Summary

Oscillations & Waves

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Contents

Chapter 1

Vibrations

1.1 Harmonic Oscillators

Common examples include the mass on a spring, simple pendulum, and LC circuit.

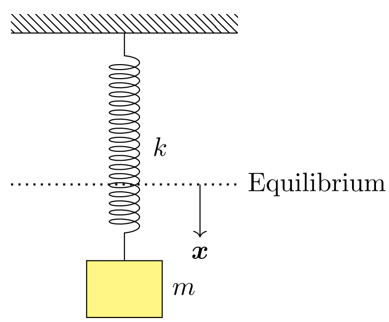


Figure 1.1: Mass on a Spring.

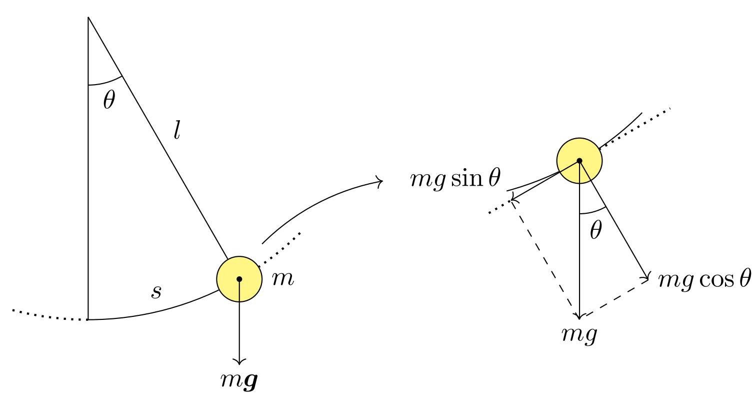


Figure 1.2: Simple Pendulum. The small angle approximation is used.

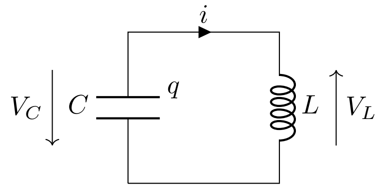


Figure 1.3: A LC circuit.

All the harmonic oscillators have the same governing equation. Take ψ as the general variable,

$$\ddot{\psi} + \omega_0^2 \psi = 0 \tag{1.1}$$

is the harmonic oscillator equation. The complex trial solution here is

$$\tilde{x} = \tilde{A} \mathrm{e}^{i\omega_0 t}, \text{ where } \tilde{A} = A\mathrm{e}^{i\phi}.$$

The general solution is

$$\psi = A\cos(\omega_0 t + \phi). \tag{1.2}$$

There are many harmonic motions in life. Consider an arbitrary force $F(x)$ with $F(0) = 0$.

$$F(x) = F(x=0) + \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=0} x + \frac{1}{2}\left.\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right|_{x=0} x^2 + \ldots.$$

For x that is small enough,

$$\begin{cases} F(x=0) & \text{Zero. This is the equilibrium point.} \\ \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=0} x & \text{Linear restoring force.} \\ \frac{1}{2}\left.\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right|_{x=0} x^2 & \text{Small when } x \text{ is small enough.} \end{cases}$$

	Mass on a Spring	Simple Pendulum	LC Circuit
Step 1	$F = -kx$	$\tau = -mgl\sin\theta \approx -mgl\theta$	$V_C + V_L = 0$
Step 2	$m\ddot{x} + kx = 0$	$ml^2\ddot{\theta} + mgl\theta = 0$	$q/C + L\ddot{q} = 0$
Natural Freq.	$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{g}{l}}$	$\omega_0 = \sqrt{\frac{1}{LC}}$

1.2 Damped Harmonic Motion

The governing equation for damped harmonic motion is:

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = 0. \tag{1.3}$$

The corresponding characteristic equation is

$$r^2 + \gamma r + \omega_0^2 = 0,$$

and this gives a discriminant of

$$\Delta = \gamma^2 - 4\omega_0^2.$$

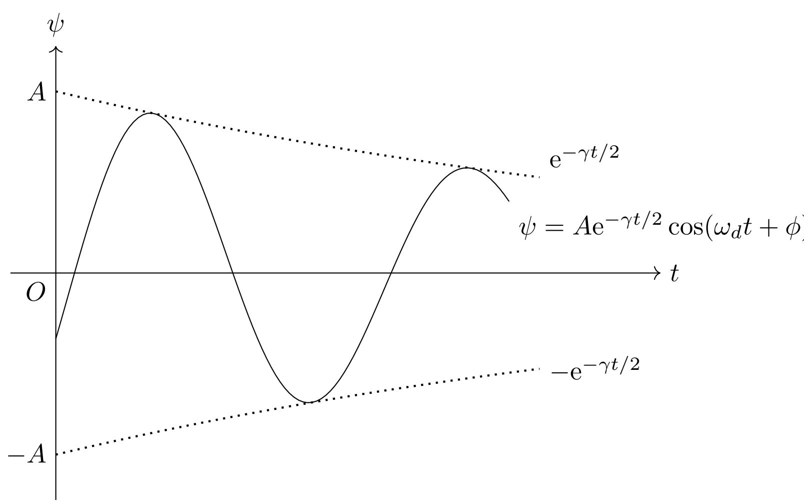
The roots here are:

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}. \tag{1.4}$$

The complex trial solution here is $\tilde{\psi} = \tilde{A}\mathrm{e}^{i\omega t}$.

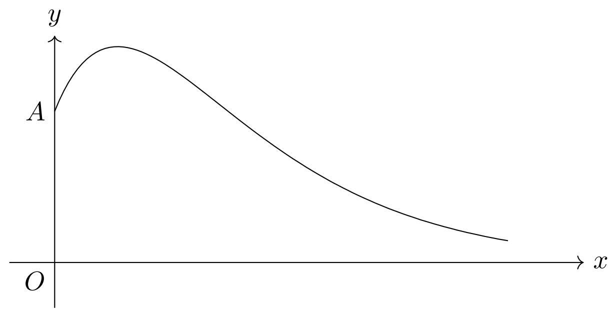
When γ is so small that $\frac{\gamma}{2} < \omega_0$, this is light damping:

$$\psi = A\mathrm{e}^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi),\ \text{where } \omega_d = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}. \tag{1.5}$$



When $\frac{\gamma}{2} = \omega_0$, this is critical damping, where the oscillator goes back to equilibrium with the shortest time:

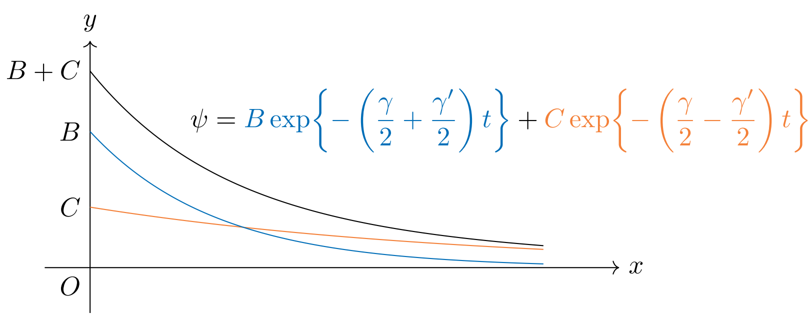
$$\psi = (A + Bt)\mathrm{e}^{-\frac{\gamma}{2}t}. \tag{1.6}$$



When $\frac{\gamma}{2} > \omega_0$, this is heavy damping:

$$\psi = Be^{-\left(\frac{\gamma}{2}+\frac{\gamma'}{2}\right)t} + Ce^{-\left(\frac{\gamma}{2}-\frac{\gamma'}{2}\right)t}. \tag{1.7}$$

The first term decays faster than $\frac{\gamma}{2}$, and the second term decays more slowly.



1.3 Driven Harmonic Motion

1.3.1 Solution

The governing equation here is:

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = f_0\cos(\omega t), \tag{1.8}$$

where the RHS is a general driving term. For example, consider the mass on a spring.

$$\begin{aligned} F_n &= -kx - b\dot{x} + F_0\cos(\omega t) \\ m\ddot{x} + b\dot{x} + kx &= F_0\cos(\omega t) \\ \ddot{x} + \gamma\dot{x} + \omega_0^2 x &= \frac{F_0}{m}\cos(\omega t). \end{aligned}$$

Also, in the LCR circuit case,

$$\begin{aligned} V_S &= V_L + V_C + V_R \\ V_0 \cos(\omega t) &= L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{q}{C} + iR \\ L\ddot{q} + R\ddot{q} + \frac{q}{C} &= V_0 \cos(\omega t) \\ \ddot{q} + \gamma\ddot{q} + \omega_0^2 q &= \frac{V_0}{L}\cos\omega t. \end{aligned}$$

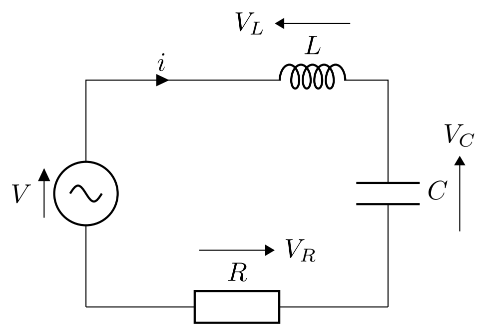


Figure 1.4: A LCR circuit.

The total solution includes a homogeneous one (damped) and a particular one (steady state).

(1) Transient: homogeneous solution

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = 0 \Rightarrow \psi_1.$$

(2) Steady State: particular solution

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = f_0\cos(\omega t) \Rightarrow \psi_2 = A\cos(\omega t + \phi).$$

To solve this differential equation, try $\tilde{\psi} = \tilde{A}\mathrm{e}^{i\omega_1 t}$.

Therefore, the total solution should be:

$$\psi = \psi_1 + \psi_2. \tag{1.9}$$

1.3.2 Resonance

From mathematical manipulation,

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \tag{1.10}$$

$$\phi = \arctan\left(-\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right). \tag{1.11}$$

Theory tells that there is a huge response near $\omega = \omega_0$, and this is called the resonance. From the first derivative of A with respect to ω, we get the value of the drive frequency ω_r with greatest amplitude:

$$\omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}}.$$

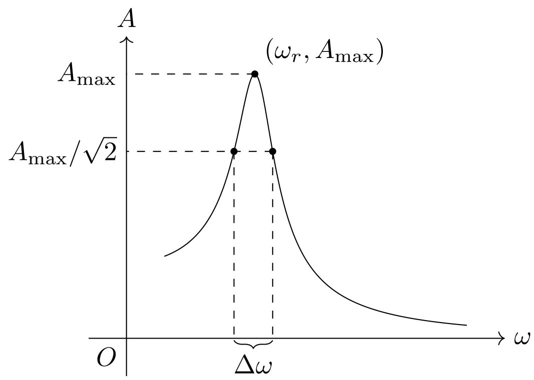


Figure 1.5: Amplitude pattern with different drive frequency.

While the greatest amplitude is at ω_r, we often approximate the greatest amplitude at $\omega = \omega_0$:

$$A(\omega_0) = \frac{f_0}{\omega_0\gamma}.$$

The width of the peak, which is defined as the difference in frequency when the amplitude is $1/\sqrt{2}$ of its maximal value, is experimentally determined as γ.

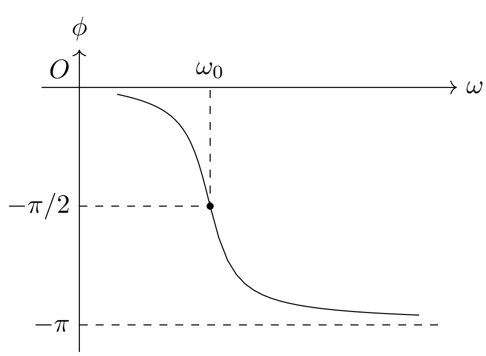


Figure 1.6: Phase difference pattern with different drive frequency.

1.3.3 Quality Factor

The quality factor, Q, is a dimensionless quantity. We see that Q tells us roughly how many oscillations the system makes before the oscillations damp away. In the context of resonance, it tells us how narrow the resonant peak will be relative to the natural frequency – larger Q means a narrower and taller peak.

$$Q = \frac{\omega_0}{\gamma}. \tag{1.12}$$

1.3.4 Energy for the Steady State

Consider the equation of motion as $x = A\cos(\omega t + \phi)$.

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 A^2\cos^2(\omega t + \phi) \Rightarrow \langle U\rangle = \frac{1}{4}m\omega_0^2 A^2.$$

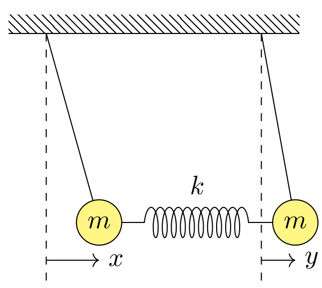
$$K = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t + \phi) \Rightarrow \langle K\rangle = \frac{1}{4}m\omega^2 A^2.$$

Therefore, the total average energy for a period is:

$$\langle E\rangle = \langle U\rangle + \langle K\rangle = \frac{1}{4}mA^2(\omega_0^2 + \omega^2). \tag{1.13}$$

1.4 Coupled Oscillators

Consider two pendulums connected by a spring, which is unstreched at equilibrium.



For free pendulums, one can write:

$$\begin{cases} m\ddot{x} = -m\omega_0^2 x \\ m\ddot{y} = -m\omega_0^2 y. \end{cases}$$

The extension of the spring is $y - x$, so the force by the spring is $F = k(y - x)$. Therefore,

$$\begin{aligned} m\ddot{x} &= -m\omega_0^2 x + k(y - x) \\ m\ddot{y} &= -m\omega_0^2 y - k(y - x) \end{aligned} \tag{1.14}$$

is the equation of motion for the coupled oscillators.

To solve the system of equations, try $\tilde{x} = \tilde{A}e^{i\omega t}$ and $\tilde{y} = \tilde{B}e^{i\omega t}$.

This yields the equation

$$\omega_0^2 - \omega^2 + \frac{k}{m} = \pm\frac{k}{m} \tag{1.15}$$

and two solutions

$$\begin{cases} \omega_1 = \omega_0 \\ \omega_2 = \sqrt{\omega_0^2 + 2\omega_s^2}, \end{cases}$$

where $\omega_s^2 = \dfrac{k}{m}$.

We have found special solutions called the normal modes. The two special frequencies are the normal mode frequencies.

(1) Normal Mode 1: $\omega = \omega_1 = \omega_0$

Given $\tilde{A} = A_1 e^{i\phi_1}$ and $\tilde{B} = \tilde{A}$,

$$x_1 = A_1\cos(\omega_1 t + \phi_1), \quad y_1 = A_1\cos(\omega_1 t + \phi_1).$$

The two pendulums oscillate together with equal amplitudes and phases at the natural frequency ω_0.

(2) Normal Mode 2: $\omega = \omega_2 = \sqrt{\omega_0^2 + 2\omega_s^2}$

Given $\tilde{A} = A_2 e^{i\phi_2}$ and $\tilde{B} = -\tilde{A}$,

$$x_2 = A_2\cos(\omega_2 t + \phi_2), \quad y_2 = -A_2\cos(\omega_2 t + \phi_2).$$

The two pendulums oscillate together with equal amplitudes but are completely out of phase. The spring extends and contracts, adding to the restoring force; this is why the angular frequency is higher.

(3) General Solution:

A general solution to the coupled oscillators is the combination of the two normal modes:

$$\begin{cases} x = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2) & (x = x_1 + x_2) \\ y = A_1\cos(\omega_1 t + \phi_1) - A_2\cos(\omega_2 t + \phi_2) & (y = y_1 + y_2). \end{cases}$$

There are four free parameters A_1, A_2, ϕ_1, and ϕ_2, as there are four initial conditions $x(0)$, $\dot{x}(0)$, $y(0)$, and $\dot{y}(0)$.

Chapter 2

Waves

2.1 Wave Equation

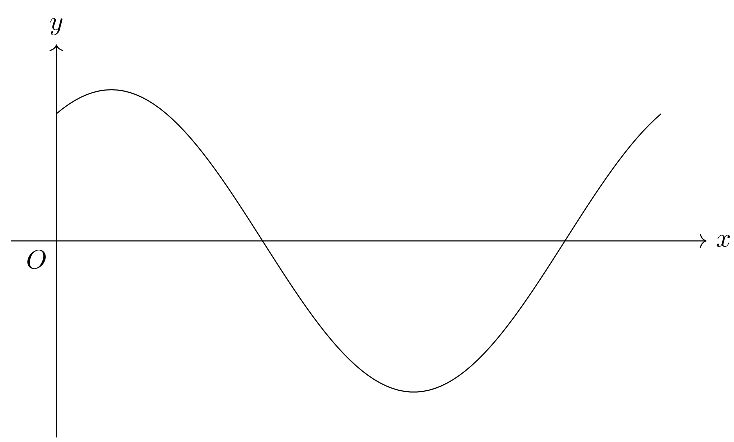
In 1D, the wave equation is:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{2.1}$$

In 3D, it becomes:

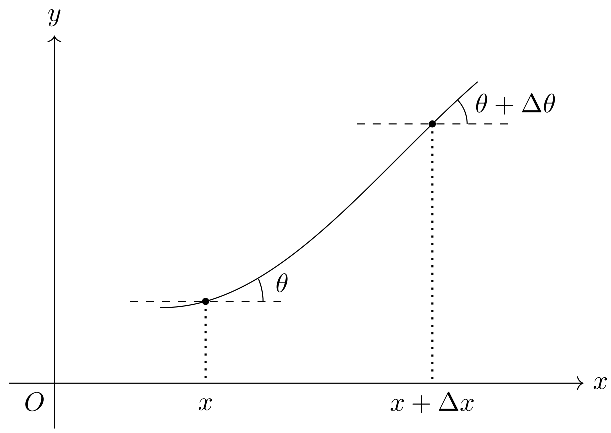
$$\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{2.2}$$

Consider the transverse waves on a string.



The string has a mass per unit length of ρ, and there is a uniform tension T inside the string.

To investigate how the displacement y depends on x and t, consider a tiny element of string of length Δx at position x.



By assuming that θ and $\Delta\theta$ are small, we get:

$$F_x = T\cos(\theta + \Delta\theta) - T\cos\theta \approx 0$$
$$F_y = T\sin(\theta + \Delta\theta) - T\sin\theta \approx T\Delta\theta.$$

As $\theta \approx \tan\theta$,

$$F_y = T\Delta\theta \approx T\Delta(\tan\theta) = T\Delta\left(\frac{\partial y}{\partial x}\right) \approx T\frac{\partial^2 y}{\partial x^2}\Delta x.$$

Therefore, we get:

$$m\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}\Delta x \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}.$$

By identifying the speed as $\frac{1}{v^2} = \frac{\rho}{T}$, we get:

$$v = \sqrt{\frac{T}{\rho}}. \tag{2.3}$$

2.2 Solutions to Wave Equation

2.2.1 General Solution in 1D

Simple derivations would prove that any form of f and g below can satisfy the wave equation:

$$\psi = f(x - vt) + g(x + vt). \tag{2.4}$$

2.2.2 Sinusoidal Solution in 1D

Plug $\psi = f(x - vt)$ into a sine wave, we get:

$$\psi = A\cos(k(x - vt)) = A\cos(kx - \omega t), \tag{2.5}$$

where k is the wave number. Also, the phase velocity of the wave is:

$$\omega = kv \Rightarrow v = \frac{\omega}{k}. \tag{2.6}$$

2.2.3 Plane Waves in 3D

$$\psi = A\cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t), \tag{2.7}$$

where $|\boldsymbol{k}| = \dfrac{2\pi}{\lambda}$.

2.2.4 Spherical Waves in 3D

$$\psi = \frac{A}{r}\cos(kr - \omega t). \tag{2.8}$$

The amplitude and intensity fall due to conservation of energy.

2.3 Energy of Waves

Consider a small section of string (length Δx) that passes a transverse wave.

$$\begin{aligned} \Delta K &= \frac{1}{2}m\dot{y}^2 \\ &= \frac{1}{2}\rho\Delta x\left(\frac{\partial y}{\partial t}\right)^2. \end{aligned}$$

The potential energy is:

$$\Delta U = T(\Delta s - \Delta x).$$

By binomial expansion,

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2} = \Delta x\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \approx \Delta x\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right).$$

Therefore,

$$\Delta U = \frac{1}{2} T \Delta x \left(\frac{\partial y}{\partial x}\right)^2.$$

Take $y = f(x - vt)$ and use $q = x - vt$:

$$\frac{\partial y}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}q}$$
$$\frac{\partial y}{\partial t} = -v\frac{\mathrm{d}f}{\mathrm{d}q}.$$

Therefore, using $v^2 = \dfrac{T}{\rho}$,

$$\left(\frac{\partial y}{\partial t}\right)^2 = v^2 \left(\frac{\partial y}{\partial x}\right)^2 = \frac{T}{\rho}\left(\frac{\partial y}{\partial x}\right)^2,$$

$$\Delta U = \frac{1}{2}\rho \Delta x \left(\frac{\partial y}{\partial t}\right)^2 = \Delta K.$$

This leads to the conclusion that the potential energy is the same as the kinetic energy, and

$$P = \frac{\Delta U + \Delta K}{\Delta t} = \rho \frac{\Delta x}{\Delta t}\left(\frac{\partial y}{\partial t}\right)^2 = \rho v \left(\frac{\partial y}{\partial t}\right)^2.$$

Returning to the sine wave case, and we get:

$$\langle P \rangle = \frac{1}{2}\rho v \omega^2 A^2. \tag{2.9}$$

2.4 Boundaries

2.4.1 Fixing at Origin (1D)

Consider the string case. Fixing at $x = 0$ means $y(0, t) = 0$ for any t. Using the general solution $y = f(x - vt) + g(x + vt)$,

$$f(-vt) + g(vt) = 0 \Rightarrow g(\psi) = -f(-\psi).$$

Therefore,

$$y = f(x - vt) + g(x + vt) = f(x - vt) - f(-x - vt). \tag{2.10}$$

If the wave is a sinusoidal wave, then, by trigonometric identities, it is a standing wave:

$$\begin{aligned} y &= f(x - vt) - f(-x - vt) \\ &= A\cos(k(x - vt)) - A\cos(k(-x - vt)) \\ &= 2A\sin(kx)\sin(\omega t). \end{aligned}$$

A travelling wave transports energy. The standing wave oscillates but does not travel, and thus it does not transport energy. The standing wave has nodes – points where there is no motion. The nodes are separated by half a wavelength.

2.4.2 Fixing at Both Ends (1D)

Now consider the sine wave case. The wave is not only fixed at $x = 0$, but also at $x = L$. This means

$$\sin(kL) = 0. \tag{2.11}$$

Therefore, we know that

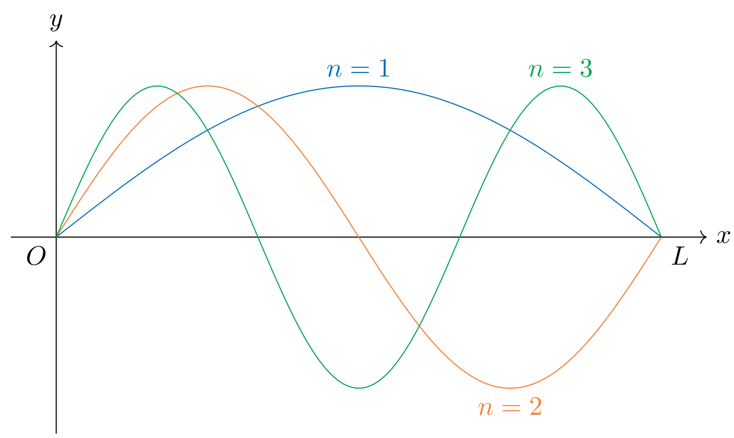
$$kL = n\pi \Rightarrow k = \frac{n\pi}{L}. \tag{2.12}$$

There are a few allowed wave numbers to satisfy the condition. We can also talk about the allowed wavelengths and allowed angular frequencies:

$$\lambda = \frac{2\pi}{k} = \frac{2L}{n} \tag{2.13}$$

$$\omega = kv = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}. \tag{2.14}$$

When $n = 1$, this is the fundamental mode or the first harmonic. $n = 2$ is the second harmonic, and so on.



When we talk about the frequencies in Hz, we refer to the following expression:

$$f = \frac{\omega}{2\pi} = \frac{v}{\lambda} = \frac{n}{2L}\sqrt{\frac{T}{\rho}}. \tag{2.15}$$

2.4.3 2D Situation

We propose a solution:

$$y(x,z,t) = A\sin(k_x x)\sin(k_z z)\sin(\omega t). \tag{2.16}$$

By the wave equation in 2D, we get

$$k_x^2 + k_z^2 = \frac{\omega^2}{v^2}. \tag{2.17}$$

With the boundaries at $x = 0$, $x = a$, $z = 0$, and $z = b$, we get the allowed wave numbers,

$$k_x = \frac{n_1\pi}{a} \qquad k_z = \frac{n_2\pi}{b}, \tag{2.18}$$

and the allowed angular frequency:

$$\omega = v\sqrt{\left(\frac{n_1\pi}{a}\right)^2 + \left(\frac{n_2\pi}{b}\right)^2}. \tag{2.19}$$

2.5 Interference

We can add waves together, and the sum is also a wave:

$$\psi = \psi_1 + \psi_2. \tag{2.20}$$

The intensity of the wave is defined as

$$I = \alpha|\psi|^2, \tag{2.21}$$

where α is the proportionality. Note that intensities cannot add; you should add the waves first and calculate the intensity. Consider the waves

$$\psi_1 = A_1 \mathrm{e}^{i(\omega t + \phi_1)} \qquad \psi_2 = A_2 \mathrm{e}^{i(\omega t + \phi_2)},$$

where the phases contain the spatial part of the phase as well as any additional phase offsets. By maths,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\phi_1 - \phi_2).$$

The red part above is the interference term.

Consider the double slit case, where $A_1 = A_2$ and $I_1 = I_2$, the separation is d, and the phase difference is $\Delta\phi = k\Delta r = kd\sin\theta$. Therefore,

$$I = 2I_1 + 2I_1\cos(kd\sin\theta) = 4I_1\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right). \tag{2.22}$$

2.6 Wave Packets

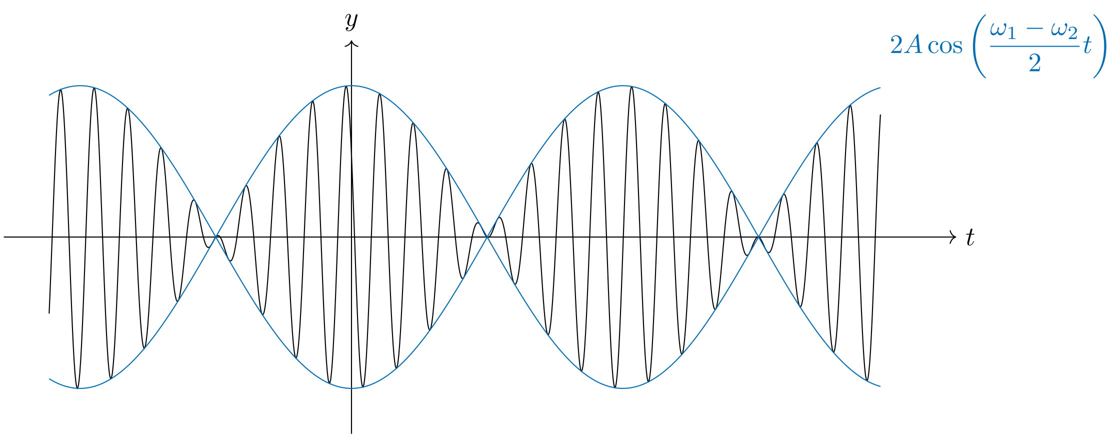
2.6.1 Beats - Two Waves of Slightly Different Freq.

When adding waves of slightly different angular frequencies, we get a beat.

For example, by adding $\psi_1 = A\cos(\omega_1 t + \phi)$ and $\psi_2 = A\cos(\omega_2 t + \phi)$, we get:

$$\psi = \psi_1 + \psi_2 = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t + \phi\right), \tag{2.23}$$

where the first argument is the half difference, and the second is the half sum.



2.6.2 More Waves

When more waves are added together, there is a more accurate approximation of a single pulse at $t = 0$.

With the total angular frequency range being $\Delta\omega$, we can get a pulse of duration:

$$\Delta t \sim \frac{1}{\Delta\omega}. \tag{2.24}$$

Similarly, when we shift to the waves with slightly different wave numbers (Δk), we get a pulse in space:

$$\Delta x \sim \frac{1}{\Delta k}. \tag{2.25}$$

2.6.3 Phase and Group Velocities

Now, we add two waves that differ in angular frequency by $\Delta\omega$ and in wave number by Δk:

$$\begin{aligned}\psi &= A\cos(kx-\omega t) + A\cos[(k+\Delta k)x - (\omega+\Delta\omega)t] \\ &= 2A\cos[(k+\Delta k/2)x - (\omega+\Delta\omega/2)t]\cos\left[\frac{1}{2}(\Delta kx - \Delta\omega t)\right] \\ &\approx 2A\cos(kx-\omega t)\cos\left[\frac{1}{2}(\Delta kx - \Delta\omega t)\right].\end{aligned}$$

Now see the first factor in equation. It moves along with speed $v = \omega/k$. This speed is called the phase velocity. It is the speed that appears in the wave equation and is the only wave speed encountered so far. Now consider one of the peaks of the slowly-varying envelope, i.e. the second factor in equation. It moves along with a speed $\Delta\omega/\Delta k$. This has to do with how a change in frequency relates to a change in wave number. This velocity is called the group velocity and is not necessarily the same as the phase velocity.

The same idea holds for adding many waves, and in general, we define the phase velocity,

$$v_p = \frac{\omega}{k}, \tag{2.26}$$

and the group velocity,

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{2.27}$$

2.6.4 Dispersion and Information Transmission

Normally, v_p does not depend on the frequency of the wave. With v_p being constant, we can see:

$$\begin{aligned}\omega &= kv_p \\ \mathrm{d}\omega &= v_p\mathrm{d}k \\ \frac{\mathrm{d}\omega}{\mathrm{d}k} &= v_p \\ v_g &= v_p.\end{aligned}$$

That is to say, when v_p does not depend on frequency, the group velocity and the phase velocity are the same.

Nevertheless, this is not the case. The phenomenon of dispersion tells us that v_p actually depends on the frequency. For EM waves,

$$\omega = \sqrt{\omega_p^2 + k^2c^2},$$

where ω_p is a constant called the plasma frequency. In this case,

$$v_p = \frac{\omega}{k} > c, \qquad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} < c.$$

As information is contained within the change of the wave, it travels at group velocity ($< c$).

Chapter 3

Basic Electronics

3.1 Circuit Theory

The currents in electronics are taken to be the flow of notional positive charges. Rarely, you may need to think about what the electrons are actually doing, and always it is the opposite direction to conventional current. Current is defined as follows:

$$I = \frac{\mathrm{d}q}{\mathrm{d}t}. \tag{3.1}$$

Drift speed is the speed of the positive charges in the conductor:

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} = ne\frac{\mathrm{d}V}{\mathrm{d}t} = neA\frac{\mathrm{d}x}{\mathrm{d}t} = neAv_d.$$

Therefore, the drift speed is:

$$v_d = \frac{I}{neA}. \tag{3.2}$$

Theory of resistors tells us that the resistance of a conductor with resistivity ρ, cross-sectional area (through which the current flows) A, and length l is:

$$R = \rho\frac{l}{A}. \tag{3.3}$$

Consider a tiny segment of conductor with cross-sectional area $\mathrm{d}S$ and length $\mathrm{d}l$. By Ohm's law, the current is voltage divided by resistance:

$$\begin{aligned}
I &= \frac{\Delta V}{R} \\
\boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} &= \frac{\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}}{\rho \mathrm{d}l / \mathrm{d}S} \\
j \mathrm{d}S &= \mathrm{d}S \frac{E \mathrm{d}l}{\rho \mathrm{d}l} \\
\boldsymbol{j} &= \sigma \boldsymbol{E}, \qquad (3.4)
\end{aligned}$$

where the conductivity σ is the reciprocal of resistivity ρ.

$$\sigma = \frac{1}{\rho}. \qquad (3.5)$$

A simple circuit diagram would look like below. The potential difference arrows point to the direction of higher potential. It is always parallel to the current for the source and always anti-parallel for circuit components.

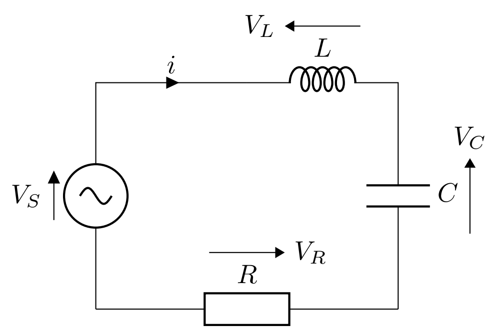


Figure 3.1: A LCR circuit.

3.2 Power

When an infinitesimal charge $\mathrm{d}q$ passes through a constant potential difference V, we can write the change in energy as

$$\mathrm{d}U = V \mathrm{d}q.$$

From here we can get the power:

$$P = \frac{\mathrm{d}U}{\mathrm{d}t} = V \frac{\mathrm{d}q}{\mathrm{d}t} = VI. \qquad (3.6)$$

Power is always the product of potential difference and current.

Considering the resistor, we can note that the current arrow is anti-parallel to the p.d. arrow. Consequently, the potential of the charges falls as they move through the resistor; $\mathrm{d}U/\mathrm{d}t$ will be negative in respect of the charge as it moves through the resistor. However, this energy must be deposited into the resistor, and as the resistor gets hot, we can be confident that this indicates positive power dissipated in the resistor. It is convention to define the power from the point of view of the component itself and hence we can write

$$P_R = V_R I.$$

At the source, notice that the current arrow and the potential difference arrow point in the same direction: positive current through an increasing potential means that the charge is gaining energy; $\mathrm{d}U/\mathrm{d}t$ will be positive which we can interpret as energy delivered from the source into the circuit, by the raising of the potential of the charges. The source itself loses energy.

$$P_S = -V_S I.$$

Consequently, the net power is zero; the circuit conserves energy. Charge is the medium by which energy is moved from the source to the components. The resistor power - proportional to the square of the p.d. across it - will always be positive whenever current is driven through it, and the source power will always be negative when it delivers current into the circuit.

3.3 Kirchhoff's Laws

3.3.1 Kirchhoff's Voltage Law: KVL

Around any circuit loop, the sum of the potential gains due to EMFs must equal the sum of the potential drops across the components.

$$\sum_{k=1}^{n} V_k = 0. \tag{3.7}$$

3.3.2 Kirchhoff's Current Law: KCL

At any junction, the sum of currents pointing into the junction must equal the sum of currents pointing out of the junction.

$$\sum_{k=1}^{n} I_k = 0. \tag{3.8}$$

3.4 Loading

Whenever a real-world voltage-source is connected to some device which causes it to supply current, then the actual p.d. observed across the terminals will be reduced and the source is said to be 'loaded'. This effect is referred to as loading.

3.5 Capacitors

Capacitors are circuit components that store charge and electrical energy. The unit of capacitance is in Farad (F). The expression of capacitance is

$$C = \frac{Q}{V}. \tag{3.9}$$

To determine the capacitance of a capacitor, we use the expression below:

$$C = \frac{\epsilon_0 \epsilon_r A}{d}, \tag{3.10}$$

where ϵ_0 is the vacuum permittivity, ϵ_r is the relative permittivity, A is the area of the capacitor, and d is the plate separation.

To derive the energy of a capacitor, imagine putting an infinitesimal charge $\mathrm{d}q$ on the capacitor.

$$\mathrm{d}U = V\mathrm{d}q = \frac{q}{C}\mathrm{d}q$$

$$U = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}. \tag{3.11}$$

The current across a capacitor is:

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = C\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{3.12}$$

Alternatively, assuming the capacitor is uncharged at the beginning, we can write:

$$v_C = \frac{1}{C}\int_0^t i(t')\mathrm{d}t'. \tag{3.13}$$

3.6 Inductors

A solenoid used as a circuit element is known as an inductor, and in this context its self-inductance is simply referred to as inductance. Inductance has units in Henry (H).

The voltage across the inductor as a component is:

$$v_L = L\frac{\mathrm{d}i}{\mathrm{d}t}. \tag{3.14}$$

To derive the energy stored in an inductor, consider the power of the inductor:

$$P = iv_L = Li\frac{\mathrm{d}i}{\mathrm{d}t} \Rightarrow P\mathrm{d}t = \mathrm{d}U = Li\mathrm{d}i,$$

$$U = \frac{1}{2}Li^2. \tag{3.15}$$

3.7 DC Circuits

3.7.1 Series and Parallel

While the resistors and inductors share the same rule, the capacitors behave the same as the mechanical springs. Impedance also adds like resistance.

	Series	Parallel
Resistance	$R_{eq} = \sum R_i$	$\frac{1}{R_{eq}} = \sum \frac{1}{R_i}$
Inductance	$L_{eq} = \sum L_i$	$\frac{1}{L_{eq}} = \sum \frac{1}{L_i}$
Impedance	$Z_{eq} = \sum Z_i$	$\frac{1}{Z_{eq}} = \sum \frac{1}{Z_i}$
Capacitance	$\frac{1}{C_{eq}} = \sum \frac{1}{C_i}$	$C_{eq} = \sum C_i$
Spring Constant	$\frac{1}{k_{eq}} = \sum \frac{1}{k_i}$	$k_{eq} = \sum k_i$

3.7.2 RC Circuits

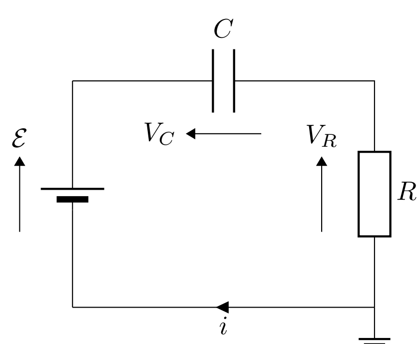


Figure 3.2: A RC Circuit.

When the capacitor is charging, by KVL,

$$
\begin{aligned}
V_S &= V_C + V_R \\
\mathcal{E} &= \frac{q}{C} + R\frac{\mathrm{d}q}{\mathrm{d}t} \\
\frac{\mathrm{d}q}{C\mathcal{E} - q} &= \frac{\mathrm{d}t}{RC}.
\end{aligned}
$$

Therefore,

$$
\int_0^q \frac{\mathrm{d}q'}{C\mathcal{E} - q'} = \int_0^t \frac{\mathrm{d}t'}{RC},
$$

and

$$
q = C\mathcal{E}\left(1 - \mathrm{e}^{-t/RC}\right). \tag{3.16}
$$

Other quantities can be easily derived, like $v = \frac{q}{C}$ and $i = \frac{\mathrm{d}q}{\mathrm{d}t}$.

The quantity

$$
\tau = RC \tag{3.17}
$$

is defined as the time constant of the circuit.

	q	v_C	i	v_R
$t = 0$	0	0	$\mathcal{E}/R$	$\mathcal{E}$
t	$C\mathcal{E}\left(1 - \mathrm{e}^{-t/\tau}\right)$	$\mathcal{E}\left(1 - \mathrm{e}^{-t/\tau}\right)$	$\mathcal{E}/R \cdot \mathrm{e}^{-t/\tau}$	$\mathcal{E}\mathrm{e}^{-t/\tau}$
$t \to \infty$	$C\mathcal{E}$	$\mathcal{E}$	0	0

When discharging, by KVL,

$$
\begin{aligned}
V_S &= V_C + V_R \\
0 &= \frac{q}{C} + R\frac{\mathrm{d}q}{\mathrm{d}t} \\
\frac{\mathrm{d}q}{q} &= -\frac{\mathrm{d}t}{RC}.
\end{aligned}
$$

Therefore,

$$
\int_{q_0}^q \frac{\mathrm{d}q'}{q'} = -\int_0^t \frac{\mathrm{d}t'}{RC},
$$

and

$$
q = q_0 \mathrm{e}^{-t/\tau}. \tag{3.18}
$$

The conventional current is now negative, which means it is now flowing back from the capacitor and into the (switched-off) source. The capacitor is returning the power back into the circuit (negative power), and this power is dissipated in the resistor.

	q	$v_C = -v_R$	i
$t = 0$	q_0	q_0/C	$-q_0/RC$
t	$q_0 e^{-t/\tau}$	$q_0/C \cdot e^{-t/\tau}$	$q_0/RC \cdot e^{-t/\tau}$
$t \to \infty$	0	0	0

3.7.3 RL Circuits

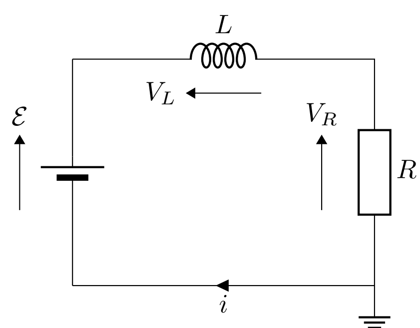


Figure 3.3: A RL Circuit.

When charging, by KVL,

$$\begin{aligned} V_S &= V_L + V_R \\ \mathcal{E} &= L\frac{\mathrm{d}i}{\mathrm{d}t} + iR \\ \frac{\mathrm{d}i}{\mathcal{E}/R - i} &= \frac{R}{L}\mathrm{d}t. \end{aligned}$$

Therefore,

$$\int_0^i \frac{\mathrm{d}i'}{\mathcal{E}/R - i'} = \int_0^t \frac{R}{L}\mathrm{d}t',$$

and

$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right). \tag{3.19}$$

The quantity

$$\tau = \frac{L}{R} \tag{3.20}$$

is defined as the time constant here.

The voltage across the resistor is $v_R = iR$, and the voltage across the inductor is $v_L = \mathcal{E} - v_R$.

Unlike the equivalent RC circuit, the source continues to deliver power $p_S = -\mathcal{E}i$ even for time $t \gg \tau$.

	i	v_L	v_R
$t=0$	0	$\mathcal{E}$	0
t	$\mathcal{E}/R\cdot\left(1-\mathrm{e}^{-t/\tau}\right)$	$\mathcal{E}\mathrm{e}^{-t/\tau}$	$\mathcal{E}\left(1-\mathrm{e}^{-t/\tau}\right)$
$t\to\infty$	$\mathcal{E}/R$	0	$\mathcal{E}$

When discharging, by KVL,

$$\begin{aligned}V_S &= V_L + V_R\\ 0 &= L\frac{\mathrm{d}i}{\mathrm{d}t} + iR\\ \frac{\mathrm{d}i}{i} &= -\frac{R}{L}\mathrm{d}t.\end{aligned}$$

Therefore,

$$\int_{i_0}^{i}\frac{\mathrm{d}i'}{i'} = -\int_0^t\frac{R}{L}\mathrm{d}t',$$

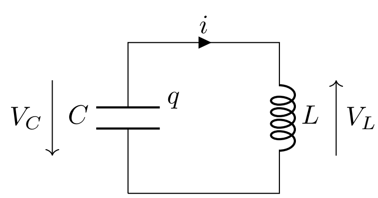
and

$$i = i_0\mathrm{e}^{-t/\tau}. \tag{3.21}$$

	i	$v_R=-v_L$
$t=0$	i_0	i_0R
t	$i_0\mathrm{e}^{-t/\tau}$	$i_0R\mathrm{e}^{-t/\tau}$
$t\to\infty$	0	0

The inductor is now delivering stored energy back into the circuit, where it is dissipated in the resistor. The inductor now has negative power.

3.7.4 LC Circuits



By KVL,

$$\begin{aligned}V_C + V_L &= 0\\ \frac{q}{C} + L\frac{\mathrm{d}i}{\mathrm{d}t} &= 0\\ \ddot{q} + \frac{q}{LC} &= 0.\end{aligned}$$

Ideally, the charge on the capacitor would exhibit simple harmonic motion with the angular frequency being the natural frequency $\omega_0 = \sqrt{1/LC}$.

3.8 AC Analysis

3.8.1 Phasors

We can represent a signal (voltage or current) by a vector in the complex plane which represents just the amplitude and phase of the quantity.

We create a phasor representation of of a signal as follows:

(1) Express the signal as a complex exponential.

(2) Remove the time-dependent part.

(3) The result will be a vector in the complex plane with the length of the signal's amplitude and the angle of the signal's phase. For example,

$$v = V_0 \cos(\omega t + \phi) \Rightarrow \tilde{V} = V_0 \mathrm{e}^{j\phi}.$$

To recover the signal as a function of time,

(1) Multiply by $\mathrm{e}^{j\omega t}$ (all voltages and currents oscillate at the same frequency).

(2) Take the real part.

3.8.2 Complex Impedance

For any linear component, or network of components, where the current phasor is $\tilde{I}$ and the potential difference phasor is $\tilde{V}$, the complex impedance is

$$\tilde{Z} = \frac{\tilde{V}}{\tilde{I}}. \tag{3.22}$$

Since impedance is a ratio of voltage to current, it also has units of ohms. Noting that all voltages and currents oscillate at the same frequency, we assume

$$\tilde{i} = \tilde{I}\mathrm{e}^{j\omega t}, \qquad \tilde{v} = \tilde{V}\mathrm{e}^{j\omega t}.$$

The following equations help us to derive the impedances for each circuit component:

$$v_L = L\frac{\mathrm{d}i}{\mathrm{d}t} \Rightarrow \tilde{Z}_L = j\omega L \tag{3.23}$$

$$i = C\frac{\mathrm{d}v_C}{\mathrm{d}t} \Rightarrow \tilde{Z}_C = \frac{1}{j\omega C} = \frac{-j}{\omega C} \tag{3.24}$$

$$v_R = iR \Rightarrow \tilde{Z}_R = R. \tag{3.25}$$

Equivalent impedance follows the same rule as the resistance and inductance.

3.9 Filters and Gain

The frequency-dependent behaviour of RL and RC circuits makes them useful for a type of circuit known as a filter which is used to select (or remove) a range of frequencies from a signal.

Note that since the signal is assumed to come from some source (e.g. a sensor of some kind) and assumed to then go on to some measurement device (e.g. a scope) then typically the source and destination are not shown. Of course, in reality there will always be complete circuits, however for the filter circuit we are mainly interested in the relation between the signal input to the filter and the output from the filter. The use of complex quantities will make the frequency-dependence of this input / output relationship much easier to visualise.

The filter is characterised by the gain, which is the ratio of output to input voltage phasors. Therefore, gain is a complex quantity:

$$\tilde{G} = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = G\mathrm{e}^{j\phi}. \tag{3.26}$$

By algebra, we can see

$$\tilde{V}_{\text{out}} = \tilde{G}\tilde{V}_{\text{in}}.$$

It will always be the case that

(1) The amplitude of the output will be scaled by a factor G.

(2) The phase of the output will be shifted by ϕ.

For all first order passive filters (RL and RC), G is a positive quantity that is smaller than 1.

To visualise $\tilde{G}$, it is convenient to plot the magnitude G versus frequency on a log-log plot. The argument or phase ϕ is usually shown in degrees (by convention) on a log-linear plot. Together, these two plots are commonly known as the Bode Plot.

The logarithm of the gain magnitude is, by convention, expressed in dB (decibels):

$$G_{\text{dB}} = 20\log_{10} G. \tag{3.27}$$

Since G is a ratio of amplitudes, the above equation expresses the relative power in the output of the filter to the input of the filter.

By convention, we choose the frequency where

$$G = \frac{1}{\sqrt{2}} \tag{3.28}$$

to be the cut-off frequency.

An ideal low-pass filter would pass all frequencies below the cut-off and block all frequencies above. This is impossible to achieve in real-world filters. For the RC low-pass filter, it can be seen that for $\omega \gg \omega_C$,

$$G \approx \frac{\omega_C}{\omega}, \tag{3.29}$$

which results in a gradient of -20 dB/decade, where a decade is a factor of 10 in frequency. For example, the gain falls by -20 dB between the frequencies 10 and 100 rad/s.

3.9.1 Low Pass Filters

All of the four RC and RL filter circuits are described as first order passive filters since they are composed of 'passive' components (resistors, capacitors and inductors) and their behaviours are governed by first-order differential equations.

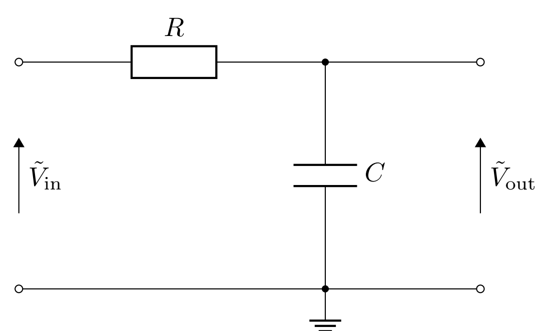


Figure 3.4: A RC low pass filter.

Assuming no current flows to the V_{out} terminals,

$$\tilde{I} = \frac{\tilde{V}_{\text{in}}}{\tilde{Z}_R + \tilde{Z}_C}$$

$$\tilde{V}_{\text{out}} = \tilde{I}\tilde{Z}_C = \tilde{V}_{\text{in}} \times \frac{\tilde{Z}_C}{\tilde{Z}_R + \tilde{Z}_C}$$

$$\tilde{G} = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{\tilde{Z}_C}{\tilde{Z}_R + \tilde{Z}_C} = \frac{1}{1 + j\omega RC}$$

$$G = \frac{1}{\sqrt{1+(\omega RC)^2}} \text{ (blocks signals with high } \omega)$$

$$G = \frac{1}{\sqrt{2}} \Rightarrow \omega_C = \frac{1}{RC}.$$

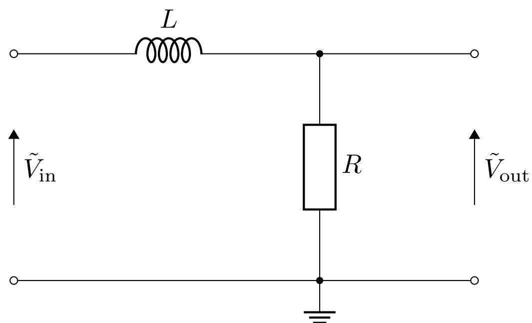


Figure 3.5: A RL low pass filter.

$$\tilde{I} = \frac{\tilde{V}_{\text{in}}}{\tilde{Z}_R + \tilde{Z}_L}$$

$$\tilde{V}_{\text{out}} = \tilde{I}\tilde{Z}_R = \tilde{V}_{\text{in}} \times \frac{\tilde{Z}_R}{\tilde{Z}_R + \tilde{Z}_L}$$

$$\tilde{G} = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{\tilde{Z}_R}{\tilde{Z}_R + \tilde{Z}_L} = \frac{1}{1 + j\omega L/R}$$

$$G = \frac{1}{\sqrt{1+(\omega L/R)^2}} \text{ (blocks signals with high } \omega)$$

$$G = \frac{1}{\sqrt{2}} \Rightarrow \omega_C = \frac{R}{L}.$$

3.9.2 High Pass Filters

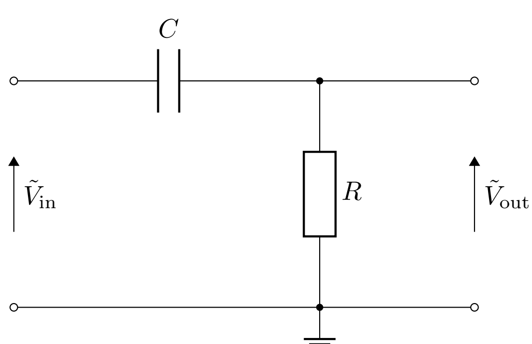


Figure 3.6: A RC high pass filter.

$$\tilde{I} = \frac{\tilde{V}_{\text{in}}}{\tilde{Z}_C + \tilde{Z}_R},$$

$$\tilde{V}_{\text{out}} = \tilde{I}\tilde{Z}_R = \tilde{V}_{\text{in}} \times \frac{\tilde{Z}_R}{\tilde{Z}_C + \tilde{Z}_R}.$$

$$\tilde{G} = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{\tilde{Z}_R}{\tilde{Z}_C + \tilde{Z}_R} = \frac{j\omega RC}{1 + j\omega RC}$$

$$G = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \text{ (blocks signals with low } \omega\text{)}$$

$$G = \frac{1}{\sqrt{2}} \Rightarrow \omega_C = \frac{1}{RC}.$$

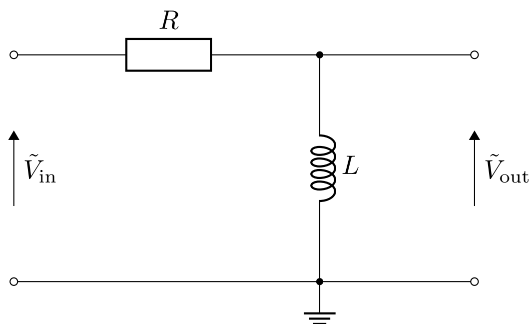


Figure 3.7: A RL high pass filter.

$$\tilde{I} = \frac{\tilde{V}_{\text{in}}}{\tilde{Z}_R + \tilde{Z}_L},$$

$$\tilde{V}_{\text{out}} = \tilde{I}\tilde{Z}_L = \tilde{V}_{\text{in}} \times \frac{\tilde{Z}_L}{\tilde{Z}_R + \tilde{Z}_L}.$$

$$\tilde{G} = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{\tilde{Z}_L}{\tilde{Z}_R + \tilde{Z}_L} = \frac{j\omega L/R}{1 + j\omega L/R}$$

$$G = \frac{\omega L/R}{\sqrt{1 + (\omega L/R)^2}} \text{ (blocks signals with low } \omega\text{)}$$

$$G = \frac{1}{\sqrt{2}} \Rightarrow \omega_C = \frac{R}{L}.$$

Chapter 4

Fourier Analysis

4.1 Dirac Delta Function

4.1.1 Definition

The Dirac Delta function is an ideal function that is defined by the criteria:

$$\delta(x) = 0 \text{ for } x \neq 0, \tag{4.1}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-t)\mathrm{d}x = f(t), \tag{4.2}$$

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1. \tag{4.3}$$

The criteria tell us that the Dirac Delta function has the sifting property. Note the difference between the Dirac Delta and the Kronecker Delta:

$$\begin{cases} \text{Kronecker Delta: } \delta_{mn} = 1 \text{ iff } m = n; \\ \text{Dirac Delta: } \displaystyle\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1, \text{ and } \delta(x) \neq 0 \text{ iff } x = 0. \end{cases} \tag{4.4}$$

There are three common types of functions that are used to approximate the Dirac Delta: the top hat function, a complex exponential, and a Gaussian. In each case, we consider

$$\delta(x) = \lim_{n\to\infty} \delta_n(x). \tag{4.5}$$

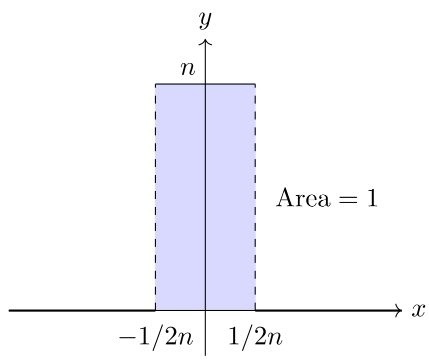
4.1.2 Top Hat Function

A top hat function is a constant near the origin. Therefore, to approximate the Dirac Delta, we ask

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = lw = 1, \qquad w \to 0.$$

Therefore, we get:

$$\delta_n(x) = \begin{cases} n & \text{for } |x| \le \dfrac{1}{2n} \\ 0 & \text{for } |x| > \dfrac{1}{2n}. \end{cases} \tag{4.6}$$



4.1.3 Complex Exponential

One could notice that a sinc function squeezed to the origin can approximate the delta function. Therefore, we can try the function

$$f(x) = n\,\mathrm{sinc}(nx) \equiv \frac{\sin(nx)}{x}.$$

We should check the value of the integral.

$$\begin{aligned} I = \int_{-\infty}^{\infty} f(x)\mathrm{d}x &= n\int_{-\infty}^{\infty} \frac{\sin(nx)}{nx}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{\sin(nx)}{nx}\,\mathrm{d}(nx) \\ &= \int_{-\infty}^{\infty} \frac{\sin u}{u}\mathrm{d}u \quad (u = nx). \end{aligned}$$

The above derivation also tells us that these integrals with different angular frequencies yield the same answer.

(1) $\dfrac{1}{u} = \displaystyle\int_0^{\infty} \mathrm{e}^{-ut}\mathrm{d}t,$

(2) $\mathrm{sinc}(u)$ is even: $I = \displaystyle\int_{-\infty}^{\infty} \frac{\sin u}{u}\mathrm{d}u = 2\int_0^{\infty} \frac{\sin u}{u}\mathrm{d}u,$

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \frac{\sin u}{u} \mathrm{d}u \\
&= 2\int_{0}^{\infty} \frac{\sin u}{u} \mathrm{d}u \\
&= 2\int_{0}^{\infty} \sin u \left(\int_{0}^{\infty} \mathrm{e}^{-ut} \mathrm{d}t \right) \mathrm{d}u \\
&= 2\int_{0}^{\infty} \mathrm{d}t \int_{0}^{\infty} \mathrm{e}^{-ut} \sin u \mathrm{d}u.
\end{aligned}
$$

Using integration by parts,

$$
\begin{aligned}
J &= \int_{0}^{\infty} \mathrm{e}^{-ut} \sin u \mathrm{d}u \\
&= -\int_{0}^{\infty} \mathrm{e}^{-ut} \,\mathrm{d}(\cos u) \\
&= -\left(\mathrm{e}^{-ut} \cos u \Big|_{0}^{\infty} - \int_{0}^{\infty} \cos u \,\mathrm{d}\big(\mathrm{e}^{-ut}\big) \right) \\
&= 1 - t\int_{0}^{\infty} \mathrm{e}^{-ut} \cos u \mathrm{d}u \\
&= 1 - t\int_{0}^{\infty} \mathrm{e}^{-ut} \,\mathrm{d}(\sin u) \\
&= 1 - t\left(\mathrm{e}^{-ut} \sin u \Big|_{0}^{\infty} - \int_{0}^{\infty} \sin u \,\mathrm{d}\big(\mathrm{e}^{-ut}\big) \right) \\
&= 1 + t^2 J.
\end{aligned}
$$

So,

$$
J = 1 + t^2 J \Rightarrow J = \frac{1}{1+t^2}.
$$

Therefore,

$$
I = 2\int_{0}^{\infty} J \mathrm{d}t = 2\int_{0}^{\infty} \frac{1}{1+t^2} \mathrm{d}t = 2\arctan t\Big|_{0}^{\infty} = \pi.
$$

For the value of the integral to be 1, we need to scale $f(x)$ by $1/\pi$. Therefore, the function $\delta_n(x)$ can be used to approximate the delta function:

$$
\delta_n(x) = \frac{\sin(nx)}{\pi x}. \tag{4.7}
$$

As $\sin(nx) = \frac{1}{2i}(\mathrm{e}^{inx} - \mathrm{e}^{-inx})$,

$$
\begin{aligned}
\delta_n(x) &= \frac{\sin(nx)}{\pi x} \\
&= \frac{1}{2\pi i x}(\mathrm{e}^{inx} - \mathrm{e}^{-inx}) \\
&= \frac{1}{2\pi i x} \cdot \mathrm{e}^{ixt}\Big|_{t=-n}^{t=n} \\
&= \frac{1}{2\pi i x} \int_{-n}^{n} \mathrm{e}^{ixt}\,\mathrm{d}(ixt) \\
&= \frac{1}{2\pi} \int_{-n}^{n} \mathrm{e}^{ixt}\mathrm{d}t.
\end{aligned}
$$

To conclude, the complex exponential approximation of Dirac delta function is:

$$
\delta_n(x) = \frac{1}{2\pi} \int_{-n}^{n} \mathrm{e}^{ixt}\mathrm{d}t. \tag{4.8}
$$

Moreover, as $n \to \infty$, we can see:

$$
\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ixt}\mathrm{d}t. \tag{4.9}
$$

When in applications, it may be a little harder to recognize an integral as a Dirac delta. For example,

$$
\delta(t-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\omega(t-x)}\mathrm{d}\omega. \tag{4.10}
$$

4.1.4 Gaussian

A Gaussian can also be used to approximate the Dirac delta. As a reminder, a Gaussian is in the form of

$$
f(x) = a\mathrm{e}^{-(x-b)^2/2c^2}, \tag{4.11}
$$

where b is the mean and c is the standard deviation.

To approximate the Dirac delta, the standard deviation should be small (a small c), the centre should be at the origin ($b = 0$), and the peak at origin should be infinitely tall ($a \to \infty$). Therefore, we can try the function

$$
f(x) = n\mathrm{e}^{-n^2x^2}.
$$

Now, check the value of integral of this function:

$$
\begin{aligned}
I = \int_{-\infty}^{\infty} f(x)\mathrm{d}x &= \int_{-\infty}^{\infty} n\mathrm{e}^{-n^2x^2}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-u^2}\mathrm{d}u \quad (u = nx).
\end{aligned}
$$

u is clearly a dummy variable. We may replace it with any variables we want. Therefore, using plane polar coordinates,

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x \cdot \int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} \mathrm{e}^{-\rho^2}\rho\mathrm{d}\rho\mathrm{d}\phi \\
&= \int_{0}^{2\pi}\mathrm{d}\phi\int_{0}^{\infty} \rho\mathrm{e}^{-\rho^2}\mathrm{d}\rho \\
&= -\frac{1}{2}\cdot 2\pi\cdot(\mathrm{e}^{-\rho^2})\Big|_0^{\infty} \\
&= \pi.
\end{aligned}$$

Therefore, the function $f(x)$ should be scaled by a factor of $1/\sqrt{\pi}$:

$$\delta_n(x) = \frac{n}{\sqrt{\pi}}\mathrm{e}^{-n^2x^2}. \tag{4.12}$$

To summarize,

$$\delta_n(x) = \begin{cases} n \text{ for } |x| \leq \dfrac{1}{2n} & \text{(Top Hat Function)} \\ \dfrac{1}{2\pi}\displaystyle\int_{-n}^{n}\mathrm{e}^{ixt}\mathrm{d}t & \text{(Complex Exponential)} \\ \dfrac{n}{\sqrt{\pi}}\mathrm{e}^{-n^2x^2} & \text{(Gaussian)} \end{cases} \tag{4.13}$$

4.2 Fourier Series

4.2.1 Orthogonal and Orthonormal Functions

We say that the set of all vectors, $\{\boldsymbol{V}_n\}$, make an orthogonal set if

$$\boldsymbol{V}_n \cdot \boldsymbol{V}_m = 0 \quad \text{for } m \neq n. \tag{4.14}$$

The set of vectors is called a complete orthogonal set if the size of the orthogonal set is the same as the number of dimensions in the space considered. The set is called orthonormal if the vectors are normalised. In that case,

$$\boldsymbol{V}_n \cdot \boldsymbol{V}_m = \delta_{mn} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m. \end{cases} \tag{4.15}$$

If we have a complete orthonormal set of N vectors $\boldsymbol{V}_n$, then we can in a unique way write an arbitrary vector $\boldsymbol{A}$ as a linear superposition of the orthonormal vectors. This is why we call $\boldsymbol{V}_n$ an orthonormal basis.

Conversely, we can decompose a given vector $\boldsymbol{A}$ using its scalar product with the basis vectors.

If we consider the vectors to be complex, then the definition of an orthonormal set is

$$\boldsymbol{V}_n \cdot \boldsymbol{V}_m^* = \delta_{mn}. \tag{4.16}$$

This definition can be further extended to the realm of functions. We define the inner product $\langle f, g\rangle$ between two (complex) functions f and g on an interval (a, b) as

$$\langle f, g\rangle = \int_a^b f(x) g^*(x) \mathrm{d}x. \tag{4.17}$$

Similarly, a set of functions, f_n, form an orthonormal set if

$$\langle f_n, f_m\rangle = \delta_{mn}. \tag{4.18}$$

4.2.2 Complex Fourier Series

The set of complex exponentials

$$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{inx} \quad \text{with } n \in \mathbb{Z} \tag{4.19}$$

form an orthonormal set on the interval $(-\pi, \pi)$.

In order to check, we should investigate both orthogonality and normality:

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{2\pi}}\mathrm{e}^{inx}, \frac{1}{\sqrt{2\pi}}\mathrm{e}^{imx} \right\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{inx}(\mathrm{e}^{inx})^* \mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i(n-m)x} \mathrm{d}x \\
&= \frac{1}{(n-m)\pi}\frac{1}{2i}\mathrm{e}^{i(n-m)x}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{(n-m)\pi}\sin((n-m)\pi) \\
&= 0 \quad \text{for any } n \neq m \Rightarrow \text{Orthogonality;}
\end{aligned}$$

$$\left\langle \frac{1}{\sqrt{2\pi}}\mathrm{e}^{inx}, \frac{1}{\sqrt{2\pi}}\mathrm{e}^{inx} \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \mathrm{d}x = 1 \Rightarrow \text{Normality.}$$

Therefore, we know that any complex function can be represented by the orthonormal set described above:

$$f(x) = \sum_n c_n \mathrm{e}^{inx}, \tag{4.20}$$

where the scale factors are included in c_n.

By doing an inner product with an complex exponential, we can find c_n from $f(x)$:

$$\left\langle f(x), \mathrm{e}^{inx} \right\rangle = \int_{-\pi}^{\pi} \left(\sum_m c_m \mathrm{e}^{imx} \right) \mathrm{e}^{-inx} \mathrm{d}x = \int_{-\pi}^{\pi} c_n \mathrm{d}x = 2\pi c_n.$$

Therefore,

$$c_n = \frac{1}{2\pi} \left\langle f(x), \mathrm{e}^{inx} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-inx} \mathrm{d}x. \tag{4.21}$$

4.2.3 Trigonometric Fourier Series

By algebra, we can see:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{inx} = c_0 + \sum_{n=1}^{\infty} \left(c_n \mathrm{e}^{inx} + c_{-n} \mathrm{e}^{-inx} \right)$$

$$\begin{aligned}
c_n \mathrm{e}^{inx} + c_{-n} \mathrm{e}^{-inx} &= \frac{c_n + c_{-n}}{2} (\mathrm{e}^{inx} + \mathrm{e}^{-inx}) + \frac{c_n - c_{-n}}{2} (\mathrm{e}^{inx} - \mathrm{e}^{-inx}) \\
&= \frac{c_n + c_{-n}}{2} \cdot 2\cos(nx) + \frac{c_n - c_{-n}}{2} \cdot 2i \sin(nx) \\
&= (c_n + c_{-n}) \cos(nx) + i(c_n - c_{-n}) \sin(nx) \\
&\equiv a_n \cos(nx) + b_n \sin(nx).
\end{aligned}$$

Therefore, the set of cosines and sines can also be used to decompose a given function $f(x)$.

$$\begin{aligned}
a_n &= c_n + c_{-n} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-inx} \mathrm{d}x + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{inx} \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot 2\cos(nx) \mathrm{d}x \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \mathrm{d}x.
\end{aligned}$$

$$\begin{aligned}
b_n &= i(c_n - c_{-n}) \\
&= i \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-inx} \mathrm{d}x - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{inx} \mathrm{d}x \right) \\
&= -i \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot \left(\mathrm{e}^{inx} - \mathrm{e}^{-inx} \right) \mathrm{d}x \\
&= -i \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot 2i \sin(nx) \mathrm{d}x \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \mathrm{d}x.
\end{aligned}$$

By identifying $a_0 = c_0 + c_{-0} = 2c_0$, we get the Fourier series in cosines and sines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \text{ where} \tag{4.22}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \mathrm{d}x, \tag{4.23}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \mathrm{d}x. \tag{4.24}$$

4.2.4 Reality Condition

Reality condition talks about the relations within the coefficients when

$$f(x) = f^*(x) :$$

$$f(x) = \sum c_n \mathrm{e}^{inx} = f^*(x) = \sum c_n^* \mathrm{e}^{-inx},$$

and therefore,

$$c_n^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(x) \mathrm{e}^{inx} \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{inx} \mathrm{d}x = c_{-n}.$$

Reality condition states that if $f(x) = f^*(x)$, then

$$c_n^* = c_{-n}. \tag{4.25}$$

This tells us that to construct a real function, we only need to specify complex coefficients c_n for $n \geq 0$.

4.2.5 Dirichlet Conditions

(1) $f(x)$ is periodic (2π);

(2) $f(x)$ is single-valued;

(3) $f(x)$ has a finite number of extrema in the interval (infinite - infinitely high frequency);

(4) $f(x)$ has a finite number of discontinuities;

(5) $\displaystyle\int_{-\infty}^{\infty} |f(x)| \mathrm{d}x$ is finite.

If the above conditions are satisfied, then the Fourier Series converges to $f(x)$ at all points where $f(x)$ is continuous. The series converges to the midpoint between the values of $f(x)$ from the left and right at any discontinuity.

4.2.6 Other Lengths of Intervals

Consider a function $f(x)$ defined in $(-l, l)$. To still represent it with the complex exponentials, we should let the phase change by a total of 2π in the interval. Therefore,

$$x = l, \quad \phi = \pi \Rightarrow \phi = \frac{\pi x}{l}.$$

So, we can write:

$$f(x) = \sum c_n \mathrm{e}^{in\pi x/l}, \text{ where} \tag{4.26}$$

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) \mathrm{e}^{-in\pi x/l} \mathrm{d}x. \tag{4.27}$$

$$f(x) = \frac{a_0}{2} + \sum \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right], \text{ where} \tag{4.28}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) \mathrm{d}x, \tag{4.29}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \tag{4.30}$$

4.2.7 Power Spectrum

By auxiliary angle formula, the trigonometric Fourier Series can be written as:

$$f(x) = \frac{a_0}{2} + \sum \alpha_n \cos(nx - \theta_n). \tag{4.31}$$

4.2.8 Parseval's Identity

Consider the complex Fourier Series $f(x) = \sum c_n \mathrm{e}^{inx}$.

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x &= \int_{-\pi}^{\pi} \left| \sum c_n \mathrm{e}^{inx} \right|^2 \mathrm{d}x \\
&= \int_{-\pi}^{\pi} \left(\sum_{n,m} c_n c_m^* \mathrm{e}^{i(n-m)x} \right) \mathrm{d}x \\
&= \sum_{n,m} c_n c_m^* \cdot 2\pi \delta_{mn} \\
&= 2\pi \sum |c_n|^2.
\end{aligned}$$

By rearranging, we get the Parseval's Identity:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 \mathrm{d}x = \sum_{-\infty}^{\infty}|c_n|^2. \tag{4.32}$$

This also leads to Parseval's Inequality for the truncated Fourier Series, where for any positive $N < \infty$, we have

$$\sum_{-N}^{N}|c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 \mathrm{d}x. \tag{4.33}$$

The equivalent form of Parseval's Identity for the trigonometric Fourier Series is:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 \mathrm{d}x = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2). \tag{4.34}$$

Parseval's Identity is very useful for considering how good an Fourier Series approximation to the function is. However, be aware that it only talks about the overall quality, but not about how good the approximation is at a given point.

4.3 Fourier Transform

4.3.1 From Series to Transform

Consider a general $f(t)$ that ranges from $-T/2$ to $T/2$:

$$f(t) = \sum c_n \mathrm{e}^{in\phi} = \sum c_n \mathrm{e}^{in\frac{2\pi}{T}t}.$$

We define $\omega_0 = \dfrac{2\pi}{T}$ to be the fundamental frequency of the complex exponentials. From the definition of c_n, we get:

$$f(t) = \sum c_n \mathrm{e}^{in\omega_0 t} = \sum \left(\frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-in\omega_0 t}\mathrm{d}t\right)\mathrm{e}^{in\omega_0 t}.$$

The difference in the angular frequency for each complex exponential is:

$$\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0.$$

Therefore, we can write:

$$f(t) = \sum_{n=-\infty}^{\infty}\left(\frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-in\omega_0 t}\mathrm{d}t\right)\mathrm{e}^{in\omega_0 t}.$$

Now consider the case when $T \to \infty$:
(1) As $\omega_0 = \dfrac{2\pi}{T} \to 0$, we can see the set of angular frequencies becoming continuous: $n\omega_0 \to \omega$;
(2) $\Delta\omega \to \mathrm{d}\omega$, and $\displaystyle\sum_{n=-\infty}^{\infty} \Delta\omega \to \int_{-\infty}^{\infty} \mathrm{d}\omega$;
(3) The integral $\displaystyle\int_{-T/2}^{T/2} \mathrm{d}t \to \int_{-\infty}^{\infty} \mathrm{d}t$.
Therefore,

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} f(t)\mathrm{e}^{-in\omega_0 t}\mathrm{d}t \right) \mathrm{e}^{in\omega_0 t} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\mathrm{d}t\mathrm{e}^{i\omega t} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\mathrm{e}^{i\omega t}\mathrm{d}t \right) \mathrm{e}^{-i\omega t}\mathrm{d}\omega.
\end{aligned}
$$

So, we define the Fourier Transform of a function $f(t)$ as

$$
g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\mathrm{e}^{i\omega t}\mathrm{d}t, \tag{4.35}
$$

and the corresponding inverse Fourier Transform as

$$
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\mathrm{e}^{-i\omega t}\mathrm{d}\omega. \tag{4.36}
$$

In general, we regard f to be a function of time, and g, a function of angular frequency. The spatial counterparts of these two functions are those of position and angular wave number:

$$
g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{ikx}\mathrm{d}x, \tag{4.37}
$$

$$
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\mathrm{e}^{-ikx}\mathrm{d}k. \tag{4.38}
$$

We may prove the inverse Fourier Transform with the help of a delta function:

$$
\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} f(x)\delta(x-t)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\omega(x-t)}\mathrm{d}\omega \right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{i\omega x}\mathrm{d}x \right) \mathrm{e}^{-i\omega t}\mathrm{d}\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\mathrm{e}^{-i\omega t}\mathrm{d}\omega.
\end{aligned}
$$

The inverse Fourier Transform really brings us back to the original function.

4.3.2 Gaussian

The Fourier Transform of a Gaussian is still a Gaussian. Consider the function

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-t^2/2\sigma^2}. \tag{4.39}$$

The Fourier Transform of this function is:

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{i\omega t} \mathrm{d}t = \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2\sigma^2} \mathrm{e}^{i\omega t} \mathrm{d}t.$$

To solve for $g(\omega)$, differentiate it with respect to ω:

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}\omega} &= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2\sigma^2} it \mathrm{e}^{i\omega t} \mathrm{d}t \\
&= -\frac{i\sigma}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\omega t} \mathrm{d}\left(\mathrm{e}^{-t^2/2\sigma^2}\right) \\
&= -\frac{\sigma\omega}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2\sigma^2} \mathrm{e}^{i\omega t} \mathrm{d}t \quad \text{(integration by parts)} \\
&= -\sigma^2 \omega g(\omega).
\end{aligned}$$

Therefore,

$$g(\omega) = g(0) \mathrm{e}^{-\sigma^2\omega^2/2}.$$

Noting that $g(0) = \frac{1}{\sqrt{2\pi}}$ and defining $\rho = \frac{1}{\sigma}$,

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\omega^2/2\rho^2}. \tag{4.40}$$

To conclude,

$$\mathscr{F}\left(\frac{1}{\sigma} \mathrm{e}^{-t^2/2\sigma^2}\right) = \mathrm{e}^{-\omega^2/2\rho^2}, \tag{4.41}$$

or alternatively,

$$\mathscr{F}\left(\mathrm{e}^{-t^2/2\sigma^2}\right) = \frac{1}{\rho} \mathrm{e}^{-\omega^2/2\rho^2}. \tag{4.42}$$

The Fourier Transform of a Gaussian is a Gaussian, and their widths are reciprocal.

4.3.3 Delta Function

The Fourier Transform of a delta function is a constant.

$$\mathscr{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) \mathrm{e}^{i\omega t} \mathrm{d}t = \frac{1}{\sqrt{2\pi}}.$$

To make sense of it, consider the function $\delta(t)$ as a Gaussian with width $\sigma \to 0$. This means $\rho \to \infty$. A Gaussian with infinite standard deviation is without doubt a constant.

4.3.4 Top Hat Function

The Fourier Transform of a top hat function is a sinc function. Consider the function

$$f(t) = \begin{cases} \dfrac{\sqrt{2\pi}}{2T} & t \leq |T| \\ 0 & t > |T|. \end{cases} \tag{4.43}$$

$$\begin{aligned} g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2\pi}}{2T} \mathrm{e}^{i\omega t} \mathrm{d}t \\ &= \frac{1}{2T} \int_{-T}^{T} \mathrm{e}^{i\omega t} \mathrm{d}t \\ &= \frac{1}{2i\omega T} \left[\mathrm{e}^{i\omega t} \right] \Big|_{-T}^{T} \\ &= \mathrm{sinc}(\omega T). \end{aligned}$$

The Fourier Transform of a top hat function is a sinc function whose width is reciprocal to the top hat width.

The width of sinc function:

$$w = \frac{2\pi}{T}.$$

To make sense of this Fourier Transform, consider the two cases where T blows up to infinity or goes to 0:

$$\begin{cases} T \to \infty & f(t) \to \text{const.}, \quad g(\omega) \to \delta(\omega) \\ T \to 0 & f(t) \to \delta(t), \quad g(\omega) \to \text{const.} \end{cases} \tag{4.44}$$

4.3.5 Properties of Fourier Transform

(1) Linearity:

$$\mathscr{F}[\alpha f_1(t) + \beta f_2(t)] = \alpha \mathscr{F}[f_1(t)] + \beta \mathscr{F}[f_2(t)]. \tag{4.45}$$

(2) Change of Sign:

$$\mathscr{F}[f(-t)] = g(-\omega). \tag{4.46}$$

(3) Translation:

$$\mathscr{F}[f(t - t_0)] = \mathrm{e}^{i\omega t_0} g(\omega). \tag{4.47}$$

Note that the Fourier Transform of a shifted delta function is no longer a constant:

$$\mathscr{F}[\delta(t-t_0)] = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{i\omega t_0}. \tag{4.48}$$

(4) Scaling:

$$\mathscr{F}[f(\alpha t)] = \frac{1}{|\alpha|} g\left(\frac{\omega}{\alpha}\right). \tag{4.49}$$

(5) Conjugation:

$$\mathscr{F}[f^*(t)] = g^*(-\omega). \tag{4.50}$$

(6) Reality Condition:

$$f(t) = f^*(t) \Rightarrow g(-\omega) = g^*(\omega). \tag{4.51}$$

To recap, for Fourier Series,

$$c_{-n} = c_n^*. \tag{4.52}$$

Negative frequencies are redundant.

(7) Derivative:

$$\mathscr{F}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = -i\omega g(\omega). \tag{4.53}$$

(8) Parseval's Identity:

$$\int_{-\infty}^{\infty} f(t) g^*(t) \mathrm{d}t = \int_{-\infty}^{\infty} F(\omega) G^*(\omega) \mathrm{d}\omega, \tag{4.54}$$

where $F(\omega) = \mathscr{F}[f(t)]$, $G(\omega) = \mathscr{F}[g(t)]$. Also, for $f(t) = g(t)$,

$$\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |F(\omega)|^2 \mathrm{d}\omega. \tag{4.55}$$

4.3.6 Solving Partial Differential Equations

A Fourier Transform can be used to change a PDE into an ODE. This comes at the price of having to make first a Fourier Transform and subsequently an inverse one.

Consider the heat equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{4.56}$$

Let $\hat{u}(k,t) = \mathscr{F}[u(x,t)]$. That is to say,

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) \mathrm{e}^{ikx} \mathrm{d}x.$$

Therefore,

$$\begin{aligned}
\frac{\partial \hat{u}}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{ikx}\mathrm{d}x\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\mathrm{e}^{ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a^2\frac{\partial^2 u}{\partial x^2}\mathrm{e}^{ikx}\mathrm{d}x \\
&= \mathscr{F}\left[a^2\frac{\partial^2 u}{\partial x^2}\right] \\
&= a^2(-ik)^2\hat{u} \\
&= -a^2k^2\hat{u}.
\end{aligned}$$

This leads us to an ODE:

$$\frac{\partial \hat{u}}{\partial t} = -a^2k^2\hat{u}. \tag{4.57}$$

Then, we can get our solution $u(x,t)$ by the inverse Fourier Transform:

$$u(x,t) = \mathscr{F}^{-1}[\hat{u}(k,t)].$$

Similarly, consider the wave equation:

$$\frac{\partial^2 \psi}{\partial t^2} = c^2\frac{\partial^2 \psi}{\partial x^2}. \tag{4.58}$$

Let $\hat{\psi}(k,t) = \mathscr{F}[\psi(x,t)]$. Therefore,

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\hat{\psi} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 \psi}{\partial t^2}\mathrm{e}^{ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c^2\frac{\partial^2 \psi}{\partial x^2}\mathrm{e}^{ikx}\mathrm{d}x \\
&= \mathscr{F}\left[c^2\frac{\partial^2 \psi}{\partial x^2}\right] \\
&= -c^2k^2\hat{\psi}.
\end{aligned}$$

4.3.7 Multi-dimensional Fourier Transform

A multi-dimensional Fourier Transform is simply a transform acting on each of the variables:

$$g(k_x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{ik_x x}\mathrm{d}x, \tag{4.59}$$

$$g(k_x, k_y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ik_y y}\mathrm{d}y\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{ik_x x}\mathrm{d}x. \tag{4.60}$$

We can make use of vectors in this situation. Introduce $\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\boldsymbol{k} = k_x\hat{\mathbf{i}} + k_y\hat{\mathbf{j}}$:

$$g(\boldsymbol{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\boldsymbol{r})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^2\boldsymbol{r}, \tag{4.61}$$

$$f(\boldsymbol{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\boldsymbol{k})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^2\boldsymbol{r}. \tag{4.62}$$

For n dimensions,

$$g(\boldsymbol{k}) = \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty} f(\boldsymbol{r})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^n\boldsymbol{r}. \tag{4.63}$$

4.4 Convolution

4.4.1 Definition

We may consider $f(t)$ as a signal, and $g(t)$ as a resolution function. Then, we define the convolution of f with g, $f * g$:

$$\begin{aligned}(f * g)(t) &= \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau \quad (f \text{ shifting through } g)\\ &= \int_{-\infty}^{\infty} g(\tau)f(t-\tau)\mathrm{d}\tau \quad (g \text{ shifting through } f).\end{aligned} \tag{4.64}$$

4.4.2 The Convolution Theorem

The complicated convolution of two functions after a Fourier Transform simply turns into multiplication. We may prove the convolution theorem by taking the inverse Fourier Transform of the following function:

$$\sqrt{2\pi}F(\omega)G(\omega), \text{ where } F(\omega) = \mathscr{F}[f(t)],\ G(\omega) = \mathscr{F}[g(t)]. \tag{4.65}$$

$$\begin{aligned}\mathscr{F}^{-1}[\sqrt{2\pi}F(\omega)G(\omega)] &= \int_{-\infty}^{\infty} F(\omega)G(\omega)\mathrm{e}^{-i\omega t}\mathrm{d}\omega\\ &= \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{i\omega\tau}\mathrm{d}\tau\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(s)\mathrm{e}^{i\omega s}\mathrm{d}s\right)\mathrm{e}^{-i\omega t}\mathrm{d}\omega\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{i\omega(\tau+s-t)}\mathrm{d}\omega\right)f(\tau)g(s)\mathrm{d}\tau\mathrm{d}s\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(s)\delta(\tau+s-t)\mathrm{d}\tau\mathrm{d}s\\ &= \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} f(t-s)g(s)\mathrm{d}s\\ &= (f*g)(t).\end{aligned}$$

Alternatively, we write:

$$\mathscr{F}[(f*g)(t)] = \sqrt{2\pi}F(\omega)G(\omega). \tag{4.66}$$

Similarly, we can easily prove that:

$$\mathscr{F}[f(t)g(t)] = \frac{1}{\sqrt{2\pi}}(F * G)(\omega). \tag{4.67}$$

Convolution can be used to shift a function:

$$f(t) * \delta(t-d) = \int_{-\infty}^{\infty} f(\tau)\delta(t-d-\tau)\mathrm{d}\tau = f(t-d). \tag{4.68}$$

With this property, we can work out that:

$$\mathscr{F}[f(t-d) + f(t+d)] = 2\cos(\omega d)g(\omega). \tag{4.69}$$

Another important point is about the convolution of two Gaussian:

$$g(\mu_1, \sigma_1) * g(\mu_2, \sigma_2) = g\left(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right). \tag{4.70}$$