Vv214 Linear Algebra

Second Midterm Exam - Review class

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Table of Contents

Table of Contents

Linear Space

Definition

A **linear space** V is a set endowed with a rule for (closed) addition and a rule for (closed) scalar multiplication such that these operations satisfy the following eight rules (for all f, g, h in V and all c, k in $\mathbb{R}$):

1. $(f+g)+h = f+(g+h)$.
2. $f+g = g+f$.
3. There exists a neutral element n in V such that $f+n = f$, for all f in V. This n is unique and denoted by 0.
4. For each f in V there exists a g in V such that $f+g=0$. This g is unique and denoted by f.
5. $k(f+g) = kf + kg$.
6. $(c+k)f = cf + kf$.
7. $c(kf) = (ck)f$.
8. $1f = f$.

Inner Product Space

Definition

An **inner product** in a linear space V is a rule that assigns a real scalar (denoted by $\langle f, g \rangle$) to any pair f, g of elements of V, such that the following properties hold for all f, g, h in V, and all c in $\mathbb{R}$:

1. $\langle f, g \rangle = \langle g, f \rangle$.
2. $\langle f + h, g \rangle = \langle h, g \rangle + \langle f, g \rangle$.
3. $\langle cf, g \rangle = c \langle f, g \rangle$.
4. $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0$ if and only if $f = 0$.

A linear space endowed with an inner product is called an **inner product space**.

Norm, Orthogonality, Distance

Norm

$$\|f\| = \sqrt{\langle f, f\rangle}.$$

Orthogonality

$$\langle f, g\rangle = 0.$$

Distance

$$dist(f, g) = \|f - g\|.$$

Table of Contents

Orthonormal Vectors

Definition

the vectors $u_1, \cdots, u_m$ in $\mathbb{R}^n$ are called **orthonormal** if they are unit vectors and orthogonal to one another:

$$u_i \cdot u_j = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}.$$

Remarks

1. Orthonormal vectors are linearly independent.
2. Orthonormal vectors form a basis.

Orthogonal Complement

Orthogonal Complement

Consider a subspace V of $\mathbb{R}^n$.

$$V^\perp = \{x \in \mathbb{R}^n : \langle v, x \rangle = 0, \forall v \in V\}$$

Orthogonal Projection

Consider a vector x in $\mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$. Then we can write in a unique way

$$x = x^\parallel + x^\perp,$$

where $x^\parallel \in V$ and $x^\perp \in V^\perp$.

Orthogonal Complement

Remarks

1. $V \cap V^{\perp} = \{0\}$.
2. $dim(V) + dim(V^{\perp}) = dim(\mathbb{R}^n) = n$.
3. $(V^{\perp})^{\perp} = V$.

QR Factorization

The Gram-Schmidt process

Given linearly independent $v_1, \cdots, v_n$, find $u_1, \cdots, u_n$, such that

- $span\{v_1, \cdots, v_n\} = span\{u_1, \cdots, u_n\}$;
- $u_1, \cdots, u_n$ is an orthonormal basis.

QR Factorization

Consider an $n \times m$ matrix M with linearly independent columns. Then there exists an $n \times m$ matrix Q whose columns are orthonormal and an upper triangular matrix R with positive diagonal entries such that

$$M = QR.$$

This representation is unique.

Table of Contents

Transpose

Properties

- $(AB)^T = B^T A^T$;
- $(A^T)^{-1} = (A^{-1})^T$;
- $rank(A) = rank(A^T)$.

Theorem

$$(imA)^{\perp} = ker(A^T).$$

Least-squares Solution

The normal equation

The least-squares solutions of the system $Ax = b$ are the exact solution of the system

$$A^T Ax = A^T b.$$

If A is invertible x is given by

$$x = (A^T A)^{-1} A^T b$$

Table of Contents

Isomorphisms

Definition

An invertible linear transformation T is called an **isomorphism**.

Properties

- T is an isomorphism from V to W $\Leftrightarrow$ $ker(T) = \{0\}$ & $im(T) = W$
- T is an isomorphism from V to W $\Rightarrow$ $dim(V) = dim(W)$.
- $dim(V) = dim(W)$ & $im(T) = W$ $\Rightarrow$ T is an isomorphism from V to W.
- $dim(V) = dim(W)$ & $ker(T) = \{0\}$ $\Rightarrow$ T is an isomorphism from V to W.

Coordinates

Definition

If $\mathcal{B} = (v_1, ..., v_m)$ is a basis of a subspace V in $\mathbb{R}^n$, and $x \in V$, then $x = c_1 v_1 + \cdots + c_m v_m$ and $\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$ is called the $\mathcal{B}$*-coordinate vector* of x, denoted $[x]_{\mathcal{B}}$.

$\mathcal{B}$-matrix of a linear transformation

Matrix of Transformation

For the basis $v_1, \dots, v_m$

$$x = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} [x]_{\mathcal{B}},$$

or

$$x = S[x]_{\mathcal{B}}.$$

Obtain $[x]_{\mathcal{B}}$ from x

If $m = n$, we can find $[x]_{\mathcal{B}}$ for arbitrary x,

$$[x]_{\mathcal{B}} = S^{-1}x.$$

The Matrix of a Linear Transformation

Definition

Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{B}$ is a basis of $\mathbb{R}^n$. Then the $\mathcal{B}$-matrix of T transforms $[x]_{\mathcal{B}}$ to $[Tx]_{\mathcal{B}}$,

$$[Tx]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

If $\mathcal{B} = (v_1, ..., v_n)$,

$$\begin{aligned} B &= \begin{bmatrix} [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \end{bmatrix} \\ &= \begin{bmatrix} S^{-1}T(v_1) & \cdots & S^{-1}T(v_n) \end{bmatrix} \\ &= S^{-1}T \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = S^{-1}TS \end{aligned}$$

Table of Contents

Determinant

Properties

- Division: dividing a row/column by a non zero scaler k, $det(A') = det(A)/k$.
- Swap: swapping two rows/columns, $det(A') = -det(A)$.
- Addition: adding a multiple of a row/column to another row/column, $det(A') = det(A)$.

Properties

- $det(AB) = det(A)det(B)$.
- $det(A^{-1}) = 1/det(A)$.

Table of Contents

Summary

Go over

- The textbook,
- Homework 4-6,
- Slides and exercises on recitation classes.