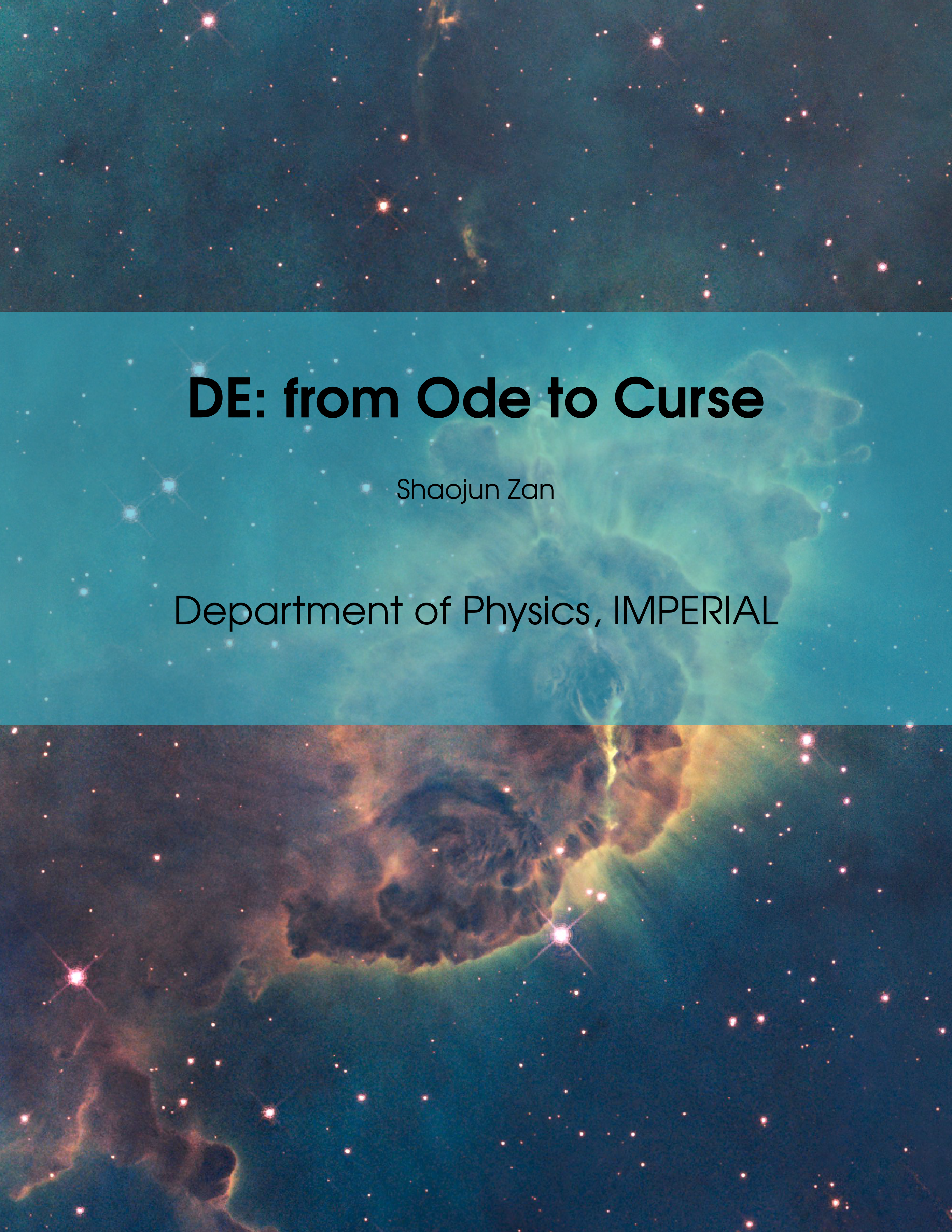


DE: from Ode to Curse

Shaojun Zan

Department of Physics, IMPERIAL

Contents

1 Technical Jargons

- Ordinary & partial differential equations: ODE & PDE

  An ODE only involves one independent variable, while a PDE involves more.

- Order:

  The order of a differential equation is the maximum number of times the dependent variable has been differentiated.

- Degree:

  The degree of a DE is the power to which the highest-order derivative is raised, when the DE is rationalized so that only integer powers of derivatives of y occur. As an example,

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{3/2} + x^2 y = 0$$

  is a 3-rd order, 2-nd degree, non-linear, homogeneous ODE.

- General ODE and solutions:

$$F(x, y, y', \ldots, y^{(n)}) = 0 \quad \Rightarrow \quad f(x, y, c_1, \ldots, c_n) = 0. \tag{1}$$

  The number of constants is the same as order.

1.1 Linearity

- Differential equations:

$$\text{Linearity}\begin{cases}\text{The variables and their derivatives must always appear as a } \underline{\text{simple first power}}.\\ \text{The coefficients depend only on the } \underline{\text{independent variables}}.\end{cases}$$

  The most general form of a linear nth-order ODE is

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = f(x). \tag{2}$$

  A 2nd-order linear PDE is in the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial y^2} + C\frac{\partial^2 u}{\partial x \partial y} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0, \tag{3}$$

where A, B, $\ldots$, and G are functions of x and y.

As what we did in conics, we classify PDE's according to the following rules:

$$\begin{cases} B^2 < 4AC & \text{Elliptical PDE} \\ B^2 = 4AC & \text{Parabolic PDE} \\ B^2 > 4AC & \text{Hyperbolic PDE.} \end{cases}$$

- Differential operators:

A linear operator is defined in the following way:

$$\hat{L}(ay_1 + by_2) = a\hat{L}(y_1) + b\hat{L}(y_2). \tag{4}$$

A linear differential operator (linear operator) has the general form

$$\hat{L} = a_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \ldots + a_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + a_0(x). \tag{5}$$

The property of linear operators allows the use of superposition principles, especially in homogeneous DEs.

1.2 Homogeneity

Fundamentally, a homogeneous problem is one where **the solution can be scaled**.

- Differential equations:

Homogeneous DE's do not involve terms consisting of only independent variables.

- Boundary Conditions:

Boundary Conditions & Initial Conditions: BC & IC

BC's are specified at different points in the domain ($y(a) = y_a$ and $y(b) = y_b$), while IC's are specified at the same point in the domain ($y(a) = y_a$ and $y'(a) = y'_a$).

Most generally, for some constants d_i,

$$d_1 y(a) + d_2 y'(a) = 0 \qquad \text{and} \qquad d_3 y(b) + d_4 y'(b) = 0 \tag{6}$$

are homogeneous BC's. This implies periodic BC's are homogeneous.

2 Important PDE's and Classification of BC's

- Wave equation:
$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{7}$$
where c is the phase speed of the wave. This is a hyperbolic PDE.

- Diffusion equation:
$$\frac{\partial u}{\partial t} = D \nabla^2 u, \tag{8}$$
where D is the diffusion coefficient. This is a parabolic PDE.

- Laplace and Poisson equations:
$$\begin{cases} \nabla^2 u = 0 & \text{Laplace} \\ \nabla^2 u = f(\mathbf{r}, t) & \text{Poisson} \end{cases} \tag{9}$$
These two are elliptic PDE's.

The following table shows the classification of BC's.

Conditions	Description
Dirichlet	u defined on boundaries
Neumann	$\boldsymbol{\nabla} u$ defined on boundaries
Cauchy	Dirichlet + Neumann
Mixed	u or $\boldsymbol{\nabla} u$ applied on different parts of boundary
Periodic BC	$u(x_r) = u(x_l + L) = u(x_l)$ and $u'(x_r) = u'(x_l)$
Reflective BC	$u(-x) = u(x)$

3 First-order ODE's

The general form is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y). \tag{10}$$

3.1 Separable Solutions

This technique works if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)g(x). \tag{11}$$

The solution is obtained by direct integration:

$$\int \frac{\mathrm{d}y}{f(y)} = \int g(x)\,\mathrm{d}x\,. \tag{12}$$

3.2 Integrating Factors (IF) for Linear ODE's

The general form for a linear first-order ODE is

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = h(x). \tag{13}$$

Written in **standard form**,

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + p(x)\right] y = q(x). \tag{14}$$

If homogeneous, then

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + p(x)\right] y = 0 \tag{15}$$

Solution: y satisfies that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -p(x) \cdot y,$$

and this is characteristic of an exponential function. Therefore,

$$y(x) = A\exp\left[-\int^x p(s)\,\mathrm{d}s\right]. \tag{16}$$

If inhomogeneous,

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + p(x)\right] y = q(x) \neq 0 \tag{17}$$

Integrating factor:

$$I = \exp\left[\int^x p(s)\,\mathrm{d}s\right] \quad \text{such that} \quad \frac{\mathrm{d}I}{\mathrm{d}x} = p(x)I \tag{18}$$

This leads to

$$I\left[\frac{\mathrm{d}}{\mathrm{d}x}+p(x)\right]y=\frac{\mathrm{d}(yI)}{\mathrm{d}x}=q(x)I,$$

and therefore,

$$y_p(x)=I^{-1}\int^x \mathrm{d}s\ \ q(s)I \tag{19}$$

is the particular solution of this inhomogeneous equation.

The general solution is the sum of both the particular solution and the complementary solution:

$$y=y_c+y_p. \tag{20}$$

3.3 Variation of Parameters

By solving the homogeneous equation (HE), we found the solution $u(x)$:

$$u(x)=C\exp\left[-\int^x p(s)\,\mathrm{d}s\right].$$

The method is also called the constant variation. The spirit is to substitute the constant C by a function of x. Now try $y=v(x)u(x)$ in the inhomogeneous equation (IE):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(vu)+pvu&=q\\ v\frac{\mathrm{d}u}{\mathrm{d}x}+u\frac{\mathrm{d}v}{\mathrm{d}x}+pvu&=q\\ u\frac{\mathrm{d}v}{\mathrm{d}x}&=q\\ v(x)&=\int^x\frac{q(s)}{u(s)}\,\mathrm{d}s+C.\end{aligned}$$

Therefore,

$$y(x)=\frac{1}{\mu(x)}\left[\int^x q(s)\mu(s)\,\mathrm{d}s+C\right] \tag{21}$$

by identifying the IF as $\mu(x)=1/u(x)$.

3.4 Substitutions

The spirit in substitution is to transform an intractable ODE into one that can be well solved with the discussed methods. One key equation here is **Euler's differential equation**:

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\alpha x\frac{\mathrm{d}y}{\mathrm{d}x}+\beta y=0. \tag{22}$$

By the substitution $t = \ln x$, we see

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (\alpha - 1)\frac{\mathrm{d}}{\mathrm{d}t} + \beta\right] y = 0.$$

Later, we shall see the series solution to this equation with Frobenius' method.

3.5 Final Notes

The general solution to a first-order, first-degree ODE represents a family of integral curves.

These solutions for different c do not intersect, but this is not true in higher order differential equations.

Existence and Uniqueness

The solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x), \quad y(x_0) = y_0 \tag{23}$$

exists in the range $\alpha < x < \beta$, and is **unique** if $f(x)$ and $g(x)$ are continuous in the same range, providing it contains x_0.

This theorem also only applies to first-degree ODE's.

For higher degree equations, two new things may happen:

- The family of solutions can cross: a breakdown of uniqueness;
- A solution arises that has no arbitrary integration constant: **the singular solution**.

4 Second-order ODE's

4.1 General Solutions

The general form of a second-order linear ODE is

$$\left[a_2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + a_0(x)\right] y = f(x). \tag{24}$$

In standard form,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = g(x). \tag{25}$$

Existence and Uniqueness

If $p(x)$, $q(x)$, and $g(x)$ are continuous over the interval $\alpha < x < \beta$ containing x_0 (initial value problems), then there exists a unique solution that satisfies the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y'_0$.

Note that this theorem strictly applies to **initial value problems** (IVP) where the two conditions are specified at a single point x_0. These so called initial conditions (IC's) are the values of y and y'.

When the two conditions are specified at different points, it is a boundary value problem (BVP). These conditions are called boundary conditions (BC's). A BVP may have a unique solution, no solution, or an infinity of solutions that meet the BC's.

The solution to any linear 2-nd order ODE is the sum of the complementary function (CF) and a particular integral (PI).

The CF is the general solution of the HE (y_{CF}):

$$y_{\mathrm{CF}}(x) = c_1 y_1(x) + c_2 y_2(x). \tag{26}$$

The functions y_1 and y_2 are called basis functions.

A PI is **any** solution to the IE (y_{PI}). The general solution is:

$$y_{\mathrm{GS}}(x) = y_{\mathrm{CF}}(x) + y_{\mathrm{PI}}(x). \tag{27}$$

The PI does not need to satisfy the IC's. The CF ensures that the general solution satisfies the IC's.

4.2 Second-order ODE's with Constant Coefficients

4.2.1 Complementary Functions

The HE is now

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0. \tag{28}$$

Using the trial function $y = \exp(mx)$, we get the **characteristic equation**:

$$a_2 m^2 + a_1 m + a_0 = 0. \tag{29}$$

- $a_1^2 > 4a_0 a_2$:

  We see two real distinct roots, m_+ and m_-. Therefore, the CF is

  $$y = c_1 \exp(m_+ x) + c_2 \exp(m_- x). \tag{30}$$

- $a_1^2 < 4a_0 a_2$:

  We see two complex roots which are complex conjugates:

  $$m = p \pm iq.$$

  Therefore,

  $$y = \exp(px) \left[c_1 \exp(iqx) + c_2 \exp(-iqx)\right],$$

  or equivalently,

  $$y = \exp(px) \left[c_3 \cos(qx) + c_4 \sin(qx)\right]. \tag{31}$$

- $a_1^2 = 4a_0 a_2$:

  There is only one root, but we need two solutions. Let's try the trick of variation of parameters:

  $$y = u(x)\mathrm{e}^{mx}. \tag{32}$$

  This gives us $u'' = 0$, so $u = a + bx$. This means that the full solution is

  $$y = (a + bx) \exp(mx). \tag{33}$$

  The basis functions here are $\exp(mx)$ and $x \exp(mx)$.

Using variation of parameters to obtain the second solution from a known first solution is known as **reduction of order**. It is applicable to all linear ODE's.

4.2.2 Particular Integrals

In this section we seek to find PI's for the IE

$$\left[a_2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+a_1\frac{\mathrm{d}}{\mathrm{d}x}+a_0\right]y=g(x). \tag{34}$$

- Exponentials:

  If $g(x)=A\exp(\alpha x)$, then $y(x)=c\exp(\alpha x)$ is a solution.

- Sinusoidal Functions:

  If $g(x)=a\cos(\beta x)+b\sin(\beta x)$, then the PI is

$$y=c\cos(\beta x)+d\sin(\beta x). \tag{35}$$

- Polynomials:

  If $g(x)=ax^n$, then the PI is

$$y=\sum d_n x^n. \tag{36}$$

4.3 General Properties for Linear ODE's

The following properties hold for any linear inhomogeneous ODE's:

- Additivity:

  If g is a sum of terms $g=g_1+g_2$, then the PI is the sum of the PI's for g_1 and g_2 individually.

- Not unique:

  There are infinitely many PI's.

4.4 Linear Independence and the Wronskian

Consider a 2-nd order HE with IC's. By the theorem of IVP, we know there should be a unique solution. Suppose we somehow get the solution:

$$y=c_1y_1+c_2y_2.$$

Satisfying the IC's implies

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y_0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = y_0'$$

Written in matrix form,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}.$$

if we want to find c_1 and c_2, we invert the matrix, if it is invertible.

Wronskian

Through this we define the Wronskian

$$W_{y_1,y_2}(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}. \tag{37}$$

The unique solution to the IVP can be found if $W(x_0) \neq 0$. This is because y_1 and y_2 are linearly independent when $W \neq 0$.

For just two functions, being linearly dependent means being proportional to each other.

4.4.1 Wronskian Test for Linear Dependence

The Wronskian test can be used to check for linear independence if the functions being tested are differentiable. As above,

$$W(x) \begin{cases} = 0 \text{ everywhere in the interval:} & \text{linear dependence} \\ \neq 0 \text{ somewhere in the interval:} & \text{linear independence} \end{cases} \tag{38}$$

4.4.2 Generalization

If these functions are sufficiently differentiable ($n-1$ times), then a Wronskian test can be used to check for linear independence:

$$W(x) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \dots & y_n^{n-1} \end{vmatrix} \tag{39}$$

As a gentle reminder, linear dependence is defined as

$$\sum k_i y_i(x) = 0. \tag{40}$$

4.5 Obtaining the Second Solution

From the equation $y'' + p(x)y' + q(x)y = 0$, we may find only one solution y_1. However, there is still a way through which we can find the second linear independent solution:

$$y_2(x) = y_1(x) \int^x \frac{1}{y_1^2(s)} \exp\left(-\int^s p(t)\,\mathrm{d}t\right) \mathrm{d}s\,. \tag{41}$$

The proof is through variation of parameters. To check that this is indeed linearly independent, we shall evaluate the Wronskian determinant:

$$W(x) = \exp\left(-\int^x p(s)\,\mathrm{d}s\right) \neq 0.$$

4.6 Higher-order Equations

The concepts seen in this chapter extend to n-th order linear ODE's in a natural way, such as the superposition of CF's and PI's and the characteristic equations.

5 Separation of Variables

This technique works for linear homogeneous PDE's without mixed partial derivatives.

The individual solutions found by the separation of variables method are **not general**: most solutions are not separable. However, they form a basis in terms of which the general solution may be expanded. Note that the linear sum of separable solutions is not in general separable.

The general procedure of the separation of variables technique is as follows:

- Separate variables;
- Solve the ODE's for the normal modes (eigenstates or eigenfunctions);
- Form the general solution by superposition and impose the initial conditions.

The process of variable separation sometimes needs more attention than expectation.

Diffusion equation in 2D

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Let us try the function $u(x,y,t) = X(x)Y(y)T(t)$. This leads us to

$$XY\frac{\mathrm{d}T}{\mathrm{d}t} = D\left(YT\frac{\mathrm{d}X}{\mathrm{d}x} + XT\frac{\mathrm{d}Y}{\mathrm{d}y}\right) \quad \Rightarrow \quad \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = D\left(\frac{1}{X}\frac{\mathrm{d}X}{\mathrm{d}x} + \frac{1}{Y}\frac{\mathrm{d}Y}{\mathrm{d}y}\right).$$

The terms in color each depend on only one variable, so they must be constants. Nevertheless, we cannot simply claim that

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{X}\frac{\mathrm{d}X}{\mathrm{d}x} = \frac{1}{Y}\frac{\mathrm{d}Y}{\mathrm{d}y} = \text{const.}$$

Instead, we should say

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = c = D\left(\frac{1}{X}\frac{\mathrm{d}X}{\mathrm{d}x} + \frac{1}{Y}\frac{\mathrm{d}Y}{\mathrm{d}y}\right),$$

from this we get the **secondary separation**

$$\frac{1}{X}\frac{\mathrm{d}X}{\mathrm{d}x} = k, \qquad \frac{1}{Y}\frac{\mathrm{d}Y}{\mathrm{d}y} = \frac{c}{D} - k.$$

5.1 Preliminaries: Series Solution

Consider a second order homogeneous linear differential equation:

$$\left[a_2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + a_0(x)\right] y = 0. \tag{42}$$

By rearrangement, we see its standard form:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right] y = 0. \tag{43}$$

Analytic and Singular Points

A function $f(x)$ that has a Taylor expansion about $x = x_0$ with a radius of convergence $R > 0$ is said to be **analytic** at $x = x_0$.

- If $p(x)$ and $q(x)$ are analytic functions at x_0, then x_0 is classified as an **ordinary point** of the ODE.
- If $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0, then x_0 is called a regular singular point.
- If neither of the above is true for x_0, then x_0 is a singular point.

For an ordinary point, the power series solution exists:

$$y(x) = \sum_{n=0} c_n (x - x_0)^n, \tag{44}$$

with the radius of convergence not smaller than the distance from x_0 to the nearest singular point.

Frobenius' Method

If x_0 is a regular singular point, then there exists at least one solution of the equation in the form of a generalized power series:

$$y(x) = (x - x_0)^r \sum_{n=0} c_n (x - x_0)^n = \sum_{n=0} c_n (x - x_0)^{n+r}, \tag{45}$$

where r is a parameter to be determined.
It is important to note that c_0 **must not be zero**, or we can absorb one more factor of $(x - x_0)$ into $(x - x_0)^r$. This leads to the **indicial equation** (equation for the index).

The roots of indicial equations are pivotal in the determination of the general solutions. In general, there are three cases:

- $\Delta r \notin \mathbb{N}$:

  This is the most simple case, and the roots $r = \pm m$ yield two independent solutions as we want:

$$y = c_0 y_0 + c_1 y_1 . \tag{46}$$

- $r_1 = r_2$:

  The second linearly independent solution is given by variation of parameters:

$$y_2(x) = y_1(x) \int^x \frac{1}{y_1^2(s)} \exp\left(- \int^s p(t)\,\mathrm{d}t \right) \mathrm{d}s \,. \tag{47}$$

- $\Delta r \in \mathbb{N}$:

  In this situation, the method of Frobenius may or may not produce the second solution to the differential equation (Don't you love this case).

Before progressing, we shall also meet the derivatives in series solution:

$$\begin{aligned} y &= \sum_{n=0} c_n (x - x_0)^n \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \sum_{n=1} n c_n (x - x_0)^{n-1} \\ &= \sum_{n=0} (n+1) c_{n+1} (x - x_0)^n \\ \Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \sum_{n=2} n(n-1) c_n (x - x_0)^{n-2} \\ &= \sum_{n=0} (n+2)(n+1) c_{n+2} (x - x_0)^n \end{aligned}$$

One important note here is that we're doing all of this for the sake of equalizing powers of x, not for beauty or confusion.

5.2 Legendre's Equation

5.2.1 Origin

Legendre's equation comes from Laplace's equation in spherical coordinates:

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{48}$$

Plugging the trial solution $u(r,\theta,\phi)=R(r)T(\theta)F(\phi)$ while multiplying r^2/u, we get

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1}{T}\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}T}{\mathrm{d}\theta}\right)+\frac{1}{F\sin^2\theta}\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2}=0.$$

Consider the substitution $\mu=\cos\theta$. By chain rule we see

$$\frac{\mathrm{d}}{\mathrm{d}\theta}=\frac{\mathrm{d}\mu}{\mathrm{d}\theta}\frac{\mathrm{d}}{\mathrm{d}\mu}=-\sin\theta\frac{\mathrm{d}}{\mathrm{d}\mu}=-\sqrt{1-\mu^2}\frac{\mathrm{d}}{\mathrm{d}\mu}. \tag{49}$$

Stupid Mistake

A futile trial leads to this color box. While we see

$$\frac{\mathrm{d}}{\mathrm{d}\theta}=-\sqrt{1-\mu^2}\frac{\mathrm{d}}{\mathrm{d}\mu},$$

but

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}=(1-\mu^2)\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}$$

is not true. The correct way is

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}=-\sqrt{1-\mu^2}\frac{\mathrm{d}}{\mathrm{d}\mu}\left(-\sqrt{1-\mu^2}\frac{\mathrm{d}}{\mathrm{d}\mu}\right)=-\mu\frac{\mathrm{d}}{\mathrm{d}\mu}+(1-\mu^2)\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}. \tag{50}$$

Therefore, the substitution leads to

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}T}{\mathrm{d}\mu}\right]+\frac{1}{F(1-\mu^2)}\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2}=0.$$

- First separation: ϕ

  Multiplying by $(1-\mu^2)$, we get

$$\frac{1-\mu^2}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1-\mu^2}{T}\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}T}{\mathrm{d}\mu}\right]=-\frac{1}{F}\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2}=m^2, \tag{51}$$

where we shall solve for $F(\phi)$.

- Second separation:

  Dividing by $(1-\mu^2)$, we get

$$\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}T}{\mathrm{d}\mu}\right]-\frac{m^2}{1-\mu^2}=-\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-k, \tag{52}$$

  where we shall solve for $R(r)$.

Multiplying T on both sides, we finally get the **associated Legendre's equation**:

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}T}{\mathrm{d}\mu}\right]+\left(k-\frac{m^2}{1-\mu^2}\right)T=0. \tag{53}$$

It arises naturally from the angular part of the solution of Laplace's equation when we apply separation of variables.

5.2.2 Series Solution

With the above discussion, by assuming $m=0$, we recognize $x=0$ as an ordinary point of Legendre's equation. Therefore, we try

$$\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+ky=0 \quad\Rightarrow\quad y=\sum_{n=0}c_nx^n.$$

To equalize all the powers of x, we see

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\sum_{n=0}(n+2)(n+1)c_{n+2}x^n, \qquad x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\sum_{n=0}n(n-1)c_nx^n.$$

By grouping all terms together, we get

$$\sum_{n=0}x^n\left[(n+2)(n+1)c_{n+2}-n(n-1)c_n-2nc_n+kc_n\right]=0 \tag{54}$$

This must hold for all n, so we get

$$c_{n+2}=\frac{n(n+1)-k}{(n+2)(n+1)}c_n. \tag{55}$$

The relation separates all odd indexes from the even ones, so we get the general solution as the linear combination of two independent solutions (bases):

$$y = c_0 y_0 + c_1 y_1 = c_0 \left[1 - \frac{k}{2!} x^2 - \frac{k(6-k)}{4!} x^4 + \dots \right] + c_1 \left[x - \frac{2-k}{3!} x^3 + \frac{(2-k)(12-k)}{5!} x^5 + \dots \right]. \tag{56}$$

The grouping of parity terms implies that y_0 is even, and y_1 is odd.

In physics, x should be in range $[-1, 1]$, but Gauss' test of convergence shows that the solutions diverges at $x = \pm 1$. This is potentially a problem. In order to get solutions that do not diverge at $x = \pm 1$, we need either to set the coefficients to 0, or we need a series to stop at some point. In general, we demand that one series is not infinite (transcendental), and the coefficient of the other is zero.

We shall notice that for some k, one of the series is truncated. This is given by

$$k = l(l+1), \quad l = 0, 1, 2, \dots \tag{57}$$

The solution diverges unless $k = l(l+1)$ and one coefficient is zero.

We shall also notice the radial part of the differential equation:

$$r^2 \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + 2r \frac{\mathrm{d}R}{\mathrm{d}r} - kR = 0, \tag{58}$$

which is in the form of Euler's differential equation. By identifying $r = 0$ as a regular singular point, the series solution gives

$$\left[(n+s)(n+s+1) - k \right] c_n = 0. \tag{59}$$

The requirement $c_0 \neq 0$ gives $k = s(s+1)$, and only $n = 0$ and $n = -2s - 1$ satisfies the equation. Therefore, the general solution is

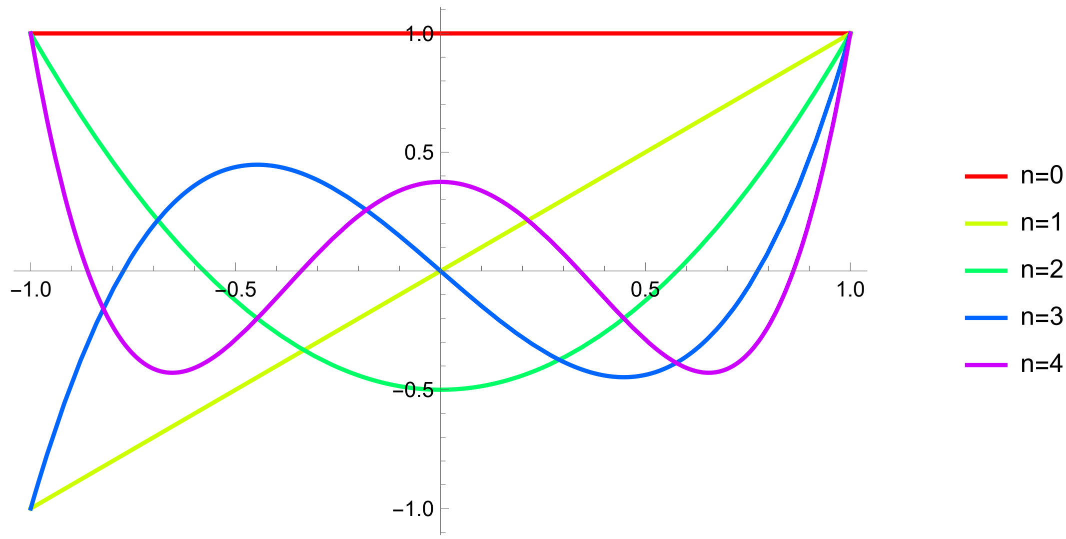
$$R(r) = Ar^s + Br^{-(s+1)}. \tag{60}$$

5.2.3 Legendre Polynomials

We get polynomial solutions $P_l(x)$ from the truncated power series. They are normalized, unusually, by setting $P_l(1) = 1$:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{3x^2 - 1}{2} \\ P_3(x) &= \frac{5x^3 - 3x}{2} \end{aligned} \tag{61}$$

The following graph shows the plots of $P_n(x)$, where n ranges from 0 to 4:



5.2.4 Alternative Definitions

- Rodriguez' formula:

$$P_l(x) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} \left[(x^2 - 1)^l \right]. \tag{62}$$

- Generating functions:

Generating Functions

A generating function is a representation of an infinite sequence of numbers as the coefficients of a formal power series. In this situation, we aim to find some function $g(x, t)$ such that

$$g(x, t) = \sum_{l=0} P_l(x) t^l. \tag{63}$$

We can notice with ease that

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{l=0} P_l(x) t^l \tag{64}$$

serves the role. This is the generating function definition of Legendre polynomials.

5.2.5 Properties

- Orthogonality:

$$\int_{-1}^{1} \mathrm{d}x\, P_l(x) P_n(x) = \frac{2}{2l+1}\delta_{ln}. \tag{65}$$

> The Legendre polynomials are normalized in an exotic way due to historical reasons. The normalization condition here is
>
> $$P_l(1) = 1. \tag{66}$$

- Completeness:

  The Legendre polynomials form a complete set, so all functions defined on $[-1, 1]$ that satisfy Dirichlet conditions can be broken down into the form

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x). \tag{67}$$

  The coefficients are called Legendre-Fourier coefficients:

$$a_l = \frac{2l+1}{2}\int_{-1}^{1} \mathrm{d}x\, f(x) P_l(x). \tag{68}$$

  From here we can see the completeness relation here:

$$\mathbb{1} = \sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x) \int_{-1}^{1} \mathrm{d}x\, P_l(x). \tag{69}$$

To make up the rest of youth, let's meet the completeness relation of complex Fourier series:

$$\mathbb{1} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \exp(in\theta) \int_{-\pi}^{\pi} \mathrm{d}\theta\, \exp(-in\theta). \tag{70}$$

5.2.6 Application: Multipole Expansion

Poisson's equation in electrostatics reads

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}. \tag{71}$$

For a point source with unit charge, the potential is

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{1}{\|\mathbf{r} - \mathbf{r}'\|}; \tag{72}$$

this implies that for a charge distribution given by $\rho(\mathbf{r}')$,

$$\phi = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathrm{d}^3\mathbf{r}' \, \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \rho(\mathbf{r}'). \tag{73}$$

What we would do here is to expand the $1/\|\mathbf{r} - \mathbf{r}'\|$ term:

$$\begin{aligned}
\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} &= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} \\
&= \frac{1}{r} \frac{1}{\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta}} \\
&\equiv \frac{1}{r} \frac{1}{\sqrt{1 + t^2 - 2xt}},
\end{aligned}$$

if $t \equiv \left(\frac{r'}{r}\right)$ and $x \equiv \cos\theta$. From here we see the generating functions of Legendre polynomials. Therefore,

$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \frac{1}{r} \frac{1}{\sqrt{1 + t^2 - 2xt}} = \frac{1}{r} \left[1 + xt + \left(\frac{3x^2}{2} - \frac{1}{2} \right) t^2 + \dots \right] \tag{74}$$

We shall divide the contributions of ϕ according to the Legendre polynomials:

- The first term is the contribution from charge as if all the charge is concentrated at the origin:

$$\phi_0 = \frac{1}{4\pi\epsilon_0 r} \int_{V'} \mathrm{d}^3\mathbf{r}' \, \rho(\mathbf{r}') \equiv \frac{1}{4\pi\epsilon_0 r} q. \tag{75}$$

- The second term is the contribution from the dipole moment.

$$\frac{xt}{r} = \frac{r'\cos\theta}{r^2} = \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3},$$

$$\phi_1 = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3} \cdot \int_{V'} \mathbf{r}'\rho(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}' \equiv \frac{1}{4\pi\epsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3}. \tag{76}$$

- The third term is the quadrupole potential.

$$\frac{1}{r}\left(\frac{3x^2}{2} - \frac{1}{2}\right)t^2 = \frac{r'^2}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

To understand this, in the basis of $\mathbf{r}$,

$$r'\cos\theta = r'_i\hat{r}_i \tag{77}$$

by Einstein notation. Therefore,

$$r'^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) = \hat{r}_i\hat{r}_j\left(\frac{3}{2}r'_ir'_j - \frac{1}{2}\|\mathbf{r}'\|^2\delta_{ij}\right). \tag{78}$$

From this we define the charge quadrupole tensor Q_{ij}:

$$Q_{ij} = \int_{V'} \mathrm{d}^3\mathbf{r}'\,\rho(\mathbf{r}')\left(\frac{3}{2}r'_ir'_j - \frac{1}{2}\|\mathbf{r}'\|^2\delta_{ij}\right). \tag{79}$$

So, this leads us to the quadrupole potential:

$$\phi_2 = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\hat{r}_iQ_{ij}\hat{r}_j. \tag{80}$$

The point is we want to find a quantity that doesn't depend on the position coordinates.

5.3 Bessel's Equation

5.3.1 Origin

Legendre's equation comes from Laplace's equation in cylindrical coordinates:

$$\nabla^2 \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}. \tag{81}$$

Consider the solution $u = R(\rho)F(\phi)Z(z)$. Dividing through u gives

$$\frac{1}{R}\left(\frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + \frac{1}{\rho^2 F}\frac{\mathrm{d}^2 F}{\mathrm{d}\phi^2} + \frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = 0.$$

- First separation: z
$$\frac{1}{R}\left(\frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + \frac{1}{\rho^2 F}\frac{\mathrm{d}^2 F}{\mathrm{d}\phi^2} = -\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = -k^2,$$
from which we can solve for $Z(z)$.

- Second separation: ϕ Multiplying by ρ^2, we get
$$\frac{\rho^2}{R}\left(\frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + k^2\rho^2 = \frac{1}{F}\frac{\mathrm{d}^2 F}{\mathrm{d}\phi^2} = -m^2,$$
so we can solve for ϕ. Multiplying through R and we get
$$\rho^2\frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \rho\frac{\mathrm{d}R}{\mathrm{d}\rho} + (k^2\rho^2 - m^2)R = 0.$$

Changing to a scaled variable $x = k\rho$, we get the Bessel's equation of order m:

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - m^2)y = 0. \tag{82}$$

Bessel's equation arises from the radial part of the solution of Laplace's equation in cylindrical coordinates.

5.3.2 Series Solution

Written in standard form, we see Bessel's equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 - \frac{m^2}{x^2}\right)y = 0. \tag{83}$$

Clearly, $x = 0$ is a regular singular point of Bessel's equation. This means we should use Frobenius' method here:

$$y = \sum_{n=0} c_n x^{n+r}, \qquad c_0 \neq 0.$$

However, it would still be easier if we solve

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - m^2)y = 0.$$

This leads us to

$$\begin{cases} x^2 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \displaystyle\sum_{n=0} (n+r)(n+r-1)c_n x^{n+r} \\ x\dfrac{\mathrm{d}y}{\mathrm{d}x} = \displaystyle\sum_{n=0} (n+r)c_n x^{n+r} \\ x^2 y = \displaystyle\sum_{n=0} c_n x^{n+r+2} = \sum_{n=2} c_{n-2} x^{n+r} \\ m^2 y = m^2 \displaystyle\sum_{n=0} c_n x^{n+r}. \end{cases}$$

There are some terms not beginning at $n = 0$. Combined with the fact that $c_0 \neq 0$, this gives the indicial equation:

$$\begin{cases} n = 0: \quad [r(r-1) + r - m^2]c_0 = 0 \\ n = 1: \quad [r(r+1) + r + 1 - m^2]c_1 = 0. \end{cases} \tag{84}$$

The first equation gives us $r^2 = m^2$, so $r = \pm m$. Then, the second equation reduces to

$$(2r+1)c_1 = 0 \Rightarrow r = -\frac{1}{2} \text{ or } c_1 = 0. \tag{85}$$

Firstly, let us consider the case that $c_1 = 0$, so r is not limited to $-1/2$. The series solution is now

$$(n+r)(n+r-1)c_n + (n+r)c_n + c_{n-2} - m^2 c_n = 0,$$

and this gives us

$$c_n = -\frac{1}{n^2 + 2rn} c_{n-2}. \tag{86}$$

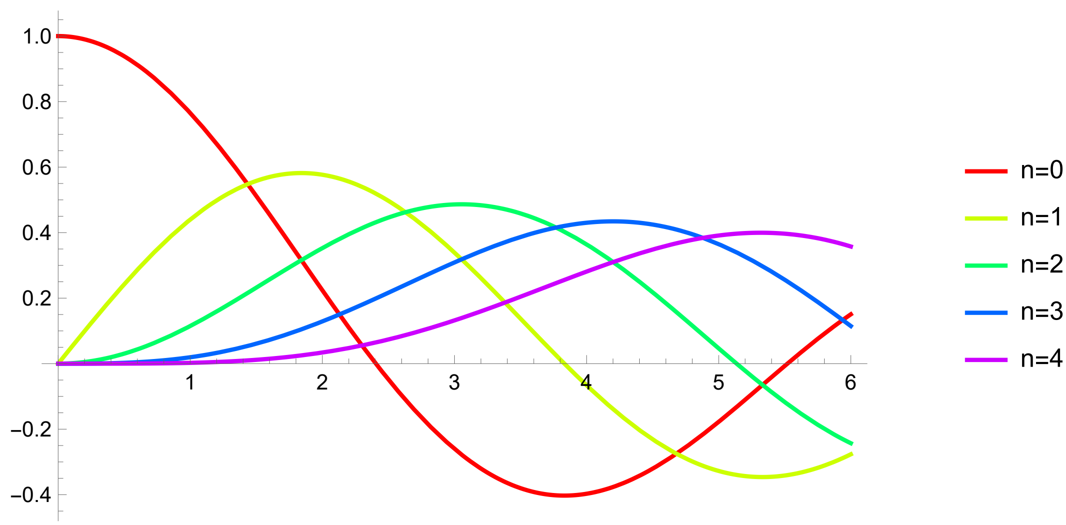
Ratio test shows that the radius of convergence is infinity. In the context of Bessel functions, we denote our solutions as Bessel functions of the first kind of order m, $J_m(x)$, when $\Delta r \notin \mathbb{N}$:

$$y(x) = aJ_m(x) + bJ_{-m}(x). \tag{87}$$

When $\Delta r \in \mathbb{N}$, auspiciously, using variation of parameters, we can get the Bessel functions of the second kind of order m, $Y_m(x)$:

$$y(x) = aJ_m(x) + bY_m(x). \tag{88}$$

This is also the general solution to Bessel's equation. The following plot shows the Bessel functions of the first kind, $J_n(x)$ where n ranges from 0 to 4.



We shall, particularly, investigate the case $r = -1/2$ where c_1 is not necessarily 0. In this case, we see

$$c_n = -\frac{1}{n(n-1)}c_{n-2}. \tag{89}$$

$r = -1/2$ is a special case of the equation when $m = 1/2$. Therefore, we should also investigate the behavior of the equation at $r = 1/2$:

$$c_n = -\frac{1}{n(n+1)}c_{n-2} \quad \Rightarrow \quad y(x) = c_0\frac{\sin x}{\sqrt{x}}.$$

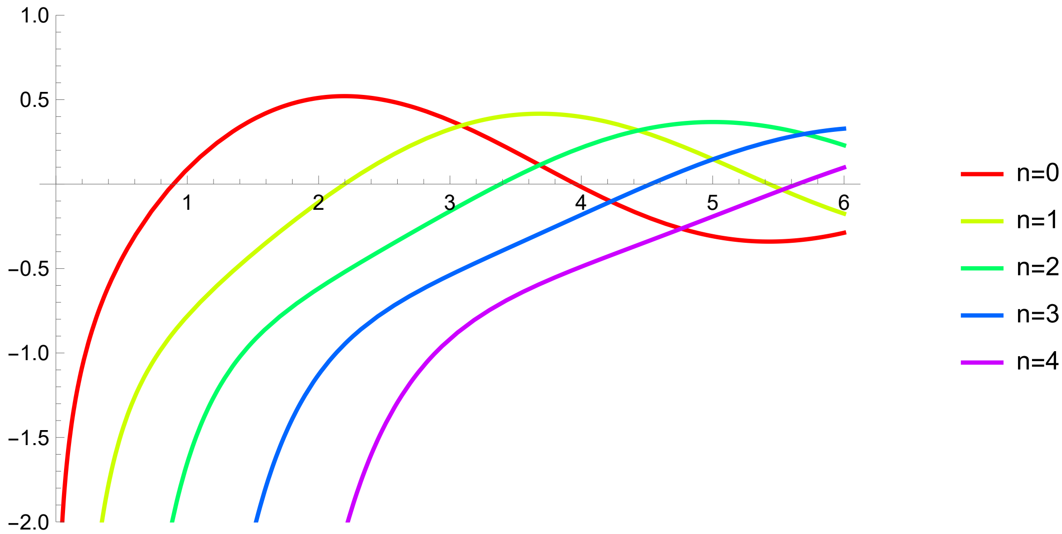
For the $r = -1/2$ case,

$$\begin{aligned} y(x) &= x^r \sum_{n=0} c_n x^n \\ &= x^{-1/2}c_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right) + x^{-1/2}c_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right) \\ &= c_0\frac{\cos x}{\sqrt{x}} + c_1\frac{\sin x}{\sqrt{x}}. \end{aligned}$$

The second term just duplicates the solution we found for $r = 1/2$, so the general solution is just

$$y(x) = c_0 \frac{\cos x}{\sqrt{x}} + c_1 \frac{\sin x}{\sqrt{x}}. \tag{90}$$

The following plot shows Bessel functions of the second kind, $Y_n(x)$ where n ranges from 0 to 4:



5.3.3 Alternative Definitions

- Rodriguez' formula:

$$J_n(x) = x^n \left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sin x}{x}\right), \tag{91}$$

$$Y_n(x) = -x^n \left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\cos x}{x}\right). \tag{92}$$

- Generating function:

$$\exp\left[\left(\frac{x}{2}\right)\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x) t^n. \tag{93}$$

- Integral representation:

  While the direct substitution $t = \exp(i\theta)$ and identification as a Fourier series arrives at the same result, we identify the generating function definition as a Laurent series (if you don't, then you have a problem).

Therefore, by the expansion coefficients of Laurent series,

$$c_n = \frac{1}{2\pi i} \oint_C \mathrm{d}z \, \frac{f(z)}{(z-a)^{n+1}}, \tag{94}$$

we know

$$J_n = \frac{1}{2\pi i} \oint_C \frac{\mathrm{d}z}{z^{n+1}} \exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right]. \tag{95}$$

Now we substitute $z = \exp(i\theta)$ and get

$$J_n = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \exp[i(x \sin\theta - n\theta)] = \frac{1}{\pi} \int_0^{\pi} \mathrm{d}\theta \cos(x \sin\theta - n\theta). \tag{96}$$

Particularly, when $n = 0$,

$$J_0 = \frac{1}{\pi} \int_0^{\pi} \mathrm{d}\theta \cos(x \sin\theta) = \frac{1}{\pi} \int_0^{\pi} \mathrm{d}\theta \sin(x \sin\theta). \tag{97}$$

5.3.4 Properties

- Orthogonality:

  To simplify the notation, define

$$\lambda_{pj} = \frac{\alpha_{pj}}{a}, \tag{98}$$

  so λ_{pj} is the value of the jth positive zero of J_p scaled by a fixed factor $1/a$. This leads us to

$$\int_0^a x \,\mathrm{d}x J_p(\lambda_{pj} x) J_p(\lambda_{pk} x) = \delta_{jk} \frac{a^2}{2} J_{p+1}^2(\alpha_{pj}). \tag{99}$$

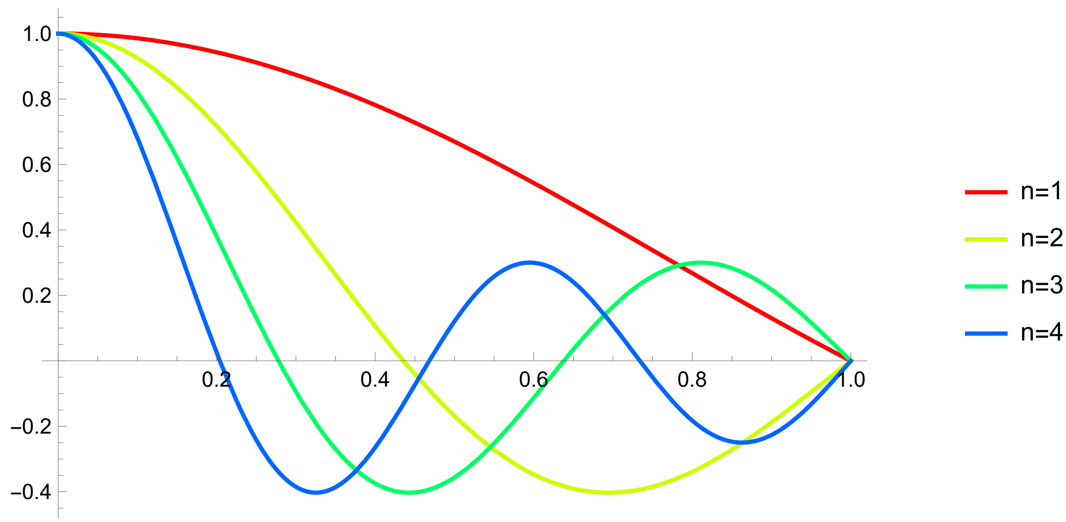
  The following plot shows the rescaling of $J_0(x)$ by its nth zero.

  There is a weighting function $\rho(x) = x$. This arises naturally, as the Bessel functions are actually defined in the context of cylindrical polars.

- Completeness:

  The Bessel functions form a complete set, so all functions defined on $[0, a]$ that satisfy Dirichlet conditions can be broken down into the form

$$f(x) = \sum_{j=1}^{\infty} A_j J_p(\lambda_{pj} x). \tag{100}$$



The related coefficients A_j are called Bessel-Fourier coefficients:

$$A_j = \frac{2}{a^2 J_{p+1}^2(\alpha_{pj})} \int_0^a x \,\mathrm{d}x \, f(x) J_p(\lambda_{pj} x). \tag{101}$$

The completeness relation here is given by the formula

$$\mathbb{1} = \sum_{j=1}^{\infty} J_p(\lambda_{pj} x) \frac{2}{a^2 J_{p+1}^2(\alpha_{pj})} \int_0^a x \,\mathrm{d}x \, J_p(\lambda_{pj} x). \tag{102}$$

- For integer p,

$$J_{-p}(x) = (-1)^p J_p(x). \tag{103}$$

This means that $J_{-p}(x)$ is not an independent solution from $J_p(x)$ when p is an integer.

- For any p,

$$Y_p(x) = \frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin(\pi p)}. \tag{104}$$

5.4 Summary

	Legendre Polynomials	Bessel Functions
Physical Situation	Spherical Laplace's Equation (θ)	Cylindrical Helmholtz's Equation (ρ)
ODE Definition	$(1-x^2)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\dfrac{\mathrm{d}y}{\mathrm{d}x} + ky = 0$	$x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}y}{\mathrm{d}x} + (x^2-m^2)y = 0$
Rodriguez' Formula	$P_l(x) = \dfrac{1}{2^l l!}\dfrac{\mathrm{d}^l}{\mathrm{d}x^l}\left[(x^2-1)^l\right]$	$J_n(x) = x^n\left(-\dfrac{1}{x}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\dfrac{\sin x}{x}\right)$ $Y_n(x) = -x^n\left(-\dfrac{1}{x}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\dfrac{\cos x}{x}\right)$
Generating Function	$\dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0} P_l(x)t^l$	$\exp\left[\left(\dfrac{x}{2}\right)\left(t-\dfrac{1}{t}\right)\right] = \sum_{n=-\infty} J_n(x)t^n$
Orthogonality	$\int_{-1}^{1}\mathrm{d}x\, P_l(x)P_n(x) = \dfrac{2}{2l+1}\delta_{ln}$	$\int_0^a x\,\mathrm{d}x\, J_p(\lambda_{pj}x)J_p(\lambda_{pk}x) = \delta_{jk}\dfrac{a^2}{2}J_{p+1}^2(\alpha_{pj})$
Fourier Series	$\mathbb{1} = \sum_{l=0}^{\infty}\dfrac{2l+1}{2}P_l(x)\int_{-1}^{1}\mathrm{d}x\, P_l(x)$	$\mathbb{1} = \sum_{j=1}^{\infty} J_p(\lambda_{pj}x)\dfrac{2}{a^2J_{p+1}^2(\alpha_{pj})}\int_0^a x\,\mathrm{d}x\, J_p(\lambda_{pj}x)$

Particularly, the Bessel functions has integral representation:

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta \exp[i\,(x\sin\theta - n\theta)]. \tag{105}$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\mathrm{d}\theta\cos(x\sin\theta - n\theta) = \frac{1}{\pi}\int_0^{\pi}\mathrm{d}\theta\cos(n\theta - x\sin\theta). \tag{106}$$

For $n=0$,

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\mathrm{d}\theta\cos(x\sin\theta) = \frac{1}{\pi}\int_0^{\pi}\mathrm{d}\theta\cos(x\cos\theta). \tag{107}$$

The separable solution to Laplace's equation in spherical coordinates is:

$$u_{lm} = \left[A_{lm}r^l + B_{lm}r^{-(l+1)}\right]P_{lm}(\cos\theta)\mathrm{e}^{im\phi} \quad\Rightarrow\quad u = \sum_l\sum_{m=-l}^{l}u_{lm}. \tag{108}$$

In cylindrical coordinates, we have

$$u_{mk} = [CJ_m(kr) + DY_m(kr)]\,\mathrm{e}^{im\phi}\left[A\mathrm{e}^{kz} + B\mathrm{e}^{-kz}\right] \quad\Rightarrow\quad u = \sum_{m,k}u_{mk}. \tag{109}$$

6 Fourier Methods

By defining Fourier transform pairs

$$\tilde{f}(k) = \mathcal{F}\left[f(x)\right] = \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\mathrm{e}^{-ikx} \tag{110}$$

and

$$f(x) = \mathcal{F}^{-1}\left[\tilde{f}(k)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \tilde{f}(k)\mathrm{e}^{ikx}, \tag{111}$$

we can transform PDE's into ODE's. This is because

$$\mathcal{F}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) = ik\tilde{f}(k). \tag{112}$$

This definition can be safely extend to higher dimensions:

$$\tilde{f}(\mathbf{k}) = \mathcal{F}\left[f(\mathbf{r})\right] = \int \mathrm{d}^3\mathbf{r}\, f(\mathbf{r})\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{113}$$

$$f(\mathbf{r}) = \mathcal{F}^{-1}\left[\tilde{f}(\mathbf{k})\right] = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\, \tilde{f}(\mathbf{k})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}, \tag{114}$$

where the differential operator $\boldsymbol{\nabla}$ transforms to $i\mathbf{k}$.

7 Green's Functions

7.1 Definition

Consider a DE with a linear differential Operator $\hat{\mathcal{L}}$:

$$\hat{\mathcal{L}}y = f(x). \tag{115}$$

Green's Function

The Green's function (GF), $G(x,z)$, is defined to be the solution to the equation above, but with the RHS set to a Dirac delta function located at $x = z$:

$$\hat{\mathcal{L}}G(x,z) = \delta(x-z). \tag{116}$$

Green's functions are a way to solve inhomogeneous ODE's and PDE's.

The solution, by the sifting property of Dirac delta, is

$$y(x) = \int \mathrm{d}z\, f(z)G(x,z). \tag{117}$$

7.2 Properties

- $G(x,z)$ satisfies the auxiliary conditions, like BC's in x:

$$G(a,z) = y_a \qquad \text{and} \qquad G(b,z) = y_b.$$

- Derivatives of $G(x,z)$:

  The nth derivative at $x = z$ is infinite, as needed for the LHS to yield a Dirac delta function demanded by the RHS.

  The $(n-1)$th derivative at $x = z$ is discontinuous. To show this, consider $\hat{\mathcal{L}} = \sum_{i=0}^{n} a_i(x)\frac{\mathrm{d}^i}{\mathrm{d}x^i}$. Integration around $x = z$ gives

$$\int_{z-\epsilon}^{z+\epsilon} \mathrm{d}x\, \hat{\mathcal{L}}y = \int_{z-\epsilon}^{z+\epsilon} \mathrm{d}x\, \delta(x-z) = 1. \tag{118}$$

  By shrinking $\epsilon \to 0$ and assuming all $y^{(i)}$'s are finite $(i = 1, 2, \ldots, n-1)$, we get

$$\int_{z-\epsilon}^{z+\epsilon} \mathrm{d}x\, a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 1 \quad \Rightarrow \quad \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{n-1}\Bigg|_{z+\epsilon} - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{n-1}\Bigg|_{z-\epsilon} = \frac{1}{a_n(z)}. \tag{119}$$

- Adjoint symmetry:

$$G(x,z) = G^\dagger(z,x) \tag{120}$$

where $\hat{\mathcal{L}}^\dagger G^\dagger = \delta(x-z)$, and $\hat{\mathcal{L}}^\dagger$ is the adjoint of $\hat{\mathcal{L}}$.

Adjoint

For a linear operator T, its formal adjoint is defined as follows:

$$Tu = \sum_{k=0}^{n} a_k(x) \frac{\mathrm{d}^k}{\mathrm{d}x^k} u \qquad \Rightarrow \qquad T^\dagger = \sum_{k=0}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left[a_k^*(x) u\right]. \tag{121}$$

For a second order linear operator

$$\hat{\mathcal{L}} = a_2(x) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x) \frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x),$$

its adjoint is

$$\hat{\mathcal{L}}^\dagger \equiv \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[a_2(x) y\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[a_1(x) y\right] + a_0(x). \tag{122}$$

The immediate corollary is that if $\hat{\mathcal{L}}$ is **self-adjoint** ($\hat{\mathcal{L}} = \hat{\mathcal{L}}^\dagger$), then G is symmetric:

$$G(x,z) = G(z,x). \tag{123}$$

7.3 Green's Functions for Boundary Value Problems (BVP)

The general procedure is

- Find the CF for the HE.

- The GF is formed from the CF in two pieces, where the point of transition is $x = z$.

Except at $x = z$, the GF satisfies

$$\hat{\mathcal{L}} G = 0,$$

so away from $x = z$ the GF will have the form of the CF (with particular constants to fit the BC's).

Note that $\hat{\mathcal{L}}$ is a differential operator that only involves x, but $G(x,z)$ involves both x and z. This means that the coefficients of independent solutions should be functions of z.

- Apply the BC's. For the left piece, only consider the left BC.

- Match the two pieces at $x = z$ using the continuity properties of GF's there. This fixes the z dependence of GF.

To use the GF, note that it is split into two parts:

$$y(x) = \int \mathrm{d}z\, f(z)G(x,z) = \int^{x} \mathrm{d}z\, f(z)G_{z<x}(x,z) + \int_{x} \mathrm{d}z\, f(z)G_{z>x}(x,z). \tag{124}$$

7.4 Green's Functions in Multiple Dimensions

One thing to note here is the Dirac delta in multiple dimensions. By

$$\int_V \mathrm{d}^3\mathbf{r}\, \delta(\mathbf{r} - \mathbf{r}') = 1, \tag{125}$$

we can get

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{\delta(r - r')}{r^2} \frac{\delta(\theta - \theta')}{\sin\theta} \delta(\phi - \phi'). \tag{126}$$

It's easy to show the Green's function for Poisson equation:

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}'\|}. \tag{127}$$

7.5 Method of Images

If one wants to solve $\hat{\mathcal{L}}u(\mathbf{r}) = f(\mathbf{r})$ subject to the BC $u(\mathbf{r}) = 0$ on some finite boundary S, the procedure is to build a GF from the superposition of an appropriate fundamental GF (GF with BC infinitely far away) inside the domain of the problem plus fundamental GF's located outside the domain.

Mathematically,

$$G(\mathbf{x}, \mathbf{z}) = F(\mathbf{x}, \mathbf{z}) + H(\mathbf{x}, \mathbf{z}), \tag{128}$$

where $F(\mathbf{x}, \mathbf{z})$ and $H(\mathbf{x}, \mathbf{z})$ each is the GF inside and outside the domain. $H(\mathbf{x}, \mathbf{z})$ can be thought of as images or reflections of the GF inside the domain.

Since the image GF's are located outside the domain, then $\hat{\mathcal{L}}H = 0$ for all $\mathbf{x}$ inside the domain. Thus, H serves to correct F so as to enforce the homogeneous BC's, but does not interfere with the singular behavior of F.

8 Boundary Value Problems and Eigenvalue Problems

8.1 Eigenvalue Problems (EVP)

An eigenvalue problem (EVP) is a BVP with linear differential operators, homogeneous ODE's, and BC's. Generally, we can write

$$\hat{\mathcal{L}}y = \lambda h(x)y, \tag{129}$$

where $\hat{\mathcal{L}}$ is a linear operator, λ is the eigenvalue, $h(x)$ is some function, and y is the eigenfunction.

When the linear operator is self-adjoint, then we write

$$\hat{\mathcal{L}}y = \lambda \rho(x)y, \tag{130}$$

where $\rho(x)$ is a real-valued 'weight function' which is strictly positive. In this case, we choose

$$\int_a^b \rho(x)|y(x)|^2 \, \mathrm{d}x = 1 \tag{131}$$

as the normalization condition.

8.2 Self-Adjoint Problems (SAP)

An EVP is said to be **self-adjoint** if

$$\int_a^b u^* \hat{\mathcal{L}} v \, \mathrm{d}x = \int_a^b \left[\hat{\mathcal{L}}u\right]^* v \, \mathrm{d}x \, . \tag{132}$$

Borrowing the heritage of quantum mechanics, we shall write

$$\left\langle u \middle| \hat{\mathcal{L}} v \right\rangle = \left\langle \hat{\mathcal{L}} u \middle| v \right\rangle . \tag{133}$$

When $\hat{\mathcal{L}}$ satisfies this condition, we say that $\hat{\mathcal{L}}$ is Hermitian. The importance of SAP's lies in the following:

- SAP's have real eigenvalues with orthogonal eigenfunctions;
- Solutions of SAP's form a complete basis, so any function that satisfies Dirichlet conditions can be expanded. Examples include Fourier and Legendre series.

8.3 Sturm-Liouville Problems (SLP)

Sturm-Liouville Problems (SLP's) are EVP's involving a linear operator $\hat{\mathcal{L}}$ in a particular form, together with special BC's. The key point is that SLP's are self-adjoint.

Sturm-Liouville Problems

The specific form of $\hat{\mathcal{L}}$ is $\hat{\mathcal{L}}y = [p(x)y']' + q(x)y$, which results in the ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)y = \lambda\rho(x)y. \tag{134}$$

- $p(x)$, $p'(x)$, $q(x)$, and $\rho(x)$ are continuous and real;
- $\rho(x) > 0$ and $p(x) > 0$ ($p = 0$ is allowed at the ends of the interval).

The required BC's are that, for any solutions u and v,

$$\left[u^* p\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=a}^{x=b} = 0. \tag{135}$$

The beauty of this problem is that it covers both regular and singular EVP's. The BC here extends the definition of homogeneous BC to include singular problems where one or both of the boundaries are singular points of the ODE.

If an EVP is not in Sturm-Liouville form, it is sometimes possible to get it into Sturm-Liouville form by using an integrating factor. Consider the linear ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = \lambda h(x)y.$$

The trick is to apply the IF to $z \equiv \dfrac{\mathrm{d}y}{\mathrm{d}x}$ instead of y:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x} + f(x)z.$$

By the integrating factor

$$\mu(x) \equiv \exp\left[\int^x f(s)\,\mathrm{d}s\right], \tag{136}$$

$$\mu(x)\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)z\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right].$$

Therefore, multiplying the IF gives us

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \mu(x)g(x)y = \lambda\mu(x)h(x)y. \tag{137}$$

Conversion to Sturm-Liouville form does not change the solutions, but it does reveal the **weight function** needed in normalization (and in generalized Fourier series).