IT24644

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□ Prove that the set of rational numbers $\mathbb{Q}$, equiped with the two binary operations of addition and multiplication forms a field.

Soln : A set F with two binary operations + and • is a field if the following hold :

1. (F, +) is an abelian (commutative) group :

(a) closure under + ,
(b) associativity of + ,
(c) identity element 0,
(d) additive inverses
(e) commutativity of +

2. $(F \setminus \{0\}, \cdot)$ is an abelian group :

(a) closure under •
(b) associativity of •
(c) identity element 0
(d) inverse multiplicative for every non zero element.
(e) commutivity of •

3. Distributivity : $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in F$

Finally, $0 \neq 1$ must hold (so the two ~~id~~ identifiers are distinct)

<u>Verification for $\mathbb{Q}$:</u>

Every rational number can be written as $\frac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{Z} \setminus \{0\}$.

1. $(\mathbb{Q}, +)$ is an abelian group.

• Closure under addition :

if $x = a/b$ and $y = c/d$ then

$$x+y = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

and $ad+bc$ and bd are integers with $bd \neq 0$

Thus $x+y \in \mathbb{Q}$

→ <u>Associativity :</u> addition of rationals is associative because it follows from associativity of integer addition.

for rationals x, y, z $(x+y)+z = x+(y+z)$.

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→ Additive Identity : 0 satisfies $x+0=x$ for every rational x.

→ Additive inverse : for $x = a/b$, the additive inverse is $-x = \frac{-a}{b}$, which is rational and satisfies $x+(-x)=0$

→ Commutativity :

$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$$

Hence, $(\mathbb{Q}, +)$ is an abelian group.

2. $(\mathbb{Q} \setminus \{0\}$ is an abelian group

• Closure under multiplication :

with $x = a/b$, $y = c/d$

$$x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

and ac, bd, are integers with $bd \neq 0$, So the product is in $\mathbb{Q}$. If neither x nor y is zero then $ac \neq 0$. so the product is non zero.

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- **Associativity:** Multiplication of rationals is associative.
- **Multiplicative identity:** 1 satisfies $1 \cdot x = x$ for all $x \in \mathbb{Q}$
- **Multiplicative inverse:** For a non-zero rationale $x = \frac{a}{b}$ with $a \neq 0$, the inverse is b/a (an element of $\mathbb{Q}$) and $\frac{a}{b} \cdot \frac{b}{a} = 1$
- **Commutativity:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$ because integer multiplication is commutative. Thus $(\mathbb{Q} \setminus \{0\}, \cdot)$ is an abelian group.

3. **Distributivity:** For rationals $x = \frac{a}{b}$, $y = \frac{c}{d}$, $z = \frac{e}{f}$

$$x \cdot (y + z) = \frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{cf + de}{df} = \frac{acf}{bdf} + \frac{ade}{bdf}$$

$$= \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$$

$$= x \cdot y + x \cdot z$$

9. $0 \neq 1$

In $\mathbb{Q}$, 0 is $0/1$ and 1 is $1/1$. These are different rationals, so $0 \neq 1$. This prevents the degenerate one-element ring.

All field axioms hold for $\mathbb{Q}$: $(\mathbb{Q}, +)$ is a abelian group, $(\mathbb{Q} \setminus \{0\}, \cdot)$ is an abelian group multiplication distributes over addition, and $0 \neq 1$. Therefore $\mathbb{Q}$ with usual addition and multiplication is a field.