# INDEPENDENT READING: REPRESENTATION THEORY NOTES

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## 1. Introduction

This note is for a 2-semester-long reading project themed introduction to representation theory. The reference book is *Representation theory: a first course* [1].

It is not a complete note for the reading note but rather serve as a compliment to the reading process. I hope you could also enjoy the book along with my notes. :)

## 2. Chapter 10: Lie Algebras in Dimensions One, Two, and Three

### 2.1. Simply connected form.

In §10.3 (p.141), it mentions that *the simply-connected forms do form a family.*

**Definition 2.1** (Simply connected form $G_{sc}$)**.** Given a Lie algebra $\mathfrak{g}$, there exists a unique connected simply connected Lie group $G_{sc}$ with Lie algebra $\mathfrak{g}$, which is denoted as the *simply connected form.*

**Remark 2.2.** $G_{sc}$ is also the universal cover of any connected Lie group with Lie algebra $\mathfrak{g}$.

**Definition 2.3** (Universal covering group $\widetilde{G}$)**.** Let $G$ be a connected Lie group. A *universal covering group* of $G$ is a simply connected Lie group $\widetilde{G}$ together with a Lie group homomorphism that is a covering map

$$p : \widetilde{G} \to G. \tag{1}$$

The kernel of $p$ is discrete and central, and

$$\ker p \subset Z(\widetilde{G}), \ker p \cong \pi_1(G). \tag{2}$$

**Remark 2.4.** Their Lie algebra agrees: $\mathrm{Lie}(G) = \mathrm{Lie}(\widetilde{G}) = \mathfrak{g}$.

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**Lemma 2.5.** *For any connected Lie group $G$ with Lie algebra $\mathfrak{g}$, $G_{sc} = \widetilde{G}$.*

2.2. **Adjoint form.** Related concept of *simply connected form*.

**Definition 2.6.** Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. The adjoint representation is

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}).$$

Its kernel is the center $Z(G)$. The *adjoint form* of $G$ is

$$G_{\mathrm{ad}} := \mathrm{Ad}(G) \cong G/Z(G).$$

**Remark 2.7.** $Z(G_{ad}) = \{e\}$.

**Theorem 2.8.** *Let $\mathfrak{g}$ be a semisimple Lie algebra and $G_{\mathrm{sc}}$ the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then every connected Lie group $G$ with Lie algebra $\mathfrak{g}$ is of the form*

$$G \cong G_{\mathrm{sc}}/\Gamma, \qquad \Gamma \subset Z(G_{\mathrm{sc}}) \text{ discrete.}$$

*Moreover,*

$$G_{\mathrm{ad}} = G_{\mathrm{sc}}/Z(G_{\mathrm{sc}}).$$

**Theorem 2.9.** *Let $G$ be a connected complex reductive group. Then*

$$G \cong (G_{\mathrm{sc}} \times (\mathbb{C}^*)^r)/F,$$

*where $G_{\mathrm{sc}}$ is the simply connected semisimple group with the same semisimple Lie algebra as $G$, $(\mathbb{C}^*)^r$ is a central torus, and $F \subset Z(G_{\mathrm{sc}} \times (\mathbb{C}^*)^r)$ is finite.*

Here are some diagrams illustrating the relation between any connected Lie group $G$, its simply connected form $G_{sc}$ and adjoint form $G_{ad} := G/Z(G)$:

$$\begin{tikzcd}
G_{\mathrm{sc}} \arrow[d, two heads] \arrow[dd, bend right, two heads, swap, "\text{universal cover}"] \\
G \arrow[d, two heads] \\
G_{ad}
\end{tikzcd}
\qquad
\begin{tikzcd}
G_{\mathrm{sc}} \arrow[d] \arrow[dd, bend right] \\
G \times (\mathbb{C}^\times)^r \arrow[d, two heads] \\
G_{ad}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{SL}_n(\mathbb{C}), \pi_1 = 1 \arrow[d, two heads] \arrow[dd, bend right] \\
\mathrm{GL}_n(\mathbb{C}) \arrow[d, two heads] \\
\mathrm{PGL}_n(\mathbb{C}), \pi_1 = \mu_n
\end{tikzcd}$$

**Example 2.10.** The group $\mathrm{SL}_n(\mathbb{C})$ is simply connected and has center

$$Z(\mathrm{SL}_n(\mathbb{C})) = \mu_n = \{\zeta I : \zeta^n = 1\}.$$

Since

$$\mathrm{PGL}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C})/\mu_n,$$

the quotient map is a covering with kernel $\mu_n$. Hence,

$$\pi_1(\mathrm{PGL}_n(\mathbb{C})) \cong \mu_n.$$

2.3. **Why rank(ad($X$)) must be 2?** In §10.4 (p.141), it mentions that *for any nonzero* $X \in \mathfrak{g}$*, the rank of* $\mathrm{ad}(X)$ *must be 2.*

**Proposition 2.11.** *$\mathfrak{g}$ is a Lie algebra with dimension three, rank 3. For any nonzero $X \in \mathfrak{g}$, the rank of* $\mathrm{ad}(X)$ *must be 2.*

*Proof.* First of all, since $[X, \mathbb{C}X] = 0$, then $\mathbb{C}X \subset \mathrm{Ker}(\mathrm{ad}(X))$. Suppose for contradiction that $\mathrm{ad}(X)$ also kills some $Y$ not in $\mathrm{span}\{X\}$. Since $\mathfrak{g}$ is dimension 3, we can find $Z$ to complete the basis for $\mathfrak{g}$. However, $\{[X,Y] = 0, [X,Z], [Y,Z]\}$ forms a basis for $\mathscr{D}\mathfrak{g}$, meaning that $3 = \mathrm{rank}\mathfrak{g} = \dim\mathscr{D}\mathfrak{g} \le 2$. Oops! □

2.4. **Simply connected form for $\mathfrak{sl}_2\mathbb{C}$ is $\mathrm{SL}_2\mathbb{C}$.** In §10.4 (p.142), it mentions that *the map*

$$\mathrm{SL}_2\mathbb{C} \to \mathbb{C}^2 - \{(0,0)\}$$

*sending a matrix to its first row expresses the topological space* $\mathrm{SL}_2\mathbb{C}$ *as a bundle with fiber* $\mathbb{C}$ *over* $\mathbb{C}^2 - \{(0,0)\}$.

**Remark 2.12.**

$$\mathrm{SL}_2\mathbb{C}/\left\{\begin{pmatrix} 1 & \\ \mathbb{C} & 1 \end{pmatrix}\right\} \cong \mathbb{C}^2 - \{(0,0)\}$$

**Proposition 2.13.** *$\mathrm{SL}_n\mathbb{C}$ is simply connected(, then it must be the unique simply connected form for $\mathfrak{sl}_2\mathbb{C}$).*

*Proof.* (cite from Wiki)

It follows that the topology of the group $\mathrm{SL}(n,\mathbb{C})$ is the product of the topology of $\mathrm{SU}(n)$ and the topology of the group of Hermitian matrices of unit determinant with positive eigenvalues. A Hermitian matrix of unit determinant and having positive eigenvalues can be uniquely expressed as the exponential of a traceless Hermitian matrix, and therefore the topology of this is that of $(n^2-1)$-dimensional Euclidean space. Since $\mathrm{SU}(n)$ is simply connected, then $\mathrm{SL}(n,\mathbb{C})$ is also simply connected, for all $n \ge 2$ (Section 2.5, Proposition 13.11 in [2]). □

2.5. **Real projective line $\mathbb{P}^1\mathbb{R}$.** In §10.4 (p.143), *real projective line* $\mathbb{P}^1\mathbb{R}$ is mentioned.

$$\mathbb{P}^1\mathbb{R} = (\mathbb{R}^2 \setminus \{0\})/\sim, \quad (x, y) \sim (\lambda x, \lambda y),\ \lambda \in \mathbb{R}^\times.$$
$$\mathbb{P}^1\mathbb{R} \cong S^1/\{\pm 1\}.$$
$$\mathbb{P}^1\mathbb{R} \cong \mathbb{R} \cup \{\infty\}.$$

## 3. Chapter 11: Representation of $\mathfrak{sl}_2\mathbb{C}$

## 4. Acknowledgment

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## References

[1] William Fulton, Joe Harris, and SpringerLink (Online service). *Representation theory: a first course*, volume 129. New York, NY: Springer-Verlag, 1 edition, 1999.

[2] Brian C Hall. *Lie groups, Lie algebras, and representations: An elementary introduction*, volume 222 of *Graduate Texts in Mathematics*. Springer, 2 edition, 2015.