Vv214 Linear Algebra

Second Midterm Exam - Review class

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- Overview
- Eigenvalue & Diagonalization
- Discrete Dynamical System
- Cayley-Hamilton theorem
- Singular value decomposition

Summary

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Summary

Overview

Covered in the Final Exam

1. Materials before Mid-2.
2. Eigenvalue & Diagonalization.
3. Spectral theorem & Quadratic form.
4. Discrete Dynamical Systems.
5. Cayley-Hamilton theorem.
6. Singular value decomposition.

Eigenvalue

Comments

- Eigenvalues are zeros of a characteristic polynomial,

$$f(\lambda) = \det(A - \lambda I) = 0.$$

- Eigenvalues can be real or complex.
- Eigenspaces are the kernels of $A - \lambda I$. Eigenspaces are also subspaces of $\mathbb{R}^n/\mathbb{C}^n$.

$$E_{\lambda_k} = \ker(A - \lambda_k I).$$

- The dimension of an eigenspace E_{λ_k} is called geometric multiplicity for the eigenvalue λ_k.

Diagonalization

Comments

- For all eigenvalues of a matrix,

$$\text{Geometric Multiplicity} \leq \text{Algebraic Multiplicity}.$$

- A matrix is diagonalizable iff for all eigenvalues

$$\text{Geometric Multiplicity} = \text{Algebraic Multiplicity}.$$

- A matrix is not diagonalizable iff there exists a eigenvalue

$$\text{Geometric Multiplicity} < \text{Algebraic Multiplicity}.$$

Exercises

(*VV214_Final Exam Practice.*) Determine algebraic and geometric multiplicities for each eigenvalue. Check if the following matrices are diagonalizable and if yes, find their diagonal forms.

$$\begin{pmatrix} 3 & 2 \\ -5 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Exercises

Solution

1.

$$\begin{pmatrix} i\sqrt{\frac{2}{5}} & -i\sqrt{\frac{2}{5}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3-i\sqrt{10} & 0 \\ 0 & 3+i\sqrt{10} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{5}{2}}i & \frac{1}{2} \\ \frac{1}{2}\sqrt{\frac{5}{2}}i & \frac{1}{2} \end{pmatrix}$$

2.

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

Exercises

(*vv214_Assignment_7.*) Let A be a 3×3 matrix and let u, v and w be nonzero vectors with

$$Au = -5u, \qquad Av = 0, \qquad Aw = 5w.$$

1. Find eigenvalues of A^2.
2. Compute $\det A$ and $\det(A - 2I)$

Exercises

Solution

1. 0 and 25.
2. 0 and 42.

Comments

1. Eigenvalues of A^k are λ_i^k where λ_i is eigenvalue of A.
2. $\det(A - \lambda I) = f(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$
   where f is the characteristic polynomial for A.

Discrete Dynamical System

Exercise

(*vv214_Assignment_7.*) Two interacting populations of hares and foxes can be modeled by the recursive equations

$$\begin{aligned} h(t+1) &= 4h(t) - 2f(t) \\ f(t+1) &= h(t) + f(t). \end{aligned}$$

For the initial populations given by $f(0) = f, h(0) = h$, find closed formulas for $h(t)$ and $f(t)$.

Discrete Dynamical System

Solution

$$\begin{pmatrix} h(n) \\ f(n) \end{pmatrix} = \begin{pmatrix} -2^n + 2 \cdot 3^n & 2^{n+1} - 2 \cdot 3^n \\ -2^n + 3^n & 2^{n+1} - 3^n \end{pmatrix} \begin{pmatrix} h \\ f \end{pmatrix}$$

Cayley-Hamilton theorem

Comments

Any matrix satisfies its own characteristic polynomial,

$$f(A) = 0.$$

Exercises

(*VV214_Final Exam Practice.*) Find the inverse matrix A^{-1} using Cayley-Hamilton theorem

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$$

Cayley-Hamilton theorem

Exercises

(*VV214_Final Exam Practice.*) Simplify $-A^3 + 4A^2 + 3A - 4I$ with

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Cayley-Hamilton theorem

Method of order reduction

Find $\sinh A$ where

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$

Singular value

Comments

- The singular values of an $n \times m$ matrix A are the square roots of the eigenvalues of the symmetric $m \times m$ matrix $A^T A$.
- If A is an $n \times m$ matrix of rank r, then the singular values $\sigma_1, \cdots, \sigma_r$ are nonzero, while $\sigma_{r+1}, \cdots, \sigma_m$ are zero.

Singular value decomposition

Comments

$$A = U\Sigma V^T$$

- A: any $n \times m$ matrix,
- U: orthogonal $n \times n$ matrix,
- V: orthogonal $m \times m$ matrix,
- Σ: a $n \times m$ matrix, whose first $r = rank(A)$ diagonal entries are the nonzero singular values of A, and all other entries are zero.

Singular value decomposition

Comments

If we have

$$A = U\Sigma V^T,$$

then

$$A^T = (U\Sigma V^T)^T = (V^T)^T\Sigma^T U^T = V\Sigma^T U^T$$

Exercises

(*VV214_Final Exam Practice.*) Find SVD of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}.$$

Find dim(ker A), dim(imA), dim(ker A^T), dim(imA^T).

Singular value decomposition

Solution

1. $$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{15}} \\ \frac{1}{\sqrt{6}} & 0 & \sqrt{\frac{5}{6}} \end{pmatrix}^T$$

2. $1, 2, 0, 2$

Singular value decomposition

Exercises

(*vv214_Assignment_7.*) Let A be a matrix with the singular value decomposition

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

1. Find the characteristic polynomials and eigenvalues of AA^T and A^TA.
2. Find the largest possible value of $\|Av\|$, for the corresponding unit vectors v.
3. Sketch the image, under A, of the unit sphere in the corresponding linear space $\mathbb{R}^3$.

Singular value decomposition

Solution

1. A^TA: $3, 1, 0$, $f(\lambda) = -\lambda^3 + 4\lambda^2 - 3\lambda$;
   AA^T: $3, 1$, $f(\lambda) = \lambda^2 - 4\lambda + 3$.
2. $\|Av\| = \sqrt{3}$, $v = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}^T$

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Go over

- The textbook,
- Homework 7 and exam guild questions,
- Slides and exercises on recitation classes.