

1. Consider the following table of data points/nodal points:

Time (sec) t	Velocity (ms <sup>-1</sup> ) v(t)
2	10
4	20
6	25

- a) [4+1 marks] Find an interpolating polynomial of velocity that goes through the above data points by using **Vandermonde Matrix** method. Also compute an approximate value of acceleration at Time, t=7 sec.  
 b) [4 marks] Find an interpolating polynomial of velocity that goes through the above data points by using **Lagrange** method.  
 c) [1 mark] If a new data point is added in the above scenario, which method you should use in finding a new interpolating polynomial. Also what will be the degree of that new polynomial?

1. a) Here,  $x_0 = 2$ ,  $x_1 = 4$ ,  $x_2 = 6 \therefore$  Degree will be 2

$$\text{Now, } V = \begin{bmatrix} 1 & 2^1 & 2^2 \\ 1 & 4^1 & 4^2 \\ 1 & 6^1 & 6^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}$$

$$Y = \begin{bmatrix} 10 \\ 20 \\ 25 \end{bmatrix}$$

We know,  $V \cdot A = Y$

$$\Rightarrow A = V^{-1} \cdot Y$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 10 \\ 20 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 35/4 \\ -5/8 \end{bmatrix}$$

$$\text{Now, } P_2(x) = a_0 + a_1 x^1 + a_2 x^2$$

$$= -5 + \frac{35}{4}x - \frac{5}{8}x^2$$

$$\text{Now, } v(7) = P_2(7) = -5 + \frac{35 \times 7}{4} - \frac{5 \times 7^2}{8}$$

$$= 25.625 \text{ ms}^{-1}$$

$$\therefore \text{Acceleration} = \frac{d}{dx}(P_2(x)) = \frac{35}{4} - \frac{5}{4}x \Rightarrow a(7) = \frac{35}{4} - \frac{5}{4} \times 7 = 0 \text{ ms}^{-2}$$

b) Using lagrange method,

$$P_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$

$$\text{Now, } l_0(x) = \frac{x-x_1}{x_0-x_1} \times \frac{x-x_2}{x_0-x_2} = \frac{x-4}{2-4} \times \frac{x-6}{2-6} = \frac{1}{8}(x^2 - 10x + 24)$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0} \times \frac{x-x_2}{x_1-x_2} = \frac{x-2}{4-2} \times \frac{x-6}{4-6} = -\frac{1}{4}(x^2 - 8x + 12)$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0} \times \frac{x-x_1}{x_2-x_1} = \frac{x-2}{6-2} \times \frac{x-4}{6-4} = \frac{1}{8}(x^2 - 6x + 8)$$

$$\therefore P_2(x) = \frac{1}{8}(x^2 - 10x + 24)x10 - \frac{1}{4}(x^2 - 8x + 12)x20 + \frac{1}{8}(x^2 - 6x + 8)x25$$

$$= -\frac{5x^2}{8} + \frac{35x}{4} - 5$$

c) If new data point is added, Newton's divided difference method should be used. Degree of polynomial will then be 3.

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2. Read the following and answer accordingly:

- (4 marks) Consider the nodes  $[-\pi/2, 0, \pi/2]$ . Find an interpolating polynomial of appropriate degree by using Newton's divided-difference method for  $f(x) = x \sin(x)$ .
- (2 marks) Use the interpolating polynomial to find an approximate value at  $\pi/4$ , and compute the percentage relative error at  $\pi/4$ .
- (4 marks) Add a new node  $\pi$  to the above nodes, and find the interpolating polynomial of appropriate degree.

2. a) Given,  $f(x) = x \sin x$

$$x_0 = -\frac{\pi}{2}$$

$$f[x_0] = -\frac{\pi}{2} \times \sin(-\frac{\pi}{2}) = \frac{\pi}{2}$$

$$x_1 = 0$$

$$f[x_1] = 0 \times \sin(0) = 0$$

$$x_2 = \frac{\pi}{2}$$

$$f[x_2] = \frac{\pi}{2} \times \sin(\frac{\pi}{2}) = \frac{\pi}{2}$$

Using Newton's Divided Difference form,

$$\begin{aligned} x_0 &= -\frac{\pi}{2} & f[x_0] &= \frac{\pi}{2} \\ x_1 &= 0 & f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{0 - \frac{\pi}{2}}{0 - (-\frac{\pi}{2})} = \frac{-\frac{\pi}{2}}{\frac{\pi}{2}} = -1 \\ x_2 &= \frac{\pi}{2} & f[x_1, x_2] &= \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{\frac{\pi}{2} - 0}{\frac{\pi}{2} - 0} = 1 \\ f[x_0] &= \frac{\pi}{2} & f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0]}{x_2 - x_0} \\ &&&= \frac{1 - (-1)}{\frac{\pi}{2} - (-\frac{\pi}{2})} \\ &&&= \frac{1 + 1}{\frac{\pi}{2} + \frac{\pi}{2}} \\ &&&= \frac{2}{\pi} \end{aligned}$$

Since 3 nodes,

$$\begin{aligned} P_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= \frac{\pi}{2} + (-1)(x + \frac{\pi}{2}) + \frac{2}{\pi}(x + \frac{\pi}{2})(x - 0) \\ &= \frac{\pi}{2} - x - \frac{\pi}{2} \rightarrow \frac{2}{\pi}(x + \frac{\pi}{2}x) \\ &= -\frac{\pi}{2} + \frac{2}{\pi}x^2 + x = \frac{2}{\pi}x^2 \end{aligned}$$

b) At  $x = \frac{\pi}{4}$ ,

$$P_2(\frac{\pi}{4}) = \frac{2}{\pi} \times (\frac{\pi}{4})^2 = \frac{2}{\pi} \times \frac{\pi^2}{16} = \frac{\pi}{8}$$

$$\text{Now, } f(\frac{\pi}{4}) = \frac{\pi}{4} \times \sin(\frac{\pi}{4}) = \frac{\pi}{4} \times \frac{1}{\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$$

$$\begin{aligned} \text{Percentage relative error} &= \left| \frac{f(\frac{\pi}{4}) - P(\frac{\pi}{4})}{f(\frac{\pi}{4})} \right| \times 100\% \\ &= 29.29\% \end{aligned}$$

c) Adding new node  $\pi$ ,

$$\begin{aligned} x_0 &= -\frac{\pi}{2} & f[x_0] &= \frac{\pi}{2} \\ x_1 &= 0 & f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{0 - \frac{\pi}{2}}{0 - (-\frac{\pi}{2})} = \frac{-\frac{\pi}{2}}{\frac{\pi}{2}} = -1 \\ x_2 &= \frac{\pi}{2} & f[x_1, x_2] &= \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{\frac{\pi}{2} - 0}{\frac{\pi}{2} - 0} = 1 \\ x_3 &= \pi & f[x_2, x_3] &= \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{0 - \frac{\pi}{2}}{\pi - \frac{\pi}{2}} = -1 \\ f[x_0] &= \frac{\pi}{2} & f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0]}{x_2 - x_0} = \frac{1 - \frac{\pi}{2}}{\frac{\pi}{2} - (-\frac{\pi}{2})} = \frac{1 + \frac{\pi}{2}}{\frac{\pi}{2} + \frac{\pi}{2}} = \frac{2}{\pi} \\ f[x_1] &= 0 & f[x_0, x_1, x_2, x_3] &= \frac{f[x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-1 - \frac{2}{\pi}}{\pi - (-\frac{\pi}{2})} = \frac{-1 - \frac{2}{\pi}}{\frac{3\pi}{2}} = -\frac{2}{\pi} \end{aligned}$$

$$f[x_0, x_1, x_2] = \frac{2}{\pi} \quad \rightarrow \quad f[x_0, x_1, x_2, x_3] = \frac{f[x_0, x_1, x_2] - f[x_0, x_1, x_3]}{x_3 - x_0}$$

$$f[x_1, x_2, x_3] = -\frac{2}{\pi}$$

$$= \frac{-\frac{2}{\pi} - \frac{2}{\pi}}{\pi + \frac{\pi}{2}} = \frac{-\frac{4}{\pi}}{\frac{3\pi}{2}} = -\frac{8}{3\pi}$$

Since there are 4 nodes,

$$\begin{aligned} P_3(x) &= P_2(x) + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2) \\ &= \frac{2}{\pi}x^2 + \left(-\frac{8}{3\pi}\right)(x+\frac{\pi}{2})(x-0)(x-\frac{\pi}{2}) \\ &= \frac{2}{\pi}x^2 - \frac{8}{3\pi}x\left(x-\frac{\pi}{4}\right) \\ &= \frac{2}{\pi}x^2 - \frac{8}{3\pi}x^3 + \frac{8\pi}{12\pi}x \\ &= \underline{\underline{\frac{2}{\pi}x^2 - \frac{8}{3\pi}x^3 + \frac{2}{3}x}} \end{aligned}$$

