Sixth Recitation Class
Linear Algebra

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April 24, 2019

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Cayley-Hamilton Theorem

Theorem

Any $A \in M_{n\times n}(\mathbb{F})$ satisfies its own characteristic equation, i.e.

$$f_A(A) = (-A)^n + (\mathrm{tr}A)(-A)^{n-1} + \cdots + (\det A)I_n.$$

Exmpales

(i) Order reduction. For $m > n$,

$$g(A) = \sum_{k=0}^{m} a_k A^k = \sum_{k=0}^{n-1} b_k A^k.$$

(ii) Analytic function. For analytic function $g(x)$,

$$g(A) = \sum_{k=0}^{\infty} a_k A^k = \sum_{k=0}^{n-1} b_k A^k.$$

Cayley-Hamilton Theorem

Comment:

For the vector space defined to be the polynomial of matrix A, the set

$$\mathcal{A} = \{I_n, A, ..., A_{k-1}\}$$

seems to be a **basis**. Is this true?

Definition

The smallest degree polynomial $m_A(t) \neq 0$ such that $m_A(A) = 0$ is called the ***minimal polynomial*** of A.

Orthogonally Diagonalizable Matrices

Definition

A matrix A is ***orthogonally diagonalizable*** if there exists an orthonormal eigenbasis for A.

Comment:

Note that for the matrix that contains orthonormal basis in its column,

$$Q^{-1} = Q^T.$$

Then we have

$$A = Q \Lambda Q^T$$

where Λ is a diagonal matrix, as the eigenvalue decomposition of A.

Orthogonally Diagonalizable Matrices

Note:

Note that if A is orthogonally diagonalizable, then A is symmetric

$$A^T = (Q\Lambda Q^T)^T = (Q^T)^T \Lambda^T Q^T = Q\Lambda Q^T = A.$$

Is the converse true?

Theorem

(Spectral Theorem) A matrix A is orthogonally diagonalizable iff A is symmetric ($A^T = A$).

Proof

We have known that the eigenspaces for different eigenvalues of symmetric matrix are orthogonal to each other (Example in last RC class). It is sufficient to show that

$$A.M. \text{ of } \lambda = G.M. \text{ of } \lambda.$$

Orthogonally Diagonalizable Matrices

Proof(Contd.)

If λ is an eigenvalue, we assume there is an orthonormal basis $\{v_1, ..., v_r\}$ for E_λ, we extend this orthonormal basis to an orthonormal basis $\mathcal{B}$ for $\mathbb{F}^n$,

$$S = [v_1 \quad \cdots \quad v_n] .$$

Then we have the $\mathcal{B}$-matrix of A

$$\begin{aligned} S^{-1}AS &= [[Av_1]_{\mathcal{B}} \quad \cdots [Av_n]_{\mathcal{B}}] \\ &= \begin{bmatrix} \lambda I_r & 0 \\ C_1 & C_2 \end{bmatrix} . \end{aligned}$$

Here we used the block matrix notation and I_r is the $r \times r$ identity matrix.

Orthogonally Diagonalizable Matrices

Proof(Contd.)

We note that S is an orthogonal matrix, then

$$S^{-1} = S^T,$$

so we have $S^{-1}AS = S^TAS$. We recall that A is symmetric and have

$$(S^TAS)^T = S^TA^TS = S^TAS,$$

so the matrix S^TAS is symmetric. Furthermore, we have

$$C_1 = 0$$

which means

$$S^TAS = \begin{bmatrix} \lambda I_r & 0 \\ 0 & C_2 \end{bmatrix}.$$

Orthogonally Diagonalizable Matrices

Proof(Contd.)

We can assume that the algebraic multiplicity is greater than r, then

$$\begin{aligned} \det(A - \mu I_n) &= \det(S^T AS - \lambda I_n) \\ &= (\lambda - \mu)^r \det(C_2 - \mu I_{n-r}). \end{aligned}$$

We need $\det(C_2 - \lambda I_{n-r}) = 0$ to make the algebraic multiplicity of λ greater than r. There will be a vector $\overline{u} \in \ker(C_2 - \lambda I_{n-r})$ such that

$$C_2 \overline{u} = \lambda \overline{u}.$$

Orthogonally Diagonalizable Matrices

Proof(Contd.)

Now we can construct a new vector as the eigenvector of A with respect to eigenvalue λ,

$$\begin{aligned} AS\begin{bmatrix} 0 \\ \overline{u} \end{bmatrix} &= SS^TAS\begin{bmatrix} 0 \\ \overline{u} \end{bmatrix} \\ &= S\begin{bmatrix} \lambda I_r & 0 \\ 0 & C_2 \end{bmatrix}\begin{bmatrix} 0 \\ \overline{u} \end{bmatrix} \\ &= \lambda S\begin{bmatrix} 0 \\ \overline{u} \end{bmatrix} \end{aligned}$$

However, this vector is not an element of E_λ (Why?), this contradicts the assumption that E_λ is the eigenspace for λ.

Orthogonally Diagonalizable Matrices

Example

Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

Solution

We can find $\lambda_1 = 1, \lambda_2 = 5$, then

$$\overline{v_1} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad \overline{v_1} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

And we have

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Orthogonally Diagonalizable Matrices

Example

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with their multiplicities. Note that the algebraic multiplicity agrees with the geometric multiplicity.

Orthogonally Diagonalizable Matrices

Example

If an $n \times n$ matrix A is both symmetric and orthogonal, what can you say about the eigenvalues and eigenspaces of A? Interpret the linear transformation $T(\overline{x}) = A\overline{x}$ geometrically for the case $n = 3$ and $n = 2$.

Solution

We can write $A = Q\Lambda Q^T$, then

$$A^T A = Q\Lambda^2 Q^T = I_n.$$

We can conclude that $\Lambda^2 = I_n$. Since eigenvalues of A are real, they can be either 1 or -1.

Orthogonally Diagonalizable Matrices

Solution(Contd.)

For $n = 3$, we have four cases:

(i) $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$, the identity map.

(ii) $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$, reflection with respect to a plane through the origin.

(iii) $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -1$, rotation with 180° about a line through the origin.

(iv) $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1$, reflection with respect to the origin.

Orthogonally Diagonalizable Matrices

Example

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find an orthonormal eigenbasis for A. (Hint: Note that this is a permutation.)

Orthogonally Diagonalizable Matrices

Solution

Note that this matrix maps standard basis to

$$A\overline{e_1} = \overline{e_4}, A\overline{e_2} = \overline{e_3}, A\overline{e_3} = \overline{e_2}, A\overline{e_4} = \overline{e_1}.$$

Then we can pick eigenvalues as

$$\overline{v_1} = \frac{1}{\sqrt{2}}(e_1 + e_4), \quad \overline{v_2} = \frac{1}{\sqrt{2}}(e_1 - e_4),$$
$$\overline{v_3} = \frac{1}{\sqrt{2}}(e_2 + e_3), \quad \overline{v_4} = \frac{1}{\sqrt{2}}(e_2 - e_3).$$

Quadratic Form

Definition

A ***quadratic form*** is a function $q(\overline{x}) : \mathbb{R}^n \to \mathbb{R}$ with form

$$q(\overline{x}) = \sum_{i,j=1,\ldots,n} a_{ij} x_i x_j.$$

Comment:

A quadratic form can also be written as

$$q(\overline{x}) = (\overline{x}, A\overline{x}) = \overline{x}^T A\overline{x}.$$

Quadratic Form

Question:

For $\overline{x} = [x \; x^2 \; \cdots \; x^n]^T$, the quadratic form $q(\overline{x})$ is always zero iff A is skew-symmetric. Why? (Hint: consider $q(\overline{x}) = \sum a_{ij} x^i x^j$.)

Example

Consider a quadratic form

$$q(\overline{x}) = (\overline{x}, A\overline{x}),$$

where A is a symmetric $n \times n$ matrix.

(i) Find $q(\overline{e_1})$. Express the answer in terms of entries of A.

(ii) Find $q(\overline{v_1})$ if $\overline{v}$ is a unit eigenvector of A, with associated eigenvalue λ.