# Third Recitation Class Linear Algebra

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### Linear Combination of Vectors

For a tuple of vectors  $(v_1, ..., v_m)$ ,

$$\lambda_1 v_1 + \cdots + \lambda_m v_m$$

with  $\lambda_1, ..., \lambda_m \in \mathbb{F}$ , is called *linear combination*.

## Span

The set

$$\{v = \lambda_1 v_1 + \cdots + \lambda_m v_m : \lambda_1, ..., \lambda_m \in \mathbb{F}\}$$

is called the **span** of  $(v_1, ..., v_m)$ .

## Independence of Vectors

$$\lambda_1 v_1 + \cdots + \lambda_m v_m = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_m$$



### Subspace

For a linear space V, a subset  $U \subseteq V$  is called a *subspace* if

$$\alpha_1 u_1 + \alpha_2 u_2 \in U$$

for  $u_1, u_2 \in U$  and  $\alpha_1, \alpha_2 \in \mathbb{F}$ . In other word, this set is **closed** under linear combination.

## Image and Kernel

If T is a linear transformation from V to U

- ightharpoonup im T is subspace in U,
- ▶ ker T is subspace in V.

### **Basis**

For a linear space V, a set  $\mathcal{B} = \{v_1,...,v_m\}$  is called **basis** if there is **unique**  $\lambda_1,...,\lambda_m$ 

$$v = \lambda_1 v_1 + \dots + \lambda_m v_m$$

for any  $v \in V$ .

# Basis for a Linear Space

In V,  $\{v_1, ..., v_m\}$  is a basis iff

- 1.  $\{v_1, ..., v_m\}$  is independent,
- 2. span $\{v_1, ..., v_m\} = V$ .

#### Dimension

True or false For a linear space V, a set of vectors in  $\{v_1, ..., v_m\}$ .

- ▶ If this set is a basis, then dim V = m.
- ▶ If  $\{v_1, ..., v_m\}$  is independent, then dim  $V \ge m$ .
- ▶ If dim V = m and  $\{v_1, ..., v_m\}$  is independent, then  $\{v_1, ..., v_m\}$  is a basis.

Consider the matrix  $A = [v_1 \ v_2 \ \cdots \ v_m]$ .

- ▶ If rankA = m, then  $\{v_1, ..., v_m\}$  are independent,
- ▶ If rankA < m, then  $\{v_1, ..., v_m\}$  are dependent.

### Coordinates

### Definition

If  $\mathcal{B} = (v_1,...,v_m)$  is a basis of a subspace V in  $\mathbb{R}^n$ , and  $x \in V$ , then

$$x = c_1 v_1 + \cdots + c_m v_m$$

and

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

is called the  $\mathcal{B}$ -coordinate vector of x, denoted  $[x]_{\mathcal{B}}$ .

#### Note:

Be careful  $m \le n$  so we may not find  $[x]_{\mathcal{B}}$  for arbitrary x.

#### Coordinates

 $\mathcal{B}$ -matrix of a linear transformation

The Matrix of a Linear Transformation Linear Transformations not in  $\mathbb{R}^n$ 

## $\mathcal{B}$ -matrix of a linear transformation

### Matrix of Transformation

For the basis  $v_1, ..., v_m$ 

$$x = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} [x]_{\mathcal{B}},$$
  
 $x = S[x]_{\mathcal{B}}.$ 

or

Obtain 
$$[x]_{\mathcal{B}}$$
 from  $x$ 

If m = n, we can find  $[x]_{\mathcal{B}}$  for arbitrary x,

$$[x]_{\mathcal{B}} = S^{-1}x.$$

### Comment:

What about m < n?

## $\mathcal{B}$ -matrix of a linear transformation

## Example

Find  $[x]_{\mathcal{B}}$  for x with vectors  $v_1, v_2, v_3$  in  $\mathbb{R}^3$ ,

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

### Solution

The inverse of the matrix is

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

#### Coordinates

B-matrix of a linear transformation

The Matrix of a Linear Transformation

Linear Transformations not in  $\mathbb{R}^n$ 

### Definition

Consider  $T: \mathbb{R}^n \to \mathbb{R}^n$  and  $\mathcal{B}$  is a basis of  $\mathbb{R}^n$ . Then the  $\mathcal{B}$ -matrix of T transforms  $[x]_{\mathcal{B}}$  to  $[Tx]_{\mathcal{B}}$ ,

$$[Tx]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

If 
$$\mathcal{B} = (v_1, ..., v_n)$$
,

$$B = \begin{bmatrix} [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \end{bmatrix}$$

### Example

For vector v, the cross product is a linear map  $v \times (\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ , for basis  $v_1, v_2, v_3$ , find B,

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

### Solution

$$S^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} T(v_1) & \cdots & T(v_n) \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & -3 & -1 \end{bmatrix}.$$

#### Theorem

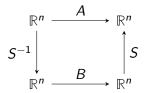
If the matrix of  $\mathcal{B}$  is S and the matrix of T is A, then

$$B=S^{-1}AS.$$

### Proof.

For  $x \in \mathbb{R}^n$ ,

$$S^{-1}AS[x]_{\mathcal{B}} = S^{-1}Ax = S^{-1}Tx = [Tx]_{\mathcal{B}}.$$



#### Note:

The inverse form of the theorem will be more.

$$A = SBS^{-1}$$

The meaning of this equation is that: A linear transformation can be expressed by its effect on any basis.

## Example

Find the matrix of reflection with respect to line y = 2x in  $\mathbb{R}^2$ .

### Solution

$$S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, S = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

## Similar Matrix

### Definition

Consider two  $n \times n$  matrices A, B, they are similar if there exists an invertible matrix S such that

$$AS = SB$$
, or  $B = S^{-1}AS$ .

## Example

Is A similar to B?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

### Comment:

Very important algebraic structure!

#### Coordinates

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### Linear Transformations not in $\mathbb{R}^n$

#### **Theorem**

For a linear space V, if dim V = n, then there exists a **bijective linear transformation** from V to  $\mathbb{R}^n$ . This map is called **isomorphisim**.

## Example

For  $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ , we know that  $\dim \mathcal{P}_2 = 3$ , there is a an isomorphism  $\varphi : \mathcal{P}_2 \to \mathbb{R}^3$  such that

$$\varphi(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \varphi(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \varphi(x^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

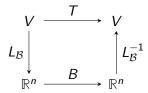
## $\mathcal{B}$ -matrix of a linear transformation

#### Definition

Consider a linear transformation  $T:V\to V$  and dim V=n. Then for a basis  $\mathcal B$  of V, we define an isomorphism  $L_B$  that maps  $\mathcal B$  to standard basis. The matrix B is called  $\mathcal B$ -matrix of transformation T if

$$[T(f)]_{\mathcal{B}} = B[f]_{\mathcal{B}}, \text{ for all } f \in V.$$

Here  $[f]_{\mathcal{B}} = L_{\mathcal{B}}(f) \in \mathbb{R}^n$ .



## **B**-matrix of Linear Transformation

### **Theorem**

The columns of B are

$$B = [[T(v_1)]_{\mathcal{B}} \cdots [T(v_n)]_{\mathcal{B}}].$$

#### **Theorem**

We can write B by  $L_{\mathcal{B}}$  and T,

$$B=L_{\mathcal{B}}\circ T\circ L_{\mathcal{B}}^{-1}.$$

### Example

Find the  $\mathcal{B}$ -matrix of  $\frac{d}{dx}: \mathcal{P}_2 \to \mathcal{P}_2$  with  $\mathcal{B} = (1, x, x^2)$ .

## **B**-matrix of Linear Transformation

## Example

Let V be the space of all upper triangle  $2 \times 2$  matrices. Consider the linear transformation

$$T\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = aI_2 + bP + cP^2$$

from V to V, where  $P=\begin{bmatrix}1&2\\0&3\end{bmatrix}$ . Find the matrix A of T with respect to the basis

$$\mathcal{B} = ( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} ).$$

# About Infinite Dimensional Linear Space

### **Linear Combination**

Only linear combination of *finite* number of elements is meaningful.

### Basis

There is no *countable* basis.