Vibrations and Waves

Course Notes

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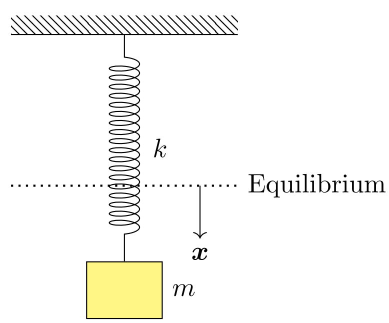
Chapter 1

Harmonic Oscillators

1.1 The Harmonic Oscillator Equation

As the first example, consider a mass hanging on a spring.



When displaced from equilibrium, the restoring force is linearly proportional to the displacement:

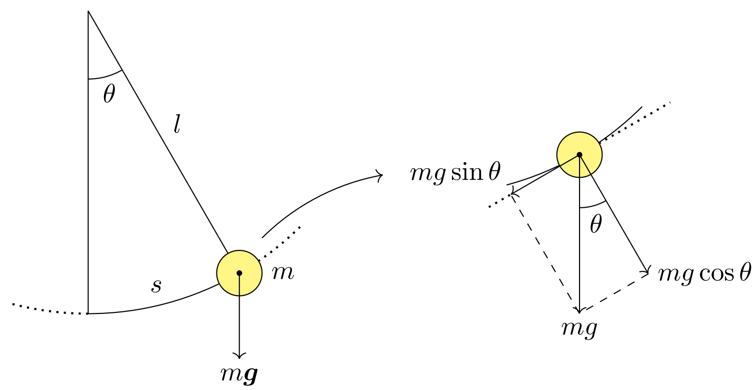
$$\boldsymbol{F} = -k\boldsymbol{x}. \tag{1.1}$$

In the above equation, the minus sign corresponds to the word "restoring", k is the proportionality (spring constant), and $\boldsymbol{x}$ is the displacement from the equilibrium.

Using Newton's 2nd law, we get:

$$m\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = -k\boldsymbol{x} \Rightarrow \frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = -\frac{k}{m}\boldsymbol{x}.$$

Now consider a simple pendulum as the second example.



In the above figure, note that the arc length s is

$$s = l\theta.$$

The restoring force, in this case, is given by the component of gravity tangential to the track of the mass, or $mg\sin\theta$. Therefore, we write

$$F_s = -mg\sin\theta,$$

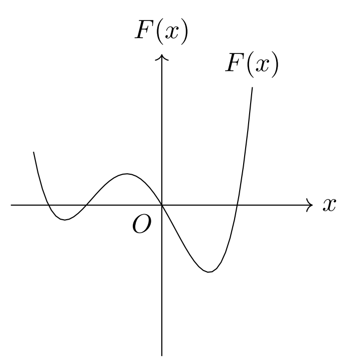
where the minus sign is for the word "restoring". For very small angles, we know

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \approx \theta.$$

According to Newton's 2nd law,

$$\begin{aligned} m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} &= -mg\sin\theta \\ ml\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} &\approx -mg\theta \\ \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} &= -\frac{g}{l}\theta. \end{aligned}$$

Actually, harmonic oscillators are more common in life than we might expect. Mathematically, consider an arbitrary force as a function of the position, $F(x)$. Assume that the force is zero at the origin $x = 0$.



Now, Taylor expand the function at the origin and we get:

$$F(x) = F(x=0) + \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=0} x + \frac{1}{2}\left.\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right|_{x=0} x^2 + \dots.$$

For x that is small enough,

$$\begin{cases} F(x=0) & \text{Zero. This is the equilibrium point.} \\ \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=0} x & \text{Linear restoring force.} \\ \frac{1}{2}\left.\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right|_{x=0} x^2 & \text{Small when } x \text{ is small enough.} \end{cases}$$

1.2 Solutions to the Harmonic Oscillator Equation

From the above examples, we get the equation for the mass on the spring and the pendulum in similar forms:

$$\ddot{x} = -\frac{k}{m}x, \qquad \ddot{\theta} = -\frac{g}{l}\theta.$$

In fact, all the harmonic oscillators have the same governing equation.

1.2.1 Trial Solution

To find out the solution to these group of equations, we can begin with a trial solution:

$$\begin{aligned} x &= a\cos(\omega_0 t) \\ \Rightarrow \dot{x} &= -a\omega_0 \sin(\omega_0 t) \\ \Rightarrow \ddot{x} &= -a\omega_0^2 \cos(\omega_0 t) \\ &= -\omega_0^2 x. \end{aligned}$$

This indicates that our proposed solution is a solution provided that ω_0^2 take the right values. That is to say,

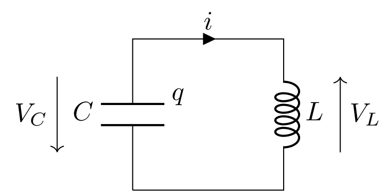
$$\begin{cases} \ddot{x} = -\frac{k}{m}x \Rightarrow \omega_0 = \sqrt{\frac{k}{m}} \\ \ddot{\theta} = -\frac{g}{l}\theta \Rightarrow \omega_0 = \sqrt{\frac{g}{l}}. \end{cases}$$

It is obvious that the trial solution $x = b\sin(\omega_0 t)$ works just as well. Providing that $x = a\cos(\omega_0 t)$ and $x = b\sin(\omega_0 t)$ are clearly linearly independent, the general solution to this second order differential equation is

$$x = a\cos(\omega_0 t) + b\sin(\omega_0 t) \Rightarrow x = A\cos(\omega_0 t + \phi), \tag{1.2}$$

where a and b are determined by the initial conditions, A, also determined by the initial conditions, is the amplitude of the oscillation, and ω_0 is called the natural frequency of this harmonic oscillator.

Another common harmonic oscillator is the LC circuit, a circuit that consists of an inductor and a capacitor:



Using Kirchhoff's Voltage Law (KVL), we can write:

$$\begin{aligned} V_C + V_L &= 0 \\ \frac{q}{C} + L\frac{\mathrm{d}i}{\mathrm{d}t} &= 0 \\ L\frac{\mathrm{d}i}{\mathrm{d}t} &= -\frac{q}{C} \\ \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} &= -\frac{1}{LC}q. \end{aligned}$$

Therefore, we can get that the natural frequency of a LC circuit is:

$$\omega_0 = \sqrt{\frac{1}{LC}}.$$

To sum up, the three most common types of harmonic oscillators have the natural frequencies listed below:

$$\begin{cases} \text{Mass on a Spring} & \omega_0 = \sqrt{\dfrac{k}{m}} \\ \text{Simple Pendulum} & \omega_0 = \sqrt{\dfrac{g}{l}} \\ \text{LC Circuit} & \omega_0 = \sqrt{\dfrac{1}{LC}}. \end{cases} \tag{1.3}$$

1.2.2 Analytic Solution

We may begin with the equation

$$\ddot{x} = -\omega_0^2 x. \tag{1.4}$$

Rearranging gives

$$\ddot{x} + \omega_0^2 x = 0,$$

and the corresponding characteristic equation is

$$r^2 + \omega_0^2 = 0.$$

The two roots of this characteristic equation is

$$r = \pm i\omega_0.$$

Therefore, the general solution to the differential equation $\ddot{x} = -\omega_0^2 x$ should be:

$$\begin{aligned} x &= C_1 \mathrm{e}^{r_1 t} + C_2 \mathrm{e}^{r_2 t} \\ &= C_1 \mathrm{e}^{i\omega_0 t} + C_2 \mathrm{e}^{-i\omega_0 t}. \end{aligned}$$

According to Euler's equation,

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta,$$

we can rewrite the above solution as:

$$\begin{aligned} x &= C_1 \mathrm{e}^{i\omega_0 t} + C_2 \mathrm{e}^{-i\omega_0 t} \\ &= C_1(\cos\omega_0 t + i\sin\omega_0 t) + C_2(\cos(-\omega_0 t) + i\sin(-\omega_0 t)) \\ &= C_1(\cos\omega_0 t + i\sin\omega_0 t) + C_2(\cos\omega_0 t - i\sin\omega_0 t) \\ &= (C_1 + C_2)\cos\omega_0 t + i(C_1 - C_2)\sin\omega_0 t \\ &\equiv K_1\cos\omega_0 t + K_2\sin\omega_0 t. \end{aligned}$$

That is to say, the general solution is $x = K_1\cos\omega_0 t + K_2\sin\omega_0 t$.

1.2.3 Complex Trial Solution

Using complex notations, we can introduce a complex variable $\tilde{x} = x + iy$ satisfying the equation

$$\ddot{\tilde{x}} = -\omega_0^2\tilde{x}.$$

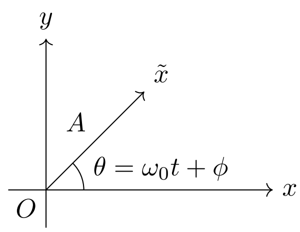
Clearly, the real part of the above function is the harmonic oscillation equation we are working with. Now, try the solution $\tilde{x} = \tilde{A}\mathrm{e}^{i\omega_0 t}$, where $\tilde{A} = A\mathrm{e}^{i\phi}$:

$$\begin{aligned} \dot{\tilde{x}} &= i\omega_0\tilde{A}\mathrm{e}^{i\omega_0 t} \\ \ddot{\tilde{x}} &= -\omega_0^2\tilde{A}\mathrm{e}^{i\omega_0 t} = -\omega_0^2\tilde{x}. \end{aligned}$$

The above derivation indicates that $\tilde{x} = \tilde{A}\mathrm{e}^{i\omega_0 t}$ is also a solution of the differential equation. The real part is the solution of the harmonic oscillator equation:

$$x = \mathrm{Re}\{\tilde{x}\} = \mathrm{Re}\big\{A\mathrm{e}^{i\phi}\mathrm{e}^{i\omega_0 t}\big\} = A\cos(\omega_0 t + \phi).$$

Using an argand diagram, we can plot $\tilde{x} = \tilde{A}\mathrm{e}^{i\omega_0 t}$ out:



Noting that A is fixed and θ advances uniformly in time, we can recognize $\tilde{x}$ as a uniform circular motion. And by recalling that the real part is the solution to the harmonic oscillator equation, we say that circular motion projected onto an axis is the harmonic motion.

1.3 Energy of the Harmonic Oscillator

1.3.1 Potential Energy

Let's return to the mass on the spring harmonic oscillator. The potential energy satisfies the equation

$$U = -W,$$

where W is the work done in stretching the spring.

$$\begin{aligned}
\mathrm{d}W &= F(x)\mathrm{d}x \\
U = -W &= -\int \mathrm{d}W \\
&= -\int_0^x F(x')\mathrm{d}x' \\
&= \int_0^x kx'\mathrm{d}x' \\
&= \frac{1}{2}kx^2 \\
&= \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi).
\end{aligned}$$

1.3.2 Kinetic Energy

We begin with the expression of kinetic energy:

$$K = \frac{1}{2}m\dot{x}^2.$$

According to the general solution $x = A\cos(\omega_0 t + \phi)$, we know that the velocity of the harmonic oscillator is given by:

$$\dot{x} = -\omega_0 A \sin(\omega_0 t + \phi).$$

Therefore,

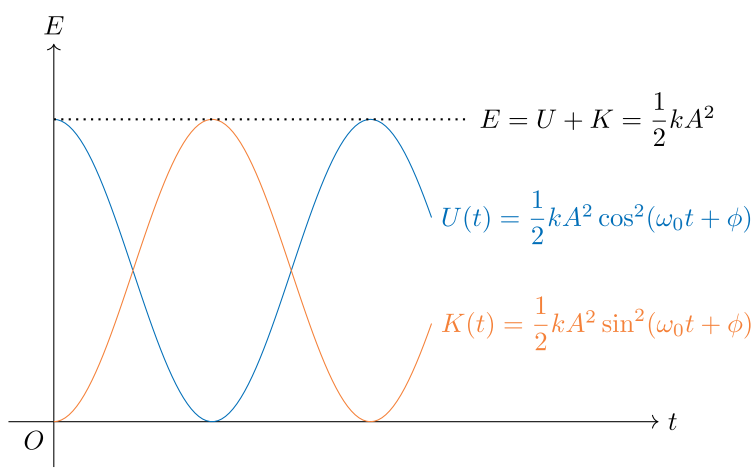
$$K = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi).$$

1.3.3 Total Energy

The total mechanical energy of the system is the sum of the potential energy and the kinetic energy. Recalling that $\omega_0^2 = \dfrac{k}{m}$,

$$\begin{aligned} E &= U + K \\ &= \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) + \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) + \frac{1}{2}kA^2 \sin^2(\omega_0 t + \phi) \\ &= \frac{1}{2}kA^2. \end{aligned}$$

The total energy in an ideal harmonic oscillator is always a constant.



The energy oscillates back and forth between potential energy and kinetic energy with a period of $\dfrac{T}{2}$.

Chapter 2

Damped Harmonic Oscillators

The damped harmonic oscillators are a more complicated version of harmonic oscillators. Apart from the force that is proportional to the displacement, there is an extra force ("damping term") that is proportional to the velocity:

$$\boldsymbol{F}_d = -b\dot{\boldsymbol{x}}. \tag{2.1}$$

In this case, the total force, according to Newton's 2nd law, is

$$\begin{aligned} F &= -kx - b\dot{x} \\ m\ddot{x} &= -kx - b\dot{x} \\ \ddot{x} &= -\frac{k}{m}x - \frac{b}{m}\dot{x}. \end{aligned}$$

We recognize $\dfrac{k}{m} \equiv \omega_0^2$, define $\gamma = \dfrac{b}{m}$, and we can rewrite the above equation as

$$\ddot{x} = -\omega_0^2 x - \gamma\dot{x}.$$

This equation can be generalised to many other variables, like the angle for the pendulum and the current in a circuit, etc. So, we now use ψ to represent the general variable. The equation for damped harmonic oscillation is:

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = 0. \tag{2.2}$$

In order to solve this equation, we apply the complex notation method again. To begin with, we introduce a complex variable $\tilde{\psi}$, whose real part is ψ, satisfying the equation

$$\ddot{\tilde{\psi}} + \gamma\dot{\tilde{\psi}} + \omega_0^2\tilde{\psi} = 0.$$

Then, consider a trial solution $\tilde{\psi} = \tilde{A}e^{i\omega t}$, with $\tilde{A} = Ae^{i\phi}$. Substituting this trial solution into the differential equation gives:

$$\begin{aligned}(i\omega)^2\tilde{A}e^{i\omega t} + \gamma(i\omega)\tilde{A}e^{i\omega t} + \omega_0^2\tilde{A}e^{i\omega t} &= 0\\ -\omega^2 + i\gamma\omega + \omega_0^2 &= 0\\ \omega^2 - i\gamma\omega - \omega_0^2 &= 0.\end{aligned}$$

This is a quadratic equation, and we can easily derive the roots for ω:

$$\omega = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

This indicates that our trial solution is acceptable, and there are two possible values of ω.

According to the value of the discriminant,

$$\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2},$$

we define three regimes of damping: light damping, critical damping, and heavy damping.

2.1 Light Damping

The damping term is simply the damping force,

$$\boldsymbol{F}_d = -b\dot{\boldsymbol{x}},$$

and light damping means a small b, and thus a small γ. In this case, γ is small enough so that the discriminant

$$\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} > 0.$$

Alternatively, we know that

$$\frac{\gamma}{2} < \omega_0.$$

In this case, for simplicity, we write

$$\omega = \frac{i\gamma}{2} \pm \omega_d,$$

where $\omega_d = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$ is real.

Therefore, the solution to the differential equation should be:

$$\begin{aligned}\tilde{\psi} &= \tilde{A}\mathrm{e}^{i\omega t} \\ &= \tilde{A}\mathrm{e}^{i(i\gamma/2\pm\omega_d)t} \\ &= A\mathrm{e}^{i\phi}\mathrm{e}^{-\gamma t/2}\mathrm{e}^{\pm i\omega_d t} \\ &= A\mathrm{e}^{-\gamma t/2}\mathrm{e}^{i(\pm\omega_d t+\phi)}.\end{aligned}$$

Then, the real part should be:

$$\psi = \mathrm{Re}\left\{\tilde{\psi}\right\} = A\mathrm{e}^{-\gamma t/2}\cos(\pm\omega_d t + \phi).$$

It seems that there are two solutions,

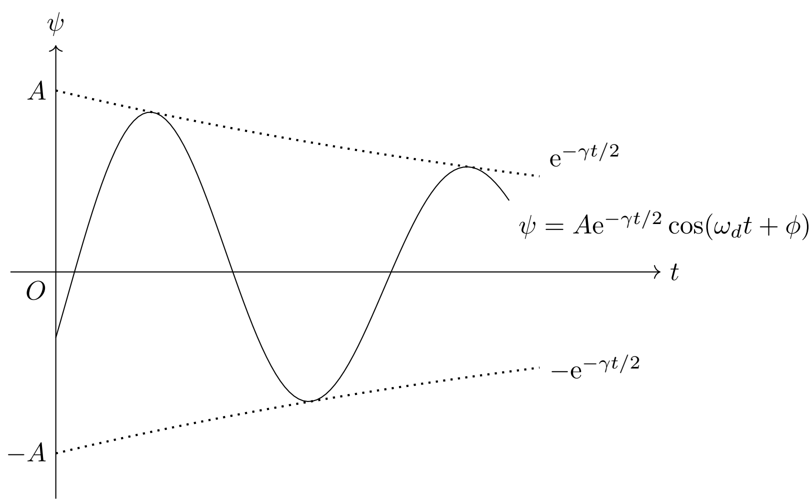
$$\begin{cases}\psi = A\mathrm{e}^{-\gamma t/2}\cos(\omega_d t + \phi) \\ \psi = A\mathrm{e}^{-\gamma t/2}\cos(-\omega_d t + \phi).\end{cases}$$

However, by recalling that

$$\cos(-\omega_d t + \phi) = \cos(\omega_d t - \phi)$$

and noting that ϕ isn't determined yet, these two expressions are actually the same solution. The general solution, thus, is:

$$\psi = A\mathrm{e}^{-\gamma t/2}\cos(\omega_d t + \phi).$$



This is an oscillation at angular frequency ω_d with exponentially decaying amplitude. As the two exponential decay curves serve as the envelope of the amplitude, a larger γ means a faster decay to equilibrium.

2.2 Heavy Damping

Heavy damping is the situation in which $\frac{\gamma}{2} > \omega_0$. At this time, the discriminant

$$\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

is imaginary. In this case, for simplicity, we introduce

$$\frac{\gamma'}{2} = \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} < \frac{\gamma}{2}.$$

The discriminant is imaginary as the term inside the square root is negative, so γ' is real, as the term inside the square root must be positive. Therefore,

$$\begin{aligned} \omega &= \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} \\ &= \frac{i\gamma}{2} \pm i\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} \\ &= \frac{i\gamma}{2} \pm \frac{i\gamma'}{2} \\ &= i\left(\frac{\gamma}{2} \pm \frac{\gamma'}{2}\right). \end{aligned}$$

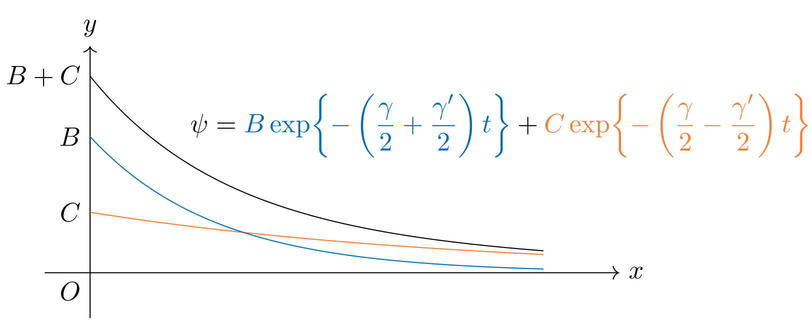
Then, the complex exponential would be:

$$\exp\{i\omega t\} = \exp\left\{i\left[i\left(\frac{\gamma}{2} \pm \frac{\gamma'}{2}\right)\right]t\right\} = \exp\left\{-\left(\frac{\gamma}{2} \pm \frac{\gamma'}{2}\right)t\right\}.$$

Therefore, the general solution to heavy damping is:

$$\psi = Be^{i\omega_1 t} + Ce^{i\omega_2 t} = B\exp\left\{-\left(\frac{\gamma}{2} + \frac{\gamma'}{2}\right)t\right\} + C\exp\left\{-\left(\frac{\gamma}{2} - \frac{\gamma'}{2}\right)t\right\}.$$

The first term decays faster than $\frac{\gamma}{2}$, and the second term decays more slowly.



As the second term decays more slowly, on a long time scale, this term will dominate. Also, increasing γ' decreases the value of $\frac{\gamma}{2} - \frac{\gamma'}{2}$; therefore, increasing γ' makes the system decay more slowly to equilibrium.

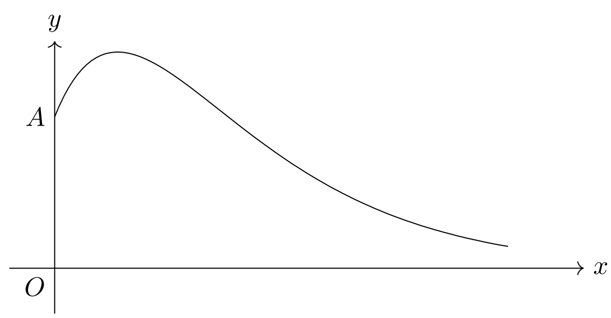
2.3 Critical Damping

The critical damping describes the situation where $\frac{\gamma}{2} = \omega_0$. As this time, the dicriminant

$$\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} = 0.$$

The general solution to this equation is

$$\psi = (A + Bt)\mathrm{e}^{-\gamma t/2}.$$

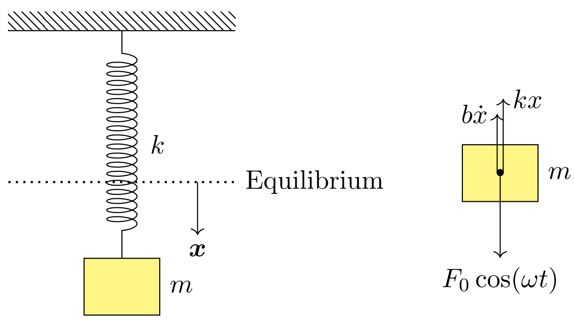


Chapter 3

Driven Harmonic Oscillators

3.1 Equation of Motion

We are to study the motion of a mass on a spring again. However, this time, in addition to the force from the spring ($-k\boldsymbol{x}$) and the damping force ($-b\dot{\boldsymbol{x}}$), the mass is now driven by a time varying external force $F_0 \cos(\omega t)$.



By Newton's 2nd law,

$$F = m\ddot{x} = -kx - b\dot{x} + F_0 \cos(\omega t),$$

and the equation of motion is given by the following expression:

$$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x} + \frac{F_0}{m}\cos(\omega t).$$

By identifying the natural frequency $\omega_0^2 = \dfrac{k}{m}$ and defining the damping coefficient $\gamma = \dfrac{b}{m}$, which has the unit of rad/s, we get:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t). \tag{3.1}$$

3.2 Solution

We apply the complex notation method again. Introduce a complex variable $\tilde{x}$ that $\mathrm{Re}\{\tilde{x}\} = x$ and satisfies the equation

$$\ddot{\tilde{x}} + \gamma\dot{\tilde{x}} + \omega_0^2\tilde{x} = \frac{F_0}{m}\mathrm{e}^{i\omega t}.$$

Now, begin with the trial solution

$$\tilde{x} = \tilde{A}\mathrm{e}^{i\omega_1 t}.$$

Plugging this back to the original equation, we get:

$$-\omega_1^2\tilde{A}\mathrm{e}^{i\omega_1 t} + i\omega_1\gamma\tilde{A}\mathrm{e}^{i\omega_1 t} + \omega_0^2\tilde{A}\mathrm{e}^{i\omega_1 t} = \frac{F_0}{m}\mathrm{e}^{i\omega t}.$$

Multiply both sides by $\mathrm{e}^{-i\omega t}$:

$$\tilde{A}\mathrm{e}^{i(\omega_1-\omega)t}\left(-\omega_1^2 + i\omega_1\gamma + \omega_0^2\right) = \frac{F_0}{m}.$$

As the RHS is independent of t, we know that LHS must also be independent of t. Therefore, $\omega_1 = \omega$. This makes sense, as the system oscillates at the frequency of the drive. Then,

$$\tilde{A}\left(-\omega^2 + i\omega\gamma + \omega_0^2\right) = \frac{F_0}{m} \Rightarrow \tilde{A} = \frac{F_0/m}{\omega_0^2 - \omega^2 + i\omega\gamma}.$$

Let $\tilde{A} = A\mathrm{e}^{i\phi}$, and we get:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2}} \tag{3.2}$$

$$\phi = \arctan\left(-\frac{\omega\gamma}{\omega_0^2-\omega^2}\right). \tag{3.3}$$

Therefore, we get the solution to the original real equation:

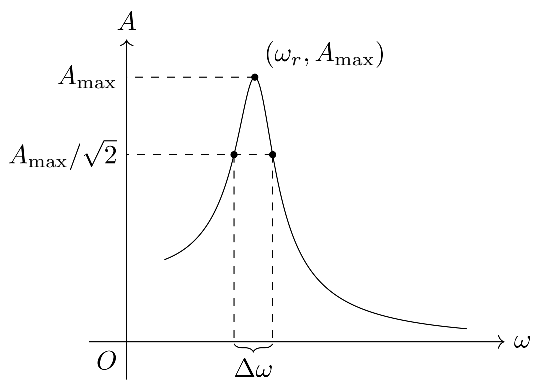
$$\begin{aligned} x = \mathrm{Re}\{\tilde{x}\} &= \mathrm{Re}\left\{\tilde{A}\mathrm{e}^{i\omega t}\right\} = \mathrm{Re}\left\{A\mathrm{e}^{i(\omega t+\phi)}\right\} \\ &= A\cos(\omega t + \phi). \end{aligned}$$

This means the mass oscillates at the drive frequency with the amplitude given by equation (3.2) and phase given by equation (3.3).

3.3 Resonance

Theory tells that there is a huge response near $\omega = \omega_0$, and this is called the resonance. From the first derivative of A with respect to ω, we get the value of the drive frequency ω_r with greatest amplitude:

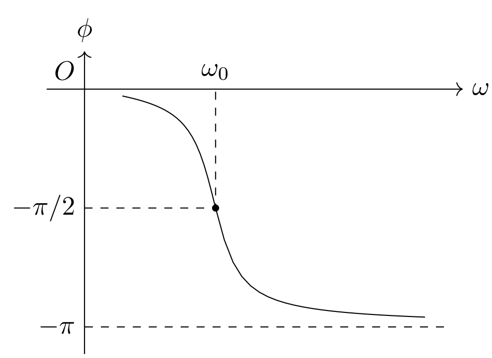
$$\omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}}.$$



While the greatest amplitude is at ω_r, we often approximate the greatest amplitude at $\omega = \omega_0$:

$$A(\omega_0) = \frac{F_0}{m\omega_0\gamma}.$$

The width of the peak, which is defined as the difference in frequency when the amplitude is $1/\sqrt{2}$ of its maximal value, is experimentally determined as γ.



The mass is in phase with the drive below resonance, as it easily keeps up with the drive. When high above drive frequency, the drive is too fast for the system to keep up. At the resonant frequency, the oscillator is $-\pi/2$ out of phase with the drive.

3.4 Quality Factor

The quality factor is a dimensionless quantity characterizing the sharpness of the resonance curve. It is defined as

$$Q = \frac{\omega_0}{\gamma}. \tag{3.4}$$

A larger Q means a narrower and taller peak.

3.5 Energy Stored in a Driven Oscillator

First, think about the potential energy. With the equation of motion given by $x = A\cos(\omega t + \phi)$,

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega t + \phi).$$

Therefore, the average potential energy over a period is:

$$\langle U \rangle = \frac{1}{T}\int_0^T U \mathrm{d}t = \frac{1}{4}m\omega_0^2 A^2.$$

Now, the kinetic energy is given by the equation $K = \frac{1}{2}m\dot{x}^2$, where $\dot{x} = -\omega A \sin(\omega t + \phi)$. So,

$$\begin{aligned} K &= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \\ \langle K \rangle &= \frac{1}{T}\int_0^T K \mathrm{d}t = \frac{1}{4}m\omega^2 A^2. \end{aligned}$$

Therefore, the total average energy over a period is:

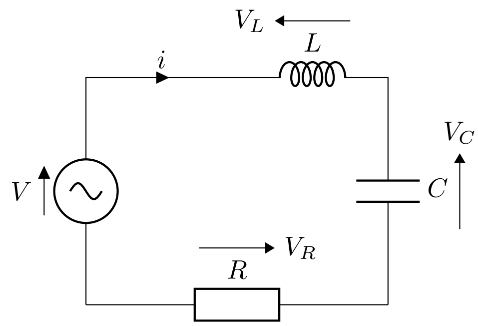
$$\langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{4}mA^2(\omega^2 + \omega_0^2).$$

3.6 Resonant Electric Circuits

A common resonant circuit is a driven LCR circuit, with all components connected in series. The voltage source provides a sinusoidal signal ($V_0 \sin(\omega t)$).

Using Kirchhoff's Voltage law, one can write:

$$\begin{aligned} V &= V_C + V_R + V_L \\ V_0 \sin(\omega t) &= \frac{q}{C} + iR + L\frac{\mathrm{d}i}{\mathrm{d}t} \\ V_0 \omega \cos(\omega t) &= \frac{i}{C} + R\frac{\mathrm{d}i}{\mathrm{d}t} + L\frac{\mathrm{d}^2 i}{\mathrm{d}t^2}. \end{aligned}$$



By defining $\omega_0 = \sqrt{1/LC}$ and $\gamma = R/L$, the above equation can be rearranged as:

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}i}{\mathrm{d}t} + \omega_0^2 i = \frac{V_0\omega}{L}\cos(\omega t).$$

So, we can conclude that the resonant frequency is set by choices of L and C, and the resistor R is the source of damping.

3.7 Transients

The general solution to a non-homogeneous differential equation involves both a solution to the corresponding homogeneous equation and a particular equation. To sum up, there are three steps to arrive at the general solution. Look again at the equation below:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t).$$

(1) Find the Particular Solution:

$$x_1 = A\cos(\omega t + \phi).$$

(2) Find the General Solution to the Homogeneous Equation:

$$\begin{aligned} &\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \\ &\qquad\Rightarrow x_2 = B\exp\Big\{-\frac{\gamma}{2}t\Big\}\cos(\omega_d t + \theta). \end{aligned}$$

(3) Add the Solutions Together:

The general solution to the differential equation is:

$$x = A\cos(\omega t + \phi) + B\exp\Big\{-\frac{\gamma}{2}t\Big\}\cos(\omega_d t + \theta), \tag{3.5}$$

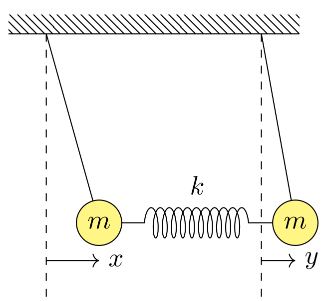
where the first part represents the steady-state part, and the second, transient part. The steady-state part involves no free parameters, oscillating at the drive frequency, ω. The transient part decays away over time, and B and θ are free parameters.

Chapter 4

Coupled Oscillators

4.1 Equation of Motion

Consider two pendulums connected by a spring, which is unstreched at equilibrium.



For free pendulums, one can write:

$$\begin{cases} m\ddot{x} = -m\omega_0^2 x \\ m\ddot{y} = -m\omega_0^2 y. \end{cases}$$

The extension of the spring is $y - x$, so the force by the spring is $F = k(y - x)$. Therefore,

$$\begin{aligned} m\ddot{x} &= -m\omega_0^2 x + k(y - x) \\ m\ddot{y} &= -m\omega_0^2 y - k(y - x) \end{aligned} \tag{4.1}$$

is the equation of motion for the coupled oscillators.

4.2 Trial Solutions

To solve this equation, try a solution where both pendulums oscillate with the same frequency:

$$\tilde{x} = \tilde{A}e^{i\omega t}, \quad \tilde{y} = \tilde{B}e^{i\omega t}.$$

Substitute these two solutions back to the above equations and we get:

$$\begin{aligned} m\ddot{x} &= -m\omega_0^2 x + k(y-x) \\ m(i\omega)^2\tilde{A}e^{i\omega t} &= -m\omega_0^2\tilde{A}e^{i\omega t} + k(\tilde{B}-\tilde{A})e^{i\omega t} \\ \left(\omega_0^2 - \omega^2 + \frac{k}{m}\right)\tilde{A} &= \frac{k}{m}\tilde{B}. \end{aligned}$$

$$\begin{aligned} m\ddot{y} &= -m\omega_0^2 y - k(y-x) \\ m(i\omega)^2\tilde{B}e^{i\omega t} &= -m\omega_0^2\tilde{B}e^{i\omega t} - k(\tilde{B}-\tilde{A})e^{i\omega t} \\ \left(\omega_0^2 - \omega^2 + \frac{k}{m}\right)\tilde{B} &= \frac{k}{m}\tilde{A}. \end{aligned}$$

Multiply the two equations together:

$$\begin{aligned} \left(\omega_0^2 - \omega^2 + \frac{k}{m}\right)^2\tilde{A}\tilde{B} &= \left(\frac{k}{m}\right)^2\tilde{A}\tilde{B} \\ \omega_0^2 - \omega^2 + \frac{k}{m} &= \pm\frac{k}{m}. \end{aligned}$$

Therefore, we have found two solutions for ω:

$$\begin{cases} \omega_0^2 - \omega^2 + \dfrac{k}{m} = \dfrac{k}{m} & \Rightarrow \omega = \omega_0 \\ \omega_0^2 - \omega^2 + \dfrac{k}{m} = -\dfrac{k}{m} & \Rightarrow \omega = \sqrt{\omega_0^2 + 2\omega_s^2} \text{ where } \omega_s^2 = \dfrac{k}{m}. \end{cases}$$

Then, we can substitute the ω back to get $\tilde{A}$ and $\tilde{B}$:

$$\begin{cases} \omega = \omega_0 & \Rightarrow \tilde{A} = \tilde{B} \\ \omega = \sqrt{\omega_0^2 + 2\omega_s^2} & \Rightarrow \tilde{A} = -\tilde{B}. \end{cases}$$

4.3 Normal Modes and the General Solution

We have found special solutions called the normal modes. The two special frequencies are the normal mode frequencies.

(1) Normal Mode 1: $\omega = \omega_1 = \omega_0$

Given $\tilde{A} = A_1 \mathrm{e}^{i\phi_1}$ and $\tilde{B} = \tilde{A}$,

$$x_1 = A_1 \cos(\omega_1 t + \phi_1), \quad y_1 = A_1 \cos(\omega_1 t + \phi_1).$$

The two pendulums oscillate together with equal amplitudes and phases at the natural frequency ω_0.

(2) Normal Mode 2: $\omega = \omega_2 = \sqrt{\omega_0^2 + 2\omega_s^2}$

Given $\tilde{A} = A_2 \mathrm{e}^{i\phi_2}$ and $\tilde{B} = -\tilde{A}$,

$$x_2 = A_2 \cos(\omega_2 t + \phi_2), \quad y_2 = -A_2 \cos(\omega_2 t + \phi_2).$$

The two pendulums oscillate together with equal amplitudes but are completely out of phase. The spring extends and contracts, adding to the restoring force; this is why the angular frequency is higher.

(3) General Solution:

A general solution to the coupled oscillators is the combination of the two normal modes:

$$\begin{cases} x = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) & (x = x_1 + x_2) \\ y = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) & (y = y_1 + y_2). \end{cases}$$

There are four free parameters A_1, A_2, ϕ_1, and ϕ_2, as there are four initial conditions $x(0)$, $\dot{x}(0)$, $y(0)$, and $\dot{y}(0)$.

Now consider an example where $x(0) = x_0$, $\dot{x}(0) = 0$, $y(0) = 0$, and $\dot{y}(0) = 0$. This means:

$$\begin{cases} x(0) = A_1 \cos(\phi_1) + A_2 \cos(\phi_2) = x_0 \\ y(0) = A_1 \cos(\phi_1) - A_2 \cos(\phi_2) = 0 \\ \dot{x}(0) = -\omega_1 A_1 \sin(\phi_1) - \omega_2 A_2 \sin(\phi_2) = 0 \\ \dot{y}(0) = -\omega_1 A_1 \sin(\phi_1) + \omega_2 A_2 \sin(\phi_2) = 0. \end{cases}$$

The last two equations indicate that $\phi_1 = \phi_2 = 0$. Then, we can get $A_1 = A_2 = \dfrac{x_0}{2}$. Therefore,

$$\begin{aligned} x &= \frac{x_0}{2}(\cos(\omega_1 t) + \cos(\omega_2 t)) \\ &= x_0 \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \end{aligned}$$

$$\begin{aligned} x &= \frac{x_0}{2}(\cos(\omega_1 t) - \cos(\omega_2 t)) \\ &= -x_0 \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \sin\left(\frac{\omega_1 - \omega_2}{2} t\right), \end{aligned}$$

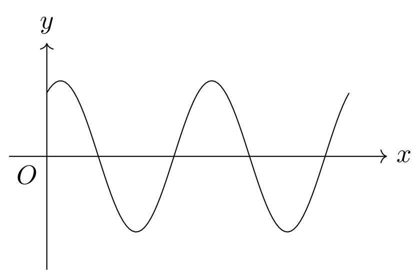
where $\dfrac{\omega_1 + \omega_2}{2}$ is identified as the half sum, and $\dfrac{\omega_1 - \omega_2}{2}$ as the half difference.

Chapter 5

Wave Equation

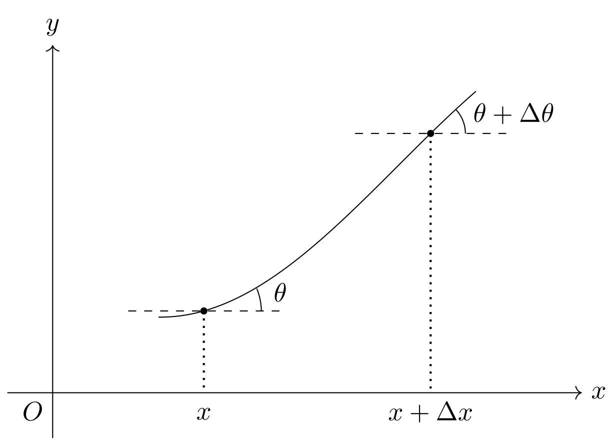
5.1 Derivation

Consider the transverse waves on a string.



The string has a mass per unit length of ρ, and there is a uniform tension T inside the string.

To investigate how the displacement y depends on x and t, consider a tiny element of string of length Δx at position x.



The line segment has tension on both ends:

By assuming that θ is small, we can get:

$$\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \frac{1}{2}\theta^2$$

$$\begin{aligned} F_x &= T\cos(\theta+\Delta\theta) - T\cos\theta \\ &\approx T\left[1 - \frac{1}{2}(\theta+\Delta\theta)^2\right] - T\left(1 - \frac{1}{2}\theta^2\right) \\ &= T\left(\theta\Delta\theta - \frac{1}{2}\Delta\theta^2\right) \\ &\approx 0 \end{aligned}$$

$$\begin{aligned} F_y &= T\sin(\theta+\Delta\theta) - T\sin(\theta) \\ &\approx T(\theta+\Delta\theta) - T\theta \\ &= T\Delta\theta. \end{aligned}$$

As the gradient of string at point x is $\tan\theta = \left(\frac{\partial y}{\partial x}\right)_x$, using $\theta \approx \tan\theta$ for small angles,

$$\begin{aligned} \Delta\theta &= (\theta+\Delta\theta) - \theta \\ &= \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x . \end{aligned}$$

As we we know $\Delta f(x) \approx \frac{\mathrm{d}f}{\mathrm{d}x}\Delta x$,

$$\Delta\theta = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x \approx \frac{\partial^2 y}{\partial x^2}\Delta x.$$

Therefore, using Newton's 2nd law,

$$F_y = T\Delta\theta = ma$$
$$T\frac{\partial^2 y}{\partial x^2}\Delta x = \rho\Delta x\frac{\partial^2 y}{\partial t^2}$$
$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2},$$

and the wave equation is derived. The standard form of wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \tag{5.1}$$

where v is the speed of the wave. So, we say the transverse oscillations on a stretched string obey the wave equation with

$$v = \sqrt{\frac{T}{\rho}}. \tag{5.2}$$

5.2 Longitudinal Waves in a Rod

By assuming that the section is streched by ϵ and changes size by $\Delta\epsilon$, we look for an equation that describes the displacement ϵ. The result is

$$\frac{\partial^2 \epsilon}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \epsilon}{\partial t^2},$$

where $v = \sqrt{\frac{\gamma}{\rho}}$, γ is Young's modulus, and ρ is the density.

5.3 Wave Equation in 3D

We call the quantity that's oscillating by ψ. It's a function of all three spatial coordinates and time: $\psi(x, y, z, t)$. So, the 3D wave equation is:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{5.3}$$

5.4 Solutions to the Wave Equation

We may begin with a trial solution $y = f(x - vt)$. It would also help to define $q = x - vt$, so the trial solution is:

$$y = f(q) = f(x - vt).$$

Therefore,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\mathrm{d}f}{\mathrm{d}q}\frac{\partial q}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}q} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\mathrm{d}f}{\mathrm{d}q}\right) = \frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\mathrm{d}f}{\mathrm{d}q}\right)\frac{\partial q}{\partial x} \\
&= \frac{\mathrm{d}^2 f}{\mathrm{d}q^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\mathrm{d}f}{\mathrm{d}q}\frac{\partial q}{\partial t} = -v\frac{\mathrm{d}f}{\mathrm{d}q} \\
\frac{\partial^2 f}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left(-v\frac{\mathrm{d}f}{\mathrm{d}q}\right) = -v\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\mathrm{d}f}{\mathrm{d}q}\right)\frac{\partial q}{\partial t} \\
&= v^2\frac{\partial^2 f}{\partial q^2}.
\end{aligned}$$

The above derivation suggests that the trial solution satisfies the wave equation:

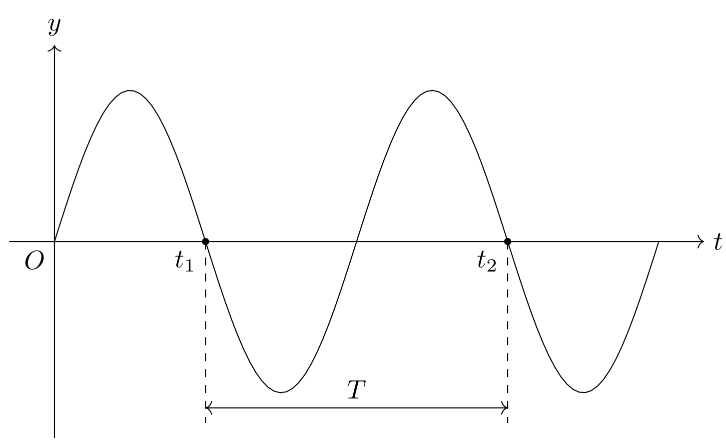
$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$

Therefore, the wave equation is satisfied for any form of f. Moreover, the trial solution $g(x+vt)$ works as well. Therefore, the general solution to the wave equation is:

$$y = f(x - vt) + g(x + vt). \tag{5.4}$$

Consider a function $y = A\cos[k(x - vt)] = A\cos(kx - \omega t)$, where $\omega = kv$.

Firstly, we can choose a fixed point (e.g. $x = 0$) and look at the wave as only a function of t.



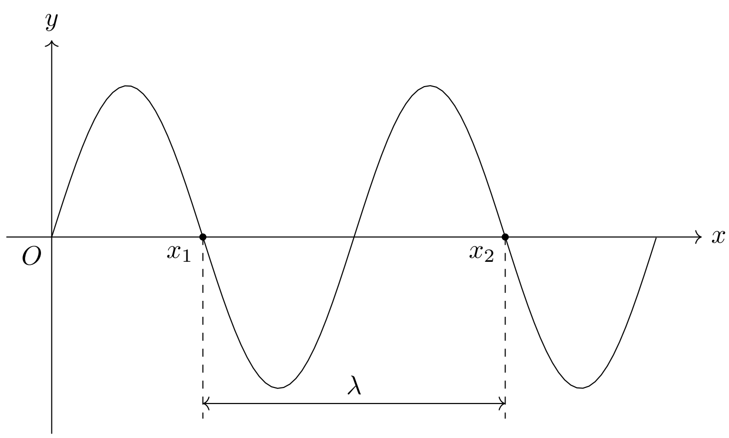
With x being kept constant, we can see that

$$\omega t_2 - \omega t_1 = 2\pi$$
$$\omega(t_1 + T) - \omega t_1 = \omega T = 2\pi$$
$$\Rightarrow \omega = \frac{2\pi}{T}.$$

$\omega = \frac{2\pi}{T}$ is defined as the angular frequency of the wave. Also, the frequency of the wave is:

$$f = \frac{\omega}{2\pi} = \frac{1}{T}.$$

Secondly, we can take a snapshot of the wave at fixed time (e.g. $t = 0$).



Similarly, we can write:

$$kx_2 - kx_1 = 2\pi$$
$$k(x_1 + \lambda) - kx_1 = k\lambda = 2\pi$$
$$\Rightarrow k = \frac{2\pi}{\lambda}.$$

$k = \frac{2\pi}{\lambda}$ is defined as the wave number of the wave.

The speed of the wave can be easily derived as the wavelength per period. So,

$$v = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k}.$$

This again confirms our interpretation of v as the speed of the wave.

5.5 3D Plane Waves

A plane wave is a wave that does not vary in the directions perpendicular to the propagation direction. $\psi = A\cos(kx - \omega t)$ is a solution representing a plane wave moving in the $+x$ direction. A more general solution is

$$\psi(\boldsymbol{r}, t) = A\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t),$$

where

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}.$$

$\boldsymbol{k}$ is called the wavevector, which defines the direction of propagation and has a magnitude of $|\boldsymbol{k}| = \dfrac{2\pi}{\lambda}$. Note that the wavevector is perpendicular to the wave fronts.

5.6 3D Spherical Waves

A spherical wave propagates equally in all directions and has an intensity that drops as the inverse-square of the distance. A general equation of the spherical wave is

$$\psi(\boldsymbol{r}, t) = \frac{A}{r}\cos(kr - \omega t).$$

It is important to note that the amplitude falls by a factor of $\dfrac{1}{r}$, and the intensity falls by $\dfrac{1}{r^2}$.

To sum up, we have covered four solutions to the wave equation:

$$\begin{cases} \text{Very General Solution:} & y = f(x - vt) + g(x + vt) \\ \text{Sinusoidal Solution:} & y = A\cos(kx - \omega t) \\ \text{Plane Wave Solution:} & y = A\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) \\ \text{Spherical Wave Solution:} & y = \dfrac{A}{r}\cos(kr - \omega t). \end{cases}$$

5.7 Energy and Power

Consider a tiny segment of the string with length Δs and linear density ρ. It has a horizontal displacement of Δx and a vertical displacement of Δy. Therefore,

$$\Delta K = \frac{1}{2}mv^2 = \frac{1}{2}\rho\Delta x\left(\frac{\partial y}{\partial t}\right)^2.$$

To find out the potential energy, think about the string with original length Δx. It is then stretched to

Δs. So,

$$\Delta U = \text{Force} \times \text{Extension} = T \times (\Delta s - \Delta x).$$

By binomial expansion, we can see

$$\begin{aligned}\Delta s &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \Delta x \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \\ &\approx \Delta x \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right].\end{aligned}$$

Therefore,

$$\Delta U = \frac{1}{2} T \Delta x \left(\frac{\partial y}{\partial x}\right)^2.$$

As $\dfrac{\partial y}{\partial x} = -\dfrac{1}{v}\dfrac{\partial y}{\partial t}$ and $v = \sqrt{\dfrac{T}{\rho}}$,

$$\Delta U = \frac{1}{2}\frac{T}{v^2}\Delta x \left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\rho \Delta x \left(\frac{\partial y}{\partial t}\right)^2 = \Delta K.$$

By recalling that power is the energy per unit time,

$$P = \frac{\Delta K + \Delta U}{\Delta t} = \rho \left(\frac{\partial y}{\partial t}\right)^2 \frac{\Delta x}{\Delta t} = \rho v \left(\frac{\partial y}{\partial t}\right)^2.$$

For sinusoidal waves,

$$P = \rho v \omega^2 A^2 \sin^2(kx - \omega t).$$

The cycle-average power is:

$$\langle P \rangle = \frac{1}{T}\int_0^T P \mathrm{d}t = \frac{1}{2}\rho v \omega^2 A^2.$$

5.8 Reflections

Some waves exist with some boundaries. For example, consider a string fixed at $x = 0$. As it is fixed at $x = 0$, $y(x = 0) = 0$ for any t. Using the general solution,

$$y = f(x - vt) + g(x + vt) \Rightarrow y(x = 0) = f(-vt) + g(vt) = 0.$$

The above expression suggests that $g(x + vt) = -f(-x - vt)$. Therefore, with the string fixed at $x = 0$, the general solution is:

$$y = f(x - vt) - f(-x - vt).$$

Consider the sinusoidal wave $f(x - vt) = A\cos(kx - \omega t)$:

$$\begin{aligned} y &= A\cos(kx - \omega t) - A\cos(-kx - \omega t) \\ &= -2A\sin(-\omega t)\sin(kx) \\ &= 2A\sin(\omega t)\sin(kx). \end{aligned}$$

This is a standing wave. A standing wave is produced by the sum of an incident wave and a reflected wave. It oscillates but does not travel. The standing wave also has nodes - points where there is no motion. The nodes are separated by half a wavelength.