-Z)^2}{A}. \tag{74}$$

8.3.4 Pairing Term

A pair of identical nucleons will have zero total spin and thus maximal overlap in spatial wavefunctions. This gives more binding energy. Hence, the nucleus will be more strongly bound for ones with an even number of nucleons of either type. Therefore,

$$\Delta B_E = a_p \frac{1}{A^{1/2}}, \tag{75}$$

where

$$a_p = \begin{cases} +\text{ve} & \text{even-even nuclei,} \\ 0 & \text{even-odd nuclei,} \\ -\text{ve} & \text{odd-odd nuclei.} \end{cases} \tag{76}$$

The pairing term implies even-even nuclei always have the spins totaling zero for the nucleons in the same spatial state, so all such nuclei would be expected to have ground states with a total spin of zero.

8.4 The Beta-Stability Curve

We may express B_E in terms of only two variables from A, N, Z. The easiest way to analyze is through lines of constant A:

$$B_E(A, Z) = \left(a_v A - a_s A^{2/3} - a_a A + \frac{a_p}{A^{1/2}}\right) + \left(\frac{a_c}{A^{1/3}} + 4a_a\right) Z - \left(\frac{a_c}{A^{1/3}} + \frac{4a_a}{A}\right) Z^2. \tag{77}$$

This is a quadratic function in Z.

Given fixed even A, as Z changes by one, the pairing term changes sign. (For a fixed odd A, Z and N will always be odd-even combination, so it is free from this problem.) Hence, this shifts the quadratic curve up and down by $a_p/A^{1/2}$.

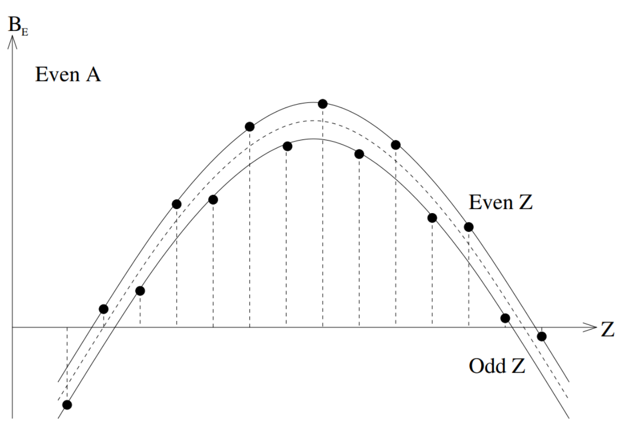


Figure 18: Binding energy for constant A as a function of Z for even-A nuclei.

Even within the region of positive binding energy, the non-maximum values of Z are not necessarily stable; as we will see, beta decay allows them to change protons to neutrons and vice-versa, and so move along the curve to the maximum value. Hence, we only see the reasonably long-lived or stable nuclei which are at or near the maximum.

The only two terms determining this quadratic are Coulomb and asymmetry. It is more obvious if we return to $B_E(A, N, Z)$. The asymmetry term vanishes if $N = Z = A/2$, and $Z = 0$ or 1 minimizes Coulomb repulsion.

Considering the A dependence, we expect $Z \approx A/2$ for small A and $Z < A/2$ for large A.

The line of the most stable nuclei is called the "**beta-stability curve**". There can be several stable nuclei for a given A.

8.5 The Shell Model

This is a very distinct model based on quantum energy levels.

8.5.1 Magic Numbers

The strongly-bound states occur when Z or N have one of a set of so-called "**magic numbers**":

$$2, 8, 20, 28, 50, 82, 126. \tag{78}$$

The elements ${}^{4}_{2}\mathrm{He}$ and ${}^{208}_{82}\mathrm{Pb}$ are called doubly-magic.

8.5.2 Nuclear Potentials

Let us return to our liquid drop model.

- All water molecules feel the total force of zero if they are not on the surface.
- The range of the attractive residual strong force is extremely short.

These observations are enough to make a physical guess — the **Woods-Saxon potential**:

$$V(r) = -\frac{V_0}{1 + \exp\left[(r - a)/d\right]}. \tag{79}$$

This function virtually the same as the Fermi-Dirac distribution, so a functions as the chemical potential (nuclear radius), and d functions as $k_{\mathrm{B}}T$ (the distance over which the potential rises).

It is not trivial to solve this potential. However, simpler cases like the infinite square well and quadratic potential cannot reproduce magic numbers over 20 ($2 \times (2\ell + 1)$ states for a given angular momentum). We need to add the spin-orbit coupling.

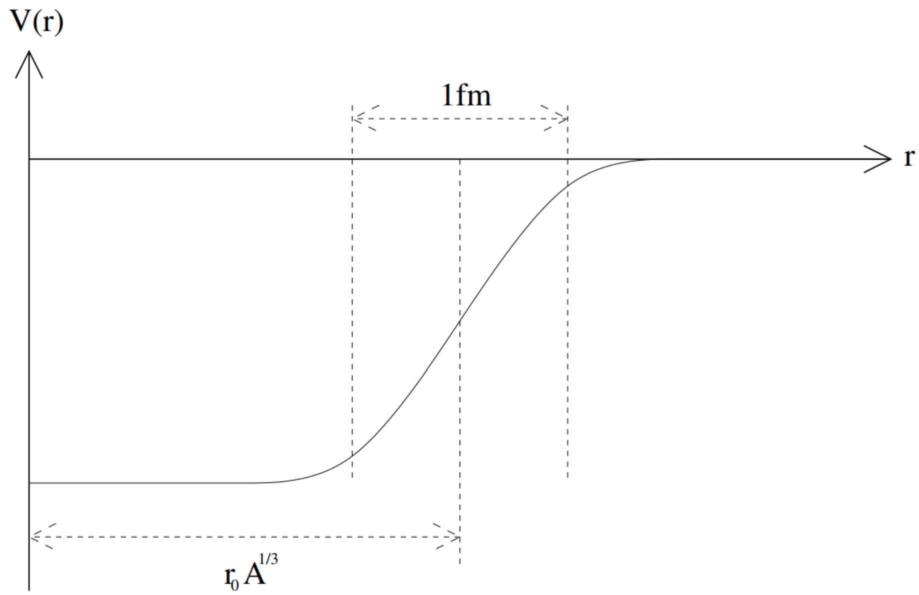


Figure 19: The Woods-Saxon nuclear potential.

8.5.3 Spin-Orbit Coupling

The energy term $\propto \mathbf{l}\cdot\mathbf{s}$ helps split the $4\ell+2$ degeneracy. By $\mathbf{j}=\mathbf{l}+\mathbf{s}$,

$$\langle \mathbf{l}\cdot\mathbf{s}\rangle = \frac{\hbar^2}{2}\left[j(j+1)-\ell(\ell+1)-s(s+1)\right].$$

By $j=\ell\pm 1/2$,

$$\langle \mathbf{l}\cdot\mathbf{s}\rangle = \begin{cases} \dfrac{\hbar^2}{2}\ell & j=\ell+\dfrac{1}{2}, \\ -\dfrac{\hbar^2}{2}(\ell+1) & j=\ell-\dfrac{1}{2}. \end{cases} \tag{80}$$

The larger j states have more binding energy and are thus more tightly bound.

9 Radioactive Decay

9.1 Gamma Decay

The gamma-rays observed in nuclear radiation are high-energy photons, so gamma decays are **EM decays**. As the EM force cannot change quark flavors, so neither Z nor N is changed after a gamma decay.

Gamma decay only happens from an excited state to a lower state.

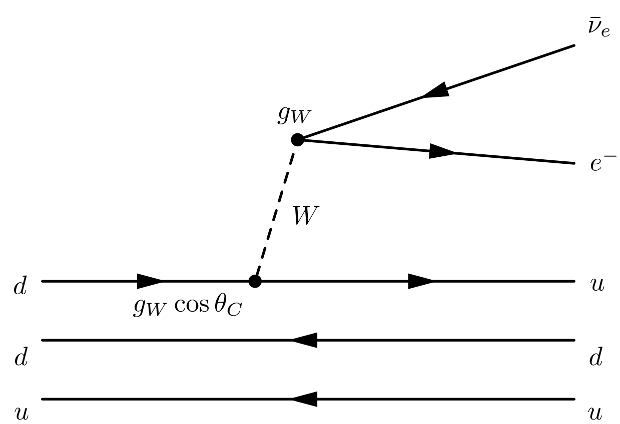
A typical EM decay timescale is $10^{-16}\,\mathrm{s}$, while for larger changes in angular momentum $\Delta J \sim 4$ or 5, this can increase to $10^{3}\,\mathrm{s}$. This suppression is due to the transition having to occur as a higher radiation multipole.

A related process is called **internal conversion**, where a virtual photon is emitted and absorbed by one of the atomic electrons.

9.2 Beta Decay

Beta decays are a **weak force process**. The beta particle which is emitted is simply an electron produced from a virtual W boson along with a (normally unobserved) electron antineutrino. The prototypical beta decay (β^-) is that of the isolated neutron:

$$n \to p + e^- + \bar{\nu}_e. \tag{81}$$



This decay is a three-body decay concerning the d quark, so the electron can have a continuous spectrum.

There is a reverse process (β^+):

$$p \to n + e^+ + \nu_e, \tag{82}$$

which does not happen for free protons due to energy conservation.

Note also that since quarks only change to other flavor quarks in weak interactions, this means there must be a baryon in the final state, and so any decay process to a lighter meson is not possible. This is sometimes referred to as **baryon number conservation**. Anyway, this means a free proton is absolutely stable as far as we know.

While free neutrons may decay with $\tau_n = 879\,\mathrm{s}$, neutrons bound in nuclei may be energetically forbidden from decay. This argument also applies for bound protons, where β^+ is sometimes possible.

9.2.1 Odd-A Nuclei

Obviously, nuclear mass = constituent mass − binding energy, so the mass plot looks inverted, where the most stable nucleus has the least mass given constant constituent mass.

A competing process to positron emission is **electron capture** (EC), as atomic electrons in an $\ell = 0$ state have a nonzero probability of being near the origin and hence being inside the nucleus.

$$e^- + p \to n + \nu_e \tag{83}$$

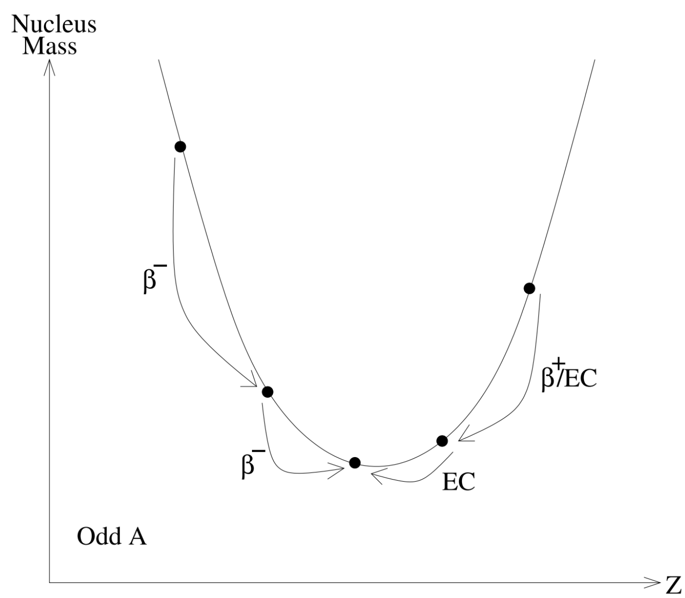


Figure 20: Beta decays increasing stability by changing Z at constant (odd) A.

EC can occur even when positron emission is energetically forbidden. Ignoring the mass of the neutrino, we have

$$\begin{cases} \beta^+: & m(Z,N) > m(Z-1,N+1) + m_e, \\ \text{EC:} & m_e + m(Z,N) > m(Z-1,N+1). \end{cases} \tag{84}$$

Therefore, if the nuclei are different in mass by less than m_e, only EC is possible.

The reverse process, called positron capture,

$$e^+ + n \to p + \bar{\nu}_e, \tag{85}$$

is impossible as there are no naturally occurring positrons orbiting around the nucleus.

9.2.2 Even-A Nuclei

The pairing term now adding complications again.

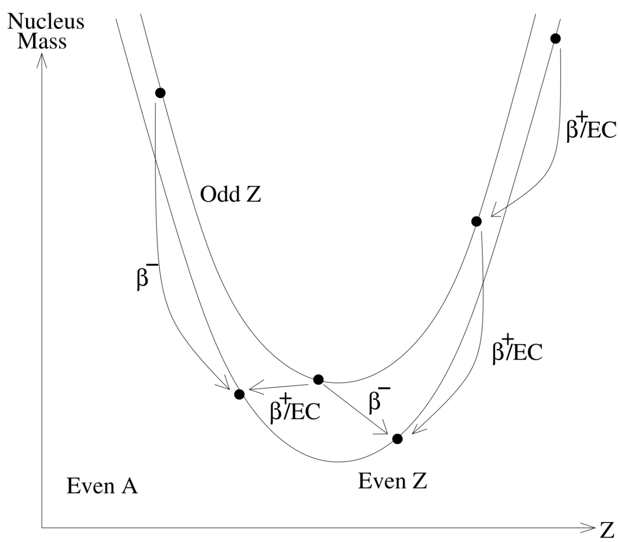


Figure 21: Beta decays increasing stability by changing Z at constant (even) A.

Beta decays on either side of the minimum alternate between even-even (the lower mass curve) and odd-odd (the higher mass curve) nuclei.

This alternation increases the possibility of **double beta decay**. If the heavier even-even nucleus can convert two neutrons to protons simultaneously, then it does not have to go via the heavier intermediate odd-odd state:

$$^{A}_{Z}\mathrm{X} \to {}^{\ A}_{Z+2}\mathrm{Y} + 2e^- + 2\bar{\nu}_e. \tag{86}$$

However, it is very unlikely for two weak decays to happen at the same time, and correspondingly, the lifetime can be up to 10^{22} years.

9.3 Alpha Decay

Alpha decay is due to **strong force**. It occurs by emission of nucleons, specifically an alpha particle ${}^4_2\mathrm{He}$ is ejected:

$$ {}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X} \to {}^{\mathrm{A}-4}_{\mathrm{Z}-2}\mathrm{Y} + {}^4_2\mathrm{He}. \tag{87} $$

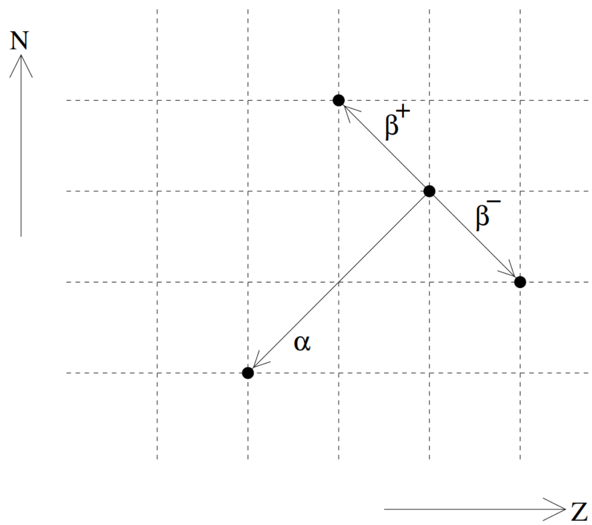


Figure 22: Change in nucleon numbers by $\beta^{\pm}$ and α decays. γ decays do not change Z, N, or A.

"The energy release is quite small unless the products are particularly strongly bound", says the doubly magic ${}^4_2\mathrm{He}$. Alpha decays are only energetically possible for $A > 150$ and typically seen for $A > 200$. For $A > 200$, $A - 4 > 200$, and so the product can decay again. A sequence of alpha decays is often seen that can be many decays long.

However, a pure alpha decay sequence would leave a lower A nucleus with a higher and higher fraction of neutrons, whereas the beta-stability curve requires the fraction of neutrons to become lower as A is reduced. Hence, the beta decays bring the intermediate nuclei back closer to the beta-stability curve.

As Alpha decays always change A by 4, so there are effectively only four such decay sequences.

We can consider a simple model where the alpha particle has an independent existence within the nucleus before the decay. This then looks like a quantum tunneling process. The probability of transmission through the whole barrier is given by the **Gamow factor** G:

$$ \exp\left(-2\int \mathrm{d}r\, k(r)\right) = \mathrm{e}^{-2G}. \tag{88} $$

The Gamow factor is given by

$$ G = \frac{\sqrt{2mQ}}{\hbar}\frac{Ze^2}{4\epsilon_0 Q}. \tag{89} $$

10 Nuclear Fission & Nuclear Fusion

10.1 Fission Energetics

Alpha decay may be viewed as a special case of **fission**, which refers to the general process of splitting a nucleus into two or more fragments.

The maximum binding energy per nucleon is reached around $^{56}_{26}$Fe, so we expect that fission is energetically possible for nuclei larger than around twice this size. It turns out that splitting a nucleus into two equal nuclei normally gives the largest energy release, Q:

$$Q = m(Z,N)c^2 - 2m(Z/2,N/2)c^2 = 2B_E(Z/2,N/2) - B_E(Z,N). \tag{90}$$

One may estimate Q via the SEMF. Energy release is when $Q > 0$:

$$\frac{Z^2}{A} > 0.702\frac{a_s}{s_c} \approx 18. \tag{91}$$

This condition is actually satisfied for $A > 100$, which in turn gives $Z > 42$.

Spontaneous fission is possible if it is energetically favored. The two terms concerned are again the surface area term and the Coulomb term. This fission begins with infinitesimal deformation, where surface area correction increases, while Coulomb repulsion is decreased. For the Coulomb term to dominate, we get

$$\frac{Z^2}{A} > \frac{2a_s}{a_c} \approx 51. \tag{92}$$

This gives $A > 407$ and $Z > 144$, the nuclei of which do not exist by now.

Therefore, for $18 < Z^2/A < 51$, the fission overall is energetically allowed, but activation energy is needed to overcome the potential barrier.

10.2 Induced Fission

The potential barrier to be tunneled through makes the fission slow. However, it may be sped up by exciting the nuclei.

The most obvious method is to bombard the nuclei with gamma radiation. However, this is an EM reaction and so does not have as large a cross section as a nuclear force reaction. In addition, there are no simple ways to make intense-enough gamma sources in practice.

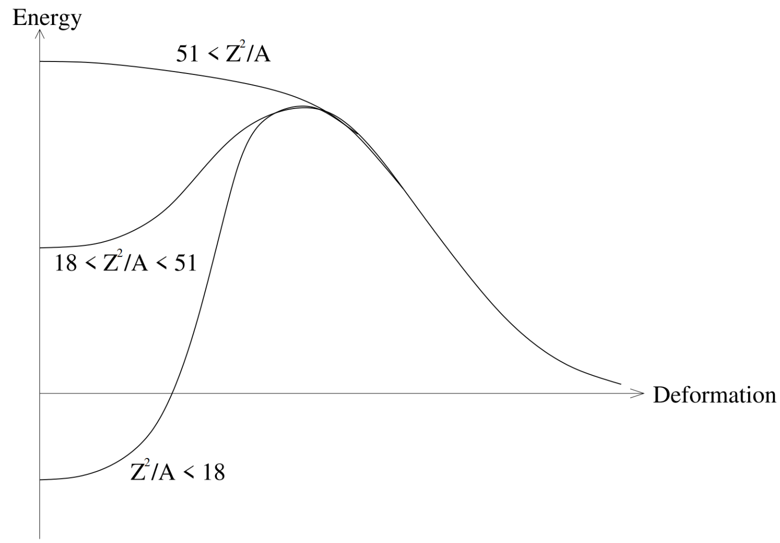


Figure 23: Potential energy of a deforming nucleus.

A large cross section requires interaction via nuclear force, so we think of protons, neutrons, and alpha particles. Of the three, neutrons are special as they don't carry EM charge and are thus free from Coulomb repulsion.

If the nucleus in which we want to induce fission is ${}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X}$, we start with isotope ${}^{\mathrm{A}-1}_{\mathrm{Z}}\mathrm{X}$:

$$ {}^{\mathrm{A}-1}_{\mathrm{Z}}\mathrm{X} + n \to {}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X} \to \text{fission}. \tag{93} $$

A simple energy calculation tells us that we must have

$$ \Delta E = B_{Ef} - B_{Ei} + T_n \tag{94} $$

to cross the energy barrier, where T_n is the kinetic energy of the neutron.

10.3 Chain Reactions

Fission results in nuclei with more neutrons than are needed to lie on the beta-stability curve. These are often ejected, and fission commonly results in between one and four neutrons being emitted. If the neutrons produced in a nuclear fission can be used to cause further fissions, then a "chain reaction" can occur.

Given the mean time required for a neutron to induce a fission event τ_f and the average number of neutrons from each fission which react further m, in a time δt,

$$ \delta n = (m-1)\, n \frac{\delta t}{\tau_f}. \tag{95} $$

The solution is

$$n = n_0 \exp\left[(m-1)\frac{t}{\tau_f}\right]. \tag{96}$$

Clearly, $m \geq 1$ is the condition for a chain reaction:

$$m \begin{cases} < 1 & \text{subcritical;} \\ = 1 & \text{critical;} \\ > 1 & \text{supercritical.} \end{cases} \tag{97}$$

10.4 Fusion Energetics

We shall meet the nuclear binding energy per nucleon again.

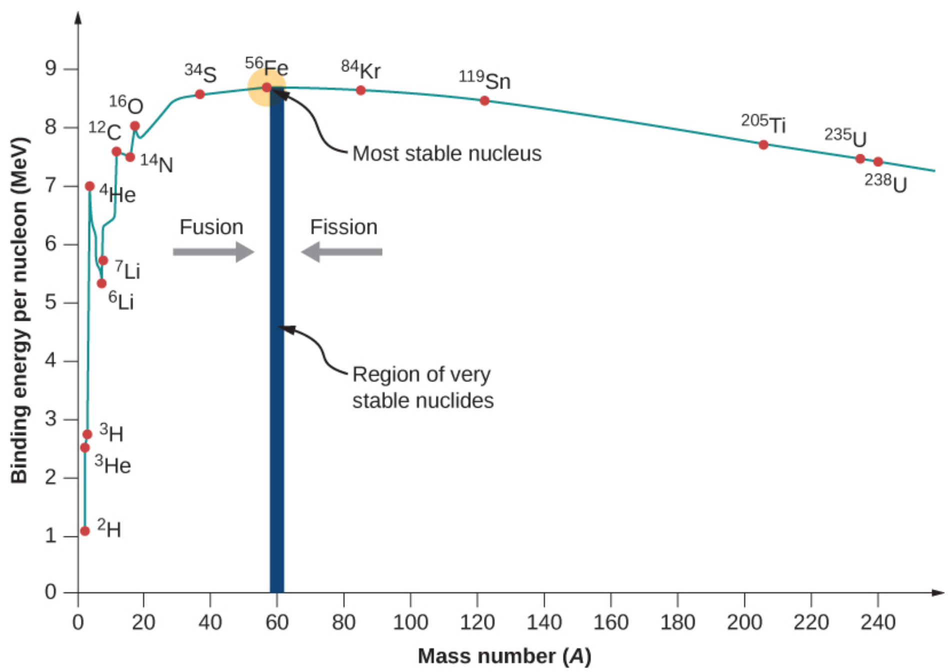


Figure 24: Nuclear binding energy per nucleon.

Therefore, for low A nuclei, energy can be released through fusion. The general fusion process is:

$$^{A_X}_{Z_X}\mathrm{X} + {}^{A_Y}_{Z_Y}\mathrm{Y} \to {}^{A_X+A_Y}_{Z_X+Z_Y}\mathrm{F}^*. \tag{98}$$

F^* must have less total mass than the initial two nuclei is energy is to be released. This means F must actually be in an excited state. A possible decay mode is a gamma decay.

11 Miscellaneous

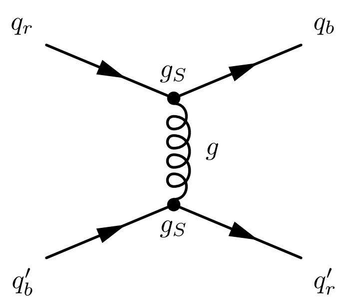
Undoubtedly, this is a stupid course of phenomenology. I literally vomited when given fragmented pieces of information from the Dirac equation in order to solve some problems. Let's make it simple by emphasizing some key qualitative ideas.

11.1 The Seeming Ambiguity of Virtual Particles

Both color charge and EM charge must be conserved, regardless of the type of vertex. However, I've wondered before why the virtual particles sometimes appear with no explicit direction or charge labels.

> My observation, by now, is that both directions make sense, and it is unnecessary to stick to one of them.

Consider a strong reaction of quarks.



There are two ways to interpret the charge of the gluon due to the ambiguity in directions: q_r decaying to q_b while emitting a gluon, or q_r receiving a gluon from q'_b. These two interpretations give different color charges on the gluon:

$$q_r \to q_b + g_{r\bar{b}}, \qquad q_r + g_{\bar{r}b} \to q_b.$$

More specifically, these two gluons are antiparticles of the other. And as one can see, these two different gluons, with directions specified, both are acceptable color combinations of the gluon.

> As we know antiparticles as traveling in the reverse direction of the associated particles, both processes may effectively describe a downward traveling virtual gluon ($g_{r\bar{b}}$), but we can also say that its antiparticle ($g_{\bar{r}b}$) is propagating upwards. These processes are physically indistinguishable, as there is no broken symmetry that distinguishes matter from antimatter. Therefore, we do not assign arrows to gluon lines in Feynman diagrams, as their direction has no intrinsic physical meaning and doing so would falsely suggest a matter-antimatter distinction.

This argument, to some extent, also applies to virtual W particles: we won't spare time to study if it is W^+ or W^-, as they are really virtual and unobservable in every sense.

In Feynman diagrams, only arrows of real particles make physical sense. The so-called direction of virtual particles, in essence, is just a bookkeeping convention, rather than a reflection of any physical propagation in spacetime.

11.2 Spectator Quarks

If you have ever wondered, why would one include the "spectator quarks" in the Feynman diagrams?

The answer relates to confinement. No quarks can exist in solitude due to confinement, so particles that participate in strong interaction are always bound states of several quarks. It is not professional to only draw out the quark that participate in the interaction. The spectators also determine the final baryon.

As an aside, asymptotic freedom allows us to treat quarks as nearly free in high-energy processes, but the presence of spectators ensures a physically consistent, confined description.

11.3 Common Facts

The simplicity and elegance of physics cannot refrain us completely from memorizing key facts.

11.3.1 Protons & Neutrons

Those which authentically need memorization won't appear on the formula sheet.

These tiny particles have a size of order

$$r = 1\,\text{fm} = 10^{-15}\,\text{m}. \tag{99}$$

And as a contrast, the atomic size is much larger — about $10^{-10}\,\text{m}$.

11.3.2 Forces of SM

The dimensionless constants tell us EM force is weakest of the three:

$$\alpha_{\text{EM}} < \alpha_W < \alpha_S, \qquad \frac{1}{137} < \frac{1}{30} < (0.1 \sim 1). \tag{100}$$

However, the ranges come into play here. The characteristic range derived from the Yukawa potential tells us that

$$r_0 = \frac{\hbar}{mc}, \tag{101}$$

so weak interaction suffer from an extremely short range. Also, as W and Z are so massive, the propagators,

$$\frac{1}{q^2 - m^2}, \tag{102}$$

are suppressed in low energy scenarios.

In summary, the weak interaction is called weak, not because it is really weaker than EM, but because of the heavy propagators and the short ranges.

As for the cross section,

$$\sigma_{\text{Strong}} > \sigma_{\text{EM}} > \sigma_{\text{Weak}}. \tag{103}$$

In terms of typical decay timescales,

$$\begin{cases} \text{Strong:} & 10^{-23}\,\text{s}, \\ \text{EM:} & 10^{-18} \sim 10^{-16}\,\text{s}, \\ \text{Weak:} & 10^{-10} \sim 10^{-8}\,\text{s}. \end{cases} \tag{104}$$

Also, in weak vertices where quark flavors are changed, by Cabibbo, the quarks in the same generation have coupling $\cos\theta_C$. If one generation is crossed, u and s have coupling $\sin\theta_C$. As for c and d, which I haven't met so far, it is $-\sin\theta_C$.