Second Recitation Class
Linear Algebra

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March 13, 2019

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Matrix Algebra

Definition

We define the sum of matrices and scalar multiples of matrices,

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{bmatrix}$$

and scalar multiples of matrices,

$$k \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1m} \\ \vdots & \ddots & \vdots \\ ka_{n1} & \cdots & ka_{nm}. \end{bmatrix}$$

Question:

Can you verify the set of all $n \times m$ matrices is a vector space?

Calculte $A\overline{x}$

Defition

Dot product between each ***row*** vector in A and $\overline{x}$.

Theorem

Linear combination of ***column*** vectors in A.

Theorem

Algebraic rule for $A\overline{x}$,

1. $A(\overline{x} + \overline{y}) = A\overline{x} + A\overline{y}$, and
2. $A(k\overline{x}) = k(A\overline{x})$.

Linear Transformations

Definition

A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a ***linear transformation*** if there exits an $n \times m$ matrix A such that

$$T(\overline{x}) = A\overline{x},$$

for all $\overline{x}$ in the vector space $\mathbb{R}^m$

Question:

Why we can always find a matrix A?

Example

Find the matrix I of identity map such that $Ix = x$ for $x \in \mathbb{R}^m$.

Find the matrix of linear transformation

Theorem

The matrix of a linear transformation T form $\mathbb{R}^m$ to $\mathbb{R}^n$ is

$$A = \begin{bmatrix} | & | & \cdots & | \\ T(\overline{e_1}) & T(\overline{e_1}) & \cdots & T(\overline{e_m}) \\ | & | & \cdots & | \end{bmatrix},$$

here we have $\overline{e_i}$ be the ith standard basis in $\mathbb{R}^m$.

Quesiton:

There is a subtle problem. This theorem is to find a matrix A for T, but we need a A to make it a linear transformation first.

Redefine Linear Transformation

Theorem

A transformation T form $\mathbb{R}^m$ to $\mathbb{R}^n$ is linear if and only if

1. $T(\overline{v} + \overline{w}) = T(\overline{v}) + T(\overline{w})$, for all vectors $\overline{v}$ and $\overline{w}$ in $\mathbb{R}^m$, and
2. $T(k\overline{v}) = kT(\overline{v})$, for all vectors $v \in \mathbb{R}^m$ and all scalars k.

Proof.

With these two properties, we can expand $T(\overline{x})$ and find A using previous theorem.

Comment:

We notice that there are two equivalent definition of a linear transformation.

1. These is an $n \times m$ matrix A which satisfies $T(\overline{x}) = A\overline{x}$.
2. The transformation T is linear.

Linear Transformations in Geometry

Summary

In this part, we focus on linear transformations defined on $\mathbb{R}^2$ or $\mathbb{R}^3$. The main task is to find the matrix of a linear transformation in geometry.

Procedure

1. Verity this linear transformation in geometry in linear.
2. Find the output of standard basis e_i, $i = \overline{1, m}$.
3. Write down the matrix as

$$A = \begin{bmatrix} | & | & \cdots & | \\ T(\overline{e_1}) & T(\overline{e_1}) & \cdots & T(\overline{e_m}) \\ | & | & \cdots & | \end{bmatrix},$$

Scaling

Definiton

Scaling of $\overline{x}$ with factor k is the map such that

$$\overline{x} \to k\overline{x}.$$

Solution

This transformation is linear from geometry.
For input e_1 and e_2, we have

$$e_1 \to ke_1, \quad e_2 \to ke_2,$$

here we suppose the scalar is k.
Then the matrix is

$$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$$

Orthogonal Projections

Definition

Orthogonal projection of $\overline{x}$ on L is $proj_L(\overline{x})$ along L such that

$$(\overline{x} - proj_L(\overline{x})) \perp L.$$

Solution

Take a vector $\overline{w}$ along line L, the orthogonal projection of $\overline{x}$ on L is

$$proj_L(\overline{x}) = (\frac{\overline{x} \cdot \overline{w}}{\overline{w} \cdot \overline{w}})\overline{w}.$$

If we take $\overline{u} = \overline{w}/\|\overline{w}\|$, then

$$proj_L(\overline{x}) = (\overline{x} \cdot \overline{u})\overline{u}.$$

This transformation is linear.

Solution(Contd).

For input $\overline{e_1}$ and $\overline{e_2}$, we have

$$\overline{e_1} \rightarrow \begin{pmatrix} u_1^2 \\ u_1 u_2 \end{pmatrix}, \quad \overline{e_2} \rightarrow \begin{pmatrix} u_2 u_1 \\ u_2^2 \end{pmatrix}.$$

Then the matrix is

$$A = \begin{bmatrix} u_1^2 & u_2 u_1 \\ u_1 u_2 & u_2^2 \end{bmatrix}$$

Reflections

Definiotn

The reflection of $\overline{x}$ with respect to line L is $ref_L(\overline{x})$ such that

$$\overline{x} - proj_L(\overline{x}) = -(ref_L(\overline{x}) - proj_L(\overline{x})).$$

Solution

We can find $ref_L(\overline{x})$ using the orthogonal projection,

$$ref_L(\overline{x}) = 2proj_L(\overline{x}) - \overline{x}.$$

This is a linear transformation.

Solution(Contd.)

For input e_1 and e_2,

$$e_1 \to \binom{2u_1^2-1}{2u_1u_2}, \quad e_2 \to \binom{2u_2u_1}{2u_2^2-1}.$$

Then

$$A = \begin{bmatrix} 2u_1^2 - 1 & 2u_2u_1 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix},$$

and we can write A as

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad a^2 + b^2 = 1.$$

Question:

Why we have this form? (Hint: Consider $T(\overline{e_1})$ and $T(\overline{e_2})$ directly.)

Projection and Reflection

Example

Find a nonzero matrix A such that

$$A^2 = A,$$

and a nonzero matrix B such that

$$B^3 = B.$$

Orthogonal Projections and Reflections in Space

Defintion

In $\mathbb{R}^3$, orthogonal projections and reflections are defined similar to $\mathbb{R}^2$.

Example

Find the matrix of reflection with respect to plane $2x_1 + x_2 - 2x_3 = 0$.

Example

Find the matrix of orthogonal projection on plane $2x_1 + x_2 - 2x_3 = 0$.

Rotation

Definition

The rotation in $\mathbb{R}^2$ with angle θ is a rotation centered at the origin in the counterclockwise direction.

Theorem

The matrix of a counterclockwise rotation in $\mathbb{R}^2$ through an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Rotation

Example

Check that if $A(\theta_1), A(\theta_2)$ is the matrix of counterclockwise rotation with angle θ_1, θ_2, then

$$A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2).$$

Rotations Combined with a Scaling

For a scaling with scalar k and a counterclockwise rotation with angle θ, the combination of these two transformation is

$$\begin{bmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{bmatrix}.$$

Note that these two transformations are commutative.

Theorem

A matrix of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

represents a rotation combined with a scaling.

Shears

Definition

A horizontal shear keep x_2 unchanged but change x_1 to $x_1 + kx_2$.
A vertical shear keep x_1 unchanged but change x_2 to $kx_1 + x_2$.

Theorem

The matrix of a horizontal shear is of the form

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix},$$

and the matrix of a vertical shear is of the form

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix},$$

where k is an arbitrary constant.

Shears

Example

If A is a shear matrix, what is $(A - I_2)^2$? Consider $A^2\overline{x} - 2A\overline{x} + \overline{x}$.

Matrix Products

Motivation

The product of matrices is motivated by the composition of linear transformation. For linear transformations $T_2 : \mathbb{R}^m \to \mathbb{R}^k$ and $T_1 : \mathbb{R}^k \to \mathbb{R}^n$, we want to find the matrix of linear transformation $T_1 \circ T_2 : \mathbb{R}^m \to \mathbb{R}^n$, defined as

$$(T_1 \circ T_2)(\cdot) = T_1(T2(\cdot)).$$

Question:

The composition of linear transformations is still linear transformation. Why?

Matrix Products

Solution

If the matrices of T_1, T_2 are A, B, we then have

$$T_1(T_2(e_j)) = A(Be_j) = Ab_j,$$

here e_j is the jth standard basis of $\mathbb{R}^m$, and the matrix of $T_1(T_2(\cdot))$ is

$$\begin{bmatrix} | & | & \cdots & | \\ Ab_1 & Ab_2 & \cdots & Ab_k \\ | & | & \cdots & | \end{bmatrix},$$

here b_j is the jth column of matrix B, $j = \overline{1, m}$.

Matrix Product

Example

Find the matrix product

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Alternative Ways to Calculate Matrix Product

We can expand Ab_j in two ways.

Expand by Row (1)

If the ith row of A is a_i', then

$$Ab_j = \begin{bmatrix} a_1' \cdot b_j \\ \vdots \\ a_n' \cdot b_j \end{bmatrix}.$$

Then the product of A and B is

$$AB = \begin{bmatrix} a_1' \cdot b_1 & \cdots & a_1' \cdot b_m \\ \vdots & \ddots & \vdots \\ a_n' \cdot b_1 & \cdots & a_n' \cdot b_m \end{bmatrix}.$$

Alternative Ways to Calculate Matrix Product

Expand by Row (2)

Consider the product of a row vector and a matrix

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{k1} & \cdots & b_{km} \end{bmatrix} = \begin{bmatrix} x_1 b_{11} & & x_1 b_{1m} \\ + & & + \\ \vdots & , \cdots , & \vdots \\ + & & + \\ x_k b_{k1} & & x_k b_{km} \end{bmatrix}.$$

If we denote the ith row of B as b_i', then

$$xB = \sum_{i=1}^{k} x_i b_i'.$$

This inspired us to use row vectors to calculate matrix product,

$$AB = \begin{bmatrix} \sum_{i=1}^{k} a_{1i} b_i' \\ \vdots \\ \sum_{i=1}^{k} a_{ni} b_i' \end{bmatrix} = \begin{bmatrix} a_1' B \\ \vdots \\ a_n' B \end{bmatrix}.$$

The result is individual ***row*** of A times B.

Alternative Ways to Calculate Matrix Product

Exmaple

Find the matrix product

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Alternative Ways to Calculate Matrix Product

Expand by Column (1)

If the ith column of A is a_i, then

$$Ab_j = \sum_{i=1}^{k} a_i b_{ij}.$$

Expand by Column (2)

Inspired by expansion by column, we can split the value that only relate to a_i,

$$AB = \sum_{i=1}^{k} \begin{bmatrix} | & | & \cdots & | \\ a_i b_{i1} & a_i b_{i2} & \cdots & a_i b_{im} \\ | & | & \cdots & | \end{bmatrix} = \sum_{i=1}^{k} a_i b_i'.$$

This gives us a way to consider the matrix as the sum of product of ***column vector in*** A and ***row vector in*** B.

Properties of Matrix Product

Row Times Column

We have defined the product of a row vector and a column vector as

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$$

Column Times Row

For a column vector times a row vector, we use the definition of matrix product,

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} = [\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} x_1, \cdots, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} x_m] = \begin{bmatrix} x_1 y_1 & \cdots & x_m y_1 \\ \vdots & \ddots & \vdots \\ x_1 y_n & \cdots & x_m y_y \end{bmatrix}.$$

Properties of Matrix Product

Associativity

The product between matrices is associative, i.e.,

$$(AB)C = A(BC).$$

Commutative

The product between matrices is generally not commutative,

$$AB \neq BA.$$

Question: Can you find a matrix D that is commutative to any matrix?

Block Matrix

Theorem

If we separate A and B into blocks with appropriate sizes,

$$
\begin{aligned}
AB &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\
&= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}
\end{aligned}
$$

Gass-Jordan in Matrix Product

Motivation

Each operation we perform in Gauss-Jordan elimination is to times a matrix on the original system.

$$E_n \cdots E_2 E_1 A = I$$

Example

Find all E to perform Gauss-Jordan on

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix}.$$

Inverse of Matrix

Definition

For a linear transformation T, T^{-1} is its inverse if

$$T(T^{-1}(\cdot)) = id(\cdot), \qquad \text{and} \qquad T^{-1}(T(\cdot)) = id(\cdot),$$

here $id(\cdot)$ is the identity map.

Definion

The inverse of matrix A is the matrix of the inverse of its linear transformation, we denote it A^{-1}.

Theorem

The inverse of matrix A has properties

$$A^{-1}A = AA^{-1} = I.$$

Here I is the identity matrix.

Find the Inverse of a Matrix

Motivation

Recall in Gauss-Jordan, we find a series of matrices $E_n, ..., E_1$ such that

$$E_n \cdots E_2 E_1 A = I,$$

then the product of $E_n \cdots E_1$ is the inverse of A.

Comment:

Note that this process may not always work.

Inverse of Matrix

Solution

To calculate easily, we write down the matrices A and I in an augmented matrix,

$$\begin{bmatrix} A & I \end{bmatrix}$$

when we times do row manipulations on A, the same operation is performed on I, so

$$\begin{aligned} &\begin{bmatrix} E_n \cdots E_1 A & E_n \cdots E_1 I \end{bmatrix} \\ =&\begin{bmatrix} I & A^{-1} \end{bmatrix} \end{aligned}$$

Invertibility

Note that we may not always find the inverse of a matrix.

Theorem

A $n \times n$ matrix has inverse if and only if

- $\text{rank}A = n$,
- $rref(A) = I_n$.

Comment:

Note that the inverse exists if Gauss-Jordan elimination works.

Theorem

If B and A are $n \times n$ matrices such that

$$BA = I_n,$$

then B and A are both invertible and $A^{-1} = B, B^{-1} = A$.

Properties of Invertible Matrices

Theorem

If BA is invertible and A, B are $n \times n$ matrices, we have

$$(BA)^{-1} = A^{-1}B^{-1}.$$