Vv214 Linear Algebra

First Midterm Exam - Review class

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Linear Equation

Definition

In mathematics, a **linear equation** is an equation that may be put in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + b = 0,$$

where $x_1, \cdots, x_n$ are the variables (or unknowns or indeterminates), and $b, a_1, \cdots, a_n$ are the coefficients, which are often real numbers.

System of linear equations

Definition

In mathematics, a **system of linear equations** has the form

$$\left| \begin{array}{r} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = b_2 \\ \vdots \quad \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n \end{array} \right|$$

where here a_{ij} , b_i are coefficients and x_i are unknowns.

Matrix

We can write a system of linear equations in to a matrix form.

Coefficient Matrix

$$\begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1m}x_m \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2m}x_m \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nm}x_m \end{bmatrix} \in \mathbb{R}^{n\times m}$$

Augmented Matrix

$$\begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1m}x_m & b_1 \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2m}x_m & b_2 \\ \vdots & \vdots & \ddots & \vdots & \ldots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nm}x_m & b_n \end{bmatrix} \in \mathbb{R}^{n\times (m+1)}$$

Vector and Vector Space

Definition

A matrix with only one column is called a **column vector**, or simply a **vector**. A matrix with only one row is called a **row vector**. The entries of a vector are called its components. The set of all column vectors with n components is denoted by $\mathbb{R}^n$; we will refer to $\mathbb{R}^n$ as a **vector space**.

Linear combination of vectors

A vector b in $\mathbb{R}^n$ is called a **linear combination** of the vectors $v_1, \cdots, v_n$ in $\mathbb{R}^n$, if there exist scalars $x_1, \cdots, x_m$ such that

$$b = x_1 v_1 + \cdots + x_n v_m = 0.$$

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Span

Definition

Consider the vectors $v_1, \cdots, v_m$ in $\mathbb{R}^n$. The set of all linear combinations of the vectors $v_1, \cdots, v_m$ is called their **span**:

$$span(v_1, \cdots, v_m) = \{c_1 v_1 + \cdots + c_m v_m : c_1, \cdots, c_m \in \mathbb{R}\}.$$

Linear Independence

Definition

Consider vectors $v_1, \cdots, v_m$ in $\mathbb{R}^n$.

- We say that a vector v_i in the list $v_1, \cdots, v_m$ is **redundant** if v_i is a linear combination of the preceding vectors $v_1, \cdots, v_{i-1}$.
- The vectors $v_1, \cdots, v_m$ are called **linearly independent** if none of them is redundant. Otherwise, the vectors are called **linearly dependent** (meaning that at least one of them is redundant).

Remark

The vectors $v_1, \cdots, v_m$ are linearly independent if and only if

$$c_1 v_1 + \cdots + c_m v_m = 0 \qquad \Rightarrow \qquad c_1 = \cdots = c_m = 0.$$

Subspace of $\mathbb{R}^n$

Definition

A subset W of the vector space $\mathbb{R}^n$ is called a **(linear) subspace** of $\mathbb{R}^n$ if it has the following three properties:

1. W contains the zero vector in $\mathbb{R}^n$.
2. W is closed under addition: If w_1 and w_2 are both in W, then so is $w_1 + w_2$.
3. W is closed under scalar multiplication: If w is in W and k is an arbitrary scalar, then kw is in W.

Basis

Definition

We say that the vectors $v_1, \cdots, v_m$ form a **basis** of a subspace V of $\mathbb{R}^n$ if they span V **and** are linearly independent. (Also, it is required that vectors $v_1, \cdots, v_m$ be in V.)

Unique representation

Every vector v in V can be expressed **uniquely** as a linear combination of basis,

$$v = c_1 v_1 + \cdots + c_m v_m.$$

Dimension

Consider a subspace V of $\mathbb{R}^n$. The number of vectors in a basis of V is called the **dimension** of V, denoted by $\dim(V)$,

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Reduced Row-Echelon Form

A matrix is in **reduced row-echelon form(rref)** if it satisfies all of the following conditions:

- If a row has nonzero entries, then the first nonzero entry is a 1, called the **leading 1** (or **pivot**) in this row.
- If a column contains a leading 1, then all the other entries in that column are 0.
- If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

Definition

The **rank** of a matrix A is the number of leading 1's in rref(A).

Elementary Row Operation

Types of elementary row operations

- Divide a row by a nonzero scalar.
- Subtract a multiple of a row from another row.
- Swap two rows.

Remarks

- The elementary row operations will NOT change the rank of a matrix, and will NOT change the solution of a system of linear equations.
- Rank(A)= Max number of independent row vectors of A = Max number of independent column vectors of A.

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Linear Operation

Definition

A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a **linear transformation** if there exists an $n \times m$ matrix A such that

$$T(x) = Ax$$

for all x in the vector space $\mathbb{R}^m$.

Properties

If A is an $n \times m$ matrix; x and y are vectors in $\mathbb{R}^m$ and k is a scalar, then

1. $A(x + y) = Ax + Ay$, and
2. $A(kx) = k(Ax)$.

Column Vectors of a Matrix

Remarks

- The i^{th} column vector of the **identical matrix** in $\mathbb{R}^n$ is called **the i^{th} vector of the elementary basis**, denoted by e_i.
- For example, e_2 in $\mathbb{R}^4$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.
- A matrix is a linear transformation which maps e_i to the i^{th} column vector of the matrix. Thinking: Why we only need to map basis?

Inverse of Matrices

Definition

A $n \times n$ matrix A is **invertible** if and only if

- $rref(A) = I_n$ or
- $rank(A) = n$ or
- $det(A) \neq 0$.

Inverse

A matrix A^{-1} is the **inverse** of A if $AA^{-1} = A^{-1}A = I$.

Theorem

$(AB)^{-1} = B^{-1}A^{-1}$.

Find the Inverse

Gauss-Jordan method

$$\begin{bmatrix} A & I \end{bmatrix} \xrightarrow{\text{row elimination}} \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

Adjugate matrix method

$A^* := (cof\ A)^T$, the transpose of the cofactor matrix of A is called an **adjugate matrix** of A. Then

$$A^{-1} = \frac{1}{det(A)} A^*.$$

Geometric Meaning

Orthogonal Projection Matrix

- $A^2 = A$.
- Column vectors are on a line.

Reflection Matrix

- $A^2 = I$, where I is the identity.
- $A = A^{-1}$.
- The eigenvalues of A equal ± 1.

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Kernel and Image

Image

The **image** of a function (not necessarily linear) consists of all the values the function takes in its target space. If f is a function from X to Y ,then

$$image(f) = \{f(x) : x \in X\}.$$

Kernel

The **kernel** of a linear transformation (matrix) A from $\mathbb{R}^m$ to $\mathbb{R}^n$ consists of all zeros of the transformation, that is, the solutions of the equation $Ax = 0$.

Kernel and Image

Remarks

- The image of a linear transformation A is the span of the column vectors of A.
- If A is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then $ker(A)$ is a subspace of $\mathbb{R}^m$ and $image(A)$ is a subspace of $\mathbb{R}^n$.

Dimension Formula

Remarks

- $dim(image(A)) = rank(A)$.

Rank-Nullity Theorem

For any $n \times m$ matrix A, or equivalently a linear transform A from $\mathbb{R}^m$ to $\mathbb{R}^n$, we always have

$$dim(ker(A)) + dim(image(A)) = m.$$

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Summary

Go over

- The textbook,
- Homework 1-3,
- Slides and exercises on recitation classes.