# Fifth Recitation Class
# Linear Algebra

YAO Shaoxiong

UM-SJTU Joint Institute

April 16, 2019

# Table of contents

# Eigenvalue Problem

## Motivation

For discrete time dynamic system, if the state between two states have relation

$$\overline{x_{n+1}} = A\overline{x_n}$$

for $n \in \mathbb{N}$. What is $\overline{x_n}$ if we know $\overline{x_0}$?

## Solution

If $A\overline{x_0} = \lambda\overline{x_0}$ for $\overline{x_0} \neq \overline{0}$, then

$$\overline{x_n} = \lambda^n\overline{x_0}.$$

## Question:

Is this always possible?

# Eigenvalues and Eigenvectors

## Definition

For $\overline{v} \neq \overline{0}$ and an $n \times n$ matrix $A$, if there exists $\lambda \in \mathbb{F}$ such that

$$A\overline{v} = \lambda\overline{v}$$

we call $\lambda$ an ***eigenvalue*** and $\overline{v}$ an ***eigenvector***.

## Question

We assume $\overline{v}$ is an eigenvector for both matrix $A$ and $B$.

(i) Can we have $\lambda = 0$? ***Yes.*** We only need $\overline{v} \neq \overline{0}$.

(ii) Is $\overline{v}$ an eigenvector for $A^2, A + B, AB$? ***Yes.***

(iii) If $\overline{u}$ is an eigenvector of $AB$ and $B\overline{u} \neq \overline{0}$, find an eigenvector for $BA$.

# Find Eigenvalues and Eigenvectors

**Theorem**

A scalar $\lambda \in \mathbb{R}$ is an eigenvalue of $A$ if and only if

$$\det(A - \lambda I_n) = 0.$$

We call $f_A(\lambda) = \det(A - \lambda I_n)$ the ***characteristic equation*** of the matrix $A$.

**Comment:**

If we consider $\lambda$ as a variable, $f_A(\lambda)$ is a polynomial

$$f_A(\lambda) = (-\lambda)^n + (\mathrm{tr}A)(-\lambda)^{n-1} + \cdots + \det A.$$

Note that $f_A(\lambda)$ always has $n$ roots but may be **repeated** or **complex** value.

# Find Eigenvalues and Eigenvectors

We find eigenvalues for some typical matrices.

### Example

Find eigenvalues for

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

### Example

If we have $q \in \mathbb{R}^n$, find an eigenvalue for

$$A = qq^T.$$

# Find Eigenvalues and Eigenvectors

## Quesiton:

(i) If $P$ is projection matrix, what are eigenvalues of $P$?

(ii) If $Q$ is an orthogonal matrix, what are eigenvalues of $Q$?

## Example

Find a $2 \times 2$ matrix $A$ such that

$$\overline{x}(t) = \begin{bmatrix} 2^t - 6^t \\ 2^t + 6^t \end{bmatrix}$$

is a trajectory of the dynamical system

$$\overline{x}(t+1) = A\overline{x}(t).$$

# Eigenvalues and Eigenvectors of a Matrix

### Theorem

For an $n \times n$ matrix $A$, the eigenvectors of different eigenvalues are **independent**.

### Proof

We use induction here. We denote eigenvalues $\lambda_1, ..., \lambda_n$ and eigenvectors $v_1, ..., v_n$.

(i) For $n = 1$, the statement is true.

(ii) If we have $v_1, ..., v_n$ are independent, then for

$$\mu_1 v_1 + \cdots + \mu_n v_n + \mu_{n+1} v_{n+1} = 0 \quad (\star)$$

we must have $\mu_{n+1} \neq 0$.

# Eigenvalues and Eigenvectors of a Matrix

## Proof(Contd.)

(ii) We times $(\star)$ by $A$ and we have

$$\begin{aligned} & \mu_1 A v_1 + \cdots + \mu_n A v_n + \mu_{n+1} A v_{n+1} = 0 \\ \Rightarrow\ & \mu_1 \lambda_1 v_1 + \cdots + \mu_n \lambda_n v_n + \mu_{n+1} \lambda_{n+1} v_{n+1} = 0. \end{aligned}$$

We subtract $(\star) \times \lambda_{n+1}$

$$\mu_1(\lambda_1 - \lambda_{n+1}) v_1 + \cdots + \mu_n(\lambda_n - \lambda_{n+1}) v_n = 0.$$

Since $\lambda_1, ..., \lambda_{n+1}$ are distinct, we have $\mu_1 = \cdots = \mu_n = 0$. We then also have $\mu_{n+1} = 0$ and have $v_1, ..., v_{n+1}$ are independent.

(iiii) By induction $v_1, ..., v_n$ are independent if we have $\lambda_1, ..., \lambda_n$ are distinct.

# Eigenvalues and Eigenvectors of a Matrix

## Exmaple

If $A$ is a symmetric $n \times n$ matrix, show the following results.

(i) If $\overline{v}$ and $\overline{w}$ are two vectors in $\mathbb{R}^n$, then

$$(A\overline{v}, \overline{w}) = (\overline{v}, A\overline{w}).$$

(ii) If $\overline{v}$ and $\overline{w}$ are two eigenvectors of $A$, with distinct eigenvalues, then $\overline{w}$ is orthogonal to $\overline{v}$.

## Comment:

The eigenvectors for distinct eigenvalues of a symmetric matrix are not only independent but also **orthogonal**.

# Algebraic Multiplicity

## Definition

An eigenvalue $\lambda_0$ of a square matrix $A$ has ***algebraic multiplicity*** $k$ if $\lambda_0$ appears exactly $k$ times in the roots of $f_A(\lambda)$, i.e.

$$f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda),\ g(\lambda_0) \neq 0.$$

## Example

Find eigenvalues and their algebraic multiplicity of

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

# Eigenspace and Geometric Multiplicity

Definition

The kernel of the matrix $A - \lambda I_n$ is called the ***eigenspace*** $E_\lambda$ associated with $\lambda$

$$E_\lambda = \ker(A - \lambda I_n) = \left\{ v \in \mathbb{R}^n : A\overline{v} = \lambda \overline{v},\ \overline{v} \neq \overline{0} \right\}.$$

Comment:

If $A$ is an $n \times n$ matrix and $\lambda_1, ..., \lambda_n$ are distinct, then

$$E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n} = \mathbb{R}^n.$$

# Eigenspace and Geometric Multiplicity

## Definition

The dimension of eigenspace $E_\lambda$ is called the ***geometric multiplicity*** of the eigenvalue $\lambda$.

$$G.M. = \dim\ker(A - \lambda I_n) = n - \operatorname{rank}(A - \lambda I_n).$$

## Comment:

From previous result, if $\lambda_1, ..., \lambda_n$ are distinct, then

$$\dim E_{\lambda_1} + \cdots + \dim E_{\lambda_n} = n.$$

# Eigenspace and Geometric Multiplicity

## Example

For a rotation $T(\overline{x}) = A\overline{x}$ in $\mathbb{R}^3$. (That is, $A$ is an orthogonal matrix and has determinant equal to 1.) Show that $T$ has a nonzero fixed point [i.e., a vector $\overline{v}$ with $T(\overline{v}) = v$]. This result is known as *Euler's theorem*.

## Solution

We note that

$$\det(A - I) = \det(A^T)\det(A - I)$$

because we have $\det(A) = 1$, then

$$\det(A^T A - A^T) = \det(I - A^T) = (-1)^3 \det(A - I).$$

So we have $\det(A - I) = 0$ and there is at least one nonzero eigenvector associated with eigenvalue 1.

# Eigenbasis

### Definition

We call a basis $\overline{v_1}, \ldots, \overline{v_n}$ of $\mathbb{R}^n$ an ***eigenbasis*** if they are eigenvectors of an $n \times n$ matrix $A$.

### Theorem

An $n \times n$ matrix $A$ has an eigenbasis iff

$$\sum_{i=1}^{s} \dim E_{\lambda_i} = n.$$

### Question:

Why we use eigenvectors as basis?

# Diagonalizable Matrices

## Theorem

Consider a linear transformation $T\overline{x} = A\overline{x}$ and $A$ is an $n \times n$ matrix. Let $\mathcal{D} = \{\overline{v_1}, ..., \overline{v_n}\}$ be an eigenbasis for $T : A\overline{v_i} = \lambda_i \overline{v_i}$. Then the $\mathcal{D}$-matrix $D$ of $T$ is

$$D = S^{-1}AS = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

## Comment:

If we find $[\overline{x}]_{\mathcal{D}}$, we have

$$\overline{x} = c_1\overline{v_1} + \cdots + c_n\overline{v_n},$$

then

$$A\overline{x} = c_1\lambda_1\overline{v_1} + \cdots + c_n\lambda_n\overline{v_n}.$$

# Diagonalizable Matrices

### Definition

An $n \times n$ matrix $A$ is called ***diagonalizable*** if $A$ is similar to a diagonal matrix $D$.

### Theorem

An $n \times n$ matrix is diagonalizable if and only if there exists an eigenbasis for $A$.

### Example

Find the eigenvalue decomposition of

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

# Diagonalizable Matrices

## Solution

We have

$$\det(A - \lambda I_4) = (-\lambda)^2(1-\lambda)^2,$$

so two eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 1.$$

Then for $\lambda_1 = 0$

$$\overline{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{v_2} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

# Diagonalizable Matrices

## Solution(Contd.)

For $\lambda_2 = 1$

$$\overline{v_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{v_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We have the diagonalized matrix

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Diagonalizable Matrices

### Question:

Why we want $A = SDS^{-1}$?

### Solution

We want to calculate $A^k$ for $k \in \mathbb{N} \backslash \{0\}$,

$$\begin{aligned} A^k &= SDS^{-1} \cdots SDS^{-1} \\ &= SD^k S^{-1}. \end{aligned}$$

### Comment:

Please note that eigenvalues may be complex. For $k \to \infty$,

(i) when will we have $A^k$ go to 0?

(ii) when will we have $A^k$ go to infinity?

(iii) when will we have $A^k$ diverge?

# Diagonalizable Matrices

## Example

What is the limit of

$$\begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}^k \begin{bmatrix} a \\ b \end{bmatrix}$$

when $k \to \infty$?

## Comment:

If each column add up to 1 and each entry is greater than 0, we will have convergent result. Why? Hint: Consider $A - I_n$.

# Similar Matrices

## Motivation

Recall that two matrices are similar if there exist an invertible matrix $S$ such that

$$A = SBS^{-1}.$$

We notice that

$$\det(A - \lambda I_n) = \det(SBS^{-1} - \lambda SS^{-1}) = \det(B - \lambda I_n),$$

so we have $f_A(\lambda) = f_B(\lambda)$.

## Theorem

Let $B = S^{-1}AS$, i.e. $A, B$ be similar matrices. Then they have same

(i) $f_A(\lambda) = f_B(\lambda)$, $AM$, $GM$ for same eigenvalue,

(ii) $\operatorname{rank} A = \operatorname{rank} B$, $\operatorname{nullity} A = \operatorname{nullity} B$,

(iii) $\det A = \det B$, $\operatorname{tr} A = \operatorname{tr} B$.