Third Recitation Class
Linear Algebra

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Linear Combination of Vectors

For a tuple of vectors $(v_1, ..., v_m)$,

$$\lambda_1 v_1 + \cdots + \lambda_m v_m$$

with $\lambda_1, ..., \lambda_m \in \mathbb{F}$, is called ***linear combination***.

Span

The set

$$\{v = \lambda_1 v_1 + \cdots + \lambda_m v_m : \lambda_1, ..., \lambda_m \in \mathbb{F}\}$$

is called the ***span*** of $(v_1, ..., v_m)$.

Independence of Vectors

$$\lambda_1 v_1 + \cdots + \lambda_m v_m = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_m$$

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Subspace

For a linear space V, a subset $U \subseteq V$ is called a ***subspace*** if

$$\alpha_1 u_1 + \alpha_2 u_2 \in U$$

for $u_1, u_2 \in U$ and $\alpha_1, \alpha_2 \in \mathbb{F}$.
In other word, this set is ***closed*** under linear combination.

Image and Kernel

If T is a linear transformation from V to U

- $\operatorname{im} T$ is subspace in U,
- $\ker T$ is subspace in V.

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Basis

For a linear space V, a set $\mathcal{B} = \{v_1, ..., v_m\}$ is called ***basis*** if there is ***unique*** $\lambda_1, ..., \lambda_m$

$$v = \lambda_1 v_1 + \cdots + \lambda_m v_m$$

for any $v \in V$.

Basis for a Linear Space

In V, $\{v_1, ..., v_m\}$ is a basis iff

1. $\{v_1, ..., v_m\}$ is independent,
2. $\text{span}\{v_1, ..., v_m\} = V$.

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Dimension

True or false For a linear space V, a set of vectors in $\{v_1, ..., v_m\}$.

- If this set is a basis, then $\dim V = m$.
- If $\{v_1, ..., v_m\}$ is independent, then $\dim V \geq m$.
- If $\dim V = m$ and $\{v_1, ..., v_m\}$ is independent, then $\{v_1, ..., v_m\}$ is a basis.

Consider the matrix $A = [v_1 \ v_2 \ \cdots \ v_m]$.

- If $\text{rank} A = m$, then $\{v_1, ..., v_m\}$ are independent,
- If $\text{rank} A < m$, then $\{v_1, ..., v_m\}$ are dependent.

Coordinates

Definition

If $\mathcal{B} = (v_1, ..., v_m)$ is a basis of a subspace V in $\mathbb{R}^n$, and $x \in V$, then

$$x = c_1 v_1 + \cdots + c_m v_m$$

and

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

is called the $\mathcal{B}$-*coordinate vector* of x, denoted $[x]_{\mathcal{B}}$.

Note:

Be careful $m \leq n$ so we may not find $[x]_{\mathcal{B}}$ for arbitrary x.

$\mathcal{B}$-matrix of a linear transformation

Matrix of Transformation

For the basis $v_1, ..., v_m$

$$x = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} [x]_{\mathcal{B}},$$

or

$$x = S[x]_{\mathcal{B}}.$$

Obtain $[x]_{\mathcal{B}}$ from x

If $m = n$, we can find $[x]_{\mathcal{B}}$ for arbitrary x,

$$[x]_{\mathcal{B}} = S^{-1}x.$$

Comment:

What about $m < n$?

$\mathcal{B}$-matrix of a linear transformation

Example

Find $[x]_{\mathcal{B}}$ for x with vectors v_1, v_2, v_3 in $\mathbb{R}^3$,

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solution

The inverse of the matrix is

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

The Matrix of a Linear Transformation

Definition

Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{B}$ is a basis of $\mathbb{R}^n$. Then the $\mathcal{B}$-matrix of T transforms $[x]_{\mathcal{B}}$ to $[Tx]_{\mathcal{B}}$,

$$[Tx]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

If $\mathcal{B} = (v_1, ..., v_n)$,

$$B = \begin{bmatrix} [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \end{bmatrix}$$

Example

For vector v, the cross product is a linear map $v \times (\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$, for basis v_1, v_2, v_3, find B,

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Solution

$$S^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} T(v_1) & \cdots & T(v_n) \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & -3 & -1 \end{bmatrix}.$$

The Matrix of a Linear Transformation

Theorem

If the matrix of $\mathcal{B}$ is S and the matrix of T is A, then

$$B = S^{-1}AS.$$

Proof.

For $x \in \mathbb{R}^n$,

$$S^{-1}AS[x]_{\mathcal{B}} = S^{-1}Ax = S^{-1}Tx = [Tx]_{\mathcal{B}}.$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "A"] \arrow[d, "S^{-1}"'] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[r, "B"] & \mathbb{R}^n \arrow[u, "S"']
\end{tikzcd}$$

The Matrix of a Linear Transformation

Note:

The inverse form of the theorem will be more.

$$A = SBS^{-1}$$

The meaning of this equation is that: *A linear transformation can be expressed by its effect on any basis.*

Example

Find the matrix of reflection with respect to line $y = 2x$ in $\mathbb{R}^2$.

Solution

$$S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, S = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Similar Matrix

Definition

Consider two $n \times n$ matrices A, B, they are similar if there exists an invertible matrix S such that

$$AS = SB, \quad \text{or} \quad B = S^{-1}AS.$$

Example

Is A similar to B?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Comment:

Very important algebraic structure!

Review before Mid 1

Coordinates

- $\mathcal{B}$-matrix of a linear transformation
- The Matrix of a Linear Transformation
- Linear Transformations not in $\mathbb{R}^n$

Linear Transformations not in $\mathbb{R}^n$

Theorem

For a linear space V, if $\dim V = n$, then there exists a ***bijective linear transformation*** from V to $\mathbb{R}^n$. This map is called ***isomorphisim***.

Example

For $\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$, we know that $\dim \mathcal{P}_2 = 3$, there is a an isomorphism $\varphi : \mathcal{P}_2 \to \mathbb{R}^3$ such that

$$\varphi(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \varphi(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \varphi(x^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$\mathcal{B}$-matrix of a linear transformation

Definition

Consider a linear transformation $T : V \to V$ and $\dim V = n$. Then for a basis $\mathcal{B}$ of V, we define an isomorphism L_B that maps $\mathcal{B}$ to standard basis. The matrix B is called $\mathcal{B}$-matrix of transformation T if

$$[T(f)]_{\mathcal{B}} = B[f]_{\mathcal{B}}, \text{ for all} f \in V.$$

Here $[f]_{\mathcal{B}} = L_{\mathcal{B}}(f) \in \mathbb{R}^n$.

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "L_{\mathcal{B}}"'] & V \\
\mathbb{R}^n \arrow[r, "B"] & \mathbb{R}^n \arrow[u, "L_{\mathcal{B}}^{-1}"']
\end{tikzcd}$$

$\mathcal{B}$-matrix of Linear Transformation

Theorem

The columns of B are

$$B = \begin{bmatrix} [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \end{bmatrix}.$$

Theorem

We can write B by $L_{\mathcal{B}}$ and T,

$$B = L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1}.$$

Example

Find the $\mathcal{B}$-matrix of $\frac{d}{dx} : \mathcal{P}_2 \to \mathcal{P}_2$ with $\mathcal{B} = (1, x, x^2)$.

$\mathcal{B}$-matrix of Linear Transformation

Example

Let V be the space of all upper triangle 2×2 matrices. Consider the linear transformation

$$T \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = aI_2 + bP + cP^2$$

from V to V, where $P = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Find the matrix A of T with respect to the basis

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

About Infinite Dimensional Linear Space

Linear Combination

Only linear combination of ***finite*** number of elements is meaningful.

Basis

There is no ***countable*** basis.