Condensed Summary

Oscillations & Waves

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Contents

Chapter 1

Vibrations and Waves

1.1 Harmonic Oscillators

$$\ddot{\psi} + \omega_0^2 \psi = 0 \tag{1.1}$$

Complex trial solution:

$$\tilde{x} = \tilde{A} \mathrm{e}^{i\omega_0 t}, \text{ where } \tilde{A} = A\mathrm{e}^{i\phi}. \tag{1.2}$$

Harmonic oscillations on small scales: consider arbitrary $F(x)$ with $F(0) = 0$.

$$F(x) = F(x=0) + \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=0} x + \frac{1}{2}\left.\frac{\mathrm{d}^2 F}{\mathrm{d}x^2}\right|_{x=0} x^2 + \ldots.$$

For x that is small enough,

$$\begin{cases} F(x=0) & \text{Zero. This is the equilibrium point.} \\ \left.\dfrac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=0} x & \text{Linear restoring force.} \\ \dfrac{1}{2}\left.\dfrac{\mathrm{d}^2 F}{\mathrm{d}x^2}\right|_{x=0} x^2 & \text{Small when } x \text{ is small enough.} \end{cases}$$

	Mass on a Spring	Simple Pendulum	LC Circuit
Step 1	$F = -kx$	$\tau = -mgl\sin\theta \approx -mgl\theta$	$V_C + V_L = 0$
Step 2	$m\ddot{x} + kx = 0$	$ml^2\ddot{\theta} + mgl\theta = 0$	$q/C + L\ddot{q} = 0$
Natural Freq.	$\omega_0 = \sqrt{\dfrac{k}{m}}$	$\omega_0 = \sqrt{\dfrac{g}{l}}$	$\omega_0 = \sqrt{\dfrac{1}{LC}}$

1.2 Damped Harmonic Oscillators

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = 0 \tag{1.3}$$

Complex trial solution:

$$\tilde{\psi} = \tilde{A}\mathrm{e}^{i\omega t}. \tag{1.4}$$

(1) Light Damping: $\frac{\gamma}{2} < \omega_0$

$$\psi = A\mathrm{e}^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi),\ \text{where } \omega_d = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}. \tag{1.5}$$

(2) Critical Damping: $\frac{\gamma}{2} = \omega_0$

$$\psi = (A + Bt)\mathrm{e}^{-\frac{\gamma}{2}t}. \tag{1.6}$$

(3) Heavy Damping: $\frac{\gamma}{2} > \omega_0$

$$\psi = B\mathrm{e}^{-\left(\frac{\gamma}{2}+\frac{\gamma'}{2}\right)t} + C\mathrm{e}^{-\left(\frac{\gamma}{2}-\frac{\gamma'}{2}\right)t}. \tag{1.7}$$

1.3 Driven Harmonic Oscillators

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = f_0\cos(\omega t) \tag{1.8}$$

(1) Transient: homogeneous solution

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = 0 \Rightarrow \psi_1.$$

(2) Steady State: particular solution

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = f_0\cos(\omega t) \Rightarrow \psi_2 = A\cos(\omega t + \phi).$$

To solve this differential equation, try $\tilde{\psi} = \tilde{A}\mathrm{e}^{i\omega_1 t}$.

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \tag{1.9}$$

$$\phi = \arctan\left(-\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right). \tag{1.10}$$

Therefore, the total solution should be:

$$\psi = \psi_1 + \psi_2. \tag{1.11}$$

The quality factor is

$$Q = \frac{\omega_0}{\gamma}, \tag{1.12}$$

and a larger Q means a sharper and taller peak (γ is smaller).

By $x = A\cos(\omega t + \phi)$, the energy in the steady state is:

$$\begin{aligned}\langle E\rangle &= \langle U\rangle + \langle K\rangle \\ &= \frac{1}{4}m\omega_0^2 A^2 + \frac{1}{4}m\omega^2 A^2 \\ &= \frac{1}{4}mA^2(\omega_0^2 + \omega^2)\end{aligned} \tag{1.13}$$

There is resonance in driven harmonic oscillators, and that occurs near the natural frequency ω_0.

1.4 Coupled Oscillators

$$\begin{cases} m\ddot{x} &= -m\omega_0^2 x + k(y - x) \\ m\ddot{y} &= -m\omega_0^2 y - k(y - x) \end{cases} \tag{1.14}$$

Complex trial solution:

$$\tilde{x} = \tilde{A}\mathrm{e}^{i\omega t}, \qquad \tilde{y} = \tilde{B}\mathrm{e}^{i\omega t}.$$

This yields the equation

$$\omega_0^2 - \omega^2 + \frac{k}{m} = \pm\frac{k}{m} \tag{1.15}$$

and two solutions

$$\begin{cases} \omega_1 = \omega_0 \\ \omega_2 = \sqrt{\omega_0^2 + 2\omega_s^2}, \end{cases}$$

where $\omega_s^2 = \dfrac{k}{m}$.

The two frequencies are two normal mode frequencies. A system of coupled oscillators with N degrees of freedom will have N normal modes. The solutions are:

$$\begin{cases} x = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \\ y = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2). \end{cases} \tag{1.16}$$

There are four free parameters A_1, A_2, ϕ_1, and ϕ_2, as there are four initial conditions $x(0)$, $\dot{x}(0)$, $y(0)$, and $\dot{y}(0)$.

1.5 Wave Equation

The wave equation in 3D is:

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}. \tag{1.17}$$

For transverse waves on a string, the total force is $F_y = T\Delta\theta$, and this gives the speed of the wave:

$$v = \sqrt{\frac{T}{\rho}}. \tag{1.18}$$

(1) General Solution: 1D

$$\psi = f(x - vt) + g(x + vt). \tag{1.19}$$

(2) Sinusoidal Solution: 1D

$$\psi = A\cos(k(x - vt)) = A\cos(kx - \omega t). \tag{1.20}$$

(3) Plane Waves: 3D

$$\psi = A\cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t), \tag{1.21}$$

where $|\boldsymbol{k}| = \dfrac{2\pi}{\lambda}$.

(4) Spherical Waves: 3D

$$\psi = \frac{A}{r}\cos(kr - \omega t). \tag{1.22}$$

The amplitude and intensity fall due to conservation of energy.

1.6 Energy of Waves

Consider the transverse waves on the string again.

$$\Delta K = \frac{1}{2} m \dot{y}^2 = \frac{1}{2} \rho \Delta x \left(\frac{\partial y}{\partial t}\right)^2 \tag{1.23}$$

$$\Delta U = T(\Delta s - \Delta x) = \frac{1}{2}T\Delta x\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}\rho\Delta x\left(\frac{\partial y}{\partial t}\right)^2 = \Delta K. \tag{1.24}$$

Therefore, the power of the wave should be:

$$P = \frac{\Delta U + \Delta K}{\Delta t} = \rho\frac{\Delta x}{\Delta t}\left(\frac{\partial y}{\partial t}\right)^2 = \rho v\left(\frac{\partial y}{\partial t}\right)^2. \tag{1.25}$$

For sine waves,

$$\langle P\rangle = \frac{1}{2}\rho v\omega^2 A^2. \tag{1.26}$$

1.7 Waves with Boundaries

$$y = f(x - vt) + g(x + vt) = f(x - vt) - f(-x - vt) \tag{1.27}$$

for a wave fixed at the origin. If the wave is a sinusoidal wave, then, by trigonometric identities, it is a standing wave:

$$\begin{aligned} y &= f(x - vt) - f(-x - vt) \\ &= A\cos(k(x - vt)) - A\cos(k(-x - vt)) \\ &= 2A\sin(kx)\sin(\omega t). \end{aligned}$$

A travelling wave transports energy. The standing wave oscillates but does not travel, and thus it does not transport energy. The standing wave has nodes – points where there is no motion. The nodes are separated by half a wavelength.

For the sine waves also fixed at $x = L$, we write

$$\sin(kL) = 0. \tag{1.28}$$

There are a few allowed wave numbers to satisfy the condition.

$$kL = n\pi \Rightarrow k = \frac{n\pi}{L}. \tag{1.29}$$

There are also allowed wavelengths and allowed angular frequencies:

$$\lambda = \frac{2\pi}{k} = \frac{2L}{n} \tag{1.30}$$

$$\omega = kv = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}. \tag{1.31}$$

When $n = 1$, this is the fundamental mode. $n = 2$ case is called the first harmonic.

For a 2D plane wave, consider its equation of motion to be

$$y(x, z, t) = A\sin(k_x x)\sin(k_z z)\sin(\omega t). \tag{1.32}$$

By the wave equation,

$$k_x^2 + k_z^2 = \frac{\omega^2}{v^2}. \tag{1.33}$$

With the boundaries at $x = 0$, $x = a$, $z = 0$, and $z = b$, we get the allowed wave numbers,

$$k_x = \frac{n_1\pi}{a} \qquad k_z = \frac{n_2\pi}{b}, \tag{1.34}$$

and the allowed angular frequency:

$$\omega = v\sqrt{\left(\frac{n_1\pi}{a}\right)^2 + \left(\frac{n_2\pi}{b}\right)^2}. \tag{1.35}$$

1.8 Interference

The intensity of the wave is defined as

$$I = \alpha|\psi|^2, \tag{1.36}$$

where α is the proportionality. Note that intensities cannot add; you should add the waves first and calculate the intensity.

Consider the waves

$$\psi_1 = A_1 e^{i(\omega t + \phi_1)} \qquad \psi_2 = A_2 e^{i(\omega t + \phi_2)},$$

where the phases contain the spatial part of the phase as well as any additional phase offsets. By maths,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\phi_1 - \phi_2).$$

The red part above is the interference term.

Consider the double slit case, where $A_1 = A_2$ and $I_1 = I_2$, the separation is d, and the phase difference is $\Delta\phi = k\Delta r = kd\sin\theta$. Therefore,

$$I = 2I_1 + 2I_1\cos(kd\sin\theta) = 4I_1\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right). \tag{1.37}$$

1.9 Wave Packets

When adding waves of slightly different angular frequencies, we get a beat.

For example, by adding $\psi_1 = A\cos(\omega_1 t + \phi)$ and $\psi_2 = A\cos(\omega_2 t + \phi)$, we get:

$$\psi = \psi_1 + \psi_2 = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t + \phi\right), \tag{1.38}$$

where the first argument is the half difference, and the second is the half sum.

When more waves are added together, there is a more accurate approximation of a single pulse at $t = 0$.

With the total angular frequency range being $\Delta\omega$, we can get a pulse of duration:

$$\Delta t \sim \frac{1}{\Delta\omega}. \tag{1.39}$$

Similarly, when we shift to the waves with slightly different wave numbers (Δk), we get a pulse in space:

$$\Delta x \sim \frac{1}{\Delta k}. \tag{1.40}$$

The phase velocity is the normal velocity we talk about throughout. However, there is another velocity called the group velocity, which is the velocity at which the envelope of the waves moves.

We define the phase velocity,

$$v_p = \frac{\omega}{k}, \tag{1.41}$$

and the group velocity,

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{1.42}$$

When v_p does not depend on frequency, the group velocity and the phase velocity are the same.

The phenomenon of dispersion tells us that v_p actually depends on the frequency. For EM waves,

$$\omega = \sqrt{\omega_p^2 + k^2c^2},$$

where ω_p is a constant called the plasma frequency. In this case,

$$v_p = \frac{\omega}{k} > c, \qquad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} < c.$$

As information is contained within the change of the wave, it travels at group velocity ($< c$).

Chapter 2

Basic Electronics

2.1 Circuit Theory

The drift speed of positive charges in a wire is:

$$v_d = \frac{I}{neA}. \tag{2.1}$$

Theory tells us that the resistance of a conductor is determined by the following formula:

$$R = \rho \frac{l}{A}. \tag{2.2}$$

Ohm's law:

$$\boldsymbol{j} = \sigma \boldsymbol{E}. \tag{2.3}$$

A source that delivers current to a circuit has negative power. The positive charges gain potential through the source. Their energy increases, which means a decrease in energy of the source.

2.2 DC Circuits

2.2.1 Kirchhoff's Voltage Law: KVL

Around any circuit loop, the sum of the potential gains due to EMFs must equal the sum of the potential drops across the components.

$$\sum_{k=1}^{n} V_k = 0. \tag{2.4}$$

2.2.2 Kirchhoff's Current Law: KCL

At any junction, the sum of currents pointing into the junction must equal the sum of currents pointing out of the junction.

$$\sum_{k=1}^{n} I_k = 0. \tag{2.5}$$

2.2.3 Loading

Whenever a real-world voltage-source is connected to some device which causes it to supply current, then the actual p.d. observed across the terminals will be reduced and the source is said to be 'loaded'. This effect is referred to as loading.

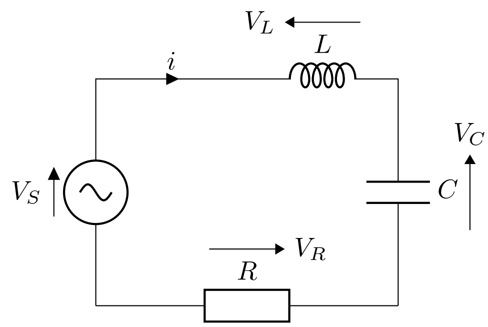
2.2.4 Series and Parallel

While the resistors and inductors share the same rule, the capacitors behave the same as the mechanical springs. Impedance also adds like resistance.

	Series	Parallel
Resistance	$R_{eq} = \sum R_i$	$\frac{1}{R_{eq}} = \sum \frac{1}{R_i}$
Inductance	$L_{eq} = \sum L_i$	$\frac{1}{L_{eq}} = \sum \frac{1}{L_i}$
Impedance	$Z_{eq} = \sum Z_i$	$\frac{1}{Z_{eq}} = \sum \frac{1}{Z_i}$
Capacitance	$\frac{1}{C_{eq}} = \sum \frac{1}{C_i}$	$C_{eq} = \sum C_i$
Spring Constant	$\frac{1}{k_{eq}} = \sum \frac{1}{k_i}$	$k_{eq} = \sum k_i$

2.2.5 General Laws in Circuits

The potential difference arrows point to the direction of higher potential. It is always parallel to the current for the source and always anti-parallel for circuit components.



Also, remember that conservation of energy asks the total power in the circuit (including the source) is always zero:

$$\sum_{i=1}^{n} p_i = 0. \tag{2.6}$$

2.2.6 RC Circuits

$$v_S = v_C + v_R$$

When charging,

$$\int_0^q \frac{\mathrm{d}q'}{C\mathcal{E} - q'} = \int_0^t \frac{\mathrm{d}t'}{RC}.$$

The quantity

$$\tau = RC \tag{2.7}$$

is defined as the time constant of the circuit.

	q	v_C	i	v_R
$t = 0$	0	0	$\mathcal{E}/R$	$\mathcal{E}$
t	$C\mathcal{E}\left(1 - \mathrm{e}^{-t/\tau}\right)$	$\mathcal{E}\left(1 - \mathrm{e}^{-t/\tau}\right)$	$\mathcal{E}/R \cdot \mathrm{e}^{-t/\tau}$	$\mathcal{E}\mathrm{e}^{-t/\tau}$
$t \to \infty$	$C\mathcal{E}$	$\mathcal{E}$	0	0

When discharging,

$$\int_{q_0}^q \frac{\mathrm{d}q'}{q'} = -\int_0^t \frac{\mathrm{d}t'}{RC}.$$

The conventional current is now negative, which means it is now flowing back from the capacitor and into the (switched-off) source. The capacitor is returning the power back into the circuit (negative power), and this power is dissipated in the resistor.

	q	$v_C = -v_R$	i
$t = 0$	q_0	q_0/C	$-q_0/RC$
t	$q_0 \mathrm{e}^{-t/\tau}$	$q_0/C \cdot \mathrm{e}^{-t/\tau}$	$q_0/RC \cdot \mathrm{e}^{-t/\tau}$
$t \to \infty$	0	0	0

2.2.7 RL Circuits

$$v_S = v_R + v_L$$

When charging,

$$\int_0^i \frac{\mathrm{d}i'}{\mathcal{E}/R - i'} = \int_0^t \frac{R}{L}\mathrm{d}t'.$$

The quantity

$$\tau = \frac{L}{R} \tag{2.8}$$

is defined as the time constant here.

	i	v_L	v_R
$t = 0$	0	$\mathcal{E}$	0
t	$\mathcal{E}/R \cdot \left(1 - \mathrm{e}^{-t/\tau}\right)$	$\mathcal{E}\mathrm{e}^{-t/\tau}$	$\mathcal{E}\left(1 - \mathrm{e}^{-t/\tau}\right)$
$t \to \infty$	$\mathcal{E}/R$	0	$\mathcal{E}$

Unlike the equivalent RC circuit, the source continues to deliver power $p_S = -\mathcal{E}i$ even for time $t \gg \tau$.

When discharging,

$$\int_{i_0}^i \frac{\mathrm{d}i'}{i'} = -\int_0^t \frac{R}{L}\mathrm{d}t'.$$

	i	$v_R = -v_L$
$t = 0$	i_0	$i_0 R$
t	$i_0 \mathrm{e}^{-t/\tau}$	$i_0 R \mathrm{e}^{-t/\tau}$
$t \to \infty$	0	0

The inductor is now delivering stored energy back into the circuit, where it is dissipated in the resistor. The inductor now has negative power.

2.2.8 LC Circuits

$$v_S = v_L + v_C$$

$$\ddot{q} + \frac{q}{LC} = 0$$

Ideally, the charge on the capacitor would exhibit simple harmonic motion with the angular frequency being the natural frequency $\omega_0 = \sqrt{1/LC}$.

2.3 AC Analysis

2.3.1 Phasors

We can represent a signal (voltage or current) by a vector in the complex plane which represents just the amplitude and phase of the quantity.

We create a phasor representation of of a signal as follows:

(1) Express the signal as a complex exponential.

(2) Remove the time-dependent part.

(3) The result will be a vector in the complex plane with the length of the signal's amplitude and the angle of the signal's phase. For example,

$$v = V_0 \cos(\omega t + \phi) \Rightarrow \tilde{V} = V_0 \mathrm{e}^{j\phi}.$$

To recover the signal as a function of time,

(1) Multiply by $\mathrm{e}^{j\omega t}$ (all voltages and currents oscillate at the same frequency).

(2) Take the real part.

2.3.2 Complex Impedance

For any linear component, or network of components, where the current phasor is $\tilde{I}$ and the potential difference phasor is $\tilde{V}$, the complex impedance is

$$\tilde{Z} = \frac{\tilde{V}}{\tilde{I}}. \tag{2.9}$$

Since impedance is a ratio of voltage to current, it also has units of ohms. Noting that all voltages and currents oscillate at the same frequency, we assume

$$\tilde{i} = \tilde{I}e^{j\omega t}, \qquad \tilde{v} = \tilde{V}e^{j\omega t}.$$

The impedances for each circuit component:

$$\tilde{Z}_L = j\omega L \tag{2.10}$$

$$\tilde{Z}_C = \frac{1}{j\omega C} = \frac{-j}{\omega C} \tag{2.11}$$

$$\tilde{Z}_R = R. \tag{2.12}$$

2.4 Filters and Gain

The frequency-dependent behaviour of RL and RC circuits makes them useful for a type of circuit known as a filter which is used to select (or remove) a range of frequencies from a signal.

Note that since the signal is assumed to come from some source (e.g. a sensor of some kind) and assumed to then go on to some measurement device (e.g. a scope) then typically the source and destination are not shown. Of course, in reality there will always be complete circuits, however for the filter circuit we are mainly interested in the relation between the signal input to the filter and the output from the filter. The use of complex quantities will make the frequency-dependence of this input / output relationship much easier to visualise.

The filter is characterised by the gain, which is the ratio of output to input voltage phasors. Therefore, gain is a complex quantity:

$$\tilde{G} = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = Ge^{j\phi}. \tag{2.13}$$

By algebra, we can see

$$\tilde{V}_{\text{out}} = \tilde{G}\tilde{V}_{\text{in}}.$$

It will always be the case that

(1) The amplitude of the output will be scaled by a factor G.

(2) The phase of the output will be shifted by ϕ.

For all first order passive filters (RL and RC), G is a positive quantity that is smaller than 1.

To visualise $\tilde{G}$, it is convenient to plot the magnitude G versus frequency on a log-log plot. The argument or phase ϕ is usually shown in degrees (by convention) on a log-linear plot. Together, these two plots are commonly known as the Bode Plot.

The logarithm of the gain magnitude is, by convention, expressed in dB (decibels):

$$G_{\mathrm{dB}} = 20 \log_{10} G. \tag{2.14}$$

Since G is a ratio of amplitudes, the above equation expresses the relative power in the output of the filter to the input of the filter.

By convention, we choose the frequency where

$$G = \frac{1}{\sqrt{2}} \tag{2.15}$$

to be the cut-off frequency.

An ideal low-pass filter would pass all frequencies below the cut-off and block all frequencies above. This is impossible to achieve in real-world filters. For the RC low-pass filter, it can be seen that for $\omega \gg \omega_C$,

$$G \approx \frac{\omega_C}{\omega}, \tag{2.16}$$

which results in a gradient of -20 dB/decade, where a decade is a factor of 10 in frequency. For example, the gain falls by -20 dB between the frequencies 10 and 100 rad/s.

Chapter 3

Fourier Analysis

3.1 Dirac Delta Function

The Dirac Delta function is an ideal function that is defined by the criteria:

$$\delta(x) = 0 \text{ for } x \neq 0, \tag{3.1}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-t)\mathrm{d}x = f(t), \tag{3.2}$$

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1. \tag{3.3}$$

The criteria tell us that the Dirac Delta function has the sifting property. Note the difference between the Dirac Delta and the Kronecker Delta:

$$\begin{cases} \text{Kronecker Delta: } \delta_{mn} = 1 \text{ iff } m = n; \\ \text{Dirac Delta: } \displaystyle\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1, \text{ and } \delta(x) \neq 0 \text{ iff } x = 0. \end{cases} \tag{3.4}$$

There are three common types of functions that are used to approximate the Dirac Delta: the top hat function, a complex exponential, and a Gaussian. In each case, we consider

$$\delta(x) = \lim_{n\to\infty} \delta_n(x). \tag{3.5}$$

To summarize,

$$\delta_n(x) = \begin{cases} n \text{ for } |x| \leq \dfrac{1}{2n} & \text{(Top Hat Function)} \\ \dfrac{1}{2\pi}\displaystyle\int_{-n}^{n} \mathrm{e}^{ixt}\mathrm{d}t & \text{(Complex Exponential)} \\ \dfrac{n}{\sqrt{\pi}}\mathrm{e}^{-n^2x^2} & \text{(Gaussian)} \end{cases} \tag{3.6}$$

For the complex exponential case, as $n \to \infty$, we can see:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ixt}\mathrm{d}t. \tag{3.7}$$

When in applications, it may be a little harder to recognize an integral as a Dirac delta. For example,

$$\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\omega(t-x)}\mathrm{d}\omega. \tag{3.8}$$

3.2 Fourier Series

The set of complex exponentials

$$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{inx} \quad \text{with } n \in \mathbb{Z} \tag{3.9}$$

form an orthonormal set on the interval $(-\pi, \pi)$.

Any complex function can be represented by the orthonormal set described above:

$$f(x) = \sum_n c_n \mathrm{e}^{inx}, \text{ where} \tag{3.10}$$

$$c_n = \frac{1}{2\pi}\left\langle f(x), \mathrm{e}^{inx}\right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\mathrm{d}x. \tag{3.11}$$

3.2.1 Trigonometric Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx), \text{ where} \tag{3.12}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x, \tag{3.13}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x. \tag{3.14}$$

We may further use the auxiliary angle formula to get:

$$f(x) = \frac{a_0}{2} + \sum \alpha_n \cos(nx - \theta_n). \tag{3.15}$$

3.2.2 General Intervals

On a general interval $(-l, l)$,

$$f(x) = \sum c_n \mathrm{e}^{in\pi x/l}, \text{ where} \tag{3.16}$$

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) \mathrm{e}^{-in\pi x/l} \mathrm{d}x. \tag{3.17}$$

$$f(x) = \frac{a_0}{2} + \sum \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right], \text{ where} \tag{3.18}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) \mathrm{d}x, \tag{3.19}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \tag{3.20}$$

3.2.3 Reality Condition

Reality condition states that if $f(x) = f^*(x)$, then

$$c_n^* = c_{-n}. \tag{3.21}$$

3.2.4 Parseval's Identity

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x &= \sum_{-\infty}^{\infty} |c_n|^2 \\ &= \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \end{aligned} \tag{3.22}$$

Parseval's Inequality for the truncated Fourier Series: for any positive $N < \infty$,

$$\sum_{-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x. \tag{3.23}$$

3.3 Fourier Transform

$$\mathscr{F} : g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{i\omega t} \mathrm{d}t \tag{3.24}$$

$$\mathscr{F}^{-1} : f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) \mathrm{e}^{-i\omega t} \mathrm{d}\omega \tag{3.25}$$

3.3.1 Delta Function

The Fourier Transform of a delta function is a constant:

$$\mathscr{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}}. \tag{3.26}$$

3.3.2 Gaussian

The Fourier Transform of a Gaussian is still a Gaussian, but with reciprocal width:

$$\mathscr{F}\left(\frac{1}{\sigma}\mathrm{e}^{-t^2/2\sigma^2}\right) = \mathrm{e}^{-\omega^2/2\rho^2}, \tag{3.27}$$

or alternatively,

$$\mathscr{F}\left(\mathrm{e}^{-t^2/2\sigma^2}\right) = \frac{1}{\rho}\mathrm{e}^{-\omega^2/2\rho^2}, \tag{3.28}$$

where $\rho = \frac{1}{\sigma}$.

When $\sigma \to 0$, $f(t) \to \delta(t)$, and $g(\omega) \to \frac{1}{\sqrt{2\pi}}$.

To derive the result, consider $\frac{\mathrm{d}g}{\mathrm{d}\omega}$.

3.3.3 Top Hat Function

The Fourier Transform of a top hat function is a sinc function:

$$\mathscr{F}\left(\frac{\sqrt{2\pi}}{2T}\right) = \mathrm{sinc}(\omega T) \text{ for } |t| \leq T. \tag{3.29}$$

The Fourier Transform of a top hat function is a sinc function whose width is reciprocal to the top hat width.

$$\begin{cases} T \to \infty & f(t) \to \text{const.}, \quad g(\omega) \to \delta(\omega) \\ T \to 0 & f(t) \to \delta(t), \quad g(\omega) \to \text{const.} \end{cases} \tag{3.30}$$

3.3.4 Properties of Fourier Transform

(1) Linearity:

$$\mathscr{F}[\alpha f_1(t) + \beta f_2(t)] = \alpha\mathscr{F}[f_1(t)] + \beta\mathscr{F}[f_2(t)]. \tag{3.31}$$

(2) Change of Sign:

$$\mathscr{F}[f(-t)] = g(-\omega). \tag{3.32}$$

(3) Translation:

$$\mathscr{F}[f(t-t_0)] = \mathrm{e}^{i\omega t_0} g(\omega). \tag{3.33}$$

Note that the Fourier Transform of a shifted delta function is no longer a constant:

$$\mathscr{F}[\delta(t-t_0)] = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{i\omega t_0}. \tag{3.34}$$

(4) Scaling:

$$\mathscr{F}[f(\alpha t)] = \frac{1}{|\alpha|} g\left(\frac{\omega}{\alpha}\right). \tag{3.35}$$

(5) Conjugation:

$$\mathscr{F}[f^*(t)] = g^*(-\omega). \tag{3.36}$$

(6) Reality Condition:

$$f(t) = f^*(t) \Rightarrow g(-\omega) = g^*(\omega). \tag{3.37}$$

To recap, for Fourier Series,

$$c_{-n} = c_n^*. \tag{3.38}$$

Negative frequencies are redundant.

(7) Derivative:

$$\mathscr{F}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = -i\omega g(\omega). \tag{3.39}$$

(8) Parseval's Identity:

$$\int_{-\infty}^{\infty} f(t)g^*(t)\mathrm{d}t = \int_{-\infty}^{\infty} F(\omega)G^*(\omega)\mathrm{d}\omega, \tag{3.40}$$

where $F(\omega) = \mathscr{F}[f(t)]$, $G(\omega) = \mathscr{F}[g(t)]$. Also, for $f(t) = g(t)$,

$$\int_{-\infty}^{\infty} |f(t)|^2\mathrm{d}t = \int_{-\infty}^{\infty} |F(\omega)|^2\mathrm{d}\omega. \tag{3.41}$$

3.3.5 Solving Partial Differential Equations

A Fourier Transform can be used to change a PDE into an ODE. This comes at the price of having to make first a Fourier Transform and subsequently an inverse one.

Consider the heat equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{3.42}$$

Let $\hat{u}(k,t) = \mathscr{F}[u(x,t)]$. That is to say,

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) \mathrm{e}^{ikx} \mathrm{d}x.$$

Therefore,

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) \mathrm{e}^{ikx} \mathrm{d}x\right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} \mathrm{e}^{ikx} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a^2 \frac{\partial^2 u}{\partial x^2} \mathrm{e}^{ikx} \mathrm{d}x \\ &= \mathscr{F}\left[a^2 \frac{\partial^2 u}{\partial x^2}\right] \\ &= a^2(-ik)^2 \hat{u} \\ &= -a^2 k^2 \hat{u}. \end{aligned}$$

This leads us to an ODE:

$$\frac{\partial \hat{u}}{\partial t} = -a^2 k^2 \hat{u}. \tag{3.43}$$

Then, we can get our solution $u(x,t)$ by the inverse Fourier Transform:

$$u(x,t) = \mathscr{F}^{-1}[\hat{u}(k,t)].$$

3.4 Convolution

$$\begin{aligned} (f * g)(t) &= \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \mathrm{d}\tau \quad (f \text{ shifting through } g) \\ &= \int_{-\infty}^{\infty} g(\tau) f(t-\tau) \mathrm{d}\tau \quad (g \text{ shifting through } f). \end{aligned} \tag{3.44}$$

The convolution theorem states that the complicated convolution of two functions after a Fourier Transform simply turns into multiplication.

$$\mathscr{F}[(f * g)(t)] = \sqrt{2\pi} F(\omega) G(\omega) \tag{3.45}$$

$$\mathscr{F}[f(t)g(t)] = \frac{1}{\sqrt{2\pi}} (F * G)(\omega) \tag{3.46}$$

Convolution can be used to shift a function:

$$f(t) * \delta(t-d) = \int_{-\infty}^{\infty} f(\tau)\delta(t-d-\tau)\mathrm{d}\tau = f(t-d). \tag{3.47}$$

With this property, we can work out that:

$$\mathscr{F}[f(t-d) + f(t+d)] = 2\cos(\omega d) g(\omega), \tag{3.48}$$

$$g(\mu_1, \sigma_1) * g(\mu_2, \sigma_2) = g\left(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right). \tag{3.49}$$